



Existence and uniqueness of weak solutions to the singular kernels coagulation equation with collisional breakage

Prasanta Kumar Barik and Ankik Kumar Giri

Abstract. The continuous coagulation equation with collisional breakage explains the dynamics of particle growth when particles experience binary collisions to form either a single particle via coalescence or two/more particles via breakup with possible transfer of matter. Each of these processes may take place with a suitably assigned probability depending on the volume of particles participating in the collision. In this article, global weak solutions to the continuous coagulation equation with collisional breakage are formulated to the collision kernels and distribution functions admitting a singularity near the origin. In particular, the proof relies on a classical weak L^1 compactness method applied to suitably chosen approximate equations. The question of uniqueness is also contemplated under more restricted class of collision kernels.

Mathematics Subject Classification. Primary: 45K05, 45G99; Secondary: 34K30.

Keywords. Collisional breakage, Existence, Uniqueness, Mass conservation, Compactness.

1. Introduction

Coagulation-fragmentation equations (CFEs) are used as models that describe the dynamics of many physical phenomena in which two or more particles can aggregate via a collision between particles to form bigger ones or break into smaller pieces. The coagulation event occurs in different chemical, biological and physical processes such as colloidal aggregation, aggregation of red blood cells and polymerization, for instance. But, in the breakage process, at least two different cases arise that depend on the breakage behaviour of particles. The breakage of particles may happen either due to the collision between a pair of particles named as *nonlinear breakage/collision-induced breakage* or due to

other than the interaction between particles (external forces or spontaneously) known as the *linear breakage*. The linear breakage may take place due to the particle-wall interaction, chemical reactions or shear fluid whereas a few examples for the occurrence of the nonlinear breakage are the rain droplet breakage, the formation of stars and planets etc. In this article, we have considered the continuous coagulation and nonlinear collisional breakage processes.

The continuous coagulation equation with collisional breakage [5, 7, 22, 25, 27] for the concentration $g = g(\mu, t)$ of particle of volume $\mu \in \mathbb{R}_+ := (0, \infty)$ at time $t \geq 0$ reads as

$$\frac{\partial g}{\partial t} = \mathcal{B}_c(g) - \mathcal{D}_{cb}(g) + \mathcal{B}_b(g), \tag{1.1}$$

with the following initial data

$$g(\mu, 0) = g_{in}(\mu) \geq 0 \text{ a.e.}, \tag{1.2}$$

where

$$\begin{aligned} \mathcal{B}_c(g)(\mu, t) &:= \frac{1}{2} \int_0^\mu E(\mu - \nu, \nu) \Psi(\mu - \nu, \nu) g(\mu - \nu, t) g(\nu, t) d\nu, \\ \mathcal{D}_{cb}(g)(\mu, t) &:= \int_0^\infty \Psi(\mu, \nu) g(\mu, t) g(\nu, t) d\nu, \\ \mathcal{B}_b(g)(\mu, t) &:= \frac{1}{2} \int_\mu^\infty \int_0^\nu P(\mu|\nu-\tau; \tau) F(\nu-\tau, \tau) \Psi(\nu-\tau, \tau) g(\nu-\tau, t) g(\tau, t) d\tau d\nu, \end{aligned}$$

and $E(\mu, \nu) + F(\mu, \nu) = 1$ with $0 \leq E(\mu, \nu) = E(\nu, \mu), F(\mu, \nu) = F(\nu, \mu) \leq 1$ for all $(\mu, \nu) \in \mathbb{R}_+^2$.

The first term, \mathcal{B}_c of (1.1) describes the formation of particles of volume μ by coalescence, and the last term, \mathcal{B}_b of (1.1) represents the birth of particles of volume μ due to the collisional breakage. The factor 1/2 appears in \mathcal{B}_c and \mathcal{B}_b to neglect the double counting for the formation of particles of volume μ due to both coalescence and collisional breakage events respectively. The second term \mathcal{D}_{cb} of (1.1) shows the death of particles of volume μ due to both coalescence and collisional breakage. On the other hand, the term \mathcal{D}_{cb} can be expressed as

$$\begin{aligned} \mathcal{D}_{cb}(g)(\mu, t) &:= \int_0^\infty E(\mu, \nu) \Psi(\mu, \nu) g(\mu, t) g(\nu, t) d\nu \\ &\quad + \int_0^\infty F(\mu, \nu) \Psi(\mu, \nu) g(\mu, t) g(\nu, t) d\nu. \end{aligned}$$

Here, the collision kernel $\Psi(\mu, \nu)$ accounts for the rate at which a particle of volume μ and a particle of volume ν collide which is symmetric with respect to μ and ν and a non-negative measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$. Since each collision must result in either coalescence or breakup. Thus, let $E(\mu, \nu)$ denotes the probability that the two colliding particles of volumes μ and ν aggregate into a single one of volume $\mu + \nu$ whereas $F(\mu, \nu)$ describes the probability that the two colliding particles of volumes μ and ν break into two or more daughter particles with possible transfer of mass or elastic collisions between two fragments during the collision. The distribution function $P(\mu|\nu; \tau)$ describes the

contribution for particles of volume μ produced from the collisional breakage event arising from the interaction between a pair of particles of volumes ν and τ which is also a nonnegative symmetric function in nature with respect to second and third variables, i.e. $P(\mu|\nu;\tau) = P(\mu|\tau;\nu) \geq 0$. Clearly, this distribution function satisfies

$$P(\mu|\nu;\tau) \begin{cases} = 0, & \text{if } \mu > \nu + \tau, \\ \geq 0, & \text{if } 0 < \mu \leq \nu + \tau. \end{cases}$$

A necessary condition for the mass conservation property during the collisional breakage event is given by

$$\int_0^{\nu+\tau} \mu P(\mu|\nu;\tau) d\mu = \nu + \tau, \quad \text{for all } \nu > 0 \text{ and } \tau > 0. \quad (1.3)$$

In addition, let us mention another property of the distribution function providing the total number of daughter particles $N > 0$ resulting from the collisional breakage process is

$$\int_0^{\nu+\tau} P(\mu|\nu;\tau) d\mu = N, \quad \text{for all } \nu > 0 \text{ and } \tau > 0, \quad (1.4)$$

where N is the size independent of ν and τ .

Now, let us consider some special cases arising from the continuous coagulation and collisional breakage equation.

- If $E \equiv 1$, then Eq. (1.1) converts into the classical continuous *Smoluchowski coagulation equation*, see [6, 9, 16, 21].
- By substituting $P(\mu|\nu;\tau) = \chi_{[\mu,\infty)}(\nu)B(\mu|\nu;\tau) + \chi_{[\mu,\infty)}(\tau)B(\mu|\tau;\nu)$ and $E \equiv 0$ into (1.1), it can easily be seen that (1.1) is transformed into the *pure nonlinear breakage equation*, see [11–13, 17].

Next, it is important to mention some physical properties i.e. moments of the concentration of particles. Let \mathcal{M}_q denotes the q^{th} moment of the concentration g of particles which is defined as

$$\mathcal{M}_q(t) = \mathcal{M}_q(g(\mu, t)) := \int_0^\infty \mu^q g(\mu, t) d\mu, \quad \text{where } q \in (-\infty, \infty).$$

The total number of particles and the total mass of particles are denoted by \mathcal{M}_0 and \mathcal{M}_1 , respectively. In coagulation process, the total number of particles, \mathcal{M}_0 decreases whereas in collisional breakage process, \mathcal{M}_0 increases with time. However, the total mass (volume) of the system may or may not be conserved during the coagulation and collisional breakage processes that depends on the nature of the coagulation kernel ($E\Psi$) and breakup kernel ($F\Psi$). It is worth to mention that the negative moments are also very useful in handling the case of some physical singular collision kernels such as Smoluchowski collision kernel for Brownian motion and Granulation kernel for fluidized bed etc. which have been discussed in [8–10, 14, 21].

1.1. Literature overview

Before getting into more details of the present work, let us first recall available literature related to the coagulation equation with collisional breakage. There is a vast literature available on the well-posedness of the continuous coagulation and linear breakage equations (CLBEs). In [4, 15, 23, 24], the authors have discussed the existence and uniqueness of solutions to the continuous CLBEs with nonsingular coagulation kernels under different growth conditions on fragmentation kernels, whereas in [3, 9, 10, 16, 21], the existence and uniqueness of solutions to the continuous CLBEs have been established for singular coagulation kernels. More precisely, the existence of self-similar solutions have been discussed in [16, 21] whereas, in [9, 10], respectively, the existence and uniqueness of weak solutions to Smoluchowski coagulation equations and CLBEs have been shown. However, there are a few number of articles in which the collisional breakage or nonlinear fragmentation model have been considered, see [11–13, 17]. In these articles, authors have demonstrated scaling solutions as well as asymptotic behaviour of solutions to the *pure nonlinear breakage models*. Moreover, they have also found analytical solutions for some specific collision and breakup kernels. In 1972, Safronov has proposed a new kinetic model which is known as the *continuous coagulation and collisional breakage model*, i.e. (1.1)–(1.2), see [22] which has been further studied by Wilkins in [27]. The Eq. (1.1) becomes the continuous nonlinear fragmentation model if $E \equiv 0$. In 2001, Laurençot and Wrzosek [20] have discussed the existence and uniqueness of weak solutions to the discrete coagulation and collisional breakage model. The existence proof was based on a weak L^1 compactness argument. They have also studied mass conservation, gelation and large time behaviour of solutions. Recently, in [5], Barik and Giri have shown the existence of weak solutions for a particular classes of nonsingular unbounded collision kernels. The main novelty of the present work is to include the singularity for small volume particles in the collision kernel in the existence and uniqueness results to the continuous coagulation and collisional breakage models. Here, the proof of the uniqueness result is motivated by [24].

In order to prove the existence result, we first consider the following basic assumptions on the collision kernel Ψ , initial data g_{in} , probability E and the distribution function P . Assume that the collision kernel Ψ satisfies

$$\Psi(\mu, \nu) \leq k \frac{(1 + \mu)^\omega (1 + \nu)^\omega}{(\mu + \nu)^\sigma} \quad \text{for all } (\mu, \nu) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad (1.5)$$

where $\sigma \in (0, 1/2)$, $0 \leq \omega < 1$ and some constant $k \geq 0$.

Next, we assume that the probability E enjoys the following relation for small volume particles

$$\frac{\eta(r) - 2}{\eta(r) - 1} \leq E(\mu, \nu) \leq 1, \quad \forall (\mu, \nu) \in (0, 1) \times (0, 1), \quad (1.6)$$

where $\eta(r) \geq N \geq 2$, for $r = 0, \sigma, 2\sigma$ and N total number of fragments is obtained after the collision between a pair of particles given in (1.4).

We further state the following four assumptions on the distribution function P : recalling $\eta(r)$ and r from (1.6), we have

$$\int_0^\nu \mu^{-r} P(\mu|\nu - \tau; \tau) d\mu \leq \eta(r) \nu^{-r}. \tag{1.7}$$

Next, for each $\lambda > 0$ and $\nu \in (0, \lambda)$ and any small measurable subset A of $(0, \lambda)$ with $|A| \leq \delta$, there exists $\theta_1 \in (0, 2\sigma]$ such that

$$\int_0^\nu \chi_A(\mu) \mu^{-\sigma} P(\mu|\nu - \tau; \tau) d\mu \leq \Omega_1(\lambda, \delta) \nu^{-\theta_1}, \quad \text{where } \lim_{\delta \rightarrow 0} \Omega_1(\lambda, \delta) = 0. \tag{1.8}$$

Here, $|A|$ denotes the Lebesgue measure of A and χ_A is the characteristic function on a set A . For $\nu + \tau > \lambda$ and $\mu \in (0, \lambda)$ for some $\tau_2 \in [0, 1 - \sigma)$, such that P satisfies

$$P(\mu|\nu; \tau) \leq k'(\lambda) \mu^{-\tau_2}, \quad \text{where } k'(\lambda) > 0. \tag{1.9}$$

For the proof of the uniqueness of weak solutions, we need the following further restriction on collision kernel Ψ given in (1.5)

$$\Psi(\mu, \nu) \leq \frac{k}{(\mu + \nu)^\sigma} \quad \text{for } k \geq 0 \quad \text{and } \sigma \in (0, 1/2). \tag{1.10}$$

Finally, let us suppose that the initial data g_{in} satisfies

$$g_{\text{in}} \in \mathcal{S}^+, \tag{1.11}$$

where \mathcal{S}^+ is the positive cone of the Banach space

$$\mathcal{S} := L^1(\mathbb{R}_+; (1 + \mu + \mu^{-2\sigma}) d\mu),$$

endowed with the norm

$$\|g\|_{\mathcal{S}} := \int_0^\infty (1 + \mu + \mu^{-2\sigma}) |g(\mu)| d\mu.$$

It can easily be seen that \mathcal{S} is a Banach space with respect to norm $\|\cdot\|_{\mathcal{S}}$, see [8].

This paper is organized as follows: In Sect. 2, some assumptions on the Ψ and g_{in} , and the definition of solution together with the main results of this article are stated. By using a weak L^1 compactness method, the existence of weak solutions to (1.1)–(1.2) is established in Sect. 3. In the last section, a uniqueness result of weak solutions to (1.1)–(1.2) is demonstrated for a class of further restricted collision kernels.

2. Preliminaries and results

Let us start this section by formulating the notion of weak solutions to (1.1)–(1.2) by means of the following definition.

Definition 2.1. A solution g of (1.1)–(1.2) is a non-negative continuous function $g : [0, T) \rightarrow \mathcal{S}^+$ such that, for a.e. $\mu \in \mathbb{R}_+$ and all $t \in [0, T)$,

(i) the following integrals are finite

$$\int_0^t \int_0^\infty \Psi(\mu, \nu)g(\nu, s)d\nu ds < \infty, \quad \text{and} \quad \int_0^t \mathcal{B}_b(g)(\mu, s)ds < \infty,$$

(ii) the function g satisfies the following weak formulation of (1.1)–(1.2)

$$g(\mu, t) = g_{\text{in}}(\mu) + \int_0^t \{\mathcal{B}_c(g)(\mu, s) - \mathcal{D}_{cb}(g)(\mu, s) + \mathcal{B}_b(g)(\mu, s)\}ds,$$

where $T \in (0, \infty]$.

Let us consider the following example of collision kernel which satisfies (1.5), see [1]:

$$\Psi(\mu, \nu) = k(\mu^{1/3} + \nu^{1/3})(\mu\nu)^{1/2}(\mu + \nu)^{-3/2}.$$

Next, we consider the following distribution function

$$P(\mu|\nu; \tau) = (\theta + 2) \frac{\mu^\theta}{(\nu + \tau)^{\theta+1}}, \quad \text{where} \quad -2 < \theta \leq 0 \quad \text{and} \quad \mu < \nu + \tau.$$

By inserting the above value of P into (1.4), it can easily be observed that we get the finite number of daughter particles, only if $-1 < \theta \leq 0$, which is denoted by N and written as $N = \frac{\theta+2}{\theta+1}$. In particular, for $\theta = 0$, we have $P(\mu|\nu; \tau) = \frac{2}{\nu}$, which gives the case of binary breakage, i.e. $N = 2$, once substituted in (1.4). On the other hand, for $\theta \in (-2, -1]$, we get an infinite number of daughter particles. The result presented in this article deals with the case of finitely many daughter particles. Therefore, it is meaningful to consider $\theta \in (2\sigma - 1, 0]$ for our settings. Now, let us verify (1.7)–(1.9) by considering the above example of distribution function P . First, we check (1.7) as

$$\begin{aligned} \int_0^\nu \mu^{-r} P(\mu|\nu - \tau; \tau) d\mu &= \frac{(\theta + 2)}{\nu^{\theta+1}} \int_0^\nu \mu^{\theta-r} d\mu \\ &= \frac{(\theta + 2)}{\nu^{\theta+1}} \left. \frac{\mu^{\theta-r+1}}{(\theta - r + 1)} \right|_0^\nu, \quad \text{provided that } \theta - r + 1 > 0 \\ &= \frac{(\theta + 2)}{\nu^{\theta+1}} \frac{\nu^{\theta-r+1}}{(\theta - r + 1)} = \frac{(\theta + 2)}{(\theta - r + 1)} \nu^{-r} = \eta(r) \nu^{-r}, \end{aligned}$$

where $\eta(r) = \frac{(\theta+2)}{(\theta-r+1)}$.

Next, applying Hölder’s inequality, for $p > 1$, to verify (1.8) as

$$\begin{aligned} &\int_0^\nu \chi_A(\mu) \mu^{-\sigma} P(\mu|\nu - \tau; \tau) d\mu \\ &\leq (\theta + 2) \frac{1}{\nu^{\theta+1}} |A|^{\frac{p-1}{p}} \left[\int_0^\nu \mu^{p(\theta-\sigma)} d\mu \right]^{\frac{1}{p}} \\ &= \frac{(\theta + 2)}{(p(\theta - \sigma) + 1)^{1/p}} \frac{1}{\nu^{\theta+1}} |A|^{\frac{p-1}{p}} \nu^{\theta-\sigma+1/p} \quad \text{where } p(\theta - \sigma) + 1 > 0 \\ &= \frac{(\theta + 2)}{(p(\theta - \sigma) + 1)^{1/p}} |A|^{\frac{p-1}{p}} \nu^{-\theta_1} = \Omega_1(\delta) \nu^{-\theta_1}, \end{aligned}$$

where $\theta_1 := 1 + \sigma - \frac{1}{p} \in (0, 2\sigma]$.

In order to check (1.9), we have

$$P(\mu|\nu; \tau) = (\theta + 2) \frac{\mu^\theta}{(\nu + \tau)^{\theta+1}} \leq (\theta + 2) \frac{\mu^\theta}{\lambda^{1+\theta}} \leq k'(\lambda)\mu^{-\tau_2} \text{ for } \lambda < \nu + \tau,$$

where $\tau_2 = -\theta \in [0, 1 - \sigma)$ and $k'(\lambda) \geq \frac{\theta+2}{\lambda^{1+\theta}}$.

Now, we are in a position to state the following existence and uniqueness results:

Theorem 2.2. *Assume that (1.5)–(1.9) hold. Then, for $g_{in} \in \mathcal{S}^+$, there exists a weak solution g to (1.1)–(1.2) on $[0, \infty)$.*

Theorem 2.3. *Suppose $g_{in} \in \mathcal{S}^+$. Assume that the collision kernel Ψ satisfies (1.10) and (1.6)–(1.9) hold. Then, (1.1)–(1.2) has a unique weak solution on $[0, \infty)$.*

3. Existence of weak solutions

In this section, we first, construct a sequence of functions (Ψ_n) with compact support for each $1 < n \in \mathbb{N}$, such that

$$\Psi_n(\mu, \nu) := \begin{cases} \Psi(\mu, \nu), & \text{if } \mu + \nu < n \text{ and } \mu, \nu \geq 1/n, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Next, we may argue as in [5, Proposition 1], by using the upper bound of $\Psi_n (\leq 4kn^{2+\sigma})$ from (3.1) or [26], to show that the truncated equation

$$\frac{\partial g^n}{\partial t} = \mathcal{B}_c^n(g^n) - \mathcal{D}_{cb}^n(g^n) + \mathcal{B}_b^n(g^n), \tag{3.2}$$

with initial data

$$g_0^n(\mu) := \begin{cases} g_{in}(\mu), & \text{if } 0 < \mu < n, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

where

$$\mathcal{B}_c^n(g^n)(\mu, t) := \frac{1}{2} \int_0^\mu E(\mu - \nu, \nu) \Psi_n(\mu - \nu, \nu) g^n(\mu - \nu, t) g^n(\nu, t) d\nu,$$

$$\mathcal{D}_{cb}^n(g^n)(\mu, t) := \int_0^{n-\mu} \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu,$$

$$\begin{aligned} \mathcal{B}_b^n(g^n)(\mu, t) &:= \frac{1}{2} \int_\mu^n \int_0^\nu P(\mu|\nu - \tau; \tau) F(\nu - \tau, \tau) \\ &\times \Psi_n(\nu - \tau, \tau) g^n(\nu - \tau, t) g^n(\tau, t) d\tau d\nu, \end{aligned}$$

has a unique non-negative solution g^n for each $n > 1$. These family of solutions $(g^n)_{n>1}$ lie in $\mathcal{C}^1([0, \infty); L^1((0, n), d\mu))$. Additionally, it enjoys the mass conserving property for all $t > 0$, i.e.

$$\int_0^n \mu g^n(\mu, t) d\mu = \int_0^n \mu g_{in}^n(\mu) d\mu. \tag{3.4}$$

Furthermore, we extend the domain of truncated unique solution g^n by zero in $\mathbb{R}_+ \times [0, \infty)$ as

$$g^n(\mu, t) := \begin{cases} \hat{g}^n(\mu, t), & \text{if } 0 < \mu < n, \\ 0, & \text{if } \mu \geq n, \end{cases} \tag{3.5}$$

for $n > 1$ and $n \in \mathbb{N}$. For notational convenient, we drop the $\hat{\cdot}$ of g^n .

In the coming section, we wish to apply a classical weak compactness technique for the family of solutions (g^n) to obtain required weak solutions.

3.1. Weak compactness

Here we first show the equi-boundedness of the family $(g^n) \subset \mathcal{S}^+$ by means of the following lemma.

Lemma 3.1. *Given any $T > 0$. Then the following holds*

$$\int_0^n (1 + \mu + \mu^{-2\sigma})g^n(\mu, t)d\mu \leq \mathcal{P}(T) \text{ for each } n > 1 \text{ and all } t \in [0, T],$$

where $\mathcal{P}(T)$ is a positive constant depending on $T, \eta(2\sigma)$ and g_{in} .

Proof. Let $t \in [0, T]$ and $1 < n \in \mathbb{N}$. Now, using (3.4), (3.3) and $g_{in} \in \mathcal{S}^+$, we estimate the following integral as

$$\int_0^n (1 + \mu + \mu^{-2\sigma})g^n(\mu, t)d\mu \leq 2 \int_0^1 \mu^{-2\sigma} g^n(\mu, t)d\mu + 3\|g_{in}\|_{\mathcal{S}}. \tag{3.6}$$

Integrating (3.2) with respect to the volume variable μ from 0 to 1, after multiplying $\mu^{-2\sigma}$, we find

$$\frac{d}{dt} \int_0^1 \mu^{-2\sigma} g^n(\mu, t)d\mu = \int_0^1 \mu^{-2\sigma} \{ \mathcal{B}_c^n(g^n)(\mu, t) - \mathcal{D}_{cb}^n(g^n)(\mu, t) + \mathcal{B}_b^n(g^n)(\mu, t) \} d\mu. \tag{3.7}$$

Next, we estimate each integral on the right-hand side to (3.7), individually. Using Fubini’s theorem, the transformation $\mu - \nu = \mu'$ and $\nu = \nu'$, then the first term on the right-hand side of (3.7) can be written as

$$\begin{aligned} & \int_0^1 \mu^{-2\sigma} \mathcal{B}_c^n(g^n)(\mu, t)d\mu \\ &= \frac{1}{2} \int_0^1 \int_0^{1-\mu} (\mu + \nu)^{-2\sigma} E(\mu, \nu) \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu d\mu. \end{aligned} \tag{3.8}$$

Again, using Fubini’s theorem, the third term on the right-hand side of (3.7) can be simplified as

$$\begin{aligned} & \int_0^1 \mu^{-2\sigma} \mathcal{B}_b^n(g^n)(\mu, t)d\mu \\ &= \frac{1}{2} \int_0^1 \int_0^\nu \int_0^\nu \mu^{-2\sigma} P(\mu|\nu - \tau; \tau) F(\nu - \tau, \tau) \Psi_n(\nu - \tau, \tau) \\ & \quad \times g^n(\nu - \tau, t) g^n(\tau, t) d\tau d\mu d\nu \end{aligned}$$

$$\begin{aligned} & + \frac{1}{2} \int_1^n \int_0^1 \int_0^\nu \mu^{-2\sigma} P(\mu|\nu - \tau; \tau) F(\nu - \tau, \tau) \Psi_n(\nu - \tau, \tau) \\ & \times g^n(\nu - \tau, t) g^n(\tau, t) d\tau d\mu d\nu =: J_1^n + J_2^n. \end{aligned} \tag{3.9}$$

Applying the repeated applications of Fubini’s theorem, (1.7), the transformation $\nu - \tau = \nu'$ and $\tau = \tau'$ and finally replacing $\nu \rightarrow \mu$ and $\tau \rightarrow \nu$, the integral J_1^n can be estimated as

$$J_1^n \leq \frac{\eta(2\sigma)}{2} \int_0^1 \int_0^{1-\mu} (\mu + \nu)^{-2\sigma} F(\mu, \nu) \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu d\mu.$$

Next, by applying (1.7), replacing ν, τ by μ, ν respectively, using Fubini’s theorem, and the transformation $\mu - \nu = \mu'$ and $\nu = \nu'$, one can simplify the second integral J_2^n as

$$\begin{aligned} J_2^n & \leq \frac{1}{2} \int_1^n \int_0^\nu \int_0^\nu \mu^{-2\sigma} P(\mu|\nu - \tau; \tau) F(\nu - \tau, \tau) \\ & \times \Psi_n(\nu - \tau, \tau) g^n(\nu - \tau, t) g^n(\tau, t) d\mu d\tau d\nu \\ & \leq \frac{\eta(2\sigma)}{2} \int_0^1 \int_{1-\mu}^{n-\mu} (\mu + \nu)^{-2\sigma} F(\mu, \nu) \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu d\mu \\ & \quad + \frac{\eta(2\sigma)}{2} \int_1^n \int_0^{n-\mu} (\mu + \nu)^{-2\sigma} F(\mu, \nu) \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu d\mu. \end{aligned}$$

Substituting the above estimates on J_1^n and J_2^n into (3.9), we obtain

$$\begin{aligned} \int_0^1 \mu^{-2\sigma} \mathcal{B}_b^n(g^n)(\mu, t) d\mu & \leq \frac{\eta(2\sigma)}{2} \int_0^n \int_0^{n-\mu} (\mu + \nu)^{-2\sigma} F(\mu, \nu) \\ & \times \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu d\mu. \end{aligned} \tag{3.10}$$

Further, inserting (3.8) and (3.10) into (3.7), then applying the symmetry of E, F and Ψ_n , and using Fubini’s theorem, (3.7) can be rewritten as

$$\begin{aligned} & \frac{d}{dt} \int_0^1 \mu^{-2\sigma} g^n(\mu, t) d\mu \\ & \leq -2 \int_0^1 \int_0^{1-\mu} \left\{ \left[1 - \frac{1}{2} E(\mu, \nu) - \frac{\eta(2\sigma)}{2} F(\mu, \nu) \right] + \frac{1}{4} E(\mu, \nu) \right\} \\ & \quad \times \mu^{-2\sigma} \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu d\mu \\ & \quad - \int_0^1 \int_1^{n-\mu} \mu^{-2\sigma} \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu d\mu \\ & \quad + \frac{\eta(2\sigma)}{2} \int_0^1 \int_1^n \mu^{-2\sigma} F(\mu, \nu) \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu d\mu \\ & \quad + \frac{\eta(2\sigma)}{2} \int_1^n \int_0^n \mu^{-2\sigma} F(\mu, \nu) \Psi_n(\mu, \nu) g^n(\mu, t) g^n(\nu, t) d\nu d\mu. \end{aligned} \tag{3.11}$$

Using the non-negativity of the first and second integrals on the right-hand side to (3.11) guaranteed from (1.6) and then applying (1.5), (3.4), (3.3) and

$g_{in} \in \mathcal{S}^+$, we obtain

$$\begin{aligned}
 & \frac{d}{dt} \int_0^1 \mu^{-2\sigma} g^n(\mu, t) d\mu \\
 & \leq k\eta(2\sigma) \int_0^1 \int_1^n \mu^{-2\sigma} \frac{(1+\mu)^\omega(1+\nu)^\omega}{(\mu+\nu)^\sigma} g^n(\mu, t) g^n(\nu, t) d\nu d\mu \\
 & \quad + k \frac{\eta(2\sigma)}{2} \int_1^n \int_1^n \mu^{-2\sigma} \frac{(1+\mu)^\omega(1+\nu)^\omega}{(\mu+\nu)^\sigma} g^n(\mu, t) g^n(\nu, t) d\nu d\mu \\
 & \leq k\eta(2\sigma) 2^{2\omega} \int_0^1 \int_1^n \mu^{-2\sigma} \nu g^n(\mu, t) g^n(\nu, t) d\nu d\mu \\
 & \quad + k \frac{\eta(2\sigma)}{2} 2^{2\omega} \int_1^n \int_1^n \mu \nu g^n(\mu, t) g^n(\nu, t) d\nu d\mu \\
 & \leq k\eta(2\sigma) 2^{2\omega} \|g_{in}\|_{\mathcal{S}} \left[\int_0^1 \mu^{-2\sigma} g^n(\mu, t) d\mu + \frac{1}{2} \|g_{in}\|_{\mathcal{S}} \right]. \tag{3.12}
 \end{aligned}$$

Then, an application of Gronwall’s inequality to (3.12) gives

$$\int_0^1 \mu^{-2\sigma} g^n(\mu, t) d\mu \leq \mathcal{P}_1(T), \tag{3.13}$$

where

$$\mathcal{P}_1(T) := e^{aT} \|g_{in}\|_{\mathcal{S}} + \frac{b}{a} (e^{aT} - 1),$$

$a := k\eta(2\sigma) 2^{2\omega} \|g_{in}\|_{\mathcal{S}}$ and $b := k \frac{\eta(2\sigma)}{2} 2^{2\omega} \|g_{in}\|_{\mathcal{S}}^2$. Finally, inserting (3.13) into (3.6), we thus have

$$\int_0^\infty (1 + \mu + \mu^{-2\sigma}) g^n(\mu, t) d\mu \leq 2\mathcal{P}_1(T) + 3 \|g_{in}\|_{\mathcal{S}} := \mathcal{P}(T).$$

This completes the proof of Lemma 3.1. □

Next, the equi-integrability of the family of functions $(g^n)_{n>1} \subset \mathcal{S}^+$ is shown in the following lemma for applying the Dunford-Pettis theorem.

Lemma 3.2. *Given any $T > 0$. Then, followings hold true:*

(i) *for all $t \in [0, T]$ and for any given $\epsilon > 0$, there exists a positive constant*

$$\lambda > \max \left\{ \left(\frac{2 \|g_{in}\|_{\mathcal{S}}}{\epsilon} \right)^{1/(1+\sigma)}, \frac{2 \|g_{in}\|_{\mathcal{S}}}{\epsilon} \right\} \text{ such that}$$

$$\sup_{n>1} \left\{ \int_\lambda^\infty (1 + \mu^{-\sigma}) g^n(\mu, t) d\mu \right\} < \epsilon,$$

(ii) *for a given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ (depending on ϵ) such that, for every small Lebesgue measurable set $A \subset \mathbb{R}_+$ with $|A| \leq \delta_\epsilon$, $n > 1$ and $t \in [0, T]$,*

$$\int_A (1 + \mu^{-\sigma}) g^n(\mu, t) d\mu < \epsilon.$$

Proof. (i) Let $\epsilon > 0$ be given. Then, by (3.4), for each $n > 1$, $g_{\text{in}} \in \mathcal{S}^+$ and for all $t \in [0, T]$, we have

$$\begin{aligned} \int_{\lambda}^{\infty} (1 + \mu^{-\sigma})g^n(\mu, t)d\mu &\leq \left[\frac{1}{\lambda} + \left(\frac{1}{\lambda} \right)^{\sigma+1} \right] \int_{\lambda}^{\infty} \mu g_{\text{in}}(\mu)d\mu \\ &\leq \left[\frac{1}{\lambda} + \left(\frac{1}{\lambda} \right)^{\sigma+1} \right] \|g_{\text{in}}\|_{\mathcal{S}} < \epsilon, \end{aligned}$$

which completes the proof of first part of Lemma 3.2.

(ii) Let $\epsilon > 0$ be given. For $A \subset \mathbb{R}_+$, we can choose λ such that $\lambda < n$ for all $n > 1$ and $t \in [0, T]$, and using Lemma 3.2 (i), we have

$$\int_{\lambda}^{\infty} (1 + \mu^{-\sigma})g^n(\mu, t)d\mu < \epsilon. \tag{3.14}$$

For $n > 1$, $\delta \in (0, 1)$ and $t \in [0, T]$, we define

$$\rho^n(\delta, t) := \sup_{t \in [0, T]} \left\{ \int_0^{\lambda} \chi_A(\mu)(1 + \mu^{-\sigma})g^n(\mu, t)d\mu : A \subset \mathbb{R}_+ \text{ with } |A| \leq \delta \right\}.$$

For $n > 1$ and $t \in [0, T]$, we estimate the following term, by employing (3.2), Leibniz’s rule, the non-negativity of g^n , Fubini’s theorem and the transformation $\mu - \nu = \mu'$ and $\nu = \nu'$, as

$$\begin{aligned} &\frac{d}{dt} \int_0^{\lambda} \chi_A(\mu)(1 + \mu^{-\sigma})g^n(\mu, t)d\mu \\ &\leq \frac{1}{2} \int_0^{\lambda} \int_0^{\lambda-\mu} \chi_A(\mu + \nu)(1 + (\mu + \nu)^{-\sigma})\Psi_n(\mu, \nu)g^n(\mu, t)g^n(\nu, t)d\nu d\mu \\ &\quad + \frac{1}{2} \left\{ \int_0^{\lambda} \int_0^{\nu} \int_0^{\nu} + \int_{\lambda}^n \int_0^{\nu} \int_0^{\lambda} \right\} \chi_A(\mu)(1 + \mu^{-\sigma})P(\mu|\nu - \tau; \tau) \\ &\quad \times \Psi_n(\nu - \tau, \tau)g^n(\nu - \tau, t)g^n(\tau, t)d\mu d\tau d\nu. \end{aligned} \tag{3.15}$$

We denote the first, second and third terms on the right-hand side of (3.15) by $J_3^n(t)$, $J_4^n(t)$ and $J_5^n(t)$, respectively. Then, we estimate each $J_i^n(t)$, for $i = 3, 4, 5$ separately. Let us first consider $J_3^n(t)$ and evaluate it as

$$\begin{aligned} J_3^n(t) &\leq k(1 + \lambda)^{2\omega} \int_0^{\lambda} \int_0^{\lambda} \chi_{(-\nu+A) \cap (0, \lambda-\nu)}(\mu)(1 + \mu^{-\sigma}) \\ &\quad \times \frac{1}{(\mu + \nu)^{\sigma}} g^n(\mu, t)g^n(\nu, t)d\mu d\nu \\ &\leq k(1 + \lambda)^{2\omega} \mathcal{P}(T) \int_0^{\lambda} \chi_{(-\nu+A) \cap (0, \lambda-\nu)}(\mu)(1 + \mu^{-\sigma})g^n(\mu, t)d\mu. \end{aligned} \tag{3.16}$$

Since $(-\nu + A) \cap (0, \lambda - \nu) \subset (0, \lambda)$ and $|(-\nu + A) \cap (0, \lambda - \nu)| \leq |-\nu + A| = |A| \leq \delta$, then from (3.16), we obtain

$$J_3^n(t) \leq k(1 + \lambda)^2 \mathcal{P}(T) \rho^n(\delta, t).$$

Next, by utilizing (1.8), $J_4^n(t)$ is evaluated, as

$$J_4^n(t) \leq \frac{1}{2} \Omega_1(\delta) \int_0^\lambda \int_0^\nu \{ \nu^{-\theta_1+\sigma} + \nu^{-\theta_1} \} \Psi_n(\nu - \tau, \tau) g^n(\nu - \tau, t) g^n(\tau, t) d\tau d\nu. \tag{3.17}$$

Applying Fubini’s theorem, then using the transformation $\nu - \tau = \nu'$ and $\tau = \tau'$, (1.5) and Lemma 3.1 into (3.17), we have

$$\begin{aligned} J_4^n(t) &\leq \frac{1}{2} k \Omega_1(\delta) \int_0^\lambda \int_0^\lambda \{ (\nu + \tau)^{-\theta_1+\sigma} + (\nu + \tau)^{-\theta_1} \} \frac{(1 + \nu)^\omega (1 + \tau)^\omega}{(\nu + \tau)^\sigma} \\ &\quad \times g^n(\nu, t) g^n(\tau, t) d\nu d\tau \\ &\leq \frac{k}{2} \Omega_1(\delta) (1 + \lambda)^2 \int_0^\lambda \int_0^\lambda \{ \nu^{-\theta_1} + \nu^{-\theta_1} \tau^{-\sigma} \} g^n(\nu, t) g^n(\tau, t) d\nu d\tau \\ &\leq k(1 + \lambda)^2 \mathcal{P}(T)^2 \Omega_1(\delta). \end{aligned}$$

Similarly, from repeated applications of Fubini’s theorem, (1.9), $\nu - \tau = \nu'$ and $\tau = \tau'$, (1.5) and Lemma 3.1, $J_5^n(t)$ can be estimated as

$$\begin{aligned} J_5^n(t) &\leq \frac{1}{2} k'(\lambda) \int_\lambda^n \int_0^\nu \int_0^\lambda \chi_A(\mu) \mu^{-\tau_2} \Psi_n(\nu - \tau, \tau) g^n(\nu - \tau, t) g^n(\tau, t) d\mu d\tau d\nu \\ &\quad + \frac{1}{2} k'(\lambda) \int_\lambda^n \int_0^\nu \int_0^\lambda \chi_A(\mu) \mu^{-\sigma-\tau_2} \Psi_n(\nu - \tau, \tau) g^n(\nu - \tau, t) g^n(\tau, t) d\mu d\tau d\nu \\ &\leq \frac{1}{2} k'(\lambda) k \int_0^\lambda \int_{\lambda-\tau}^{n-\tau} \int_0^\lambda \chi_A(\mu) \mu^{-\tau_2} \frac{(1 + \nu)^\omega (1 + \tau)^\omega}{(\nu + \tau)^\sigma} g^n(\nu, t) g^n(\tau, t) d\mu d\nu d\tau \\ &\quad + \frac{1}{2} k'(\lambda) k \int_\lambda^n \int_0^{n-\tau} \int_0^\lambda \chi_A(\mu) \mu^{-\tau_2} \frac{(1 + \nu)^\omega (1 + \tau)^\omega}{(\nu + \tau)^\sigma} \\ &\quad \times g^n(\nu, t) g^n(\tau, t) d\mu d\nu d\tau \\ &\quad + \frac{1}{2} k'(\lambda) k \int_0^\lambda \int_{\lambda-\tau}^{n-\tau} \int_0^\lambda \chi_A(\mu) \mu^{-\sigma-\tau_2} \frac{(1 + \nu)^\omega (1 + \tau)^\omega}{(\nu + \tau)^\sigma} \\ &\quad \times g^n(\nu, t) g^n(\tau, t) d\mu d\nu d\tau \\ &\quad + \frac{1}{2} k'(\lambda) k \int_\lambda^n \int_0^{n-\tau} \int_0^\lambda \chi_A(\mu) \mu^{-\sigma-\tau_2} \frac{(1 + \nu)^\omega (1 + \tau)^\omega}{(\nu + \tau)^\sigma} \\ &\quad \times g^n(\nu, t) g^n(\tau, t) d\mu d\nu d\tau \\ &\leq \frac{1}{2\lambda^\sigma} k'(\lambda) k \mathcal{P}(T)^2 \left[\int_0^\lambda \chi_A(\mu) \mu^{-\tau_2} d\mu + \int_0^\lambda \chi_A(\mu) \mu^{-\sigma-\tau_2} d\mu \right]. \end{aligned}$$

Choose $u > 1$ such that $u\tau_2 < 1$ and for $\frac{1+\sigma+\tau_2}{1-\sigma-\tau_2} > 1$, then, applying Hölder’s inequality to above estimate, we get

$$\begin{aligned} J_5^n(t) &\leq \frac{1}{2\lambda^\sigma} k'(\lambda) k \mathcal{P}(T)^2 \left\{ |A|^{\frac{u-1}{u}} \left(\int_0^\lambda \mu^{-u\tau_2} d\mu \right)^{\frac{1}{u}} \right. \\ &\quad \left. + |A|^{\frac{1-\sigma-\tau_2}{1+\sigma+\tau_2}} \left(\int_0^\lambda \mu^{-(1+\sigma+\tau_2)/2} d\mu \right)^{\frac{2(\sigma+\tau_2)}{1+\sigma+\tau_2}} \right\} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2\lambda^\sigma} k'(\lambda) k \mathcal{P}(T)^2 \left\{ \delta^{\frac{u-1}{u}} \left(\frac{\lambda^{1-u\tau_2}}{1-u\tau_2} \right)^{\frac{1}{u}} \right. \\ &\quad \left. + \delta^{\frac{1-\sigma-\tau_2}{1+\sigma+\tau_2}} \left(\frac{\lambda^{(1-(\sigma+\tau_2))/2}}{(1-\sigma-\tau_2)/2} \right)^{\frac{2(\sigma+\tau_2)}{(1+\sigma+\tau_2)}} \right\}. \end{aligned}$$

Inserting the estimates on $J_3^n(t)$, $J_4^n(t)$ and $J_5^n(t)$ into (3.15), we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^\lambda \chi_A(\mu) (1 + \mu^{-\sigma}) g^n(\mu, t) d\mu &\leq k(1 + \lambda)^2 \mathcal{P}(T) \rho^n(\delta, t) + k(1 + \lambda)^2 \Omega_1(\delta) \mathcal{P}(T)^2 \\ &\quad + \frac{1}{2\lambda^\sigma} k'(\lambda) k \mathcal{P}(T)^2 \left\{ \delta^{\frac{u-1}{u}} \left(\frac{\lambda^{1-u\tau_2}}{1-u\tau_2} \right)^{\frac{1}{u}} + \delta^{\frac{1-\sigma-\tau_2}{1+\sigma+\tau_2}} \left(\frac{\lambda^{(1-(\sigma+\tau_2))/2}}{(1-\sigma-\tau_2)/2} \right)^{\frac{2(\sigma+\tau_2)}{(1+\sigma+\tau_2)}} \right\}. \end{aligned} \tag{3.18}$$

Next, integrating (3.18) with respect to time from 0 to t and taking supremum over all A such that $A \subset \mathbb{R}_+$ with $|A| \leq \delta$ and using $g_{\text{in}} \in \mathcal{S}^+$, we estimate

$$\begin{aligned} \rho^n(\delta, t) &\leq \rho^n(\delta, 0) + k \mathcal{P}(T) (1 + \lambda)^2 \int_0^t \rho^n(\delta, s) ds + k(1 + \lambda)^2 \Omega_1(\delta) \mathcal{P}(T)^2 T \\ &\quad + \frac{k}{2\lambda^\sigma} k'(\lambda) \mathcal{P}(T)^2 T \left\{ \delta^{\frac{u-1}{u}} \left(\frac{\lambda^{1-u\tau_2}}{1-u\tau_2} \right)^{\frac{1}{u}} + \delta^{\frac{1-\sigma-\tau_2}{1+\sigma+\tau_2}} \left(\frac{\lambda^{(1-(\sigma+\tau_2))/2}}{(1-\sigma-\tau_2)/2} \right)^{\frac{2(\sigma+\tau_2)}{(1+\sigma+\tau_2)}} \right\}. \end{aligned}$$

Now, after applying Gronwall's inequality, it is obtained that

$$\rho^n(\delta, t) \leq C_1(\delta, \lambda) \exp(k \mathcal{P}(T) (1 + \lambda)^2 T), \quad t \in [0, T], \tag{3.19}$$

where

$$\begin{aligned} C_1(\delta, \lambda) &:= \rho^n(\delta, 0) + k(1 + \lambda)^2 \Omega_1(\delta) \mathcal{P}(T)^2 T + \frac{k}{2\lambda^\sigma} k'(\lambda) \mathcal{P}(T)^2 T \\ &\quad \times \left\{ \delta^{\frac{u-1}{u}} \left(\frac{\lambda^{1-u\tau_2}}{1-u\tau_2} \right)^{\frac{1}{u}} + \delta^{\frac{1-\sigma-\tau_2}{1+\sigma+\tau_2}} \left(\frac{\lambda^{(1-(\sigma+\tau_2))/2}}{(1-\sigma-\tau_2)/2} \right)^{\frac{2(\sigma+\tau_2)}{(1+\sigma+\tau_2)}} \right\}. \end{aligned}$$

From (3.19), we thus have

$$\sup_n \{ \rho^n(\delta, t) \} \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \tag{3.20}$$

Finally, adding (3.14) and (3.20), we obtain the desired result. This completes the proof to the second part of Lemma 3.2. \square

Next, we turn to show the time equicontinuity of sequences $(g^n)_{n>1}$ and $(\mu^{-\sigma} g^n)_{n>1}$.

3.2. Equicontinuity in time

Set $\Upsilon^n(\mu, t) := \mu^{-\zeta} g^n(\mu, t)$ for $\zeta = \{0, \sigma\}$, $\mu \in \mathbb{R}_+$ and $t \in [0, T]$. At $\zeta = 0$, this gives $\Upsilon^n(\mu, t) = g^n(\mu, t)$ and when $\zeta = \sigma$, $\Upsilon^n(\mu, t) := \mu^{-\sigma} g^n(\mu, t)$. Let $T > 0$. For any given $\epsilon > 0$ and $\phi \in L^\infty(\mathbb{R}_+)$ there exists $\lambda = \lambda(\epsilon) > 1$ in such way that

$$\frac{2\mathcal{P}(T)}{\lambda^{1+\zeta}} < \frac{\epsilon}{2\|\phi\|_{L^\infty}}, \tag{3.21}$$

where the constant $\mathcal{P}(T)$ is defined in Lemma 3.1. Consider $s, t \in [0, T]$ with $t \geq s$. Then, for each $n > 1$, by Lemma 3.1 and (3.21), we have

$$\begin{aligned} \int_{\lambda}^{\infty} |\Upsilon^n(\mu, t) - \Upsilon^n(\mu, s)| d\mu &\leq \frac{1}{\lambda} \int_{\lambda}^{\infty} \mu^{1-\zeta} \{g^n(\mu, t) + g^n(\mu, s)\} d\mu \\ &\leq \frac{2\mathcal{P}(T)}{\lambda^{1+\zeta}} < \frac{\epsilon}{2\|\phi\|_{L^\infty}}. \end{aligned} \tag{3.22}$$

Multiplying by $\mu^{-\zeta}\phi(\mu)$ on both sides into (3.2), integrating with respect to μ from 0 to λ , then using Leibniz’s rule and non-negativity of g^n , we simplify the following term as

$$\begin{aligned} \left| \frac{d}{dt} \int_0^\lambda \phi(\mu) \Upsilon^n(\mu, t) d\mu \right| &\leq \|\phi\|_{L^\infty} \int_0^\lambda \mu^{-\zeta} \left[\mathcal{B}_c^n(g^n)(\mu, t) \right. \\ &\quad \left. + \mathcal{D}_{cb}^n(g^n)(\mu, t) + \mathcal{B}_b^n(g^n)(\mu, t) \right] d\mu. \end{aligned} \tag{3.23}$$

Now, using Fubini’s theorem, (1.7) and applying the transformation $\mu - \nu = \mu'$ and $\nu = \nu'$ to (3.23), we estimate

$$\left| \frac{d}{dt} \int_0^\lambda \phi(\mu) \Upsilon^n(\mu, t) d\mu \right| \leq \frac{(3 + \eta(\zeta))}{2} \|\phi\|_{L^\infty} \int_0^\lambda \mu^{-\zeta} \mathcal{D}_{cb}^n(g^n)(\mu, t) d\mu. \tag{3.24}$$

By employing Fubini’s theorem, (1.5) and Lemma 3.1 to (3.24), we evaluate as

$$\left| \frac{d}{dt} \int_0^\lambda \phi(\mu) \Upsilon^n(\mu, t) d\mu \right| \leq \frac{(3 + \eta(\zeta))}{2} \|\phi\|_{L^\infty} k(1 + \lambda)^\omega \mathcal{P}(T)^2. \tag{3.25}$$

After combining the estimates in (3.22) and (3.25), finally we have

$$\begin{aligned} &\left| \int_0^\infty \phi(\mu) [\Upsilon^n(\mu, t) - \Upsilon^n(\mu, s)] d\mu \right| \\ &\leq \frac{(3 + \eta(\zeta))}{2} \|\phi\|_{L^\infty} k(1 + \lambda)^\omega \mathcal{P}(T)^2 (t - s) + \frac{\epsilon}{2}. \end{aligned} \tag{3.26}$$

Fix $\delta > 0$ and take s and t such that $t - s < \delta$. Then the estimate (3.26) implies the equicontinuity of the family $\{g^n(t), t \in [0, T]\}$ with respect to time variable t , in the topology $L^1(\mathbb{R}_+, d\mu)$. Then according to a refined version of the *Arzelà-Ascoli Theorem*, see [23, Theorem 2.1] or *Arzelà-Ascoli Theorem* [2, Appendix A8.5], we conclude that there exist a subsequence (Υ^n) (not relabeled) and a non-negative function $\Upsilon \in L^\infty((0, T); L^1(\mathbb{R}_+, d\mu))$ such that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^\infty [\Upsilon^n(\mu, t) - \Upsilon(\mu, t)] \phi(\mu) d\mu \right| = 0,$$

for all $T > 0$ and $\phi \in L^\infty(\mathbb{R}_+)$. This implies that

$$\Upsilon^n(t) \rightharpoonup \Upsilon(t) \text{ in } L^1(\mathbb{R}_+, d\mu) \text{ as } n \rightarrow \infty, \tag{3.27}$$

uniformly for all $t \in [0, T]$ to some $\Upsilon \in \mathcal{C}([0, T]; w - L^1(\mathbb{R}_+, d\mu))$, where $\mathcal{C}([0, T]; w - L^1(\mathbb{R}_+, d\mu))$ is the space of all weakly continuous functions from $[0, T]$ to $L^1(\mathbb{R}_+, d\mu)$. Applying the weak convergence of $g^n(t) - g^n(s)$ to $g(t) - g(s)$ in $L^1(\mathbb{R}_+, d\mu)$ from (3.27), Lemma 3.1, and setting $\phi(\mu) = \text{sgn}(\Upsilon^n(\mu, t) -$

$\Upsilon^n(\mu, s)$ into (3.26), we can easily improve the space from $\Upsilon \in \mathcal{C}([0, T]; w - L^1(\mathbb{R}_+, d\mu))$ to $\Upsilon \in \mathcal{C}([0, T]; L^1(\mathbb{R}_+, d\mu))$.

By considering $\zeta = 0$ and $\zeta = \sigma$, (3.27) implies that there exist two subsequences (g^n) and $(\mu^{-\sigma}g^n)$ such that

$$g^n(t) \rightharpoonup g(t) \text{ in } L^1(\mathbb{R}_+, d\mu) \text{ as } n \rightarrow \infty, \tag{3.28}$$

and

$$\mu^{-\sigma}g^n(t) \rightharpoonup \mu^{-\sigma}g(t) \text{ in } L^1(\mathbb{R}_+, d\mu) \text{ as } n \rightarrow \infty. \tag{3.29}$$

For any $a > 0, t \in [0, T]$, since $g^n \rightharpoonup g$, we thus obtain

$$\int_0^a \mu g(\mu, t) d\mu = \lim_{n \rightarrow \infty} \int_0^a \mu g^n(\mu, t) d\mu \leq \|g_{\text{in}}\|_{\mathcal{S}} < \infty.$$

and

$$\int_0^a \mu^{-\sigma}g(\mu, t) d\mu = \lim_{n \rightarrow \infty} \int_0^a \mu^{-\sigma}g^n(\mu, t) d\mu \leq \|g_{\text{in}}\|_{\mathcal{S}} < \infty.$$

Equation (3.4), the non-negativity of each g^{n_k} and g , and then $a \rightarrow \infty$ imply that

$$\int_0^\infty \mu g(\mu, t) d\mu \leq \int_0^\infty \mu g_{\text{in}}(\mu) d\mu \text{ and } g \in \mathcal{S}^+.$$

Next, to show that the limit function g constructed in (3.28) is actually a weak solution to (1.1)–(1.2) in an appropriate sense i.e. as given in Definition 2.1.

3.3. Integral convergence

In the following lemma, we wish to show that the truncated integrals on the right-hand side to (3.2) converge weakly to the original integrals on the right-hand to (1.1).

Lemma 3.3. *Let $(g^n)_{n \geq 1}$ be a bounded sequence in \mathcal{S}^+ and $g \in \mathcal{S}^+$, where $\|g^n\|_{\mathcal{S}} \leq \mathcal{P}(T)$ and $g^n \rightharpoonup g$ in $L^1(\mathbb{R}_+, d\mu)$ as $n \rightarrow \infty$. Then, for each $\lambda > 1$, we have*

$$\begin{aligned} \mathcal{B}_c^n(g^n) &\rightharpoonup \mathcal{B}_c(g), \quad \mathcal{D}_{cb}^n(g^n) \rightharpoonup \mathcal{D}_{cb}(g) \\ \text{and } \mathcal{B}_b^n(g^n) &\rightharpoonup \mathcal{B}_b(g) \text{ in } L^1((0, \lambda), d\mu) \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.30}$$

Proof. Fix $\lambda \in (1, n)$ and $\mu \in (0, \lambda)$. Suppose ϕ belongs to $L^\infty(0, \lambda)$ with compact support included in $(0, \lambda)$. We argue in the similar manner with little modifications as in Camejo and Warnecke [10] to show that the first two terms such that $\mathcal{B}_c^n(g^n) \rightharpoonup \mathcal{B}_c(g)$ and $\mathcal{D}_{cb}^n(g^n) \rightharpoonup \mathcal{D}_{cb}(g)$ in $L^1((0, \lambda), d\mu)$ as $n \rightarrow \infty$. Next, in order to show $\mathcal{B}_b^n(g^n) \rightharpoonup \mathcal{B}_b(g)$ in $L^1((0, \lambda), d\mu)$ as $n \rightarrow \infty$, it is sufficient to prove that

$$\left| \int_0^\lambda \phi(\mu) \{ \mathcal{B}_b^n(g^n)(\mu, t) - \mathcal{B}_b(g)(\mu, t) \} d\mu \right| \rightarrow 0, \tag{3.31}$$

as $n \rightarrow \infty$, for $\phi \in L^\infty(0, \lambda)$. Let us first simplify the following integral, by using triangle inequality, the repeated applications of Fubini's theorem and $\nu - \tau = \nu'$ and $\tau = \tau'$, as

$$\left| \int_0^\lambda \phi(\mu) [\mathcal{B}_b^n(g^n)(\mu, t) - \mathcal{B}_b(g)(\mu, t)] d\mu \right|$$

$$\begin{aligned}
 &\leq \left| \int_0^\lambda \int_0^\lambda \Theta(\nu, \tau) \nu^{-\sigma} \{g^n(\nu, t)g^n(\tau, t) - g(\nu, t)g(\tau, t)\} d\mu d\nu d\tau \right| \\
 &\quad + \left| \int_0^\lambda \int_\lambda^\infty \Theta(\nu, \tau) \nu^{-\sigma} \{g^n(\nu, t)g^n(\tau, t) - g(\nu, t)g(\tau, t)\} d\mu d\nu d\tau \right| \\
 &\quad + \left| \int_\lambda^\infty \int_0^\infty \Theta(\nu, \tau) \nu^{-\sigma} \{g^n(\nu, t)g^n(\tau, t) - g(\nu, t)g(\tau, t)\} d\mu d\nu d\tau \right| \\
 &:= \mathcal{I}_1^n + \mathcal{I}_2^n + \mathcal{I}_3^n, \tag{3.32}
 \end{aligned}$$

where \mathcal{I}_1^n , \mathcal{I}_2^n and \mathcal{I}_3^n denote the first, second and third terms on the right-hand side, respectively, to (3.32), and

$$\Theta(\nu, \tau) := \frac{1}{2} E'(\nu, \tau) \Psi(\nu, \tau) \nu^\sigma \int_0^{\min\{\lambda, \nu+\tau\}} \phi(\mu) P(\mu|\nu; \tau) d\mu.$$

One can infer from (3.1), (1.4) and $F \leq 1$ that

$$\Theta(\nu, \tau) \in L^\infty \left((0, \lambda) \times (0, \lambda) \right), \text{ for } (\nu, \tau) \in (0, \lambda) \times (0, \lambda).$$

Then using [18, Lemma 2.9] or the repeated application of [19, Lemma A.2], it can easily be obtained that

$$\lim_{n \rightarrow \infty} \mathcal{I}_1^n = 0. \tag{3.33}$$

Next, applying (3.1), (1.4) and Lemma 3.1, we estimate that

$$\begin{aligned}
 &\left\{ \int_0^\lambda \int_\lambda^\infty + \int_\lambda^\infty \int_0^\infty \right\} \Theta(\nu, \tau) \nu^{-\sigma} g^n(\nu, t) g^n(\tau, t) d\mu d\nu d\tau \\
 &\leq \frac{Nk}{2} \|\phi\|_{L^\infty} \left\{ \int_0^\lambda \int_\lambda^\infty \right. \\
 &\quad \left. + \int_\lambda^\infty \int_0^\infty \right\} (1 + \nu)^\omega (1 + \tau)^\omega (\nu + \tau)^{-\sigma} g^n(\nu, t) g^n(\tau, t) d\nu d\tau \\
 &\leq \frac{Nk}{\lambda^\sigma} \|\phi\|_{L^\infty} \mathcal{P}^2(T). \tag{3.34}
 \end{aligned}$$

Similarly, one can evaluate

$$\left\{ \int_0^\lambda \int_\lambda^\infty + \int_\lambda^\infty \int_0^\infty \right\} \Theta(\nu, \tau) \nu^{-\sigma} g(\nu, t) g(\tau, t) d\nu d\tau \leq \frac{Nk}{\lambda^\sigma} \|\phi\|_{L^\infty} \mathcal{P}^2(T). \tag{3.35}$$

Then, employing (3.34) and (3.35), we have

$$\mathcal{I}_2^n + \mathcal{I}_3^n \leq \frac{2Nk}{\lambda^\sigma} \|\phi\|_{L^\infty} \mathcal{P}^2(T). \tag{3.36}$$

Finally, applying (3.33) and (3.36) into (3.32), we conclude that

$$\lim_{n \rightarrow \infty} \left| \int_0^\lambda \phi(\mu) \{ \mathcal{B}_b^n(g^n)(\mu, t) - \mathcal{B}_b(g)(\mu, t) \} d\mu \right| \leq \frac{2Nk}{\lambda^\sigma} \|\phi\|_{L^\infty} \mathcal{P}^2(T). \tag{3.37}$$

Since $\lambda > 1$, thus (3.37) is true for $\lambda \in (1, \infty)$. Hence, from (3.36) and (3.37), it is clear that (3.31) holds. This completes the Proof of Lemma 3.3. \square

Now, we turn to the Proof of Theorem 2.2 with the help of above results.

Proof of Theorem 2.2. Fix $T > 0$ and $\phi \in L^\infty(\mathbb{R}_+)$. Then, for each $s \in [0, t]$, employing Lemma 3.3, we have

$$\int_0^\infty \phi(\mu)[(\mathcal{B}_c^n - \mathcal{D}_{cb}^n + \mathcal{B}_b^n)(g^n)(\mu, s) - (\mathcal{B}_c - \mathcal{D}_{cb} + \mathcal{B}_b)(g)(\mu, s)]d\mu \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.38)$$

Again, a repeated application of Fubini's theorem, the transformation $\mu - \nu = \mu'$, and $\nu = \nu'$, (1.5), (1.9), (1.4) and Lemma 3.1 confirm that there exists a positive constant $C(T)$ such that

$$\left| \int_0^\infty \phi(\mu)\{(\mathcal{B}_c^n - \mathcal{D}_{cb}^n + \mathcal{B}_b^n)(g^n)(\mu, s) - (\mathcal{B}_c - \mathcal{D}_{cb} + \mathcal{B}_b)(g)(\mu, s)\}d\mu \right| \leq C(T)\|\phi\|_{L^\infty(\mathbb{R}_+)}, \quad (3.39)$$

where

$$C(T) := k\mathcal{P}(T)^2(3 + 2N).$$

Next, one can easily check that the left-hand side of (3.39) is in $L^1((0, t), ds)$. Then from (3.38), (3.39) and the Lebesgue dominated convergence theorem, we obtain

$$\int_0^t \int_0^\infty \phi(\mu)\{(\mathcal{B}_c^n - \mathcal{D}_{cb}^n + \mathcal{B}_b^n)(g^n)(\mu, s) - (\mathcal{B}_c - \mathcal{D}_{cb} + \mathcal{B}_b)(g)(\mu, s)\}d\mu ds \rightarrow 0, \quad (3.40)$$

as $n \rightarrow \infty$. Since ϕ is arbitrary and (3.40) holds for $\phi \in L^\infty(\mathbb{R}_+)$ as $n \rightarrow \infty$, hence, by applying Fubini's theorem, we get

$$\begin{aligned} & \int_0^t (\mathcal{B}_c^n - \mathcal{D}_{cb}^n + \mathcal{B}_b^n)(g^n)(\mu, s)ds \\ & \rightarrow \int_0^t (\mathcal{B}_c - \mathcal{D}_{cb} + \mathcal{B}_b)(g)(\mu, s)ds \quad \text{in } L^1((\mathbb{R}_+), d\mu). \end{aligned} \quad (3.41)$$

Then, the definition of $(\mathcal{B}_c^n - \mathcal{D}_{cb}^n + \mathcal{B}_b^n)$ yields that

$$g^n(\mu, t) = \int_0^t (\mathcal{B}_c^n - \mathcal{D}_{cb}^n + \mathcal{B}_b^n)(g^n)(\mu, s)ds + g_{in}^n(\mu), \quad \text{for } t \in [0, T]. \quad (3.42)$$

Next, using (3.41), (3.28) and (3.42), we thus obtain

$$\begin{aligned} \int_0^\infty \phi(\mu)g(\mu, t)d\mu &= \int_0^t \int_0^\infty \phi(\mu)(\mathcal{B}_c - \mathcal{D}_{cb} + \mathcal{B}_b)(g)(\mu, s)d\mu ds \\ &+ \int_0^\infty \phi(\mu)g_{in}(\mu)d\mu, \end{aligned}$$

for any $\phi \in L^\infty(\mathbb{R}_+)$. Hence, for all $\phi \in L^\infty(\mathbb{R}_+)$, we have $g(\mu, t)$ is a solution to (1.1)–(1.2). This implies that for almost any $\mu \in \mathbb{R}_+$, we have

$$g(\mu, t) = \int_0^t (\mathcal{B}_c - \mathcal{D}_{cb} + \mathcal{B}_b)(g)(\mu, s)ds + g_{in}(\mu).$$

This completes the proof of the existence Theorem 2.2. □

In the next section, the uniqueness of weak solutions to (1.1)–(1.2) is investigated under additional growth condition (1.10) on collision kernel Ψ .

4. Uniqueness of weak solutions

Proof of Theorem 2.3. Let g and h be two weak solutions to (1.1)–(1.2) on $[0, \infty)$, with $g_{in}(\mu) = h_{in}(\mu)$. Set $Z := g - h$. For $n = 1, 2, 3, \dots$, we define

$$\begin{aligned} \Xi^n(t) &:= \int_0^n (1 + \mu^{-\sigma}) |Z(\mu, t)| d\mu \\ &= \int_0^n (1 + \mu^{-\sigma}) \operatorname{sgn}(Z(\mu, t)) \{g(\mu, t) - h(\mu, t)\} d\mu. \end{aligned} \tag{4.1}$$

Substituting the value of $g(\mu, t) - h(\mu, t)$ by using the Definition 2.1 (iii) into (4.1) and simplifying it further by applying the following identity

$$g(\mu, s)g(\nu, s) - h(\mu, s)h(\nu, s) = g(\mu, s)Z(\nu, s) + h(\nu, s)Z(\mu, s),$$

using Fubini’s theorem, symmetry of Ψ , the transformation $\mu - \nu = \mu'$ and $\nu = \nu'$ and $\nu - \tau = \nu'$ and $\tau = \tau'$, we have

$$\begin{aligned} \Xi^n(t) &= \frac{1}{2} \int_0^t \int_0^n \int_0^{n-\mu} \left[\{1 + (\mu + \nu)^{-\sigma}\} \operatorname{sgn}(Z(\mu + \nu, s)) E(\mu, \nu) \right. \\ &\quad \left. - \{1 + \mu^{-\sigma}\} \operatorname{sgn}(Z(\mu, s)) - \{1 + \nu^{-\sigma}\} \operatorname{sgn}(Z(\nu, s)) \right] \\ &\quad \times \Psi(\mu, \nu) \{g(\mu, s)Z(\nu, s) + h(\nu, s)Z(\mu, s)\} d\nu d\mu ds \\ &\quad - \int_0^t \int_0^n \int_{n-\mu}^\infty \{1 + \mu^{-\sigma}\} \operatorname{sgn}(Z(\mu, s)) \Psi(\mu, \nu) \\ &\quad \times \{g(\mu, s)Z(\nu, s) + h(\nu, s)Z(\mu, s)\} d\nu d\mu ds \\ &\quad + \frac{1}{2} \left\{ \int_0^t \int_0^n \int_0^{n-\nu} \int_0^{\nu+\tau} + \int_0^t \int_0^n \int_{n-\nu}^\infty \int_0^n \right. \\ &\quad \left. + \int_0^t \int_n^\infty \int_0^\infty \int_0^n \right\} \{1 + \mu^{-\sigma}\} F(\nu, \tau) \operatorname{sgn}(Z(\mu, s)) P(\mu|\nu; \tau) \\ &\quad \times \Psi(\nu, \tau) \{g(\nu, s)Z(\tau, s) + h(\tau, s)Z(\nu, s)\} d\mu d\tau d\nu ds. \end{aligned} \tag{4.2}$$

Next, let us define the term Q as

$$\begin{aligned} Q(\mu, \nu, s) &:= \{1 + (\mu + \nu)^{-\sigma}\} \operatorname{sgn}(Z(\mu + \nu, s)) E(\mu, \nu) \\ &\quad - \{1 + \mu^{-\sigma}\} \operatorname{sgn}(Z(\mu, s)) \\ &\quad - \{1 + \nu^{-\sigma}\} \operatorname{sgn}(Z(\nu, s)). \end{aligned}$$

Using the definition of Q and the properties of signum function, i.e. $|\Theta| = \Theta \operatorname{sgn}(\Theta)$ into (4.2), we obtain

$$\Xi^n(t) \leq \frac{1}{2} \int_0^t \int_0^n \int_0^{n-\mu} Q(\mu, \nu, s) \Psi(\mu, \nu) \{g(\mu, s)Z(\nu, s) + h(\nu, s)Z(\mu, s)\} d\nu d\mu ds$$

$$\begin{aligned}
 & + \int_0^t \int_0^n \int_{n-\mu}^\infty \{1 + \mu^{-\sigma}\} \Psi(\mu, \nu) g(\mu, s) |Z(\nu, s)| d\nu d\mu ds \\
 & - \int_0^t \int_0^n \int_{n-\mu}^\infty \{1 + \mu^{-\sigma}\} \Psi(\mu, \nu) h(\nu, s) |Z(\mu, s)| d\nu d\mu ds \\
 & + \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \int_0^{\nu+\tau} \{1 + \mu^{-\sigma}\} P(\mu|\nu; \tau) \Psi(\nu, \tau) \\
 & \times \{g(\nu, s) |Z(\tau, s)| + h(\tau, s) |Z(\nu, s)|\} d\mu d\tau d\nu ds.
 \end{aligned} \tag{4.3}$$

Due to the non-negativity of the third integral on the right-hand side to (4.3), (1.4) and $r = \sigma$ into (1.7), (4.3) can be further estimated as

$$\begin{aligned}
 \Xi^n(t) & \leq \frac{1}{2} \int_0^t \int_0^n \int_0^{n-\mu} Q(\mu, \nu, s) \Psi(\mu, \nu) g(\mu, s) Z(\nu, s) d\nu d\mu ds \\
 & + \frac{1}{2} \int_0^t \int_0^n \int_0^{n-\mu} Q(\mu, \nu, s) \Psi(\mu, \nu) h(\nu, s) Z(\mu, s) d\nu d\mu ds \\
 & + \int_0^t \int_0^n \int_{n-\mu}^\infty \{1 + \mu^{-\sigma}\} \Psi(\mu, \nu) g(\mu, s) |Z(\nu, s)| d\nu d\mu ds \\
 & + \frac{1}{2} \int_0^t \int_0^n \int_0^n \{N + \eta(\sigma)(\nu + \tau)^{-\sigma}\} \Psi(\nu, \tau) g(\nu, s) |Z(\tau, s)| d\tau d\nu ds \\
 & + \frac{1}{2} \int_0^t \int_0^n \int_0^n \{N + \eta(\sigma)(\nu + \tau)^{-\sigma}\} \Psi(\nu, \tau) h(\tau, s) |Z(\nu, s)| d\tau d\nu ds \\
 & + \int_0^t \int_0^\infty \int_n^\infty \{N + \eta(\sigma)(\nu + \tau)^{-\sigma}\} \Psi(\nu, \tau) \{g(\nu, s) |Z(\tau, s)| + h(\tau, s) |Z(\nu, s)|\} d\tau d\nu ds \\
 & =: \sum_{i=1}^6 S_i^n(t),
 \end{aligned} \tag{4.4}$$

where $S_i^n(t)$, for $i = 1, 2, \dots, 6$ are the corresponding integrals in the preceding line. We now solve each $S_i^n(t)$ individually. Using properties of the signum function, i.e. $\text{sgn}(\Theta_1)\Theta_1 = |\Theta_1|$, we consider following two bounds as

$$\begin{aligned}
 Q(\mu, \nu, s) Z(\nu, s) & = \left[\{1 + (\mu + \nu)^{-\sigma}\} \text{sgn} (Z(\mu + \nu, s)) E(\mu, \nu) \right. \\
 & \quad \left. - \{1 + \mu^{-\sigma}\} \text{sgn} (Z(\mu, s)) - \{1 + \nu^{-\sigma}\} \text{sgn} (Z(\nu, s)) \right] Z(\nu, s) \\
 & \leq 2\{1 + \mu^{-\sigma}\} |Z(\nu, s)|,
 \end{aligned} \tag{4.5}$$

and

$$Q(\mu, \nu, s) Z(\mu, s) \leq 2\{1 + \nu^{-\sigma}\} |Z(\mu, s)|. \tag{4.6}$$

Let us first estimate $S_1^n(t)$, by using (4.5), (1.10), Young’s inequality and the definition of norm, as

$$\begin{aligned}
 S_1^n(t) & \leq k \int_0^t \int_0^n \int_0^n (1 + \mu^{-\sigma})(\mu + \nu)^{-\sigma} |Z(\nu, s)| g(\mu, s) d\nu d\mu ds \\
 & \leq k \int_0^t \int_0^n \int_0^n (1 + \mu^{-\sigma})(1 + \nu^{-\sigma}) |Z(\nu, s)| g(\mu, s) d\nu d\mu ds
 \end{aligned}$$

$$\leq \frac{k}{2} \int_0^t \Xi^n(s) \int_0^n (3 + \mu^{-2\sigma})g(\mu, s)d\mu ds \leq \Gamma_1 \int_0^t \Xi^n(s)ds, \tag{4.7}$$

where

$$\Gamma_1 = \frac{3k}{2} \sup_{s \in [0,t]} \|g(s)\|_S.$$

Similarly, (1.10), (4.6), Young’s inequality and the definition of norm help to evaluate $S_2^n(t)$ as

$$S_2^n(t) \leq \Gamma_2 \int_0^t \Xi^n(s)ds, \tag{4.8}$$

where

$$\Gamma_2 = \frac{3k}{2} \sup_{s \in [0,t]} \|h(s)\|_S.$$

Again, employing the same argument as before, one can show the finiteness of the integral $S_3^n(t)$ as

$$\begin{aligned} S_3^n(t) &\leq k \int_0^t \int_0^n \int_{n-\mu}^\infty (1 + \mu^{-\sigma})(\mu + \nu)^{-\sigma} g(\mu, s)|Z(\nu, s)|d\nu d\mu ds \\ &\leq kn^{-\sigma} \int_0^t \int_0^\infty \int_0^\infty (1 + \mu^{-\sigma})g(\mu, s)[g(\nu, s) + h(\nu, s)]d\nu d\mu ds \\ &\leq 2kTn^{-\sigma} \sup_{s \in [0,t]} \left[\{\|g(s)\|_S + \|h(s)\|_S\} \|g(s)\|_S \right] < \infty. \end{aligned}$$

Thus, we obtain

$$S_3^n(t) \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{4.9}$$

Similarly, one can easily show that $S_6^n(t) \rightarrow 0$ as $n \rightarrow \infty$.

Next, $S_4^n(t)$ can be evaluated, by applying (1.10), (1.7), Young’s inequality and the definition of norm, as

$$\begin{aligned} S_4^n(t) &\leq \frac{1}{2}k\eta(\sigma) \int_0^t \int_0^n \int_0^n \{1 + (\nu + \tau)^{-\sigma}\}(\nu + \tau)^{-\sigma} g(\nu, s)|Z(\tau, s)|d\tau d\nu ds \\ &\leq \frac{k}{2}\eta(\sigma) \int_0^t \int_0^n \int_0^n (1 + \nu^{-\sigma})(1 + \tau^{-\sigma})g(\nu, s)|Z(\tau, s)|d\tau d\nu ds \\ &\leq \Gamma_3 \int_0^t \Xi^n(s)ds, \end{aligned} \tag{4.10}$$

where

$$\Gamma_3 = \frac{3k}{2}\eta(\sigma) \sup_{s \in [0,t]} \|g(s)\|_S.$$

Analogous to $S_4^n(t)$, $S_5^n(t)$ can be calculated as

$$S_5^n(t) \leq \Gamma_4 \int_0^t \Xi^n(s)ds, \tag{4.11}$$

where

$$\Gamma_4 = \frac{3}{2}k\eta(\sigma) \sup_{s \in [0,t]} \|h(s)\|_{\mathcal{S}}.$$

Now, taking $n \rightarrow \infty$ to (4.4) and then inserting (4.7), (4.8), (4.10) and (4.11), we have

$$\lim_{n \rightarrow \infty} \Xi^n(t) \leq \left(\sum_{i=1}^4 \Gamma_i \right) \lim_{n \rightarrow \infty} \int_0^t \Xi^n(s) ds. \quad (4.12)$$

The inequality (4.12) implies that

$$\int_0^\infty (1 + \mu^{-\sigma}) |Z(\mu, t)| d\mu \leq \left(\sum_{i=1}^4 \Gamma_i \right) \int_0^t \int_0^\infty (1 + \mu^{-\sigma}) |Z(\mu, s)| d\mu ds. \quad (4.13)$$

Then applying Gronwall's inequality to (4.13), we obtain

$$\int_0^\infty (1 + \mu^{-\sigma}) |Z(\mu, t)| d\mu = 0 \quad \forall t.$$

This implies $g(\mu, t) = h(\mu, t)$ a.e. $\mu \in \mathbb{R}_+$. This completes the Proof of the Theorem 2.3. \square

Acknowledgements

This work was mainly supported by Science and Engineering Research Board (SERB), Department of Science and Technology (DST), India through the project *YSS/2015/001306*. In addition, authors would like to thank University Grant Commission (UGC) India for providing the PhD fellowship to PKB through the Grant 6405/11/44.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Aldous, D.J.: Deterministic and stochastic model for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists. *Bernoulli* **5**, 3–48 (1990)
- [2] Ash, R.B.: *Measure, Integration and Functional Analysis*. Academic Press, New York (1972)
- [3] Barik, P.K.: Existence of mass-conserving weak solutions to the singular coagulation equation with multiple fragmentation. *Evol. Equ. Control Theory* **9**, 431–446 (2020)
- [4] Barik, P.K., Giri, A.K.: A note on mass-conserving solutions to the coagulation and fragmentation equation by using non-conservative approximation. *Kinet. Relat. Models* **11**, 1125–1138 (2018)

- [5] Barik, P.K., Giri, A.K.: Weak solutions to the continuous coagulation model with collisional breakage. *Discrete Contin. Dyn. Syst.* **40**, 6115–6133 (2020)
- [6] Barik, P.K., Giri, A.K., Laurençot, Ph.: Mass-conserving solutions to the Smoluchowski coagulation equation with singular kernel. *Proc. Roy. Soc. Edinburgh Sec. A* **150**, 6115–6133 (2020)
- [7] Brown, P.S.: Structural stability of the coalescence/breakage equations. *J. Atmosph. Sci.* **52**, 3857–3865 (1995)
- [8] Camejo, C.C.: The Singular Coagulation and Coagulation-Fragmentation Equation, PhD thesis (2013)
- [9] Camejo, C.C., Gröpler, R., Warnecke, G.: Regular solutions to the coagulation equations with singular kernels. *Math. Methods Appl. Sci.* **38**, 2171–2184 (2015)
- [10] Camejo, C.C., Warnecke, G.: The singular kernel coagulation equation with multifragmentation. *Math. Methods Appl. Sci.* **38**, 2953–2973 (2015)
- [11] Cheng, Z., Redner, S.: Kinetics of fragmentation. *J. Phys. A. Math. Gen.* **23**, 1233–1258 (1990)
- [12] Cheng, Z., Redner, S.: Scaling theory of fragmentation. *Phys. Rev. Lett.* **60**, 2450–2453 (1988)
- [13] Ernst, M.H., Pagonabarraga, I.: The non-linear fragmentation equation. *J. Phys. A. Math. Theor.* **40**, F331–F337 (2007)
- [14] Escobedo, M., Laurençot, Ph., Mischler, S.: On a kinetic equation for coalescing particles. *Commun. Math. Phys.* **246**, 237–267 (2004)
- [15] Escobedo, M., Laurençot, Ph., Mischler, S., Perthame, B.: Gelation and mass conservation in coagulation-fragmentation models. *J. Diff. Eqs.* **195**, 143–174 (2003)
- [16] Fournier, N., Laurençot, Ph.: Existence of self-similar solutions to Smoluchowski’s coagulation equation. *Commun. Math. Phys.* **256**, 589–609 (2005)
- [17] Kostoglou, M., Karabelas, A.J.: A study of the nonlinear breakage equation: analytical and asymptotic solutions. *J. Phys. A. Math. Gen.* **33**, 1221–1232 (2000)
- [18] Laurençot, Ph.: On a class of continuous coagulation-fragmentation equations. *J. Diff. Equ.* **167**, 245–274 (2000)
- [19] Laurençot, Ph., Mischler, S.: The continuous coagulation-fragmentation equations with diffusion. *Arch. Rational Mech. Anal.* **162**, 45–99 (2002)
- [20] Laurençot, Ph., Wrzosek, D.: The discrete coagulation equations with collisional breakage. *J. Stat. Phys.* **104**, 193–220 (2001)
- [21] Norris, J.R.: Smoluchowski’s coagulation equation: uniqueness, non-uniqueness and hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.* **9**, 78–109 (1999)

- [22] Safronov, V. S.: Evolution of the Protoplanetary Cloud and Formation of the Earth and the Planets (Israel Program for Scientific Translations Ltd. Jerusalem), (1972)
- [23] Stewart, I.W.: A global existence theorem for the general coagulation-fragmentation equation with unbounded kernels. *Math. Methods Appl. Sci.* **11**, 627–648 (1989)
- [24] Stewart, I.W.: A uniqueness theorem for the coagulation-fragmentation equation. *Math. Proc. Comb. Phil. Soc.* **107**, 573–578 (1990)
- [25] Vigil, R.D., Vermeersch, I., Fox, R.O.: Destructive aggregation: Aggregation with collision-induced breakage. *Colloid Inter. Sci.* **302**, 149–158 (2006)
- [26] Walker, C.: Coalescence and breakage processes. *Math. Methods Appl. Sci.* **25**, 729–748 (2002)
- [27] Wilkins, D.: A geometrical interpretation of the coagulation equation. *J. Phys. A* **15**, 1175–1178 (1992)

Prasanta Kumar Barik
Tata Institute of Fundamental Research Centre for Applicable Mathematics
Bangalore Karnataka 560065
India

Ankik Kumar Giri
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee Uttarakhand 247667
India
e-mail: ankik.giri@ma.iitr.ac.in

Received: 26 May 2020.

Accepted: 7 April 2021.