Nonlinear Differ. Equ. Appl. (2020) 27:24

1021-9722/20/030001-17
*published online* April 7, 2020
https://doi.org/10.1007/s00030-020-00627-x

**Nonlinear Differential Equations and Applications NoDEA**

# The direct and inverse scattering problem for the semilinear Schrödinger equation

Takashi Furuya

**Abstract.** We study the direct and inverse scattering problem for the semilinear Schrödinger equation $\Delta u + a(x,u) + k^2 u = 0$ in $\mathbb{R}^d$. We show well-posedness in the direct problem for small solutions based on the Banach fixed point theorem, and the solution has the certain asymptotic behavior at infinity. We also show the inverse problem that the semilinear function $a(x,z)$ is uniquely determined from the scattering amplitude. The idea is the linearization that by using sources with several parameters we differentiate the nonlinear equation with respect to these parameter in order to get the linear one.

**Mathematics Subject Classification.** 35R30, 35J61.

**Keywords.** Inverse scattering problem, Semilinear Schrödinger equation, Linearization, Herglotz wave function.

## 1. Introduction

In this paper, we study the direct and inverse scattering problem for the semilinear Schrödinger equation

$$\Delta u + a(x,u) + k^2 u = 0 \text{ in } \mathbb{R}^d, \tag{1.1}$$

where $d \geq 2$, and $k > 0$. Throughout this paper, we make the following assumptions for the semilinear function $a : \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$.

**Assumption 1.1.** We assume that

(i) $a(x,0) = 0$ for all $x \in \mathbb{R}^d$.
(ii) $a(x,z)$ is holomorphic at $z = 0$ for each $x \in \mathbb{R}^d$, that is, there exists $\eta > 0$ such that $a(x,z) = \sum_{l=1}^{\infty} \frac{\partial_z^l a(x,0)}{l!} z^l$ for $|z| < \eta$.
(iii) $\partial_z^l a(\cdot,0) \in L^\infty(\mathbb{R}^d)$ for all $l \geq 1$. Furthermore, there exists $c_0 > 0$ such that $\left\| \partial_z^l a(\cdot,0) \right\|_{L^\infty(\mathbb{R}^d)} \leq c_0^l$ for all $l \geq 1$.
(iv) There exists $R > 0$ such that $\mathrm{supp}\partial_z^l a(\cdot,0) \subset B_R$ for all $l \geq 1$ where $B_R \subset \mathbb{R}^d$ is a open ball with center 0 and radius $R > 0$.

The inverse scattering problems for non-linear Schrödinger equations have been studied in various ways. For the time dependent case, we refer to [21–23], and for the stationary case, we refer to [1,7,9,16–18]. In stationary case, [7,9, 17] have studied the general non-linear function of the form $a(x, |u|)u$, which does not include our no-nlinear function $a(x, u)$. The function $a(x, u)$ which satisfies Assumption 1.1 is the generalization of, in particular, the power type $q(x)u^m$ where $m \in \mathbb{N}$ where $q \in L^\infty(\mathbb{R}^d)$ with compact support. If $m = 1$, the problem is for linear Schrödinger equations, which has been well understood so far by many authors. (see e.g., [6,12,13,15])

Recently in [5,10,11], the generalization of a power type has been studied in inverse boundary value problems via using the Dirichlet-to-Neumann map. [8] also has studied the similar type of this nonlinearity. However in inverse scattering problems, only [1] has studied it in one dimension, which the non-linear function is of the form $a(x, u) = \sum_{n=1}^{\infty} q_n(x)u^n$. Motivated by these previous studies, our aim in this paper is to study the type of this nonlinearity in the case of higher dimensions $d \geq 2$, and a more general form $a(x, u)$ than [1].

We consider the incident field $u_g^{in}$ as the *Herglotz wave function*

$$u_g^{in}(x) := \int_{\mathbb{S}^{d-1}} e^{ikx\cdot\theta} g(\theta) ds(\theta),\ x \in \mathbb{R}^d,\ g \in L^2(\mathbb{S}^{d-1}), \tag{1.2}$$

which solves the free Schrödinger equation $\Delta u_g^{in} + k^2 u_g^{in} = 0$ in $\mathbb{R}^d$. The scattered field $u_g^{sc}$ corresponding to the incident field $u_g^{in}$ is a solution of the following Schrödinger equation perturbed by the semilinear function $a(x, z)$

$$\Delta u_g + a(x, u_g) + k^2 u_g = 0 \text{ in } \mathbb{R}^d, \tag{1.3}$$

where $u_g$ is total field that is of the form $u_g = u_g^{sc} + u_g^{in}$, and the scattered field $u^{sc}$ satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left( \frac{\partial u^{sc}}{\partial r} - iku^{sc} \right) = 0, \tag{1.4}$$

where $r = |x|$.

Since support of the function $a(x, z)$ is compact, the direct scattering problem (1.3)–(1.4) is equivalent to the following integral equation. (See e.g., the argument of Theorem 8.3 in [3].)

$$u_g(x) = u_g^{in} + \int_{\mathbb{R}^d} \Phi(x, y) a(y, u_g(y)) dy, \quad x \in\ \mathbb{R}^d, \tag{1.5}$$

where $\Phi(x, y)$ is the fundamental solution for $-\Delta - k^2$ in $\mathbb{R}^d$. In the following theorem, we find a small solution $u_g^{sc}$ of (1.5) for small $g \in L^\infty(\mathbb{S}^{d-1})$.

**Theorem 1.2.** *We assume that $a(x, z)$ satisfies Assumption 1.1. Then, there exists $\delta_0 \in (0, 1)$ such that for all $\delta \in (0, \delta_0)$ and $g \in L^\infty(\mathbb{S}^{d-1})$ with $\|g\|_{L^\infty(\mathbb{S}^{d-1})} < \delta^2$, there exists a unique solution $u_g^{sc} \in L^\infty(\mathbb{R}^d)$ with $\left\|u_g^{sc}\right\|_{L^\infty(\mathbb{R}^d)} \leq \delta$ such that*

$$u_g^{sc}(x) = \int_{\mathbb{R}^d} \Phi(x, y) a(y, u_g^{sc}(y) + u_g^{in}(y)) dy,\ x \in \mathbb{R}^d. \tag{1.6}$$

Theorem 1.2 is proved by the Banach fixed point theorem. By the same argument in Section 19 of [4], the solution $u_g^{sc}$ of (1.6) has the following asymptotic behavior

$$u_g^{sc}(x) = C_d \frac{\mathrm{e}^{ikr}}{r^{\frac{d-1}{2}}} u_g^{\infty}(\hat{x}) + O\left(\frac{1}{r^{\frac{d+1}{2}}}\right),\ r := |x| \to \infty,\quad \hat{x} := \frac{x}{|x|}. \tag{1.7}$$

where $C_d := k^{\frac{d-3}{2}} e^{-i\frac{\pi}{4}(d-3)}/2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}$. The function $u_g^{\infty}$ is called the *scattering amplitude*, which is of the form

$$u_g^{\infty}(\hat{x}) = \int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} a(y, u_g(y)) dy,\ \hat{x} \in \mathbb{S}^{d-1}. \tag{1.8}$$

We remark that in the standard linear case, that is, $a(x,u) = q(x)u$, the scattering amplitude corresponding to the Herglotz wave function (1.8) can be of the form

$$u_g^{\infty}(\hat{x}) = \int_{\mathbb{S}^{d-1}} \tilde{u}^{\infty}(\hat{x},\theta) g(\theta) ds(\theta),\ \hat{x} \in \mathbb{S}^{d-1}. \tag{1.9}$$

where $\tilde{u}^{\infty}(\hat{x},\theta)$ is the scattering amplitude corresponding to plane waves $e^{ikx\cdot\theta}$. This tells us that in standard linear case, the scattering amplitude of the Herglotz wave function is equivalent to that of the plane wave.

Now, we are ready to consider the inverse problem to determine the semilinear function $a(x,z)$ from scattering amplitudes $u_g^{\infty}(\hat{x})$ for all $g \in L^2(\mathbb{S}^{d-1})$ with $\|g\|_{L^2(\mathbb{S}^{d-1})} < \delta$ where $\delta > 0$ is a sufficiently small. We will show the following theorem.

**Theorem 1.3.** *We assume that $a_j(x,z)$ satisfies Assumption 1.1 ($j = 1,2$). Let $u_{g,j}^{\infty}$ be the scattering amplitude for the following problem*

$$\Delta u_{j,g} + a_j(x, u_{j,g}) + k^2 u_{j,g} = 0 \text{ in } \mathbb{R}^d, \tag{1.10}$$

$$u_{j,g} = u_{j,g}^{sc} + u_g^{in}, \tag{1.11}$$

*where $u_{j,g}^{sc}$ satisfies the Sommerfeld radiation (1.4), and $u_g^{in}$ is given by (1.2), and we assume that*

$$u_{1,g}^{\infty} = u_{2,g}^{\infty}, \tag{1.12}$$

*for any $g \in L^2(\mathbb{S}^{d-1})$ with $\|g\|_{L^2(\mathbb{S}^{d-1})} < \delta$ where $\delta > 0$ is sufficiently small. Then, we have*

$$a_1(x,z) = a_2(x,z),\ x \in \mathbb{R}^d,\ |z| < \eta \tag{1.13}$$

The idea of the proof is the linearization, which by using sources with several parameters we differentiate the nonlinear equation with respect to these parameter in order to get the linear equation. (For such ideas, we refer to [5,10,11].)

There are few previous studies that the general nonlinear function is uniquely determined from the scattering amplitude with fixed $k > 0$. [9] has shown it from behaviour of scattering amplitude corresponding to plane waves $\lambda e^{ikx\dot{\theta}}$ as $\lambda \to 0$. [16] has done from the scattering amplitude with fixed $\lambda = 1$, but the additional assumptions are needed. Our work shows it from the scattering amplitude corresponding to Herglotz wave functions $u_g^{in}$ for all small $g$ instead of using plane waves.

This paper is organized as follows. In Sect. 2, we recall the Green function for the Helmholtz equation and its properties. We also prepare the several lemmas required in the forthcoming argument. In Sect. 3, we prove Theorem 1.2 based on the Banach fixed point theorem. In Sect. 4, we consider the special solution of (1.3)–(1.4) corresponding to the incident field with several parameters in order to linearize problems. Finally in Sect. 5, we prove Theorem 1.3.

## 2. Preliminary

First, we recall the Green functions for the Helmholtz equation and its properties. We denote the Green function for $-\Delta - k^2$ in $\mathbb{R}^d$ by $\Phi(x, y)$, that is, $\Phi(x, y)$ satisfies

$$(-\Delta - k^2)\Phi(x, y) = \delta(x - y), \tag{2.1}$$

for $x, y \in \mathbb{R}^d$, $x \neq y$. In the case of $d = 2, 3$, $\Phi(x, y)$ is of the form

$$\Phi(x, y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x - y|) & \text{for } x, y \in \mathbb{R}^2, x \neq y, \\ \dfrac{e^{ik|x-y|}}{4\pi|x - y|} & \text{for } x, y \in \mathbb{R}^3, x \neq y, \end{cases} \tag{2.2}$$

respectively. Let $q \in L^\infty(\mathbb{R}^d)$ with compact support. We denote the Green function for $-\Delta - k^2 - q$ in $\mathbb{R}^d$ by $\Phi_q(x, y)$, that is, $\Phi_q(x, y)$ satisfies

$$(-\Delta - k^2 - q)\Phi_q(x, y) = \delta(x - y). \tag{2.3}$$

for $x, y \in \mathbb{R}^d$, $x \neq y$. It is well known that for every fixed $y$, $\Phi(x, y)$ and $\Phi_q(x, y)$ satisfy the Sommerfeld radiation condition.

We also recall the asymptotics behavior of $\Phi(x, y)$ as $|x| \to \infty$. In Lemma 19.3 of [4], $\Phi(x, y)$ has the following asymptotics behavior for every fixed $y$,

$$\Phi(x, y) = C_d \frac{\mathrm{e}^{ik|x-y|}}{|x - y|^{\frac{d-1}{2}}} + O\left(\frac{1}{|x - y|^{\frac{d+1}{2}}}\right), \ |x| \to \infty \tag{2.4}$$

and (see the proof of Theorem 19.5 in [4])

$$\Phi(x, y) = \begin{cases} O\left(\frac{1}{|x-y|^{d-2}}\right) & d \geq 3, x \neq y \\ O\left(\left|\ln|x - y|\right|\right) & d = 2, x \neq y \end{cases} \tag{2.5}$$

In Theorem 19.5 of [4], for every $f \in L^\infty(\mathbb{R}^d)$ with compact support, $u(x) = \int_{\mathbb{R}^d} \Phi(x, y) f(y) dy$ is a unique radiating solution. (That is, $u$ satisfies the Sommerfeld radiation condition (1.4).) Furthermore, $u$ has the following asymptotic behavior

$$u(x) = C_d \frac{\mathrm{e}^{ikr}}{r^{\frac{d-1}{2}}} u^\infty(\hat{x}) + O\left(\frac{1}{r^{\frac{d+1}{2}}}\right), \ r = |x| \to \infty, \quad \hat{x} := \frac{x}{|x|}, \tag{2.6}$$

where the scattering amplitude $u^{\infty}$ is of the form

$$u^{\infty}(\hat{x}) = \int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} f(y)dy, \ \hat{x} \in \mathbb{S}^{d-1}. \tag{2.7}$$

The following lemma is given by the same argument as in Lemma 10.4 of [3] or Proposition 2.4 of [14].

**Lemma 2.1.** *Let $q \in L^{\infty}(\mathbb{R}^d)$ with compact support in $B_R \subset \mathbb{R}^d$ where some $R > 0$. We define the Helglotz operator $H : L^2(\mathbb{S}^{d-1}) \to L^2(B_R)$ by*

$$Hg(x) := \int_{\mathbb{S}^{d-1}} e^{ikx\cdot\theta} g(\theta)ds(\theta), \ x \in B_R, \tag{2.8}$$

*and define the operator $T_q : L^2(B_R) \to L^2(B_R)$ by $T_q f := f + w\Big|_{B_R}$ where $w$ is a radiating solution such that*

$$\Delta w + k^2 w + qw = -qf \text{ in } \mathbb{R}^d. \tag{2.9}$$

*We define the subspace $V$ of $L^2(B_R)$ by*

$$V := \overline{\left\{ v\Big|_{B_R}; v \in L^2(B_{R+1}), \ \Delta v + k^2 v + qv = 0 \text{ in } \mathrm{B}_{R+1} \right\}}^{\|\cdot\|_{L^2(B_R)}}. \tag{2.10}$$

*Then, the range of the operator $T_q H$ is dense in $V$ with respect to the norm $\|\cdot\|_{L^2(B_R)}$, that is,*

$$\overline{T_q H\left(L^2(\mathbb{S}^{d-1})\right)}^{\|\cdot\|_{L^2(B_R)}} = V. \tag{2.11}$$

The following result is well known. For $d = 2$ we refer to [2], and for $d \geq 3$ we refer to [19], which corresponds to real functions. For complex functions, see Theorem 6.2 in [20].

**Lemma 2.2.** *Let $f, q_1, q_2 \in L^{\infty}(\mathbb{R}^d)$ with compact support in $B_R \subset \mathbb{R}^d$. We assume that*

$$\int_{B_R} f v_1 v_2 dx = 0, \tag{2.12}$$

*for all $v_1, v_2 \in L^2(B_{R+1})$ with $\Delta v_j + k^2 v_j + q_j v_j = 0$ in $B_{R+1}$. ($j = 1, 2$.) Then, $f = 0$ in $B_R$.*

## 3. Proof of Theorem 1.2

In Sect. 3, we will show Theorem 1.2 based on the Banach fixed point theorem. We denote the Herglotz wave function by

$$v_g(x) := \int_{\mathbb{S}^{d-1}} e^{ikx\cdot\theta} g(\theta)ds(\theta), \ x \in \mathbb{R}^d, \ g \in L^2(\mathbb{S}^{d-1}). \tag{3.1}$$

Let $q := \partial_z a(\cdot, 0)$. We define the operator $T : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ by

$$\begin{aligned} Tw(x) &:= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[a\big(y, w(y) + v_g(y)\big) - q(y)w(y)\right] dy \\ &= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[\sum_{l\geq 2} \frac{\partial_z^l a(y,0)}{l!} \big(w(y) + v_g(y)\big)^l + q(y)v_g(y)\right] dy, \ x \in \mathbb{R}^d. \end{aligned} \tag{3.2}$$

Let $X_\delta := \left\{u \in L^\infty(\mathbb{R}^d) : \|u\|_{L^\infty(\mathbb{R}^d)} \leq \delta\right\}$. We remark that $L^\infty(\mathbb{R}^d)$ is a Banach space, and $X_\delta$ is closed subspace in $L^\infty(\mathbb{R}^d)$. To find an unique fixed point of $T$ in $X$, we will show that $T : X_\delta \to X_\delta$ and $T$ is a contraction. Let $w \in X_\delta$, and let $\delta \in (0, \delta_0)$, and let $\|g\|_{L^\infty(\mathbb{S}^{d-1})} < \delta^2$. Later, we will choose a appropriate $\delta_0 > 0$.

By $\|g\|_{L^\infty(\mathbb{S}^{d-1})} < \delta^2$, we have

$$\|v_g\|_{L^\infty(\mathbb{R}^d)} \leq C \|g\|_{L^\infty(\mathbb{S}^{d-1})} \leq C\delta^2 \tag{3.3}$$

where $C > 0$ is constant only depending on $g$. By (iii) (iv) of Assumption 1.1, we have

$$\begin{aligned} |Tw(x)| &\leq \int_{B_R} |\Phi_q(x,y)| \left[\sum_{l\geq 2} \frac{c_0^l}{l!} \big(C_1\delta\big)^l + C_1\delta^2\right] dy \\ &\leq C_2\delta^2 \left(\sum_{l\geq 0} \big(C_1 c_0 \delta\big)^l\right) \int_{B_R} |\Phi_q(x,y)| dy, \end{aligned} \tag{3.4}$$

where $C_j > 0$ $(j = 1, 2)$ is constant independent of $u$ and $\delta$, and so is $\left(\sum_{l\geq 0} \big(C_1 c_0 \delta\big)^l\right)$ when $\delta > 0$ is sufficiently small. Furthermore, by the continuity of difference $\Phi(x,y) - \Phi_q(x,y)$ in $x$ and $y$ (see the proof of Theorem 31.6 in [4]), and the estimation (2.5), we have for $x \in \mathbb{R}^d$

$$\begin{aligned} \int_{B_R} |\Phi_q(x,y)| dy &\leq \int_{B_R} \big(|\Phi(x,y)| + |\Phi_q(x,y) - \Phi(x,y)|\big) dy \\ &\leq \int_{B_R} \big(|\Phi(x,y)| + C_3\big) dy \leq C_4, \end{aligned} \tag{3.5}$$

which implies that $|Tw(x)| \leq C\delta^2$ where $C, C_j > 0$ $(j = 3, 4)$ is constant independent of $u$ and $\delta$. By choosing $\delta_0 \in (0, 1/C)$, we conclude that $\|Tw\| \leq \delta$, which means $Tw \in X_\delta$.

Let $w_1, w_2 \in X_\delta$. Since we have

$$\begin{aligned}
&\big(w_1(y) + v_g(y)\big)^l - \big(w_2(y) + v_g(y)\big)^l \\
&= \sum_{m=1}^{l} \frac{l!}{(l-m)!m!} \big(w_1^m(y) - w_2^m(y)\big) v_g^{l-m}(y) \\
&\leq \sum_{m=1}^{l} \frac{l!}{(l-m)!m!} \left( \sum_{h=0}^{m-1} w_1^{m-1-h}(y) w_2^h(y) \right) \big(w_1(y) - w_2(y)\big) v_g^{l-m}(y),
\end{aligned} \tag{3.6}$$

and $|w_j(x)| \leq \delta$, then

$$\begin{aligned}
&|Tw_1(x) - Tw_2(x)| \\
&= \left| \int_{B_R} \Phi_q(x,y) \sum_{l \geq 2} \frac{\partial_z^l a(y,0)}{l!} \left[ \big(w_1(y) + v_g(y)\big)^l - \big(w_2(y) + v_g(y)\big)^l \right] dy \right| \\
&\leq \left( \int_{B_R} |\Phi_q(x,y)|\, dy \right) \sum_{l \geq 2} \\
&\qquad \frac{c_0^l}{l!} \sum_{m=1}^{l} \frac{l!}{(l-m)!m!} \left( \sum_{h=0}^{m-1} \delta^{m-1} \right) (C_1' \delta)^{l-m} \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\
&\leq C_2' \sum_{l \geq 2} \sum_{m=1}^{l} \frac{m}{(l-m)!m!} (c_0 C_1' \delta)^{l-1} \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\
&\leq C_2' \sum_{l \geq 2} \left( \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \right) (c_0 C_1' \delta)^{l-1} \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\
&\leq C_3' \sum_{l \geq 2} (c_0 C_1' \delta)^{l-1} \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\
&\leq C_3' \left( \sum_{l \geq 0} (c_0 C_1' \delta)^l \right) \delta \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)} \\
&\leq C' \delta \|u_1 - u_2\|_{L^\infty(\mathbb{R}^d)}, \ x \in \mathbb{R}^d.
\end{aligned} \tag{3.7}$$

where $C', C_j' > 0$ $(j = 1, 2, 3)$ is constant independent of $w_1, w_2$ and $\delta$. (We remark that $\left( \sum_{l \geq 0} (c_0 C_1' \delta)^l \right)$ is also constant when $\delta > 0$ is sufficiently small.) By choosing $\delta_0 \in (0, 1/C')$, we have $\|Tw_1 - Tw_2\|_{L^\infty(\mathbb{R}^d)} < \|w_1 - w_2\|_{L^\infty(\mathbb{R}^d)}$. Choosing sufficiently small $\delta_0 \in (0, \min(1/C, 1/C'))$ we conclude that $T$ has a unique fixed point in $X_\delta$.

Let $w \in X_\delta$ be a unique fixed point, that is, $w$ satisfies

$$w(x) = \int_{\mathbb{R}^d} \Phi_q(x,y) \left[ a\big(y, w(y) + v_g(y)\big) - q(y) w(y) \right] dy, \ x \in \mathbb{R}^d. \tag{3.8}$$

Since $\Phi_q(x,y)$ satisfy the Sommerfeld radiation condition (e.g., see Theorem 31.6 in [4]), $w$ is a radiating solution of $\Delta w + a(x, w + v_g) + k^2 w = 0$ in $\mathbb{R}^d$. By the same argument as in Theorem 8.3 of [3], this is equivalent to the integral equation

$$w(x) = \int_{\mathbb{R}^d} \Phi(x,y) a\big(y, w(y) + v_g(y)\big) dy, \ x \in \mathbb{R}^d, \tag{3.9}$$

which means (1.6). Therefore, Theorem 1.2 has been shown.

## 4. The special solution

In Sect. 4, we consider the special solution of (1.3)–(1.4) corresponding to the incident field with several parameters in order to linearize problems. Let $N \in \mathbb{N}$ be fixed and let $g_j \in L^2(\mathbb{S}^{d-1})$ be fixed $(j = 1, 2, \ldots, N+1)$. We set

$$v_\epsilon := \sum_{j=1}^{N+1} \epsilon_j \delta^2 v_{g_j} = v_{\left(\delta^2 \sum_{j=1}^{N+1} \epsilon_j g_j\right)}, \tag{4.1}$$

where $v_{g_j}$ is the Herglotz wave function defined by (1.2), and $\epsilon_j \in (0, \delta)$. Later, we will choose a appropriate $\delta = \delta_{g_j, N} > 0$. We remark that we can estimate that

$$\|v_\epsilon\|_{L^\infty(\mathbb{R}^d)} \leq C\delta^2 \sum_{j=1}^{N+1} \epsilon_j, \tag{4.2}$$

where $C > 0$ is constant only depending on $g_j$. We denote by $\epsilon = (\epsilon_1, \ldots, \epsilon_{N+1}) \in \mathbb{R}^{N+1}$. We will find a small solution $u_\epsilon$ of (1.6) that is of the form

$$u_\epsilon = r_\epsilon + v_\epsilon. \tag{4.3}$$

This problem is equivalent to

$$r_\epsilon(x) = \int_{\mathbb{R}^d} \Phi_q(x,y) \left[a\big(y, r_\epsilon(y) + v_\epsilon(y)\big) - q(y) r_\epsilon(y)\right] dy, \ x \in \mathbb{R}^d, \tag{4.4}$$

where $q := \partial_z a(\cdot, 0)$.

We define the space for $\delta > 0$

$$\tilde{X}_\delta := \left\{ r \in L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1}); \begin{array}{c} \text{ess.sup}_{x\in\mathbb{R}^d} |r(x,\epsilon)| \leq \sum_{j=1}^{N+1} \epsilon_j, \\ \|r\|_{L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})} \leq \delta, \end{array} \right\}, \tag{4.5}$$

where the norm $\|\cdot\|_{L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})}$ is defined by

$$\|r\|_{L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})} := \sum_{|\alpha| \leq N+1} \sup_{\epsilon \in (0,\delta)^{N+1}} \text{ess.sup}_{x\in\mathbb{R}^d} \left|\partial_\epsilon^\alpha r(x,\epsilon)\right|. \tag{4.6}$$

We remark that $L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})$ is a Banach space, and $\tilde{X}_\delta$ is closed subspace in $L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})$. We will show that following lemma in the same way of Theorem 1.2.

**Lemma 4.1.** *We assume that $a(x,z)$ satisfies Assumption 1.1. Then, there exists $\tilde{\delta}_0 = \tilde{\delta}_{0,g_j,N} \in (0,1)$ such that for all $\delta \in (0, \tilde{\delta}_0)$ there exists an unique solution $r \in \tilde{X}_\delta$ such that*

$$r(x,\epsilon) = \int_{\mathbb{R}^d} \Phi_q(x,y) \left[ a\big(y, r(y,\epsilon) + v_\epsilon(y)\big) - q(y) r(y,\epsilon) \right] dy, \;\; x \in \mathbb{R}^d, \;\; \epsilon \in (0,\delta)^{N+1}. \tag{4.7}$$

*Proof.* We define the operator $\tilde{T}$ from $L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})$ into itself by

$$\begin{aligned}
\tilde{T} r(x,\epsilon) &:= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[ a\big(y, r(y,\epsilon) + v_\epsilon(y)\big) - q(y) r(y,\epsilon) \right] dy \\
&= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[ \sum_{l \geq 2} \frac{\partial_z^l a(y,0)}{l!} \big( r(y,\epsilon) + v_\epsilon(y) \big)^l + q(y) v_\epsilon(y) \right] dy \\
&= \int_{\mathbb{R}^d} \Phi_q(x,y) \left[ \sum_{l \geq 2} \frac{\partial_z^l a(y,0)}{l!} \sum_{m=0}^{l} \frac{l!}{(l-m)!m!} r^{l-m}(y,\epsilon) v_\epsilon^m(y) \right. \\
&\quad \left. + q(y) v_\epsilon(y) \right] dy
\end{aligned} \tag{4.8}$$

Let $r \in \tilde{X}_\delta$. With (4.2) we have

$$\begin{aligned}
&\left| \tilde{T} r(x,\epsilon) \right| \\
&\leq \left( \int_{B_R} |\Phi_q(x,y)| \, dy \right) \left[ \sum_{l \geq 2} c_0^l \sum_{m=0}^{l} \frac{1}{m!} \left( \sum_{j=1}^{N+1} \epsilon_j \right)^{l-m} \left( \tilde{C}_1 \delta^2 \sum_{j=1}^{N+1} \epsilon_j \right)^m \right. \\
&\quad \left. + \tilde{C}_1 \delta^2 \sum_{j=1}^{N+1} \epsilon_j \right] \\
&\leq \tilde{C}_2 \left[ \sum_{l \geq 2} c_0^l \left( \sum_{m=0}^{\infty} \frac{\tilde{C}_1^m}{m!} \right) \left( \sum_{j=1}^{N+1} \epsilon_j \right)^l + \tilde{C}_1 \delta^2 \sum_{j=1}^{N+1} \epsilon_j \right] \\
&\leq \tilde{C}_3 \left( \sum_{j=1}^{N+1} \epsilon_j \right)^2 \sum_{l \geq 2} c_0^l \left( \sum_{j=1}^{N+1} \epsilon_j \right)^{l-2} + \tilde{C}_3 \delta \left( \sum_{j=1}^{N+1} \epsilon_j \right) \\
&\leq \tilde{C} \delta \left( \sum_{j=1}^{N+1} \epsilon_j \right),
\end{aligned} \tag{4.9}$$

where $\tilde{C}, \tilde{C}_j > 0$ $(j = 1, 2)$ is constant independent of $r$, $\delta$, $\epsilon$ (but, depending on $g_j$ and $N$). Furthermore, we consider for $\alpha \in \mathbb{N}^{N+1}$ with $|\alpha| \leq N+1$

$$\begin{aligned}&\partial_\epsilon^\alpha \tilde{T}r(x,\epsilon)\\&\quad= \int_{\mathbb{R}^d} \Phi_q(x,y)\partial_\epsilon^\alpha \left[\sum_{l\geq 2} \frac{\partial_z^l a(y,0)}{l!} \sum_{m=0}^{l} \frac{l!}{(l-m)!m!} r^{l-m}(y,\epsilon)v_\epsilon^m(y) + q(y)v_\epsilon(y)\right] dy.\end{aligned} \tag{4.10}$$

Since $|\partial_{\epsilon_j} v_\epsilon(x)| \leq \tilde{C}_1' \delta^2$ and $|\partial_\epsilon^\alpha r^{l-m}(x,\epsilon)v_\epsilon^m(x)| \leq \tilde{C}_2'(l-m)!m!\delta^{l-m}(\tilde{C}_2'\delta^2)^m$, we have

$$\begin{aligned}\left|\partial_\epsilon^\alpha \tilde{T}r(x,\epsilon)\right| &\leq \left(\int_{B_R} |\Phi_q(x,y)|\, dy\right) \left[\sum_{l\geq 2} \frac{c_0^l}{l!} \sum_{m=0}^{l} \frac{l!m!(l-m)!}{(l-m)!m!} \delta^{l+m} (\tilde{C}_2')^m + \tilde{C}_3'\delta^2\right]\\&\leq \tilde{C}_4'\delta^2 \left(\sum_{l\geq 2} (c_0\delta)^{(l-2)} \sum_{m=0}^{\infty} (\tilde{C}_2'\delta)^m\right) + \tilde{C}_4'\delta^2 \leq \tilde{C}_5'\delta^2,\end{aligned} \tag{4.11}$$

where $\tilde{C}_j' > 0$ $(j = 3, 4, 5)$ is also constant independent of $r$, $\delta$, $\epsilon$ (but depending on $\alpha$). Then, we have

$$\sum_{|\alpha|\leq N+1} \sup_{\epsilon\in(0,\delta)^{N+1}} \text{ess.sup}_{x\in\mathbb{R}^d} \left|\partial_\epsilon^\alpha \tilde{T}r(x,\epsilon)\right| \leq \tilde{C}'\delta^2, \tag{4.12}$$

where $\tilde{C}'$ is constant independent of $r$, $\delta$, $\epsilon$. (Depending on $g_j$ and $N$.) By choosing $\tilde{\delta}_0 \in \left(0, \min(1/\tilde{C}, 1/\tilde{C}')\right)$, we conclude that $\tilde{T}r \in \tilde{X}_\delta$.

Let $r_1, r_2 \in \tilde{X}_\delta$. By similar argument in (3.6) we have

$$\begin{aligned}&\tilde{T}r_1(x,\epsilon) - \tilde{T}r_2(x,\epsilon)\\&\quad= \int_{B_R} \Phi_q(x,y) \sum_{l\geq 2} \frac{\partial_z^l a(y,0)}{l!} \left[\left(r_1(y,\epsilon) + v_\epsilon(y)\right)^l - \left(r_2(y,\epsilon) + v_\epsilon(y)\right)^l\right] dy\\&\quad= \int_{B_R} \Phi_q(x,y) \sum_{l\geq 2} \frac{\partial_z^l a(y,0)}{l!} \sum_{m=1}^{l} \frac{l!}{(l-m)!m!} v_\epsilon^{l-m}(y)\\&\qquad\times \sum_{h=0}^{m-1} r_1^{m-1-h}(y,\epsilon) r_2^h(y,\epsilon) \left(r_1(y,\epsilon) - r_2(y,\epsilon)\right) dy.\end{aligned} \tag{4.13}$$

Then, we have for $\alpha \in \mathbb{N}^{N+1}$ with $|\alpha| \leq N+1$

$$\begin{aligned}&\left|\partial_\epsilon^\alpha \left(\tilde{T}r_1(x) - \tilde{T}r_2(x)\right)\right|\\&\quad\leq \int_{B_R} |\Phi_q(x,y)| \sum_{\beta\leq\alpha} \frac{\alpha!}{(\alpha-\beta)!\beta!} \sum_{l\geq 2} \frac{\left|\partial_z^l a(y,0)\right|}{l!} \sum_{m=1}^{l} \frac{l!}{(l-m)!m!}\\&\qquad\times \sum_{h=0}^{m-1} \left|\partial_\epsilon^\beta \left(v_\epsilon^{l-m}(y) r_1^{m-1-h}(y,\epsilon) r_2^h(y,\epsilon)\right)\right| \left|\partial_\epsilon^{\alpha-\beta} \left(r_1(y,\epsilon) - r_2(y,\epsilon)\right)\right| dy.\end{aligned} \tag{4.14}$$

Since

$$\left|\partial_\epsilon^\beta \left(v_\epsilon^{l-m}(y) r_1^{m-1-h}(y,\epsilon) r_2^h(y,\epsilon)\right)\right| \leq \tilde{C}_1''(l-m)!(m-1-h)!h!(\tilde{C}_1''\delta^2)^{l-m}\delta^{m-1-h}\delta^h, \tag{4.15}$$

where $\tilde{C}_1''$ is constant independent of $r_1, r_2$ and $\delta$ (depending on $\beta$), we have that

$$\begin{aligned}
&\left|\partial_\epsilon^\alpha \left(\tilde{T}r_1(x) - \tilde{T}r_2(x)\right)\right| \\
&\leq \tilde{C}_2'' \left(\sum_{\beta \leq \alpha} \frac{\alpha!}{(\alpha-\beta)!\beta!} \sum_{l \geq 2} \frac{c_0^l}{l!} \right. \\
&\quad \left. \times \sum_{m=1}^{l} \sum_{h=0}^{m-1} \frac{l!(l-m)!(m-1-h)!h!}{(l-m)!m!} \delta^{2l-m-1} (\tilde{C}_1'')^{l-m} \right) \|r_1 - r_2\| \\
&\leq \tilde{C}_3''\delta \left(\sum_{l \geq 2} (c_0\delta)^{l-2} \sum_{m=1}^{l} (\tilde{C}_1''\delta)^{l-m} \sum_{h=0}^{m-1} \frac{(m-1-h)!h!}{m!}\right) \|r_1 - r_2\| \\
&\leq \tilde{C}_4''\delta \left(\sum_{l \geq 2} (c_0\delta)^{l-2} \sum_{p=0}^{\infty} (\tilde{C}_1''\delta)^p\right) \|r_1 - r_2\| \leq \tilde{C}_5''\delta \|r_1 - r_2\|_{L^\infty(\mathbb{R}^d; C^{N+1}(0,\delta)^{N+1})},
\end{aligned} \tag{4.16}$$

which implies that

$$\sum_{|\alpha| \leq N+1} \sup_{\epsilon \in (0,\delta)^{N+1}} \mathrm{ess.sup}_{x \in \mathbb{R}^d} \left|\partial_\epsilon^\alpha \left(\tilde{T}r_1(x,\epsilon) - \tilde{T}r_2(x,\epsilon)\right)\right| \leq \tilde{C}''\delta \|r_1 - r_2\|, \tag{4.17}$$

where $\tilde{C}_j'', \tilde{C}'' > 0$ $(j = 2, 3, 4)$ is constant independent of $r_1, r_2$ and $\delta$. By choosing $\tilde{\delta}_0 \in \left(0, \min(1/\tilde{C}, 1/\tilde{C}', 1/\tilde{C}'')\right)$, we have $\|Tr_1 - Tr_2\| < \|r_1 - r_2\|$, which implies that $\tilde{T}$ has a unique fixed point in $\tilde{X}_\delta$. Lemma 4.1 has been shown. □

## 5. Proof of Theorem 1.3

In Sect. 5, we will show Theorem 1.3. Since $a(x,z)$ is holomorphic at $z = 0$ by (ii) of Assumption 1.1, it is sufficient to show that

$$\partial_z^l a_1(x,0) = \partial_z^l a_2(x,0), \ x \in \mathbb{R}^d, \tag{5.1}$$

for all $l \in \mathbb{N}$. Let $N \in \mathbb{N}$ and let $g_j \in L^2(\mathbb{S}^{d-1})$ $(j = 1, 2, \ldots, N+1)$. Let $\delta \in \left(0, \min(\delta_0, \tilde{\delta}_0)\right)$ be chosen as sufficiently small and depending on $N$ and $g_j$. ($\delta_0, \tilde{\delta}_0$ are corresponding to Theorem 1.2 and Lemma 4.1, respectively.) From Section 4, we obtain the unique solution $r_{\epsilon,j} \in \tilde{X}_\delta$ $(j = 1, 2)$ such that

$$\Delta r_{\epsilon,j} + a_j(x, r_{\epsilon,j} + v_\epsilon) + k^2 r_{\epsilon,j} = 0 \text{ in } \mathbb{R}^d, \tag{5.2}$$

where $r_{\epsilon,j}$ satisfies the Sommerfeld radiation, and $v_\epsilon$ is given by (4.1). The solution $r_{\epsilon,j}$ has the form

$$r_{\epsilon,j}(x) = \int_{\mathbb{R}^d} \Phi(x,y) a_j(y, r_{\epsilon,j}(y) + v_\epsilon(y)) dy, \ x \in \mathbb{R}^d, \ \epsilon \in (0,\delta)^{N+1}. \quad (5.3)$$

By the assumption of Theorem 1.3 we have

$$r^\infty_{\epsilon,1}(\hat{x}) = r^\infty_{\epsilon,2}(\hat{x}), \ \hat{x} \in \mathbb{S}^{d-1}, \ \epsilon \in (0,\delta)^{N+1}, \quad (5.4)$$

where $r^\infty_{\epsilon,j}$ is a scattering amplitude for $r_{\epsilon,j}$, and it has the form

$$r^\infty_{\epsilon,j}(\hat{x}) = \int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} a_j(y, r_{\epsilon,j}(y) + v_\epsilon(y)) dy, \ \hat{x} \in \mathbb{S}^{d-1}, \ \epsilon \in (0,\delta)^{N+1}. \quad (5.5)$$

In order to linearize (5.3), we will differentiate it with respect to $\epsilon_l$ ($l = 1, \ldots, N+1$), which is possible because $r_{\epsilon,j} \in \tilde{X}_\delta$. Then, we have

$$\partial_{\epsilon_l} r_{\epsilon,j}(x) = \int_{\mathbb{R}^d} \Phi(x,y) \partial_z a_j(y, r_{\epsilon,j}(y) + v_\epsilon(y))(\partial_{\epsilon_l} r_{\epsilon,j}(y) + \delta^2 v_{g_l}(y)) dy. \quad (5.6)$$

As $\epsilon \to +0$ we have by setting $q_j := \partial_z a_j(y,0)$

$$w_{l,j}(x) := \partial_{\epsilon_l} r_{\epsilon,j}\Big|_{\epsilon=0}(x) = \int_{\mathbb{R}^d} \Phi(x,y) q_j(y)(w_{l,j}(y) + \delta^2 v_{g_l}(y)) dy, \quad (5.7)$$

which implies that

$$\Delta w_{l,j} + k^2 w_{l,j} = -\, q_j(w_{l,j} + \delta^2 v_{g_l}) \text{ in } \mathbb{R}^d. \quad (5.8)$$

By setting $u_{l,j} := w_{l,j} + \delta^2 v_{g_l}$ we have

$$\Delta u_{l,j} + k^2 u_{l,j} + q_j u_{l,j} = 0 \text{ in } \mathbb{R}^d. \quad (5.9)$$

By setting $u_l := u_{l,1} - u_{l,2} (= w_{l,1} - w_{l,2})$ we have

$$\Delta u_l + k^2 u_l + q_1 u_l = (q_2 - q_1) u_{l,2} \text{ in } \mathbb{R}^d, \quad (5.10)$$

and we also have

$$(q_2 - q_1) u_{h,1} u_{l,2} = u_{h,1} \Delta u_l - u_l \Delta u_{h,1} \text{ in } \mathbb{R}^d. \quad (5.11)$$

Differentiating (5.4) with respect to $\epsilon_l$ and as $\epsilon \to 0$ we have

$$\int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} q_1(y)(w_{l,1}(y) + \delta^2 v_{g_l}(y)) dy = \int_{\mathbb{R}^d} e^{-ik\hat{x}\cdot y} q_2(y)(w_{l,2}(y) + \delta^2 v_{g_l}(y)) dy, \quad (5.12)$$

which means that $w^\infty_{l,1} = w^\infty_{l,2}$, where $w^\infty_{l,j}$ is a scattering amplitude of $w_{l,j}$. By setting $\hat{w}_l := w_{l,1} - w_{l,2}$ we have

$$\Delta \hat{w}_l + k^2 \hat{w}_l = 0 \text{ in } \mathbb{R}\backslash\overline{B_R}, \tag{5.13}$$

where $\hat{w}_l$ satisfies the Sommerfeld radiation condition, and the scattering amplitude $\hat{w}_l^\infty$ of $\hat{w}_l$ vanishes. Then, we have $\hat{w}_l = 0$ (that is, $u_l = 0$) in $\mathbb{R}\backslash\overline{B_R}$, which implies that by the Green's second theorem we have ($l, h = 1, \ldots, N{+}1$)

$$\begin{aligned} 0 &= \int_{\partial B_{R+1}} u_{h,1}\partial_\nu u_l - u_l \partial_\nu u_{h,1} ds \\ &= \int_{B_{R+1}} u_{h,1}\Delta u_l - u_l \Delta u_{h,1} dx \\ &= \int_{B_R} (q_2 - q_1) u_{h,1} u_{l,2} dx. \end{aligned} \tag{5.14}$$

By (5.8), and definition of $H$ and $T_{q_j}$ in Section 2, $u_{l,j}$ can be of the form

$$u_{l,j} = \delta^2 T_{q_j} H g_l, \tag{5.15}$$

and dividing by $\delta^4 > 0$,

$$0 = \int_{B_R} (q_2 - q_1) T_{q_1} H g_h T_{q_2} H g_l dx. \tag{5.16}$$

Combining Lemma 2.1 with Lemma 2.2, we conclude that $q_1 = q_2$.

By induction, we will show (5.1). In the first part of this section, the case of $l = 1$ has been shown. We assume that

$$\partial_z^l a_1(x, 0) = \partial_z^l a_2(x, 0), \tag{5.17}$$

for all $l = 1, 2, \ldots, N$. We will show the case of $l = N + 1$. We alredy have shown that $q_1 = q_2$ and $w_{l,1}^\infty = w_{l,2}^\infty$, which implies that by the uniqueness of the linear Schrödinger equation (5.8) we have

$$w_{l,1} = w_{l,2} \text{ in } \mathbb{R}^d, \tag{5.18}$$

for all $l = 1, \ldots, N + 1$.

We set $q := q_1 = q_2$ and $w_l := w_{l,1} = w_{l,2}$. By subinduction we will show that for all $h \in \mathbb{N}$ with $1 \leq h \leq N$

$$\partial^h_{\epsilon_{l_1}\cdots\epsilon_{l_h}} r_{\epsilon,1}\Big|_{\epsilon=0} = \partial^h_{\epsilon_{l_1}\cdots\epsilon_{l_h}} r_{\epsilon,2}\Big|_{\epsilon=0}, \tag{5.19}$$

where $l_1, \ldots, l_h \in \{1, \ldots, N+1\}$. We already have shown that (5.19) holds for $h = 1$. We assume that (5.19) holds for all $h \leq K \leq N - 1$. (If $N = 1$, this subinduction is skipped.) By differentiating (5.3) with respect to $\partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}}$ we have

$$\begin{aligned} \partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}(x) = \int_{\mathbb{R}^d} \Phi(x,y) \Bigg\{ & \partial_z^{K+1} a_j(y, r_{\epsilon,j}(y) \\ & + v_\epsilon(y)) \prod_{h=1}^{K+1} (\partial_{\epsilon_{l_h}} r_{\epsilon,j}(y) + \delta^2 v_{g_{l_h}}(y)) \\ & + \partial_z a_j(y, r_{\epsilon,j}(y) \\ & + v_\epsilon(y)) \partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}(y) + R_{K,j}(y,\epsilon) \Bigg\} dy, \end{aligned} \tag{5.20}$$

where $R_{K,j}(y,\epsilon)$ is a polynomial of $\partial_z^h a_j(y, r_{\epsilon,j}(y) + v_\epsilon(y))$ and $\partial^h_{\epsilon_{l_1}\cdots\epsilon_{l_h}}$ $(r_{\epsilon,j}(y)+v_\epsilon(y))$ for $1 \le h \le K$. As $\epsilon \to 0$ we have

$$\partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}\Big|_{\epsilon=0}(x) = \int_{\mathbb{R}^d} \Phi(x,y)\Bigg\{\partial_z^{K+1} a_j(y,0)\prod_{h=1}^{K+1}(w_{l_h}(y)+\delta^2 v_{g_{l_h}}(y)) + q(y)\partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}\Big|_{\epsilon=0}(y) + R_{K,j}(y,0)\Bigg\}dy. \tag{5.21}$$

We set $\tilde{w}_{K+1,j} := \partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}} r_{\epsilon,j}\Big|_{\epsilon=0}$ and set $\tilde{w}_{K+1} := \tilde{w}_{K+1,1} - \tilde{w}_{K+1,2}$. By assumptions of induction and subinduction we have $R_{K,1}(y,0) = R_{K,2}(y,0)$ and $\partial_z^{K+1}a_1(\cdot,0) = \partial_z^{K+1}a_2(\cdot,0)$, which implies that

$$\tilde{w}_{K+1}(x) = \int_{\mathbb{R}^d}\Phi(x,y)q(y)\tilde{w}_{K+1}(y)dy, \tag{5.22}$$

which is equivalent to

$$\Delta\tilde{w}_{K+1} + k^2\tilde{w}_{K+1} + q\tilde{w}_{K+1} = 0 \text{ in } \mathbb{R}^d, \tag{5.23}$$

where $\tilde{w}_{K+1}$ satisfies Sommerfeld radiation condition. By differentiating (5.4) with respect to $\partial^{K+1}_{\epsilon_{l_1}\cdots\epsilon_{l_{K+1}}}$ and as $\epsilon \to 0$ we have

$$\tilde{w}^\infty_{K+1,1} = \tilde{w}^\infty_{K+1,2}, \tag{5.24}$$

where $\tilde{w}^\infty_{K+1,j}$ is a scattering amplitude of $\tilde{w}_{K+1,j}$. (5.24) means that $\tilde{w}^\infty_{K+1} = 0$, which implies that by Rellich theorem, we conclude that $\tilde{w}_{K+1} = 0$ in $\mathbb{R}^d$. (5.19) for the case of $K+1$ has been shown, and the claim (5.19) holds for all $h = 1,\ldots,N$ by subinduction.

By differentiating (5.3) with respect to $\partial^{N+1}_{\epsilon_1\cdots\epsilon_{K+1}}$, and as $\epsilon \to 0$ (the same argument in (5.20)–(5.22)) we have

$$\tilde{w}_{N+1}(x) = \int_{\mathbb{R}^d}\Phi(x,y)\Bigg\{\left(\partial_z^{N+1}a_1(x,0) - \partial_z^{N+1}a_2(x,0)\right)\prod_{h=1}^{N+1}(w_h(y)+\delta^2 v_{g_h}(y)) + q(y)\tilde{w}_{N+1}(y)\Bigg\}dy. \tag{5.25}$$

where $\tilde{w}_{N+1,j} := \partial^{N+1}_{\epsilon_1\cdots\epsilon_{l_{N+1}}} r_{\epsilon,j}\Big|_{\epsilon=0}$ and set $\tilde{w}_{N+1} := \tilde{w}_{N+1,1} - \tilde{w}_{N+1,2}$. This is equivalent to

$$\Delta\tilde{w}_{N+1} + k^2\tilde{w}_{N+1} + q\tilde{w}_{N+1} = -f\prod_{h=1}^{N+1}\delta^2 T_q H g_h \text{ in } \mathbb{R}^d, \tag{5.26}$$

where $f(x) := \partial_z^{N+1}a_1(x,0) - \partial_z^{N+1}a_2(x,0)$. By differentiating (5.4) with respect to $\partial^{N+1}_{\epsilon_1\cdots\epsilon_{K+1}}$ and as $\epsilon \to 0$ (the same argument in (5.24)) we have

$$\tilde{w}^\infty_{N+1} = 0, \tag{5.27}$$

where $\tilde{w}^\infty_{N+1}$ is a scattering amplitude of $\tilde{w}_{N+1}$. Then, we have $\tilde{w}_{N+1} = 0$ in $\mathbb{R}\backslash\overline{B_R}$.

Let $\tilde{v} \in L^2(B_{R+1})$ be a solution of $\Delta\tilde{v} + k^2\tilde{v} + q\tilde{v} = 0$ in $B_{R+1}$. By the Green's second theorem and (5.26) we have

$$\begin{aligned} 0 &= \int_{\partial B_{R+1}} \tilde{v}\partial_\nu \tilde{w}_{N+1} - \tilde{v}\partial_\nu \tilde{w}_{N+1} ds \\ &= \int_{B_{R+1}} \tilde{v}\Delta\tilde{w}_{N+1} - \tilde{w}_{N+1}\Delta\tilde{v} dx \\ &= \int_{B_{R+1}} -f \prod_{h=1}^{N+1} \delta^2 T_q H g_h \tilde{v} dx, \end{aligned} \tag{5.28}$$

which implies that dividing by $\delta^2 > 0$

$$\int_{B_{R+1}} f \prod_{h=1}^{N+1} T_q H g_h \tilde{v} dx = 0. \tag{5.29}$$

Let $v \in L^2(B_{R+1})$ be a solution of $\Delta v + k^2 v + qv = 0$ in $B_{R+1}$. By Lemma 2.1 we can choose $g_{N+1}$ as $g_{N+1,j} \in L^2(B_{R+1})$ such that $T_q H g_{N+1,j} \to v$ in $L^2(B_R)$ as $j \to \infty$. Then, we have that

$$\int_{B_{R+1}} f \prod_{h=1}^{N} T_q H g_h v\tilde{v} dx = 0. \tag{5.30}$$

which implies that by Lemma 2.2

$$f \prod_{h=1}^{N} T_q H g_h = 0. \tag{5.31}$$

By Theorem 5.1 of [20], we can choose a solution $u_h \in L^2(B_{R+1})$ ($h = 1, \ldots, N$) of $\Delta u_h + k^2 u_h + q u_h = 0$ in $B_{R+1}$, which is of the form

$$u_h(x) = e^{x\cdot p_h}(1 + \psi_h(x, p_h)), \tag{5.32}$$

with $\|\psi_h(\cdot, p_h)\|_{L^2(B_{R+1})} \leq \frac{C}{|p_h|}$ where $C > 0$ is a constant, and $p_h = a_h + ib_h$, $a_h, b_h \in \mathbb{R}^d$ such that $|a_h| = |b_h|$ and $a_h \cdot b_h = 0$ (which implies that $p_h \cdot p_h = 0$), and $a_h \neq a_{h'}$, $b_h \neq b_{h'}$.

Multiplying (5.31) by $\overline{f} \prod_{h=1}^{N+1} e^{-x\cdot p_h}$ we have

$$|f|^2 \prod_{h=1}^{N} e^{-x\cdot p_h} T_q H g_h = 0, \tag{5.33}$$

which implies that

$$\int_{B_R} |f|^2 \left( \prod_{h=1}^{N-1} e^{-x\cdot p_h} T_q H g_h \right) e^{-x\cdot p_N} T_q H g_N dx = 0. \tag{5.34}$$

By Lemma 2.1, there exists a sequence $\{g_{N,j}\}_{j\in\mathbb{N}} \subset L^2(\mathbb{S}^{d-1})$ such that $T_q H g_{N,j} \to u_N = e^{x\cdot p_N}(1 + \psi_N(x, p_N))$ in $L^2(B_R)$ , which implies that

$$\int_{B_R} |f|^2 \left( \prod_{h=1}^{N-1} e^{-x\cdot p_h} T_q H g_h \right) (1 + \psi(x, p_N)) dx = 0. \tag{5.35}$$

As $|a_N| = |b_N| \to \infty$ in (5.35) we have

$$\int_{B_R} |f|^2 \prod_{h=1}^{N-1} e^{-x\cdot p_h} T_q H g_h dx = 0. \tag{5.36}$$

Repeating the operation (5.34)–(5.36) $N-1$ times, we have that

$$\int_{B_R} |f|^2 dx = 0, \tag{5.37}$$

which conclude that $f = 0$. By induction, we conclude that (5.1) for all $l \in \mathbb{N}$. Therefore, Theorem 1.3 has been shown.


## Acknowledgements

The author thanks to Professor Mikko Salo, who supports him in this study, and gives him many comments to improve this paper.




## References


[1] Aktosun, T., Papanicolau, V.G., Zisis, V.: Inverse scattering on the line for a generalized nonlinear Schrödinger equation. Inverse Probl. **20**, 1267–1280 (2004)

[2] Bukhgeim, A.: Recovering a potential from Cauchy data in the two-dimensional case. J. Inverse Ill Posed Probl. **16**, 19–33 (2008)

[3] Colton, D., Kress, R.: Inverse Acoustic and Electromagnetic Scattering Theory, Third Edition. Applied Mathematical Sciences, vol. 93. Springer, New York (2013)

[4] Eskin, G.: Lectures on Linear Partial Differential Equations, vol. 123. American Mathematical Society, New York (2011)

[5] Feizmohammadi, A., Oksanen, L.: An inverse problem for a semi-linear elliptic equation in Riemannian geometries. Preprint (2019). arXiv:1904.00608

[6] Ghosh Roy, D., Couchman, L.: Inverse Problems and Inverse Scattering of Plane Waves. Academic Press, New York (2002)

[7] Harju, M., Serov, V.: Three-dimensional direct and inverse scattering for the Schrödinger equation with a general nonlinearity. Oper. Theory Adv. Appl. **236**, 257–273 (2014)

[8] Isakov, V., Nachman, A.I.: Global uniqueness for a two-dimensional semilinear elliptic inverse problem. Trans. Am. Math. Soc. **347**, 3375–3390 (1995)

[9] Jalade, E.: Inverse problem for a nonlinear Helmholtz equation. Ann. Inst. H. Poincaré Anal. Non Linéaire **21**, 517–531 (2004)

[10] Lassas, M., Liimatainen, T., Lin, Y.-H., Salo, M.: Inverse problems for elliptic equations with power type nonlinearities. Preprint (2019). arXiv:1903.12562

[11] Lassas, M., Liimatainen, T., Lin, Y.-H., Salo, M.: Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations. Preprint (2019). arXiv:1905.02764

[12] Nachman, A.I.: Reconstructions from boundary measurements. Ann. Math. **128**, 531–576 (1988)

[13] Novikov, R.G.: Multidimensional inverse spectral problems for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. Funct. Anal. Appl. **22**, 263–272 (1989)

[14] Päivärinta, L., Salo, M., Uhlmann, G.: Inverse scattering for the magnetic Schrödinger operator. J. Funct. Anal. **259**, 1771–1798 (2010)

[15] Ramm, A.G.: Recovery of the potential from fixed-energy scattering data. Inverse Probl. **4**, 877–886 (1988)

[16] Serov, V.: Inverse fixed energy scattering problem for the generalized nonlinear Schrödinger operator. Inverse Probl. **28**, 025002 (2012)

[17] Serov, V., Harju, M., Fotopoulosc, G.: Direct and inverse scattering for nonlinear Schrödinger equation in 2D. J. Math. Phys. **53**, 123522 (2012)

[18] Serov, V., Harju, M.: A uniqueness theorem and reconstruction of singularities for a two-dimensional nonlinear Schrödinger equation. Nonlinearity **21**, 1323–1337 (2008)

[19] Sylvester, J., Uhlmann, G.: A global uniqueness theorem for an inverse boundary value problem. Ann. Math. **125**, 153–169 (1987)

[20] Uhlmann, G.: Electrical impedance tomography and Calderon's problem. Inverse Probl. **25**, 123011 (2009)

[21] Watanabe, M.: Time-dependent method for non-linear Schrödinger equations in inverse scattering problems. J. Math. Anal. Appl. **459**, 932–944 (2018)

[22] Weder, R.: Lp–Lp estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential. J. Funct. Anal. **170**, 37–68 (2000)

[23] Weder, R.: Inverse scattering for the nonlinear Schrödinger equation II. Reconstruction of the potential and the nonlinearity in the multidimensional case. Proc. Am. Math. Soc. **129**, 3637–3645 (2001)

Takashi Furuya
Graduate School of Mathematics
Nagoya University
Nagoya
Japan
e-mail: takashi.furuya0101@gmail.com

Received: 21 December 2019.
Accepted: 14 March 2020.