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The first GJMS invariant

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Abstract. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Denote by P_g the GJMS (Graham–Jenne–Mason–Sparling) operator. In this paper, we introduce the GJMS invariant μ and we define the first GJMS invariant μ_1 as the infimum of the first eigenvalue of P_g over the metrics conformal to g and of volume 1. We study when it is attained and whether is equal to μ . As an application, we show that the nonlinear GJMS equation $P_g v = \mu_1 |v|^{N-2} v$ has nodal (sign-changing) solution. When g is Einstein, the above equation has positive solutions if the scalar curvature $S_g > 0$.

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Contents

1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, and let k be an integer such that $k \geq 1$ and $2k \leq n$. In 1992, in [15] Graham–Jenne–Mason–Sparling have defined a family of conformally invariant differential operators defined for any Riemannian metric (GJMS operators for short). The construction of these operators is based on the ambient metric of Fefferman–Graham

[13]. More precisely, for any Riemannian metric g on M, there exists a local, formally self-adjoint, conformally covariant operator

$$P_g : C^\infty(M) \longrightarrow C^\infty(M),$$

such that for all $u \in C^\infty(M)$, the GJMS operator P_g is given by :

$$P_g u = \Delta_g^k u + lot \tag{1}$$

where Δ_g is the Laplace-Beltrami operator, and lot denotes differential terms of lower order. For more detail about P_g, we refer the reader to Robert [24]. This operator enjoys nice conformal invariance properties. Indeed, let $\varphi \in C^\infty(M)$ be a positive function and $N = \frac{2n}{n-2k}$. If $n \neq 2k$, then any metric $\overline{g}$ written in the form $\varphi^{\frac{4}{n-2k}} g$ is a conformal metric to g and therefore, for any metric $\overline{g}$ conformal to g, the operator P_g is conformally invariant in the following sense: for all $u \in C^\infty(M)$, we have $P_g(u\varphi) = \varphi^{N-1} P_{\overline{g}}(u)$. By taking $u \equiv 1$, we get

$$P_g \varphi = \frac{n-2k}{2} Q_{\overline{g}} \varphi^{N-1} \tag{2}$$

where

$$Q_{\overline{g}} = \frac{2}{n-2k} P_{\overline{g}}(1).$$

The scalar Q_g is called the Q-curvature and is a Riemannian invariant associated to this operator. Historically, the notion of the Q-curvature is due to Branson's 1995 article in Transactions of the AMS see [7]. He also defined it in the critical case $n = 2k$. Now when $k = 1$, P_g is the conformal Laplacian operator and Q_g is the scalar curvature S_g (up to a constant). The problem of prescribing a constant scalar curvature is known as the Yamabe problem, the classical reference for this problem is a survey by Lee-Parker [19]. When $k = 2$, P_g is the Paneitz-Branson operator introduced by Paneitz in [22] and the Q-curvature was introduced by Branson-Ørsted [8]. Results for the prescription of the Q-curvature problem for the Paneitz operator are in Djadli-Hebey-Ledoux [10], Robert [23], Esposito-Robert [12], Hang-Yang [17], Gursky-Malchiodi [16] and Benalili-Boughazi [3]. Moreover, concerning fourth-order problems, there has been also an intensive literature on the question, we refer the reader to [3,5,9,23]. Solving the problem of prescribing Q-curvature for the GJMS operator is a very difficult problem and its underlying analysis is intricate, we refer to Robert [24] and Mazumdar [21] for some particular situations. The simple case of these problems is prescribing constant Q-curvature which is equivalent to finding a positive smooth solution u of the following equation

$$P_g u = C|u|^{N-2} u \tag{3}$$

where C is a constant. In order to obtain solutions, we define the quantity

$$\mu = \inf_{u \in C^\infty(M), u>0} I(u) \tag{4}$$

where

$$I(u) = \frac{\int_M u P_g u dv_g}{(\int_M |u|^N dv_g)^{\frac{2}{N}}}. \tag{5}$$

As in the Yamabe problem, the constant μ will be called the GJMS invariant. In particular, if $u \in C^\infty(M)$, $u > 0$ and satisfy $I(u) = \mu$, clearly u is solution of (3) and $\overline{g} = u^{\frac{4}{n-2k}} g$ is the desired metric of constant Q-curvature. It is well known that the operator P_g is elliptic, self-adjoint with respect to the inner product in $L^2(M)$ [24] and has discrete spectrum with eigenvalues

$$\lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \cdots \leq \lambda_k(g) \to +\infty$$

appear with their multiplicities. The variational characterization of the first eigenvalue $\lambda_1(g)$ of P_g is given by:

$$\lambda_1(g) = \inf_{v \in H_k^2(M), u \neq 0} \frac{\int_M v P_g v dv_g}{\int_M v^2 dv_g}. \tag{6}$$

where the space $H_k^2(M)$ is the completion of $C^\infty(M)$ for the norm

$$\|u\|_{H_k^2} = \left(\int_M \sum_{l=0}^{k} |\nabla^l u|^2 dv_g \right)^{\frac{1}{2}}. \tag{7}$$

Now by referring to Ammann-Humbert [1], we introduce an invariant μ_1 that we will call the first GJMS invariant and we will define it by:

$$\mu_1 = \inf_{\overline{g} \in [g]} \lambda_1(\overline{g}) vol(M, \overline{g})^{\frac{2k}{n}} \tag{8}$$

where the set $[g] = \{\overline{g} = u^{\frac{4}{n-2k}} g,\ u \in C^\infty(M) \text{ and } u > 0\}$ is the conformal class of the metric g and $vol(M, \overline{g}) = \int_M u^N dv_g$ denotes the Riemannian volume of M with respect to the metric $\overline{g}$.

In order to find minimizers, we enlarge the conformal class $[g]$ to what we call the class of generalized metrics conformal to g. We say that $\overline{g} = u^{\frac{4}{n-2k}} g$ is a generalized metric of the Riemannian metric g if $u \in L^N(M), u \geq 0$ and u is not identically null. By the standard minimax method via Rayleigh quotients for defining eigenvalues combined with conformal covariance of P_g, one sees that for any generalized metric $\overline{g} = u^{\frac{4}{n-2k}} g$, the first eigenvalue $\lambda_1(\overline{g})$ of the GJMS opreator P_g is characterized by

$$\lambda_1(\overline{g}) = \inf_{V \in Gr_1^u(H_k^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v P_g v dv_g}{\int_M u^{N-2} v^2 dv_g} \tag{9}$$

where the Grassmannian $Gr_1^u(H_k^2(M))$ is given in the Definition (2.2).

The purpose of this paper is to study the first eigenvalue $\lambda_1(\overline{g})$ for any generalized metric $\overline{g}$ and the main problem is whether the first GJMS invariant μ_1 is attained by a generalized metric (or conformal metric) and is equal to the GJMS invariant μ. To solve this problem, we will use the ideas from [1–6,11,18]. More precisely, the method we would like to apply is introduced in [1] for studying the second Yamabe invariant μ_2 (see Definition (2.1) for μ_2) and generalized for the Paneitz-Branson operator on Einstein manifolds by Benalili and Boughazi in [3]. For clarity purposes, we state our main generic theorem and after we give some results about this method in the next section:

Theorem 1.1. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $\lambda_1(g) > 0$ and $\mu < {K_0}^{-1}$ where μ is the GJMS invariant and K_0 is the best constant in the Sobolev embedding of $H_k^2(M)$. Then there exists a nontrivial function $v \in C^{2k}(M)$ which satisfies $P_g v = \mu_1 |v|^{N-2} v$. In other words, μ_1 is attained by the generalized metric $\overline{g} = |v|^{\frac{4}{n-2k}} g$ and in particular, if $Q_g \leq 0$, v is a nodal (sign-changing) solution. Moreover, if g is Einstein and $S_g > 0$, the solution $v > 0$ and $v \in C^\infty(M)$ and this implies that $\mu = \mu_1$ and means that $\overline{g}$ is a conformal metric. Consequently, in the latter case μ_1 is attained by the desired metric $\overline{g}$ of constant Q-curvature: $Q_{\overline{g}} = \frac{2}{n-2k}\mu_1$.*

Note that this theorem is a consequence of Theorem (3.1), Proposition (3.2) and Theorems (4.1), (5.1) and (6.1). The remainder of this paper is organized as follows: In Sect. 2, we give a short motivation by recalling some results and we quote some facts which we will use in the sequel. In Sect. 3, we establish some results concerning the eigenvalues; in particular, if $\lambda_1(g) > 0$, for all generalized metric $\overline{g} = u^{\frac{4}{n-2k}} g$, the first eigenvalue $\lambda_1(\overline{g})$ is achieved, we also show that the linear equation $P_g v = \lambda_1(\overline{g}) u^{N-2} v$ has nodal (sign-changing) solution if $Q_g \leq 0$ and if $\lambda_1(g) < 0$, we show that there exists a generalized metric $\overline{g}$ such that $\lambda_1(\overline{g}) = -\infty$ which implies that $\mu_1 = -\infty$. In Sect. 4, we study the first GJMS invariant μ_1 in case $\lambda_1(g) > 0$, we will prove that μ_1 is attained by a generalized metric if $1 - \mu K_0 > 0$ where μ is the GJMS invariant and K_0 is the best constant in the Sobolev embedding see (20). In Sect. 5, we show that the nonlinear GJMS equation $P_g v = \mu_1 |v|^{N-2} v$ has a nodal solution if $Q_g \leq 0$. In Sect. 6, we deal with Einstein manifold. In particular, when $S_g > 0$, we will prove that the solution v of the latter equation is positive, $\mu_1 = \mu$ and is attained by a conformal metric $\overline{g}$ which leads to a metric with constant Q-curvature and in the case $S_g < 0$, the solution v is nodal. At the end, we show that $K_0 = (\mu(S^n, h))^{-1}$ where $\mu(S^n, h)$ is the GJMS invariant of the standard unit n-sphere of $\mathbb{R}^{n+1}$ and there are certain manifolds such that the assumption $1 - \mu K_0 > 0$ holds.

Note that when (M,g) is Einstein manifold with positive scalar curvature, the Q-curvature $Q_g = \frac{n-2k}{2}(\prod_{l=1}^k c_l)(S_g)^k$, is constant and positive [the reals c_l are given in (17)]. Therefore it is easy to see that $u = 1$ is solution of $P_g u = \mu_1 u^{N-1}$ if $\mu_1 = \frac{n-2k}{2} Q_g$, in other word μ_1 is achieved by the metric g. If $\mu_1 \neq \frac{n-2k}{2} Q_g$ and $1 - \mu K_0 > 0$ Theorem (6.1) proves the existence of $u \in C^\infty(M), u > 0$ solution to the latter equation. In particular, it follows the existence of a metric $\overline{g} = u^{\frac{4}{n-2k}} g$ which is different from the initial g and such that $Q_{\overline{g}}$ is constant. But it is not clear whether the solution u is different from the trivial constant solution. This question seems to be hard. However, when P_g is the conformal Laplacian operator the reader is refereed to [18] for more detail on the question.

2. Motivation and generality

We start by giving a short motivation by recalling some results. Indeed, in [1] Ammann and Humbert defined the Yamabe invariant of high order μ_p by

Definition 2.1.

$$\mu_p = \inf_{\overline{g}\in[g]} \lambda_p(\overline{g})\,[vol(M,\overline{g})]^{\frac{2}{n}}$$

where

$$\lambda_p(\overline{g}) = \inf_{V\in Gr_p^u(H_k^2(M))}\ \sup_{v\in V\setminus\{0\}} \frac{\int_M vP_g v dv_g}{\int_M u^{N-2}v^2 dv_g}$$

is the p^{th} eigenvalue of the conformal Laplacian P_g, $\overline{g} = u^{\frac{4}{n-2k}} g$ is a generalized metric, $p \in \mathbb{N}^*$ and the Grassmannian $Gr_p^u(H_k^2(M))$ is given in the Definition (2.2).

The authors studied the second Yamabe invariant μ_2 in the case $\mu \geq 0$ where μ is the Yamabe invariant. In particular, they obtained the following theorem:

Theorem 2.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that μ_2 is attained by a generalized metric. Then the following equation $P_g w = \mu_2 u^{N-2} w$ has a nodal solution $w \in C^2(M)$ such that $u = |w|$.*

Inspired by the previous results. In [3], Benalili and Boughazi generalize this method to the Paneitz-Branson operator on Einsteinian manifolds. Under some assumptions, they studied μ, μ_1 and μ_2 in the case $S_g > 0$ and after ten years the authors in [5] extend these results to the case $S_g < 0$. For more detail and similar work, we refer the readers to Benalili-Boughazi [4], Boughazi [6] and S. Elsayed [11]. We also specify a very interesting result proven in [11] which states that the sign of eigenvalue $\lambda_p(\overline{g})$ is conformal invariant. Clearly, in this paper we try to find similar results with the GJMS operator. More precisely, we study μ_1 and we show in which case we can get $\mu_1 = \mu$. Note that the study of μ_2 seems to be much more difficult.

In the following, we quote some facts which will be used in this paper. Put $L_+^N(M) = \{u \in L^N(M), u \geq 0 \text{ and } u \neq 0\}$ and $C_+^\infty(M) = \{u \in C^\infty(M),\ \ u > 0\}$.

Definition 2.2. For all $u \in L_+^N(M)$, $p \in \mathbb{N}^*$, the Grassmannian $Gr_p^u(H_k^2(M))$ is the set of all subspaces of $H_k^2(M)$ of dimension p and such that the restriction operator to $M\setminus u^{-1}(0)$ is injective. More explicitly, we have the subspace $V = span(v_1, \ldots, v_p) \in Gr_p^u(H_k^2(M))$ if and only if the functions $v_1, \ldots, v_p$ are linearly independent on $M\setminus u^{-1}(0)$. Sometimes it will be convenient to use the equivalent statement that the functions $u^{\frac{4}{n-2k}} v_1, u^{\frac{4}{n-2k}} v_2, \ldots, u^{\frac{4}{n-2k}} v_p$ are linearly independent.

Remark 2.1. The number $N = \frac{2n}{n-2k}$ is known as the critical exponent of the Sobolev embedding which [18], asserts that the space $H_k^2(M) \subset L^q(M)$ where $1 < q \leq N$ and this embedding is compact when $q < N$.

Definition 2.3. A generalized metric conformal to g is a metric of the form $\overline{g} = u^{\frac{4}{n-2k}} g$ such that $u \in L_+^N(M)$ where the space $L_+^N(M)$ is defined in the bottom of the previous page.

Now, we give some properties of the GJMS operator. For the proofs, the reader is refereed to Robert [24] and the references therein. The operator P_g can be written (partially) in divergence form, we precise this divergence form that will be useful in the sequel:

Proposition 2.1. *Let P_g be the conformal GJMS operator. Then for any $l \in \{1,\ldots,k-1\}$, there exists $A_{(l)}(g)$ a smooth T^0_{2l}-tensor field on M such that*

$$P_g v = \Delta_g^k v + \sum_{l=1}^{k-1} A_{l,g}(v) + \frac{n-2k}{2} Q_g v \tag{10}$$

where

$$A_{l,g}(v) = (-1)^l \nabla^{j_l \ldots j_1} (A_{(l)}(g)_{i_1\ldots i_l j_1 \ldots j_l} \nabla^{i_1 \ldots i_l} v). \tag{11}$$

Indices are raised via the musical isomorphism. In addition for any $l \in \{1,\ldots,k-1\}$, $A_{(l)}(g)$ is symmetric in the following sense: $A_{(l)}(g)(X,Y) = A_{(l)}(g)(Y,X)$ for all T^l_0-tensors X, Y on M. In particular, for all $u, v \in C^\infty(M)$ we have

$$\int_M vP_g u dv_g = \int_M uP_g v dv_g = \int_M \Delta_g^{\frac{k}{2}} u \Delta_g^{\frac{k}{2}} v + \sum_{l=0}^{k-1} A_{(l)}(g)(\nabla^l u, \nabla^l v) dv_g \tag{12}$$

where for $l = 0$, $A_{(0)}(g)(\nabla^0 u, \nabla^0 v) = \frac{n-2k}{2} Q_g uv$. Here, we have adopted the convention

$$\Delta_g^{\frac{k}{2}}(u) = \begin{cases} \Delta_g^m(u) & \text{if} \quad k = 2m \quad \text{is even} \\ \nabla\Delta_g^m(u) & \text{if} \quad k = 2m+1 \quad \text{is odd} \end{cases}$$

and, when $k = 2m + 1$ is odd, $\Delta_g^{\frac{k}{2}} u \Delta_g^{\frac{k}{2}} v = (\nabla\Delta_g^m u, \nabla\Delta_g^m v)$.

Since $A_{(l)}(g)$ are smooth, then for any $l \in \{0,\ldots,k-1\}$, there exist $C_l > 0$ such that for all $u \in H_k^2(M)$, one has

$$\left|\int_M \sum_{l=0}^{k-1} A_{(l)}(g)(\nabla^l u, \nabla^l u) dv_g\right| \le \sum_{l=0}^{k-1} C_l \int_M |\nabla^l u|^2 dv_g \le \max(C_l) \|u\|^2_{H^2_{k-1}} \tag{13}$$

As a consequence of (12), we get that the bilinear form $(u, v) \mapsto \int_M uP_g v dv_g$ extends to a continuous symmetrical bilinear form on the space $H_k^2(M) \times H_k^2(M)$.

We say that P_g is coercive if there exists $C > 0$ such that

$$\int_M vP_g v dv_g \ge C\|v\|_2^2 \quad \forall v \in H_k^2(M). \tag{14}$$

Proposition 2.2. *For all $u \in H_k^2(M)$, we define the semi-norm $\|u\|_{P_g}$ by*

$$\|u\|_{P_g} = \left(\int_M uP_g u dv_g\right)^{\frac{1}{2}}. \tag{15}$$

Assume that P_g is coercive. Then $\|.\|_{P_g}$ is a norm on $H_k^2(M)$ equivalent to the standard norm $\|.\|_{H_k^2}$. In addition, if $(v_m)_m$ is a sequence in $H_k^2(M)$ such that

$v_m \longrightarrow 0$ weakly in $H_k^2(M)$, and $v_m \longrightarrow 0$ strongly in $H_{k-1}^2(M)$, then from Bochner-Lichnerowicz-Weitzenbock type formula, one gets that

$$\int_M (|\Delta_g^{\frac{k}{2}}(v_m)|^2 dv_g = \int_M |\nabla^k v_m|^2 dv_g + o(1), \tag{16}$$

The next definition can be found in [18].

Definition 2.4. A Riemannian manifold (M, g) is Einstein, if and only if there exists a real λ such that the Ricci tensor writes $Ric_g = \lambda g$. Here $\lambda = \frac{S_g}{n}$, where S_g is the scalar curvature and is constant in this case.

The reader is refereed to [14] for the two following propositions:

Proposition 2.3. *Assume that (M, g) is Einstein, then P_g expresses as an explicit product of second-order operators with constant coefficients that depend only on the scalar curvature. In other words, the GJMS operator P_g is given by*

$$P_g = \prod_{l=1}^{k}(\Delta_g + c_l S_g) \quad where \quad c_l = \frac{(n+2l-2)(n-2l)}{4n(n-1)}. \tag{17}$$

Moreover, by calculating one can write

$$P_g = \Delta_g^k + \sum_{l=0}^{k-2} b_{k-l-1}(S_g)^{l+1}\Delta_g^{k-l-1} + b_0(S_g)^k \tag{18}$$

where $b_{k-1}, \ldots, b_1, b_0$ are positive real numbers obtained from c_l.

In addition, formula (12) implies that

$$\int_M uP_g u dv_g = \int_M (|\Delta_g^{\frac{k}{2}}(u)|^2 + \sum_{l=0}^{k-1} b_{k-l-1}(S_g)^{l+1}|\nabla^{k-l-1}u|^2) dv_g. \tag{19}$$

Proposition 2.4. *Assume that the metric g is Einstein with $S_g > 0$ and $n > 2k$, then P_g is coercive and for all $u \in C^{2k}(M)$ such that $P_g u \geq 0$, either $u > 0$ or $u \equiv 0$.*

In this definition, we are going to introduce the best constant in the Sobolev embedding $D_k^2(\mathbb{R}^n) \subset L^N(\mathbb{R}^n)$. The reader is refereed to Lions [20].

Definition 2.5. Let $D_k^2(\mathbb{R}^n)$ be the space defined as the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $\|\Delta^{\frac{k}{2}} u\|_2$. It is well know that

$${K_0}^{-1} = \inf_{u \in D_k^2 \mathbb{R}^n) - \{0\}} \frac{\int_{\mathbb{R}^n} |\Delta^{\frac{k}{2}} u|^2 dv_g}{(\int_{\mathbb{R}^n} |u|^N dv_g)^{\frac{2}{N}}} \tag{20}$$

and K_0 is the best constant in the Sobolev's continuous embedding $D_k^2(\mathbb{R}^n) \subset L^N(\mathbb{R}^n)$. It follows from Sobolev's embedding theorem that $K_0 > 0$. Moreover, the infimum is achieved by $U : x \longmapsto (1 + |x|^2)^{k-\frac{n}{2}}$, and that all minimizers are compositions of U by translations, homotheties and dilatations.

We also introduce the following results. For the proofs, the reader is refereed to Mazumdar [21].

Theorem 2.2. *Let (M, g) be a compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. For any $\epsilon > 0$, there exists $B_\epsilon > 0$ such that for all $u \in H_k^2(M)$ one has*

$$\|u\|_N^2 \leq (K_0 + \epsilon) \int_M |\Delta_g^{\frac{k}{2}}(u)|^2 dv_g + B_\epsilon \|u\|_{H_{k-1}^2}^2. \tag{21}$$

where K_0 is given by formula (20).

Moreover, for all $v \in C^\infty(M)$, there exists $C > 0$ (depend on $\|v\|_\infty$) such that

$$\int_M |\Delta_g^{\frac{k}{2}}(vu)|^2 dv_g \leq C \int_M |\Delta_g^{\frac{k}{2}}(u)|^2 dv_g + \|u\|_{H_{k-1}^2}^2 \tag{22}$$

where $u \in H_k^2(M)$.

Proposition 2.5. *Let (M, g) be a closed manifold of dimension n and let k be a positive integer such that $2k < n$. Let $f \in C^{0,\alpha}(M)$ a Hölder continuous function. Suppose that $u \in H_k^2(M)$ be a weak solution of $P_g u = f|u|^{N-2}u$. Then $u \in C^{2k}(M)$, and is a classical solution of the above equation. Further if $u > 0$ and $f \in C^\infty(M)$, then $u \in C^\infty(M)$.*

3. Generalized metrics and the first eigenvalue

Theorem 3.1. *For any generalized metric $\overline{g} = u^{\frac{4}{n-2k}} g$, assume that $u > 0$. Then any normalized minimizing sequence of $\lambda_1(\overline{g})$ is bounded in $H_k^2(M)$.*

Proof. Let $(v_m)_m$ be a minimizing sequence of $\lambda_1(\overline{g})$, in other words

$$\lambda_1(\overline{g}) = \lim_{m \longrightarrow +\infty} \lambda_{1,m} \quad \text{where} \quad \lambda_{1,m} = \frac{\int_M (|\Delta_g^{\frac{k}{2}}(v_m)|^2 + \sum_{l=0}^{k-1} A_{(l)}(g)(\nabla^l v_m, \nabla^l v_m)) dv_g}{\int_M u^{N-2} v_m^2 dv_g}.$$

It is easy to see that $(\lambda v_m)_m$ is also a minimizing sequence, then if we choose $\lambda = (\int_M u^{N-2} v_m^2 dv_g)^{-\frac{1}{2}}$, it follows that $\int_M u^{N-2} (\lambda v_m)^2 dv_g = \lambda^2 \int_M u^{N-2} v_m^2$ $dv_g = 1$, hence the sequence $(\lambda v_m)_m$ is renormalized. Without loss of generality, we assume that the sequence $(v_m)_m$ is such that

$$\int_M u^{N-2} v_m^2 dv_g = 1. \tag{23}$$

1) If $\lambda_1(\overline{g}) > 0$, then for all v in $H_k^2(M) \setminus \{0\}$, one has

$$\begin{aligned} \int_M v P_g v dv_g &\geq \lambda_1(\overline{g}) \int_M u^{N-2} v^2 dv_g \\ &\geq \underbrace{\lambda_1(\overline{g}) \min_{x \in M} u(x)^{N-2}}_{C} \int_M v^2 dv_g \quad \text{since} \quad u > 0 \\ &\geq C \|v\|_2^2 \end{aligned}$$

this means that P_g is coercive. Then Proposition (2.2) implies that $\|.\|_{P_g}$ is a norm on $H_k^2(M)$ equivalent to the standard norm $\|.\|_{H_k^2}$, then for m large enough, one has

$$\lambda_{1,m} = \int_M v_m P_g v_m dv_g = \|v_m\|_{P_g}^2 \le \lambda_1(\overline{g}) + 1,$$

hence the sequence $(v_m)_m$ is bounded in $H_k^2(M)$ and $\lambda_{1,m} \ge 0$.

2) If $\lambda_1(\overline{g}) < 0$, the GJMS operator is not necessarily coercive, then we will assume that $(v_m)_m$ is not bounded in $H_k^2(M)$, in other words $\|v_m\|_{H_k^2} \longrightarrow +\infty$ and we let

$$v'_m = \frac{v_m}{\|v_m\|_{H_k^2}}.$$

Clearly $\|v'_m\|_{H_k^2} = 1$, this means that the sequence $(v'_m)_m$ is bounded in $H_k^2(M)$ and after restriction to a subsequences still labeled $(v'_m)_m$, we may assume that there exists $v' \in H_k^2(M)$ such that $v'_m \longrightarrow v'$ weakly in $H_k^2(M)$ and $v'_m \longrightarrow v'$ strongly in $H_{k-1}^2(M)$.

On the other hand, the sequence $(v'_m)_m$ satisfies the following equation:

$$\int_M (|\Delta_g^{\frac{k}{2}}(v'_m)|^2 dv_g + \sum_{l=0}^{k-1} \int_M A_{(l)}(g)(\nabla^l v'_m, \nabla^l v'_m) dv_g = \lambda_{1,m} \int_M u^{N-2} {v'}_m^2 dv_g. \tag{24}$$

Now from the weak convergence, we have

$$\lim \int_M u^{N-2} v' v'_m dv_g = \int_M u^{N-2} (v')^2 dv_g,$$

and since

$$0 \le \int_M u^{N-2}(v' - v'_m)^2 dv_g = \int_M u^{N-2}(v')^2 dv_g - 2\int_M u^{N-2} v' v'_m dv_g + \int_M u^{N-2}(v'_m)^2 dv_g$$

one has,

$$\int_M u^{N-2}(v')^2 dv_g \le \int_M u^{N-2}(v'_m)^2 dv_g = \frac{\int_M u^{N-2} v_m^2 dv_g}{\|v_m\|_{H_k^2}^2} = \frac{1}{\|v_m\|_{H_k^2}^2} \longrightarrow 0. \tag{25}$$

Consequently,

$$\int_M u^{N-2}(v')^2 dv_g = 0$$

and since $u > 0$, it is easy to see that

$$v' \equiv 0.$$

It follows that $v'_m \longrightarrow 0$ weakly in $H_k^2(M)$ and $v'_m \longrightarrow 0$ strongly in $H_{k-1}^2(M)$ therefore,

$$\int_M \sum_{l=0}^{k-1} |\nabla^l v'_m|^2 dv_g \longrightarrow 0 \tag{26}$$

then by (26) and using (13), one has also

$$\sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l v'_m, \nabla^l v'_m)dv_g \longrightarrow 0. \tag{27}$$

Again by (26), the following equality

$$1 = \|v'_m\|^2_{H^2_k} = \int_M |\nabla^k v'_m|^2 dv_g + \int_M \sum_{l=0}^{k-1} |\nabla^l v'_m|^2 dv_g$$

leads necessarily to

$$\int_M |\nabla^k v'_m|^2 dv_g \longrightarrow 1.$$

Independently, from formula (16) i.e

$$\int_M (|\Delta_g^{\frac{k}{2}}(v'_m)|^2 dv_g = \int_M |\nabla^k v'_m|^2 dv_g + o(1),$$

this implies that

$$\int_M (|\Delta_g^{\frac{k}{2}}(v'_m)|^2 dv_g \longrightarrow 1. \tag{28}$$

Since $\lambda_1(\overline{g}) < 0$, then for m large enough $\lambda_{1,m} < 0$, it follows from (24), (25), (27) and (28) that the sequence $(v'_m)_m$ is such that

$$\underbrace{\int_M (|\Delta_g^{\frac{k}{2}}(v'_m)|^2 dv_g}_{\longrightarrow 1} + \underbrace{\sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l v'_m, \nabla^l v'_m)dv_g}_{\longrightarrow 0} = \underbrace{\lambda_{1,m}\int_M u^{N-2}(v'_m)^2 dv_g}_{\longrightarrow a},$$

where $a \leq 0$ or does not exist, in all cases this gives a contradiction. This proves that $(v_m)_m$ is bounded in $H^2_k(M)$.

Moreover, we have

$$-\int_M |\nabla^k v_m|^2 dv_g \leq \int_M (|\Delta_g^{\frac{k}{2}}(v_m)|^2 dv_g,$$

which lead to

$$-\int_M |\nabla^k v_m|^2 dv_g - \max(C_l)\|u\|^2_{H^2_{k-1}} \leq \int_M (|\Delta_g^{\frac{k}{2}}(v_m)|^2 dv_g + \sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l v_m, \nabla^l v_m)dv_g$$

where $\max(C_l)$ is given by (13), this means that

$$\min(-1, -\max(C_l))\|v_m\|^2_{H^2_k} \leq \int_M (|\Delta_g^{\frac{k}{2}}(v_m)|^2 + \sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l v_m, \nabla^l v_m)dv_g$$

In other words,

$$\lambda_{1,m} \geq \min(-1, -\max(C_l))\|v_m\|^2_{H^2_k}$$

and since $(v_m)_m$ is bounded, then there exists $M > 0$ such that

$$\lambda_{1,m} \geq \min(-1, -\max(C_l))M > -\infty.$$

□

Proposition 3.1. *Assume that* $\lambda_1(g) < 0$*, then there exists* $u \in L^N_+(M)$ *such that* $\lambda_1(\overline{g}) = -\infty$ *where* $\overline{g} = u^{\frac{4}{n-2k}} g$.

Proof. Since $\lambda_1(g) < 0$, there exist a function $v \in C^\infty(M)$ such that $\int_M vP_g v dv_g < 0$. Fix a point p in M. For $\epsilon > 0$, let ϕ_ϵ be a cut-off function adapted to our context, in other words a smooth function such that :

$$\begin{cases} 0 \leq \phi_\epsilon \leq 1 \\ \phi_\epsilon = 0 \quad \text{on} \quad B_\epsilon(p) \quad \text{and} \quad \phi_\epsilon = 1 \quad \text{on} \quad M \backslash B_{2\epsilon}(p) \\ |\nabla^l \phi_\epsilon| \leq \frac{c_l}{\epsilon^l} \quad \text{for all} \quad l \quad \text{in} \quad \{1, 2, ..., k-1\} & (28_1) \\ |\Delta_g^{\frac{k}{2}} \phi_\epsilon| \leq \frac{c_k}{\epsilon^k} & (28_2) \end{cases}$$

where $B_\epsilon(p)$ is the open ball centered at p and of radius ϵ and $c_l > 0$ are constants that do not depend on ϵ.

We claim that:

$$\lim_{\epsilon \to 0} \int_M (\phi_\epsilon v) P_g (\phi_\epsilon v) dv_g = \int_M vP_g v dv_g < 0.$$

Indeed:

Set $A_\epsilon(p) = B_{2\epsilon}(p) \backslash B_\epsilon(p)$, then one has

$$\int_M (\phi_\epsilon v_1) P_g^n (\phi_\epsilon v_1) dv_g = \underbrace{\int_{B_\epsilon(p)} (\phi_\epsilon v) P_g (\phi_\epsilon v) dv_g}_{I_1} + \underbrace{\int_{A_\epsilon(p)} (\phi_\epsilon v) P_g (\phi_\epsilon v) dv_g}_{I_2}$$
$$+ \underbrace{\int_{M \backslash B_{2\epsilon}(p)} (\phi_\epsilon v) P_g (\phi_\epsilon v) dv_g}_{I_3}.$$

Clearly the first integral $I_1 = 0$ (since $\phi_\epsilon = 0$ on the ball $B_\epsilon(p)$). For the second integral I_2 since $v \in C^\infty(M)$, we can find a constant $C > 0$ such that

$$\begin{aligned} |I_2| &\leq \int_{A_\epsilon(p)} (|\Delta_g^{\frac{k}{2}} (\phi_\epsilon v)|^2 + \sum_{l=0}^{k-1} |A_{(l)}(g)(\nabla^l \phi_\epsilon v, \nabla^l \phi_\epsilon v)|) dv_g \\ &\leq C \left(\int_{A_\epsilon(p)} (|\Delta_g^{\frac{k}{2}} (\phi_\epsilon)|^2 + \sum_{l=0}^{k-1} |\nabla^l (\phi_\epsilon)|^2) dv_g \right). \end{aligned} \tag{29}$$

The latter inequality (29) is a direct consequence of formula (22).

Using (28_1), (28_2), (29) and passing to the polar coordinates, we can easily find constants $C_k, C_{k-1}, \ldots, C_0 > 0$ such that,

$$|I_2| \leq \frac{C_k}{\epsilon^{2k}} \int_\epsilon^{2\epsilon} r^{n-1} dr + \left(\sum_{l=0}^{k-1} \frac{C_l}{\epsilon^{2l}} \right) \int_\epsilon^{2\epsilon} r^{n-1} dr \underset{\epsilon \to 0}{\longrightarrow} 0 \quad \text{since} \quad n > 2k$$

which means that the second integral:

$$I_2 \underset{\epsilon\to 0}{\longrightarrow} 0.$$

And finally since $\phi_\epsilon = 1$ on $M\backslash B_{2\epsilon}(p)$, the third integral $I_3 = \int_{M\backslash B_{2\epsilon}(p)} vP_g v dv_g$.

This implies that:

$$\lim_{\epsilon\to 0}\int_M (\phi_\epsilon v)P_g(\phi_\epsilon v)dv_g = \lim_{\epsilon\to 0}(I_1+I_2+I_3) = \lim_{\epsilon\to 0} I_3 = \int_M vP_g v dv_g < 0.$$

If we put $w = \phi_\epsilon v$, for ϵ small enough, we still have $\int_M wP_g w dv_g < 0$.

Now, let $u_\epsilon \geq 0$ be a smooth function with support in $B_\epsilon(p)$ and let $\overline{g} = u_\epsilon^{\frac{N-2}{k}} g$ since

$$\lambda_1(\overline{g}) = \inf_{v\in H_k^2(M),v\neq 0} \frac{\int_M vP_g v dv_g}{\int_M u_\epsilon^{N-2} v^2 dv_g}$$

it follows that for any real $\alpha > 0$, one has

$$\lambda_1(\overline{g}) \leqslant \lim_{\alpha\longrightarrow 0} \frac{\int_M (w+\alpha)P_g(w+\alpha)dv_g}{\int_M u_\epsilon^{N-2}(w+\alpha)^2 dv_g} = -\infty.$$

Indeed

$$\lim_{\alpha\to 0}\int_M u_\epsilon^{N-2}(w+\alpha)^2 dv_g = 0$$

and

$$\lim_{\alpha\to 0}(\int_M (w+\alpha)P_g(w+\alpha)dv_g = \int_M wP_g w dv_g < 0.$$

□

Theorem 3.2. *Let $\overline{g} = u^{\frac{4}{n-2k}} g$ be any generalized metric to g such that $u > 0$. Assume that $\lambda_1(g) > 0$. Then there exists a nontrivial function v in $H_k^2(M)$ such that, in the weak sense, v satisfy :*

$$P_g v = \lambda_1(\overline{g})u^{N-2}v \quad \textit{and} \quad \int_M u^{N-2}v^2 dv_g = 1 \tag{30}$$

Moreover, if $u \in C^\infty(M)$, then $v \in C^\infty(M)$ and if (M, g) is Einstein and $S_g > 0$, the solution $v > 0$.

Proof. Let $(v_m)_m$ be a minimizing sequence for $\lambda_1(\overline{g})$. In other words, the sequence $(v_m)_m \in H_k^2(M)$, $u^{\frac{N-2}{2}} v_m \neq 0$ and such that

$$\lim_m \frac{\int_M v_m P_g v_m dv_g}{\int_M u^{N-2} v_m^2 dv_g} = \lambda_1(\overline{g}). \tag{31}$$

Without loss of generality, we can always normalize v_m by $\int_M u^{N-2} v_m^2 dv_g = 1$.

Since $\lambda_1(g) > 0$, P_g is coercive. Then Theorem (3.1) implies that the sequence (v_m) is bounded in $H_k^2(M)$, and after restriction to a subsequence we may assume that there exists v in $H_k^2(M)$ such that $v_m \to v$ weakly in $H_k^2(M)$, strongly in $H_{k-1}^2(M)$ and almost everywhere in M. Again since P_g is coercive, Proposition (2.2) implies that $\|.\|_{P_g}$ is a norm on $H_k^2(M)$ equivalent to the standard norm $\|.\|_{H_k^2}$, then by standard argument, one has

$$\int_M vP_g v dv_g \leq \liminf \int_M v_m P_g v_m dv_g = \lambda_1(\overline{g}),$$

as in [6] from (Lemma (4)), we get

$$\int_M u^{N-2}|v^2 - v_m^2|\; dv_g \to 0 \text{ i.e } \int_M u^{N-2} v^2 dv_g = 1$$

and since $\lambda_1(\overline{g})$ is the infimum, one gets

$$\int_M vP_g v dv_g = \lambda_1(\overline{g}).$$

Consequently v is a non-trivial weak minimizer of the functional associated to $\lambda_1(\overline{g})$. Writing the Euler-Lagrange equation, we find that v satisfies in the weak sense the equation

$$P_g v = \lambda_1(\overline{g}) u^{N-2} v.$$

Moreover, since v is nontrivial, we have

$$\lambda_1(\overline{g}) = \int_M vP_g v dv_g = \|v\|_{P_g}^2 > 0. \tag{32}$$

If $u \in C_+^\infty(M)$, we get $\lambda_1(\overline{g}) u^{N-2} v \in H_k^2(M)$, then $P_g v \in H_k^2(M)$ and by regularity theorems $v \in H_{3k}^2(M)$, it follows by successive iterations that $v \in H_l^2(M)$ where l is large enough and finally if $\frac{1}{2} < \frac{l-m}{n}$, one gets

$$H_l^2(M) \subset C^m(M)$$

so we can take $m = 2k$ i.e

$$v \in C^{2k}(M), \text{ therefore } v \in C^\infty(M).$$

In particular, if (M, g) is Einstein and $S_g > 0$, from [6] (Proposition (7)), one has

$$v > 0.$$

□

Remark 3.1. Let v be the solution of the Eq. (30). Then there exists a non-trivial function w in $H_k^2(M)$ such that, in the weak sense one has :

$$P_g w = \lambda_2'(\overline{g}) u^{N-2} w$$

with the constraints $\int_M u^{N-2} w^2 dv_g = 1$ and $\int_M u^{N-2} vw dv_g = 0$ where

$$\lambda_2'(\overline{g}) = \inf \frac{\int_M vP_g v dv_g}{\int_M u^{N-2} v^2 dv_g}$$

and the infimum is taken over the set

$$E = \left\{ w \in H_k^2(M) \text{ such that } u^{\frac{N-2}{2}} w \neq 0,\; \int_M u^{N-2} w^2 dv_g = 1 \text{ and } \int_M u^{N-2} wv dv_g = 0 \right\}.$$

Proof. Let $(w_m)_m$ be a minimizing sequence for $\lambda_2'(\overline{g})$, with the same method as above, we find non trivial minimizer w to $\lambda_2'(\overline{g})$ such that $P_g w = \lambda_2'(\overline{g})u^{N-2}w$ in the weak sens with $\int_M u^{N-2}w^2 dv_g = 1$. Now writing

$$\begin{aligned}\int_M u^{N-2}wvdv_g &= \int_M u^{N-2}w_m v - u^{N-2}w_m v + u^{N-2}wvdv_g\\ &= \int_M u^{N-2}v(w-w_m)dv_g + \int_M u^{N-2}w_m v dv_g = 0.\end{aligned}$$

As the sequence $w_m \in E$, $\int_M u^{N-2}w_m v dv_g = 0$, and by using the weak convergence of w_m to w in $L^N(M)$ and since $u^{N-2}v \in L^{\frac{N}{N-1}}(M)$ where $L^{\frac{N}{N-1}}(M)$ is the dual space of $L^N(M)$, we get $\int_M u^{N-2}v(w-w_m)dv_g \to 0$ thus,

$$\int_M u^{N-2}wvdv_g = 0.$$

If $u \in C^\infty_+(M)$, one also gets $w \in C^{2k}(M)$ and finally, as in [6] it follows that $\lambda_2'(\overline{g}) = \lambda_2(\overline{g})$. □

Proposition 3.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $\overline{g}$ is a conformal metric and $\lambda_1(\overline{g}) > 0$. If $Q_g \leq 0$, then the solution v of (30) is nodal.*

Proof. By Theorem (3.2), v satisfies the equation $P_g v = \lambda_1(\overline{g})u^{N-2}v$, then from (10), one can write

$$\Delta_g^k v + \sum_{l=1}^{k-1} A_{l,g}(v) + \frac{n-2k}{2}Q_g v = \lambda_1(\overline{g})u^{N-2}v$$

Integrating over M, we get that

$$\int_M \Delta_g^k v dv_g + \sum_{l=1}^{k-1}\int_M A_{l,g}(v)dv_g + \int_M \frac{n-2k}{2}Q_g v dv_g = \lambda_1(\overline{g})\int_M u^{N-2}v dv_g.$$

Since $\overline{g}$ is conformal, again from Theorem (3.2), $v \in C^\infty(M)$ and this implies that

$$\underbrace{\int_M \Delta_g^k v dv_g}_{=0} + \underbrace{\sum_{l=1}^{k-1}\int_M A_{l,g}(v)dv_g}_{=0} + \int_M \frac{n-2k}{2}Q_g v dv_g = \lambda_1(\overline{g})\int_M u^{N-2}v dv_g.$$

Since $\lambda_1(\overline{g}) > 0$ and $Q_g \leq 0$, hence if $v \geq 0$, one has

$$\underbrace{\int_M Q_g v dv_g}_{\leq 0} = \underbrace{\int_M \lambda_1(\overline{g})u^{N-2}v dv_g}_{>0}$$

this makes a contradiction, if $v \leq 0$, one has

$$\underbrace{\int_M Q_g v dv_g}_{\geq 0} = \underbrace{\int_M \lambda_1(\overline{g})u^{N-2}v dv_g}_{<0}$$

and this is also a contradiction. Consequently, v changes the sign.

If $\lambda_1(\overline{g}) < 0$ and $Q_g \geq 0$. With the same method, we get the same thing. $\square$

4. Existence of a minimum of μ_1

In this section, we study the first GJMS invariant μ_1 in case $\lambda_1(g) > 0$. We will prove that μ_1 is attained by a generalized metric. However, if $\lambda_1(g) < 0$, Proposition (3.1) implies that μ_1 is not well defined. In other words, from the variational characterization of μ_1, one has

$$\mu_1 = -\infty.$$

In order to prove Theorem (4.1), we prove some useful lemmas.

Definition 4.1. In this definition, we precise the formula (9). Indeed, by using the definition of $\lambda_1(\overline{g})$ formula (8), the first GJMS invariant μ_1 is given by

$$\begin{aligned}\mu_1 &= \inf_{\overline{g}\in[g]} \lambda_1(\overline{g}) Vol(M,\overline{g})^{\frac{2k}{n}} \\ &= \inf_{\substack{u\in C^\infty_+(M)\\ V\in Gr_1^u(H_k^2(M))}} \sup_{v\in V^*} \frac{\int_M vP_g v dv_g}{\int_M u^{N-2}v^2 dv_g} \left(\int_M u^N dv_g\right)^{\frac{2k}{n}}.\end{aligned}$$

Lemma 4.1. *We have:*

$$\mu_1 \leq \mu \tag{33}$$

where μ is the GJMS invariant, see (4).

Proof.

$$\begin{aligned}\mu_1 &= \inf_{\overline{g}\in[g]} \lambda_1(\overline{g}) Vol(M,\overline{g})^{\frac{2k}{n}} \\ &= \inf_{\overline{g}\in[g]} \lambda_1(u^{\frac{N-2}{k}} g) Vol(M,\overline{g})^{\frac{2k}{n}} \\ &= \inf_{u\in C^\infty_+(M)} \left(\inf_{V\in Gr_1^u(H_k^2(M))} \sup_{v\in V^*} \frac{\int_M vP_g v dv_g}{\int_M u^{N-2}v^2 dv_g}\right) \left(\int_M u^N dv_g\right)^{\frac{2k}{n}}\end{aligned}$$

where $V^* = V\backslash\{0\}$.

From the embedding $C^\infty_+(M) \subset H_k^2(M)$, one can write

$$\mu_1 \leq \inf_{\substack{u\in C^\infty_+(M)\\ V\in Gr_1^u(C^\infty_+(M))}} \sup_{v\in V^*} \frac{\int_M vP_g v dv_g}{\int_M u^{N-2}v^2 dv_g} \left(\int_M u^N dv_g\right)^{\frac{2k}{n}}$$

in particular for $u = v$, one has

$$\mu_1 \leq \inf_{\substack{u\in C^\infty_+(M)\\ V\in Gr_1^u(C^\infty_+(M))}} \sup_{v\in V^*} \frac{\int_M vP_g v dv_g}{\int_M v^{N-2}v^2 dv_g} \left(\int_M v^N dv_g\right)^{\frac{2k}{n}}$$

$$\leq \inf_{\substack{u\in C_+^\infty(M)\\ V\in Gr_1^u(C_+^\infty(M))}} \sup_{v\in V^*} \frac{\int_M vP_g v dv_g}{(\int_M v^N dv_g)^{1-\frac{2k}{n}}}$$

$$\leq \inf_{\substack{u\in C_+^\infty(M)\\ V\in Gr_1^u(C_+^\infty(M))}} \sup_{v\in V^*} \frac{\int_M vP_g v dv_g}{(\int_M v^N dv_g)^{\frac{2}{N}}}.$$

Since $V \in Gr_1^u(C_+^\infty(M), V = \{\lambda v, \lambda \in \mathbb{R}^*\}$ where $v \in C_+^\infty(M)$, then we deduce that:

$$\sup_{v\in V^*} \frac{\int_M vP_g v dv_g}{(\int_M v^N dv_g)^{\frac{2}{N}}} = \sup_{\lambda\in\mathbb{R}^*} \frac{\int_M (\lambda v) P_g(\lambda v) dv_g}{\left(\int_M (\lambda v)^N dv_g\right)^{\frac{2}{N}}} = \frac{\int_M vP_g v dv_g}{(\int_M v^N dv_g)^{\frac{2}{N}}}.$$

This implies that

$$\mu_1 \leq \inf_{v\in C_+^\infty(M)} \frac{\int_M vP_g v dv_g}{\left(\int_M v^N dv_g\right)^{\frac{2}{N}}} = \mu.$$

□

Lemma 4.2. *Let* (v_m) *and* (u_m) *be two sequences such that* $v_m \to v$ *weakly in* $H_k^2(M)$, $u_m \to u$ *weakly in* $L^N(M)$ *and checking* $\int_M u_m^{N-2} v_m^2 dv_g = 1$. *Then*

$$\int_M u_m^{N-2}(v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1).$$

Proof. Writing

$$\int_M u_m^{N-2}(v_m - v)^2 dv_g = \int_M u_m^{N-2} {v_m}^2 dv_g + \int_M u_m^{N-2} v^2 dv_g - \int_M 2u_m^{N-2} v_m v dv_g$$
$$= 1 + \int_M u_m^{N-2} v^2 dv_g - \int_M 2u_m^{N-2} v_m v dv_g.$$

The sequence u_m^{N-2} is bounded in $L^{\frac{N}{N-2}}(M)$ and converges almost everywhere to u^{N-2} on M, hence $u_m^{N-2} \to u^{N-2}$ weakly in $L^{\frac{N}{N-2}}(M)$.

This means that for all ϕ in $L^{\frac{N}{2}}(M)$, one gets $\int_M u_m^{N-2}\phi dv_g \to \int_M u^{N-2}\phi dv_g$.

In particular for $\phi = v^2$, we obtain

$$\int_M u_m^{N-2} v^2 dv_g \to \int_M u^{N-2} v^2 dv_g.$$

On the other hand since

$$\int_M u_m^{N-2\,\frac{N}{N-1}} v_m^{\frac{N}{N-1}} dv_g \leq \left(\int_M {u_m}^N dv_g\right)^{\frac{N-2}{N-1}} \left(\int_M v_m^N dv_g\right)^{\frac{1}{N-1}}$$

this means that the sequence $u_m^{N-2} v_m$ is also bounded in $L^{\frac{N}{N-1}}(M)$ and since $u_m^{N-2} v_m$ goes to $u^{N-2}v$ almost everywhere, one has $u_m^{N-2} v_m \to u^{N-2}v$ weakly in $L^{\frac{N}{N-1}}(M)$, then for all $\phi \in L^N(M)$, one has $\int_M u_m^{N-2} v_m \phi dv_g \to \int_M u^{N-2} v\phi dv_g$. In particular for $\phi = v \in L^N(M)$, we obtain

$$\int_M u_m^{N-2} v_m v dv_g \to \int_M u^{N-2} v^2 dv_g$$

Consequently,

$$\int_M u_m^{N-2}(v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1). \tag{34}$$

□

Theorem 4.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $\lambda_1(g) > 0$ and*

$$1 - \mu K_0 > 0 \tag{35}$$

where μ is the GJMS invariant and K_0 is given by (20). Then there exist two nontrivial functions $u \in L_+^N(M)$ and $v \in H_k^2(M)$ such that in the weak sense, we have

$$P_g v = \mu_1 u^{N-2} v \quad \textit{and} \quad \int_M u^{N-2} v^2 dv_g = 1. \tag{36}$$

In other words, μ_1 is attained by a generalized metric.

Proof. let $g_m = u_m^{\frac{4}{n-2k}} g$ be a minimizing sequence of conformal metrics of μ_1, a sequence of metrics such that $u_m \in C^\infty(M)$, $u_m > 0$ and

$$\mu_1 = \lim_m \lambda_1(g_m) vol(M, g_m)^{\frac{2k}{n}}$$

For more clarity we set : $\lambda_1(g_m) = \lambda_{1,m}$.

Without loss of generality, we may assume that

$$vol(M, g_m) = \int_M u_m^N dv_g = 1. \tag{37}$$

Indeed, since

$$\frac{2kN}{n} = \frac{2k2n}{n(n-2k)} = \frac{2n}{n-2k} - 2 = N - 2,$$

it follows that for any $\lambda > 0$, one gets

$$I(\lambda u, v) = \frac{\int_M v P_g v dv_g}{\int_M (\lambda u)^{N-2} v^2 dv_g} \left(\int_M (\lambda u)^N dv_g \right)^{\frac{2k}{n}} = I(u, v).$$

This means that if (u_m) is a minimizing sequence, $(\lambda u_m)_m$ is also is a minimizing sequence, just choose $\lambda = (\int_M u_m^N dv_g)^{-\frac{1}{N}}$. i.e

$$\mu_1 = \lim_m \lambda_{1,m}.$$

Step 1: Firstly, (37) implies that the sequence $(u_m)_m$ is bounded in $L^N(M)$, hence there exists $u \in L^N(M)$, $u \geq 0$ such that $u_m \to u$ weakly in $L^N(M)$ and by standard argument, we get

$$\int_M u^N dv_g \leq \liminf \int_M u_m^N dv_g = 1. \tag{38}$$

Now, we are going to prove that the generalized metric $u^{\frac{4}{n-2k}} g$ with $u \in L^N(M)$, $u \geq 0$ and $u \neq 0$ minimizes μ_1.

Since $\lambda_1(g) > 0$, P_g is coercive. Then for all $u_m \in C^\infty(M)$, Theorem(3.2) implies the existence of $v_m \in C^\infty(M)$ such that

$$P_g v_m = \lambda_{1,m} u_m^{N-2} v_m \text{ and } \int_M u_m^{N-2} v_m^2 dv_g = 1.$$

Now for m large enough, we may assume that

$$\lambda_{1,m} \leq \mu_1 + 1$$

which implies that

$$\|v_m\|_{P_g}^2 = \int_M v_m P_g v_m dv_g = \lambda_{1,m} \leq \mu_1 + 1.$$

Hence the sequence $(v_m)_m$ is bounded in $H_k^2(M)$, then there exists $v \in H_k^2(M)$ such that $v_m \longrightarrow v$ weakly in $H_k^2(M)$ and $v_m \longrightarrow v$ strongly in $H_{k-1}^2(M)$. This, together with the weak convergence of $(u_m)_m$, imply that the function v is a weak solution of the following equation

$$P_g v = \mu_1 u^{N-2} v. \tag{39}$$

Step 2: we show that u, v are not identically null.

Letting $\varphi_m = v_m - v$ and

$$A = \int_M u_m^{N-2} \varphi_m^2 dv_g. \tag{40}$$

Clearly $\varphi_m \to 0$ and the strong convergence of φ_m in $H_{k-1}^2(M)$ implies that

$$\int_M \sum_{l=0}^{k-1} |\nabla^l(\varphi_m)|^2 dv_g = o(1) \quad \text{and} \quad \sum_{l=0}^{k-1} \int_M A_{(l)}(g)(\nabla^l \varphi_m, \nabla^l \varphi_m) dv_g = o(1).$$

Then by Hölder inequality, Theorem (2.2) and Brezis-Lieb lemma, one has

$$\begin{aligned}
A &\leq \left(\int_M (u_m)^{N-2\frac{N}{N-2}} dv_g \right)^{\frac{N-2}{N}} \left(\int_M (\varphi_m \right)^{\frac{N}{2}2} dv_g)^{\frac{2}{N}} \\
&\leq \|\varphi_m\|_N^2 \\
&\leq (K_0 + \varepsilon) \int_M |\Delta_g^{\frac{k}{2}} \varphi_m|^2 dv_g + B_\varepsilon \int_M \sum_{l=0}^{k-1} |\nabla^l \varphi_m|^2 dv_g \\
&\leq (K_0 + \varepsilon) \int_M |\Delta_g^{\frac{k}{2}}(v_m)|^2 - |\Delta_g^{\frac{k}{2}}(v)|^2 dv_g + o(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A \leq (K_0 + \varepsilon) &\int_M (|\Delta_g^{\frac{k}{2}}(v_m)|^2 - |\Delta_g^{\frac{k}{2}}(v)|^2) dv_g \\
&+ \sum_{l=0}^{k-1} \int_M A_{(l)}(g)(\nabla^l \varphi_m, \nabla^l \varphi_m) dv_g + o(1).
\end{aligned}$$

Again from the strong convergence in $H_{k-1}^2(M)$, one also gets

$$\sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l\varphi_m,\nabla^l\varphi_m)dv_g=\sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l v_m,\nabla^l v_m)dv_g$$
$$-\sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l v,\nabla^l v)dv_g+o(1),$$

then

$$A\leq (K_0+\varepsilon)\left[\int_M(|\Delta_g^{\frac{k}{2}}(v_m)|^2-|\Delta_g^{\frac{k}{2}}(v)|^2)dv_g\right.$$
$$\left.+\sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l v_m,\nabla^l v_m)dv_g-\sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l v,\nabla^l v)dv_g\right]+o(1).$$

Since

$$\int_M vP_gvdv_g=\int_M(|\Delta_g^{\frac{k}{2}}(v)|^2+\sum_{l=0}^{k-1}\int_M A_{(l)}(g)(\nabla^l v,\nabla^l v)dv_g,$$

We deduce that

$$\begin{aligned}A&\leq (K_0+\varepsilon)\left(\int_M v_mP_gv_m-vP_gvdv_g\right)+o(1)\\&\leq (K_0+\varepsilon)\left(\lambda_{1,m}-\int_M vP_gvdv_g\right)+o(1)\\&\leq (K_0+\varepsilon)\left(\lambda_{1,m}-\mu_1\int_M u^{N-2}v^2dv_g)\right)+o(1)\end{aligned}$$

Independently, with Lemma (4.2) formula (34), we have

$$A=1-\int_M u^{N-2}v^2dv_g+o(1)$$

then it follows that

$$1-\int_M u^{N-2}v^2dv_g\leq (K_0+\varepsilon)\left(\lambda_{1,m}-\mu_1\int_M u^{N-2}v^2dv_g)\right)+o(1).$$

Now when $m\to+\infty$, one gets

$$1-\int_M u^{N-2}v^2dv_g\leq (K_0+\varepsilon)(\mu_1-\mu_1\int_M u^{N-2}v^2dv_g)$$

therefore,

$$1-(K_0+\varepsilon)\mu_1\leq\int_M u^{N-2}v^2dv_g-(K_0+\varepsilon)\mu_1\int_M u^{N-2}v^2dv_g$$

and this leads that

$$1-K_0\mu_1\leq(1-K_0\mu_1)\int_M u^{N-2}v^2dv_g+\varepsilon\mu_1(1-\int_M u^{N-2}v^2dv_g).\tag{41}$$

Now, by using Lemma (4.1) formula (33) and the assumption (35), one easily has $1 - K_0\mu_1 > 0$ and as we can choose ε sufficiently small enough, (41) necessarily implies that

$$\int_M u^{N-2}v^2 dv_g \geq 1.$$

Fatou's lemma, implies that

$$\int_M u^{N-2}v^2 dv_g \leq \liminf \int_M u_m^{N-2}v_m^2 dv_g = 1$$

then we deduce that

$$\int_M u^{N-2}v^2 dv_g = 1.$$

This implies that v and u are not identically null which means that μ_1 is attained by the generalized metric $u^{\frac{4}{n-2k}}g$. Moreover, we obtain

$$\mu_1 = \|v\|_g^2 = \int_M vP_g v dv_g > 0. \tag{42}$$

□

5. Nonlinear GJMS equation and nodal solution

In this section, we show that the equation $P_g v = \mu_1|v|^{N-2}v$ has a nodal solution if $Q_g \leq 0$.

Theorem 5.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that μ_1 is attained by the generalized metric $u^{\frac{4}{n-2k}}g$ where $u \in L_+^N(M)$. Then $u = |v|$ where $v \in H_k^2(M)$, v is a solution weak of $P_g v = \mu_1 u^{N-2}v$ and such that $\int_M u^{N-2}v^2 dv_g = 1$. Moreover, the function $v \in C^{2k}(M)$ and if $Q_g \leq 0$, then v changes the sign.*

Proof. Let the function $h = a|v| \in L_+^N(M)$ with $a > 0$ chosen such that $\int_M h^N dv_g = 1$, by definition

$$\mu_1 \leq \frac{\int_M vP_g v dv_g}{\int_M h^{N-2}v^2 dv_g} = \frac{\mu_1 \int_M u^{N-2}v^2 dv_g}{\int_M h^{N-2}v^2 dv_g} = \frac{a^2\mu_1 \int_M u^{N-2}v^2 dv_g}{a^2 \int_M h^{N-2}v^2 dv_g}$$
$$= \frac{\mu_1 \int_M u^{N-2}(av)^2 dv_g}{\int_M (a|v|)^{N-2}(av)^2 dv_g} = \frac{\mu_1 \int_M u^{N-2}(av)^2 dv_g}{\int_M h^N dv_g} = \mu_1 \int_M u^{N-2}(a|v|)^2 dv_g.$$

By using (38) and Hölder's inequality, it follows that

$$\begin{aligned}
\mu_1 &\leq \mu_1 \left(\int_M u^{N-2\frac{N}{N-2}} dv_g\right)^{\frac{N-2}{N}} \left(\int_M (a|v|)^{2\frac{N}{2}} dv_g\right)^{\frac{2}{N}} \\
&\leq \mu_1 \left(\int_M u^N dv_g\right)^{\frac{N-2}{N}} \left(\int_M h^N dv_g\right)^{\frac{2}{N}} \\
&\leq \mu_1 \left(\int_M u^N dv_g\right)^{\frac{N-2}{N}} \leq \mu_1,
\end{aligned} \tag{43}$$

this implies that we have both equality in the Hölder inequality. The equality in the Hölder inequality implies that there exists a constant $b > 0$ such that :

$$u = b|v|.$$

From the equality $1 = \int_M u^{N-2} v^2 dv_g = b^{N-2} \int_M |v|^N dv_g$, we obtain

$$\frac{1}{b^{N-2}} = \int_M |v|^N dv_g.$$

(43) implies that $\int_M u^N dv_g = 1$, then it follows that

$$b^N \int_M |v|^N dv_g = 1$$

which leads to

$$\frac{1}{b^{N-2}} = \int_M |v|^N dv_g = \frac{1}{b^N}.$$

Therefore

$$b = 1 \quad \text{and} \quad u = |v|.$$

Hence, v is a weak solution of

$$P_g v = \mu_1 |v|^{N-2} v \tag{44}$$

and from standard regularity see Proposition(2.5), we get that $v \in C^{2k}(M)$. In addition, since $\mu_1 > 0$ and $Q_g \leq 0$, by following the same proof of Proposition (3.2), we deduce that the function v changes the sign. □

6. Case of Einsteinian manifold and positive solution

In this section, on Einstein manifold when $S_g > 0$, we will prove that the solution v of Eq. (44) is positive, $\mu_1 = \mu$ and is attained by a conformal metric which leads to the existence of a metric $\overline{g}$ conformal to g such that the Q-curvature is constant. In the case $S_g < 0$, and k is odd, the solution is nodal.

Theorem 6.1. *Let (M, g) be a compact Einstein manifold of dimension $n \geq 3$. Assume that $S_g > 0$ and $1 - \mu K_0 > 0$ where μ is the GJMS invariant and K_0 is given by formula (20). Then μ_1 is attained by the conformal metric $u^{\frac{4}{n-2k}} g$. In other words, there exists $u \in C^\infty(M), u > 0$ solution to the following equation*

$$P_g u = \mu_1 u^{N-1} \quad \textit{such that} \quad \int_M v^N dv_g = 1.$$

Proof. We follow the same proof of Theorem (4.1).

Let $g_m = u_m^{\frac{4}{n-2k}} g$ be a minimizing sequence of conformal metrics of μ_1, a sequence of metrics such that $u_m \in C^\infty(M)$, $u_m > 0$ and

$$\mu_1 = \lim_m \lambda_{1,m} \quad \text{and} \quad \int_M u_m^N dv_g = 1. \tag{45}$$

Firstly, (45) implies that the sequence $(u_m)_m$ is bounded in $L^N(M)$, hence there exists $u \in L^N(M)$, $u \geq 0$ such that $u_m \to u$ weakly in $L^N(M)$.

Since (M, g) is Einstein and $S_g > 0$, P_g is coercive. Then for all $u_m \in C^\infty(M)$, Theorem(3.2) implies the existence of $v_m \in C^\infty(M)$ such that $v_m > 0$ and

$$P_g v_m = \lambda_{1,m} u_m^{N-2} v_m \quad \text{and} \quad \int_M u_m^{N-2} v_m^2 dv_g = 1.$$

Now for m large enough, we may assume that

$$\lambda_{1,m} \leq \mu_1 + 1$$

which implies that

$$\|v_m\|_{P_g}^2 = \int_M v_m P_g(v_m) dv_g = \lambda_{1,m} \leq \mu_1 + 1.$$

Hence the sequence $(v_m)_m$ is bounded in $H_k^2(M)$, then there exists $v \in H_k^2(M)$ such that $v \geq 0$, $v_m \longrightarrow v$ weakly in $H_k^2(M)$ and $v_m \longrightarrow v$ strongly in $H_{k-1}^2(M)$. This, together with the weak convergence of $(u_m)_m$, imply that the function v is a weak solution of the following equation

$$P_g v = \mu_1 u^{N-2} v. \tag{46}$$

and in particular

$$v \geq 0.$$

Since $1 - \mu K_0 > 0$, by step (2) of the poof of Theorem (4.1), the functions u, v satisfy $\int_M u^{N-2} v^2 dv_g = 1$ and are not identically null. Since $v \geq 0$, we let the function $h = av \in L_+^N(M)$ where $a > 0$ chosen such that $\int_M h^N dv_g = 1$ and by following the same proof of Theorem (5.1), one has

$$u = v.$$

Therefore, v is a weak solution of

$$P_g v = \mu_1 v^{N-1}$$

and from standard regularity see Proposition (2.5), we get that $v \in C^{2k}(M)$. In particular since $v \geq 0$ and $\mu_1 > 0$, one has $P_g v \geq 0$ and since $v \neq 0$, it follows from Proposition (2.4) that $v > 0$ and again by regularity $v \in C^\infty(M)$.

Now since $\int_M v P_g v dv_g = \mu_1$, $\int_M |v|^N dv_g = 1$ and from the definition of μ, one has

$$\mu \leq \frac{\int_M v P_g v dv_g}{\left(\int_M |v|^N dv_g\right)^{\frac{2}{N}}} = \mu_1. \tag{47}$$

It follows that

$$\mu \leq \mu_1,$$

and by Lemma (4.1) formula (33), we get that

$$\mu_1 = \mu.$$

Therefore, the infimum μ_1 is achieved by the conformal metric $\overline{g} = u^{\frac{4}{n-2k}} g$ and this means that metric $\overline{g}$ is such that the Q-curvature

$$Q_{\overline{g}} = \frac{2}{n-2k} \mu_1.$$

□

A more interesting situation on Einstein manifold is when $S_g < 0$, this implies that $\int_M vP_g v dvg$ can be negative or positive and consequently the eigenvalues follow same thing contrary to the case $S_g > 0$ which implies only the positivity of eigenvalues.

Corollary 6.1. *Let (M, g) be a smooth compact Einstein manifold of dimension $n \geq 3$, assume that $S_g < 0$, $\lambda_1(g) > 0$ and $1 - \mu K_0 > 0$. If k is odd, the following equation*

$$P_g v = \mu_1 |v|^{N-2} v \tag{48}$$

has a nodal solution $v \in C^{2k}(M)$.

Proof. Since $\lambda_1(g) > 0$ and $1 - \mu K_0 > 0$, Theorem (4.1) implies that μ_1 is attained by a generalized metric and since (M, g) is Einstein, by using (18), (48) can be written as

$$\Delta_g^k u + \sum_{l=0}^{k-2} b_{k-l-1} (S_g)^{l+1} \Delta_g^{k-l-1} u + b_0 (S_g)^k u = \mu_1 u^{N-2} v$$

where $b_{k-1}, \ldots, b_1, b_0$ are positive real numbers. Therefore, if k is odd, $b_0(S_g)^k < 0$ and by applying Theorem (5.1) with $\frac{n-2k}{2} Q_g = b_0 (S_g)^k$, we get the result. □

Proposition 6.1. *Let $(\mathbb{S}^n, h)$ be the standard unit n-sphere of $\mathbb{R}^{n+1}$. Then the GJMS invariant of $\mathbb{S}^n$ is such that*

$$\mu(\mathbb{S}^n) = \inf_{u \in C^\infty(\mathbb{S}^n), u \neq 0} \frac{\int_{\mathbb{S}^n} u P_h(u) dv_h}{\left(\int_{\mathbb{S}^n} |u|^N dv_h\right)^{\frac{2}{N}}} = {K_0}^{-1}.$$

Proof. We follow the same proof of Proposition (1.1) in [10]. Just note here that the choice of functions φ and ϕ_ϵ must be adapted to our context, thus $\varphi \in H_k^2(M)$ and is chosen such that

$$\varphi(x) = \left(\frac{1 + |x|^2}{2}\right)^{k - \frac{n}{2}}$$

and ϕ_ϵ is given in Proposition (3.1).

Indeed, let x_0 be some point on S^n, and let $\phi : \mathbb{S}^n \backslash \{x_0\} \longrightarrow \mathbb{R}^n$ be the stereographic projection of pole x_0. If δ stands for the Euclidean metric of $\mathbb{R}^n$, then

$$(\phi^{-1})^* h = \varphi^{\frac{N-2}{k}} \delta.$$

By conformal invariance of P_g, we get that for all $u \in D_k^2(\mathbb{R}^n)$,

$$\frac{\int_{\mathbb{R}^n} u P_{\tilde{h}}(u) dv_{\tilde{h}}}{\left(\int_{\mathbb{R}^n} |u|^N dv_{\tilde{h}}\right)^{\frac{2}{N}}} = \frac{\int_{\mathbb{R}^n} |\Delta^{\frac{k}{2}}(u\varphi)|^2 dx}{\left(\int\limits_{\mathbb{R}^n} |u\varphi|^N dx\right)^{\frac{2}{N}}} \tag{49}$$

where $D_k^2(\mathbb{R}^n)$ is given in Definition(2.4) and $\tilde{h} = (\phi^{-1})^* h$. Suppose now that

$$\inf_{u \in C^\infty(\mathbb{S}^n), u \neq 0} \frac{\int_{S^n} u P_h(u) dv_h}{\left(\int_{\mathbb{S}^n} |u|^N dv_h\right)^{\frac{2}{N}}} < {K_0}^{-1} \tag{50}$$

and let $u_0 \in C^\infty(\mathbb{S}^n)$, $u_0 \neq 0$, be such that

$$\frac{\int_{S^n} u_0 P_h(u_0) dv_h}{(\int_{\mathbb{S}^n} |u_0|^N dv_h)^{\frac{2}{N}}} < K_0{}^{-1}.$$

Fix a point p in $\mathbb{S}^n$. For $\epsilon > 0$, let ϕ_ϵ be cut-off function i.e. a family of smooth functions on $\mathbb{S}^n$ such that :

$$\begin{cases} 0 \leq \phi_\epsilon \leq 1 \\ \phi_\epsilon = 0 \quad \text{on} \quad B_\epsilon(p) \quad \text{and} \quad \phi_\epsilon = 1 \quad \text{on} \quad \mathbb{S}^n \backslash B_{2\epsilon}(p) \\ |\nabla^l \phi_\epsilon| \leq \frac{c_l}{\epsilon^l} \quad \text{for all} \quad l \in \{1, 2, ..., k-1\} \\ |\Delta_g^{\frac{k}{2}} \phi_\epsilon| \leq \frac{c_k}{\epsilon^k} \end{cases}$$

where $B_\epsilon(p)$ is the open ball centered at p and of radius ϵ and c_l are constants that do not depend on ϵ. In order to get such a family, we might fix some ϕ_{ϵ_0} as above, for instance, radially symmetric, and set then, for $\epsilon \leq \epsilon_0$, $\quad \phi_\epsilon = \phi_{\epsilon_0}(\frac{r}{\epsilon})$ where r is the distance on $\mathbb{S}^n$ from x_0 to x. Let $u_\epsilon = \phi_\epsilon u_0$, one easily gets

$$\lim_{\epsilon \longrightarrow 0} \frac{\int_{\mathbb{S}^n} u_\epsilon P_h(u_\epsilon) dv_h}{(\int_{\mathbb{S}^n} |u_\epsilon|^N dv_h)^{\frac{2}{N}}} = \frac{\int_{\mathbb{S}^n} u_0 P_h(u_0) dv_h}{(\int_{\mathbb{S}^n} |u_0|^N dv_h)^{\frac{2}{N}}}.$$

As in the proof of Proposition (3.1), one has

$$\lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon^{2k}} V_h(B_{2\epsilon}(p) \backslash B_\epsilon(p)) = 0 \quad \text{since} \quad n > 2k,$$

where $V(\Omega)$ stands for the volume of Ω with respect to h. Choosing ϵ sufficiently small, it follows from (49) and (50) that there exists $\tilde{u}_\epsilon \in D_k^2(\mathbb{R}^n)$ of the form

$$\tilde{u}_\epsilon = (u_\epsilon \circ \phi^{-1})\varphi,$$

such that

$$\frac{\int_{\mathbb{R}^n} |\Delta^{\frac{k}{2}} \tilde{u}_\epsilon|^2 dx}{(\int_{\mathbb{R}^n} |\tilde{u}_\epsilon|^N dx)^{\frac{2}{N}}} < K_0{}^{-1}$$

and this contradicts (20) see the definition (2.4). Consequently,

$$\inf_{u \in C^\infty(\mathbb{S}^n), u \neq 0} \frac{\int_{\mathbb{S}^n} u P_h(u) dv_h}{(\int_{S^n} |u|^N dv_h)^{\frac{2}{N}}} = K_0{}^{-1}.$$

□

In the following proposition, we are going to show that there are certain manifolds such that the assumption $1 - \mu K_0 > 0$ holds.

Proposition 6.2. *Let* $(\mathbb{S}^n, h)$ *be the standard unit* n*-sphere of* $\mathbb{R}^{n+1}$ *and let* G_p *be the subgroup of* $O(n+1)$ *of* $\mathbb{R}^{n+1}$*. Let* $\mathbb{M}_p = \mathbb{S}^n / G_p$ *be the quotient manifold and* h_p *is the quotient metric on* $\mathbb{M}_p$*. Then the GJMS invariant of* $\mathbb{M}_p$ *satisfy the following inequality*

$$\mu(\mathbb{M}_p, h_p) < K_0{}^{-1}. \tag{51}$$

Proof. Writing $n = 2m + 1$, we let $\{z_j\}$, be the natural complex coordinates on $\mathbb{C}$ where $j = 1, \dots, m+1$. Given $p \geq 2$ integer and let G_p be the subgroup of $O(n+1)$ generated by

$$z_j \longrightarrow e^{\frac{2\pi i}{p}}.$$

It is easily seen that G_p acts freely on $\mathbb{S}^n$. We let $\mathbb{M}_p = \mathbb{S}^n / G_p$ be the quotient manifold. We let $u_p = \overline{u}/G_p$ be the quotient function induced by $\overline{u}$ on $\mathbb{M}_p$ where $\overline{u}$ is a smooth function on $\mathbb{S}^n$. Noting that

$$\int_{\mathbb{M}_p} |T u_p|^s = \frac{1}{p} \int_{\mathbb{S}^n} |T\overline{u}|^s \tag{52}$$

where s is any real number, and T is either the identity operator, the gradient operator, or the Laplace-beltrami operator. From (52) and for any $p \geq 2$, one gets

$$\begin{aligned}
\mu(\mathbb{M}_p, h_p) &= \inf_{u_p \in C^\infty(\mathbb{M}_p), u_p \neq 0} \frac{\int_{\mathbb{M}_p} u_p P_{h_p}(u_p) dv_{h_p}}{\left(\int_{\mathbb{M}_p} |u_p|^N dv_{h_p}\right)^{\frac{2}{N}}} \\
&= \frac{\frac{1}{p}}{\left(\frac{1}{p}\right)^{\frac{2}{N}}} \inf_{\overline{u} \in C^\infty(\mathbb{S}^n), \overline{u} \neq 0} \frac{\int_{\mathbb{S}^n} \overline{u} P_h(\overline{u}) dv_h}{\left(\int_{\mathbb{S}^n} |\overline{u}|^N dv_h\right)^{\frac{2}{N}}} \\
&\leq \frac{1}{2^{\frac{2k}{n}}} \inf_{\overline{u} \in C^\infty(\mathbb{S}^n), \overline{u} \neq 0} \frac{\int_{\mathbb{S}^n} \overline{u} P_h(\overline{u}) dv_h}{\left(\int_{\mathbb{S}^n} |\overline{u}|^N dv_h\right)^{\frac{2}{N}}} \\
&< \inf_{\overline{u} \in C^\infty(\mathbb{S}^n), \overline{u} \neq 0} \frac{\int_{\mathbb{S}^n} \overline{u} P_h(\overline{u}) dv_h}{\left(\int_{\mathbb{S}^n} |\overline{u}|^N dv_h\right)^{\frac{2}{N}}} \quad \text{since} \quad n > 2k.
\end{aligned}$$

By Proposition(6.1), we get that $\mu(\mathbb{M}_p, h_p) < {K_0}^{-1}$. This ends the proof of the proposition. □

It is natural to conjecture that one has the following inequality $\mu < {K_0}^{-1}$ for all compact Riemannian manifold but at our knowledge, this problem is still open and seems to be hard. However, we think that is very easy to prove the large inequality $\mu \leq {K_0}^{-1}$ by following Aubin's strategy and we have equality in this inequality if and only if (M, g) is the standard unit n-sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$ equipped with its round metric.

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