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The Strichartz estimates for the damped wave equation and the behavior of solutions for the energy critical nonlinear equation

Takahisa Inui

Abstract. For the linear damped wave equation (DW), the L^p–L^q type estimates have been well studied. Recently, Watanabe (RIMS Kôkyûroku Bessatsu B 63:77–101, 2017) showed the Strichartz estimates for DW when $d = 2, 3$. In the present paper, we give Strichartz estimates for DW in higher dimensions. Moreover, by applying the estimates, we give the local well-posedness of the energy critical nonlinear damped wave equation (NLDW) $\partial_t^2 u - \Delta u + \partial_t u = |u|^{\frac{4}{d-2}} u$, $(t, x) \in [0, T) \times \mathbb{R}^d$, where $3 \leq d \leq 5$. Especially, we show the small data global existence for NLDW. In addition, we investigate the behavior of the solutions to NLDW. Namely, we give a decay result for solutions with finite Strichartz norm and a blow-up result for solutions with negative Nehari functional.

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Contents

Birkhäuser

1. Introduction

1.1. Backgroud

We consider the damped wave equation.

$$\begin{cases} \partial_t^2\phi - \Delta\phi + \partial_t\phi = 0, & (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ (\phi(0), \partial_t\phi(0)) = (\phi_0, \phi_1), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $d \in \mathbb{N}$, (ϕ_0, ϕ_1) is given, and ϕ is an unknown complex valued function.

Matsumura [21] applied the Fourier transform to (1.1) and obtained the formula

$$\phi(t,x) = \mathcal{D}(t)(\phi_0 + \phi_1) + \partial_t\mathcal{D}(t)\phi_0,$$

where $\mathcal{D}(t)$ is defined by

$$\mathcal{D}(t) := e^{-\frac{t}{2}}\mathcal{F}^{-1}L(t,\xi)\mathcal{F}$$

with

$$L(t,\xi) := \begin{cases} \dfrac{\sinh(t\sqrt{1/4-|\xi|^2})}{\sqrt{1/4-|\xi|^2}} & \text{if } |\xi| < 1/2, \\ \dfrac{\sin(t\sqrt{|\xi|^2-1/4})}{\sqrt{|\xi|^2-1/4}} & \text{if } |\xi| > 1/2. \end{cases}$$

By this formula, Matsumura [21] proved the L^p–L^q type estimate:

$$\|\phi(t)\|_{L^p} \lesssim \langle t\rangle^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|(\phi_0,\phi_1)\|_{L^q\times L^q} + e^{-\frac{t}{4}}\left(\|\phi_0\|_{H^{[\frac{d}{2}]+1}} + \|\phi_1\|_{H^{[\frac{d}{2}]}}\right), \tag{1.2}$$

where $1 \le q \le 2 \le p \le \infty$ and $[d/2]$ denotes the integer part of $d/2$. Such L^p–L^q type estimates have been studied well. See [7,22,23] and references therein. The L^p–L^q type estimates for the heat equation and the wave equation are also well studied. We recall the L^p–L^q type estimate for the heat equation $\partial_t v - \Delta v = 0$:

$$\|\mathcal{G}(t)g\|_{L^p} \lesssim t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|g\|_{L^q},$$

where $1 \le q \le p \le \infty$ and $\mathcal{G}(t) := \mathcal{F}^{-1}e^{-t|\xi|^2}\mathcal{F}$. We also refer to the L^p–L^q type estimate for the wave equation $\partial_t^2 w - \Delta w = 0$:

$$\|\mathcal{W}(t)g\|_{L^p} \lesssim |t|^{-2d\left(\frac{1}{2}-\frac{1}{p}\right)} \|g\|_{\dot{W}^{\gamma-1,p'}},$$

for $2 \le p < \infty$ and $(d+1)(1/2-1/p) \le \gamma < d$, where p' denotes the Hölder conjugate of p and $\mathcal{W}(t) := \mathcal{F}^{-1}\sin(t|\xi|)/|\xi|\mathcal{F}$. See [1]. Matsumura's estimate (1.2) shows that the solution of (1.1) behaves like the solution of the heat equation and the wave equation in some sense. More precisely, the low frequency part of the solution to the damped wave equation behaves like the solution of the heat equation and the high frequency part behaves like the solution of the wave equation but decays exponentially (see [9] for another L^p–L^q estimate).

For the heat equation and the wave equation, by using the L^p–L^q type estimates, we obtain the space-time estimates, what we call the Strichartz estimate. The Strichartz estimates for the heat equation are

$$\|v\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} \lesssim \|v_0\|_{L^2} + \|F\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))},$$

where v satisfies $\partial_t v - \Delta v = F$ with $v(0) = v_0$ and (q,r) and $(\tilde{q},\tilde{r})$ satisfy $2/q + d/r = 2/\tilde{q} + d/\tilde{r} = d/2$. See [2,31]. We also have the Strichartz estimates for the wave equation as follows.

$$\|w\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} \lesssim \|w_0\|_{\dot{H}^1} + \|w_1\|_{L^2} + \|F\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))},$$

where w satisfies $\partial_t^2 w - \Delta w = F$ with $(w(0), \partial_t w(0)) = (w_0, w_1)$ and $1/q + d/r = d/2 - 1 = 1/\tilde{q}' + d/\tilde{r}' - 2$. See [4]. In the present paper, we give the Strichartz estimates for the damped wave equation. Recently, Watanabe [30] obtained the Strichartz estimates for the damped wave equation when $d = 2,3$ by an energy method. In this paper, we give the Strichartz estimates by a duality argument for $d = 2,3$ and higher dimensions.

We also consider the energy critical nonlinear damped wave equation.

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = |u|^{\frac{4}{d-2}} u, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ (u(0), \partial_t u(0)) = (u_0, u_1), & x \in \mathbb{R}^d, \end{cases} \tag{NLDW}$$

where $d \geq 3$, (u_0, u_1) is given, and u is an unknown complex valued function. The corresponding wave equation $\partial_t^2 w - \Delta w = |w|^{\frac{4}{d-2}} w$ is invariant under the scaling $w_\lambda(t,x) := \lambda^{(d-2)/2} w(\lambda t, \lambda x)$ for $\lambda > 0$. And the $\dot{H}^1$-norm, which is called (kinetic) energy norm, is also invariant under this scaling. Thus, the wave equation is called energy critical. Similarly, the corresponding heat equation $\partial_t v - \Delta v = |v|^{\frac{4}{d-2}} v$ is invariant under the scaling $v_\eta(t,x) := \eta^{(d-2)/2} v(\eta^2 t, \eta x)$ for $\eta > 0$. The $\dot{H}^1$-norm is also invariant under this scaling and thus the heat equation is also called energy critical. Equation (NLDW) is not invariant under the scaling. However, the power of the nonlinear term is same as the energy critical wave and heat equation. That is why we call (NLDW) energy critical.

We will show the local well-posedness for (NLDW) when $3 \leq d \leq 5$ by applying the Strichartz estimates. The existence of a local solution has been studied by Ikeda and Inui [15], Ikeda and Wakasugi [8] and Kapitanskiĭ [10] (see also [12–14]). However, the small data global existence has not been known. Using the Strichartz estimates which are proved in this paper, we can show not only the existence of a local solution but also the small data global existence for (NLDW).

Moreover, we discuss the global behavior of the solutions to (NLDW). For the energy critical nonlinear heat equation, the solution with a bounded global space-time norm decays to zero (see e.g. [6]). On the other hand, there exist finite time blow-up solutions by Levine [19]. For the energy critical nonlinear wave equation, the energy is conserved by the flow. There exist solutions which scatter to the solutions of the free wave equation and finite time blow-up solutions by Payne and Sattinger [25]. See also [16]. In the present paper, we prove that the solution to (NLDW) with a finite space-time norm decays. And we also show that there exist finite time blow-up solutions.

1.2. Main results

We state main results. First, we obtain the Strichartz estimates for (1.1). The so-called admissible pairs can be taken as same as in the heat case since the L^p–L^q type estimate of the low frequency part is similar to the heat estimate and the high frequency part decays exponentially in time. However, the derivative loss appears from the high frequency part which is wave-like part.

Proposition 1.1. (Homogeneous Strichartz estimates) *Let* $d \geq 2$, $2 \leq r < \infty$, *and* $2 \leq q \leq \infty$. *Set* $\gamma := \max\{d(1/2 - 1/r) - 1/q, \frac{d+1}{2}(1/2 - 1/r)\}$. *Assume*

$$\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \geq \frac{1}{q},$$

Then, we have

$$\begin{aligned} \|\mathcal{D}(t)f\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} &\lesssim \left\|\langle\nabla\rangle^{\gamma-1} f\right\|_{L^2}, \\ \|\partial_t\mathcal{D}(t)f\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} &\lesssim \|\langle\nabla\rangle^{\gamma} f\|_{L^2}, \\ \left\|\partial_t^2\mathcal{D}(t)f\right\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} &\lesssim \left\|\langle\nabla\rangle^{\gamma+1} f\right\|_{L^2}. \end{aligned}$$

Remark 1.1. We note that the homogeneous Strichartz estimate holds in the heat end-point case *i.e.* $(q,r) = (2, 2d/(d-2))$ when $d \geq 3$.

Proposition 1.2. (Inhomogeneous Strichartz estimates) *Let* $d \geq 2$, $2 \leq r, \tilde{r} < \infty$, *and* $2 \leq q, \tilde{q} \leq \infty$. *We set* $\gamma := \max\{d(1/2 - 1/r) - 1/q, \frac{d+1}{2}(1/2 - 1/r)\}$ *and* $\tilde{\gamma} := \max\{d(1/2 - 1/\tilde{r}) - 1/\tilde{q}, \frac{d+1}{2}(1/2 - 1/\tilde{r})\}$. *Assume that* (q,r) *and* $(\tilde{q},\tilde{r})$ *satisfies*

$$\begin{aligned} &\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) > \frac{1}{q} + \frac{1}{\tilde{q}}, \\ &\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) = \frac{1}{q} + \frac{1}{\tilde{q}} \text{ and } 1 < \tilde{q}' < q < \infty, \end{aligned}$$

or

$$(q,r) = (\tilde{q},\tilde{r}) = (\infty, 2).$$

Moreover, we exclude the wave end-point case, that is, we assume $(q,r) \neq (2, 2(d-1)/(d-3)))$ *and* $(\tilde{q},\tilde{r}) \neq (2, 2(d-1)/(d-3)))$ *when* $d \geq 4$. *Then, we have*

$$\begin{aligned} \left\|\int_0^t \mathcal{D}(t-s)F(s)ds\right\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} &\lesssim \left\|\langle\nabla\rangle^{\gamma+\tilde{\gamma}+\delta-1} F\right\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))}, \\ \left\|\int_0^t \partial_t\mathcal{D}(t-s)F(s)ds\right\|_{L_t^q(I:L_x^r(\mathbb{R}^d))} &\lesssim \left\|\langle\nabla\rangle^{\gamma+\tilde{\gamma}+\delta} F\right\|_{L_t^{\tilde{q}'}(I:L_x^{\tilde{r}'}(\mathbb{R}^d))}, \end{aligned}$$

where $\delta = 0$ *when* $\frac{1}{\tilde{q}}(1/2 - 1/r) = \frac{1}{q}(1/2 - 1/\tilde{r})$ *and in the other cases* $\delta \geq 0$ *is defined in Table 1.*

Table 1. The value of δ

δ	$\frac{1}{\tilde{q}}\left(\frac{1}{2}-\frac{1}{r}\right) < \frac{1}{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)$	$\frac{1}{\tilde{q}}\left(\frac{1}{2}-\frac{1}{r}\right) > \frac{1}{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)$
$(q,r) \in \mathcal{A}$ $(\tilde{q},\tilde{r}) \in \mathcal{A}$	0	0
$(q,r) \in \mathcal{A}$ $(\tilde{q},\tilde{r}) \notin \mathcal{A}$	$\times$	$\frac{\tilde{q}}{q}\left\{\frac{1}{\tilde{q}}-\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)\right\}$
$(q,r) \notin \mathcal{A}$ $(\tilde{q},\tilde{r}) \in \mathcal{A}$	$\frac{q}{\tilde{q}}\left\{\frac{1}{q}-\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)\right\}$	$\times$
$(q,r) \notin \mathcal{A}$ $(\tilde{q},\tilde{r}) \notin \mathcal{A}$	$\frac{1}{\tilde{q}}\frac{d-1}{2}\left\{\tilde{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)-q\left(\frac{1}{2}-\frac{1}{r}\right)\right\}$	$\frac{1}{q}\frac{d-1}{2}\left\{q\left(\frac{1}{2}-\frac{1}{r}\right)-\tilde{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)\right\}$

$\times$ means that the case does not occur and we set $\mathcal{A} := \{(q,r) : \frac{d-1}{2}(1/2-1/r) \geq 1/q\}$

Remark 1.2. If (q,r) satisfies the wave admissible condition $\frac{d-1}{2}(1/2-1/r) \geq 1/q$, then the derivative loss is same as that in the Strichartz estimates for the wave equation *i.e.* $\gamma = d(1/2-1/r)-1/q$. And thus, we need more derivative if (q,r) is the pair between the wave case and the heat case, *i.e.* $\frac{d}{2}(1/2-1/r) \geq 1/q > \frac{d-1}{2}(1/2-1/r)$.

Remark 1.3. The wave end-point case is studied in the sequel paper [11].

Applying these Strichartz estimates, we will show the following local well-posedness and small data global existence of (NLDW). For simplicity, we denote $L^q_{t,x}(I) := L^q_t(I : L^q_x(\mathbb{R}^d))$.

Definition 1.1. (*Solution*) Let $T \in (0,\infty]$. We say that u is a solution to (NLDW) on $[0,T)$ if u satisfies $(u,\partial_t u) \in C([0,T) : H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$, $\langle \nabla \rangle^{1/2} u \in L_{t,x}^{\frac{2(d+1)}{d-1}}(I)$ and $u \in L_{t,x}^{\frac{2(d+1)}{d-2}}(I)$ for any compact interval $I \subset [0,T)$, $(u(0),\partial_t u(0)) = (u_0,u_1)$, and the Duhamel's formula

$$u(t,x) = \mathcal{D}(t)(u_0+u_1) + \partial_t \mathcal{D}(t) u_0 + \int_0^t \mathcal{D}(t-s)(|u(s)|^{\frac{4}{d-2}} u(s)) ds$$

for all $t \in [0,T)$. We say that u is global if $T = \infty$.

We have the following local well-posedness result when $3 \leq d \leq 5$.

Theorem 1.3. (Local well-posedness) *Let $d \in \{3,4,5\}$ and $T \in (0,\infty]$. Let $(u_0,u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ satisfy $\|(u_0,u_1)\|_{H^1 \times L^2} \leq A$. Then, there exists $\delta = \delta(A) > 0$ such that if*

$$\|\mathcal{D}(t)(u_0+u_1) + \partial_t \mathcal{D}(t) u_0\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))} \leq \delta,$$

then there exists a unique solution u to (NLDW) *with $\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))} \leq 2\delta$. Moreover, we have the standard blow-up criterion, that is, if the maximal existence time $T_+ = T_+(u_0,u_1)$ is finite, then the solution satisfies $\|u\|_{L^{\frac{2(d+1)}{d-2}}([0,T_+))} = \infty$.*

From this, we especially get the following small data global existence.

Theorem 1.4. (Small data global existence) *Let $d \in \{3,4,5\}$ and $(u_0,u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then, there exists a small constant $\delta_0 > 0$ such that if $\|(u_0,u_1)\|_{H^1 \times L^2} \leq \delta_0$, then the solution u (constructed in Theorem 1.3) is global and satisfies $\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,\infty))} \leq C\delta_0$ for some constant $C > 0$.*

Remark 1.4. See the sequel paper [11] for the local well-posedness and small data global existence of (NLDW) when $d \geq 6$. The difficulty of $d \geq 6$ comes from the loss of differentiability of the nonlinear term. We need to pay attention to the difference estimate of the nonlinear terms.

Remark 1.5. The existence of local solution is well known (see [8,10]). However, the small data global existence has not been known except for low dimension cases (Watanabe [30] showed the small data global existence when $d = 3$).

Remark 1.6. As it is well known, we can obtain the local well-posedness of the nonlinear damped wave equation with the more general nonlinearity in the same way as Theorem 1.3. Namely, we find the local well-posedness for the following equation.

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = \mathcal{N}(u), & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ (u(0), \partial_t u(0)) = (u_0, u_1), & x \in \mathbb{R}^d. \end{cases} \tag{1.3}$$

Assume that the nonlinearity $\mathcal{N} : \mathbb{C} \to \mathbb{C}$ is continuously differentiable and obeys the power type estimates

$$\mathcal{N}(z) = O(|z|^{1+\frac{4}{d-2}}),$$
$$\mathcal{N}_z(z), \mathcal{N}_{\bar{z}}(z) = O(|z|^{\frac{4}{d-2}}),$$
$$\mathcal{N}_z(z) - \mathcal{N}_z(w), \mathcal{N}_{\bar{z}}(z) - \mathcal{N}_{\bar{z}}(w) = O(|z-w|^{\min\{1,\frac{4}{d-2}\}}(|z|+|w|)^{\max\{0,\frac{6-d}{d-2}\}}),$$

where $\mathcal{N}_z$ and $\mathcal{N}_{\bar{z}}$ are the usual derivatives

$$\mathcal{N}_z := \frac{1}{2}\left(\frac{\partial \mathcal{N}}{\partial x} - i\frac{\partial \mathcal{N}}{\partial y}\right), \quad \mathcal{N}_z := \frac{1}{2}\left(\frac{\partial \mathcal{N}}{\partial x} + i\frac{\partial \mathcal{N}}{\partial y}\right)$$

for $z = x + iy$. The typical examples are $\mathcal{N}(u) = \lambda|u|^{1+4/(d-2)}$ or $\lambda|u|^{4/(d-2)}u$ with $\lambda \in \mathbb{C}\backslash\{0\}$.

We have the energy E of (NLDW), which is defined by

$$E(u, \partial_t u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|\partial_t u\|_{L^2}^2 - \frac{d-2}{2d}\|u\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}}.$$

If u is a solution to (NLDW), then the energy satisfies

$$\frac{d}{dt}E(u(t), \partial_t u(t)) = -\|\partial_t u(t)\|_{L^2}^2$$

for all $t \in (0, T_{\max})$. This means the energy decay. This observation shows us that some global solutions may decay. Indeed, we can prove that a global solution with a finite Strichartz norm decays to 0 in the energy space as follows.

Theorem 1.5. *Let u be a global solution of* (NLDW) *and we assume that the solution u satisfies $\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,\infty))} < \infty$, then u satisfies*

$$\lim_{t\to\infty}(\|u(t)\|_{H^1} + \|\partial_t u(t)\|_{L^2}) = 0.$$

Remark 1.7. This is similar to the energy critical nonlinear heat equation. See Gustafson and Roxanas [6].

Remark 1.8. Theorem 1.5 holds for all dimensions $d \geq 3$ since we need to treat the estimate of the difference unlike the local well-posedness.

At last, we show the blow-up of the solutions to (NLDW). We set

$$J(\varphi) := \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 - \frac{d-2}{2d}\|\varphi\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}},$$
$$K(\varphi) := \|\nabla\varphi\|_{L^2}^2 - \|\varphi\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}}.$$

Then, it is well known that the minimal energy

$$\mu := \inf\left\{J(\varphi) : \varphi \in \dot{H}^1\backslash\{0\}, K(\varphi) = 0\right\}$$

is well-defined and positive (see [29] for more information). Then, we have the following blow-up result.

Theorem 1.6. (Ohta [24]) *Let* $(u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ *belong to*

$$\mathscr{B} := \{(u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) : E(u_0, u_1) < \mu, K(u_0) < 0\}.$$

Then the solution to (NLDW) *blows up in finite time.*

Remark 1.9. The proof of Theorem 1.6 is essentially given by Ohta [24]. He showed the blow-up result for abstract setting by the method of an ordinary differential inequality instead of by the so-called concavity argument which is well applied to wave or Klein-Gordon equation. We omit the proof.

Notation 1.1. We collect some notations. For the exponent p, we denote the Hölder conjugate of p by p'. The bracket $\langle\cdot\rangle$ is Japanese bracket *i.e.* $\langle a\rangle := (1 + |a|^2)^{1/2}$.

We use $A \lesssim B$ to denote the estimate $A \le CB$ with some constant $C > 0$. The notation $A \sim B$ stands for $A \lesssim B$ and $A \lesssim B$.

Let $\chi_{\le 1} \in C_0^\infty(\mathbb{R})$ be a cut-off function satisfying $\chi_{\le 1}(r) = 1$ for $|r| \le 1$ and $\chi_{\le 1}(r) = 0$ for $|r| \ge 2$ and let $\chi_{>1} = 1 - \chi_{\le 1}$.

For a function $f : \mathbb{R}^n \to \mathbb{C}$, we define the Fourier transform and the inverse Fourier transform by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx,$$

$$\mathcal{F}^{-1}[f](x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\xi} f(\xi)\,dx.$$

For a measurable function $m = m(\xi)$, we denote the Fourier multiplier $m(\nabla)$ by

$$m(\nabla)f(x) = \mathcal{F}^{-1}\left[m(\xi)\hat{f}(\xi)\right](x).$$

For $s \in \mathbb{R}$ and $1 \le p \le \infty$, we denote the usual Sobolev space by

$$W^{s,p}(\mathbb{R}^d) := \left\{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{W^{s,p}} = \|\langle\nabla\rangle^s f\|_{L^p} < \infty\right\}.$$

We write $H^s(\mathbb{R}^d) := W^{s,2}(\mathbb{R}^d)$ for simplicity. Let $\dot{W}^{s,p}(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d)$ denote the corresponding homogeneous Sobolev spaces.

We define $P_{\le 1} := \mathcal{F}^{-1}\chi_{\le 1}\mathcal{F}$, $P_{>1} := \mathcal{F}^{-1}\chi_{>1}\mathcal{F}$, and

$$P_N = \mathcal{F}^{-1}\left(\chi_{\le 1}\left(\frac{\xi}{N}\right) - \chi_{\le 1}\left(\frac{2\xi}{N}\right)\right)\mathcal{F}$$

for $N \in 2^{\mathbb{Z}}$. For a time interval I and $F : I \times \mathbb{R}^d \to \mathbb{C}$, we set

$$\|F\|_{L^q(I:L^r(\mathbb{R}^d))} := \left(\int_I \|F(t,\cdot)\|_{L^r(\mathbb{R}^d)}^q\,dt\right)^{1/q}$$

and $\|F\|_{L^q_{t,x}(I)} := \|F\|_{L^q(I:L^q(\mathbb{R}^d))}$. The space of functions with finite this norm are denoted by $L^q(I : L^r(\mathbb{R}^d))$ (or $L^q_{t,x}(I)$ if $q = r$). We sometimes use L^p_s and L^p_t to uncover time variables s and t.

This paper is structured as follows. Section 2 is devoted to show the Strichartz estimates. In particular, we give the Strichartz estimates for low frequency part in Sect. 2.1 and those for high frequency part in Sect. 2.2. In Sect. 3, we prove the local well-posedness of (NLDW) by the Strichartz estimates. Section 4 is devoted to discuss the decay of the global solutions to (NLDW) with a finite space-time norm.

2. The Strichartz estimates

We split $\mathcal{D}$ to low frequency part $\mathcal{D}_l$ and high frequency part $\mathcal{D}_h$ as follows.

$$\mathcal{D}_l(t) := \mathcal{D}(t)P_{\le 1},$$
$$\mathcal{D}_h(t) := \mathcal{D}(t)P_{>1}.$$

In this section, we prove the Strichartz estimates for low and high frequency parts respectively.

2.1. The Strichartz estimates for low frequency part

We have the L^p–L^q type estimates for low frequency part. These estimates are similar to those of the heat equation.

Lemma 2.1. (L^r–$L^{\tilde{r}}$ estimate for low frequency part [9, Proposition 2.4]) *Let* $1 \le \tilde{r} \le r \le \infty$ *and* $\sigma \ge 0$. *Then, we have*

$$\||\nabla|^\sigma \mathcal{D}_l(t) f\|_{L^r} \lesssim \langle t \rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}}-\frac{1}{r}\right)-\frac{\sigma}{2}} \|f\|_{L^{\tilde{r}}},$$

for any $t > 0$ *and* $f \in L^{\tilde{r}}(\mathbb{R}^d)$. *We also have*

$$\||\nabla|^\sigma \partial_t \mathcal{D}_l(t) f\|_{L^r} \lesssim \langle t \rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}}-\frac{1}{r}\right)-\frac{\sigma}{2}-1} \|f\|_{L^{\tilde{r}}},$$
$$\left\||\nabla|^\sigma \partial_t^2 \mathcal{D}_l(t) f\right\|_{L^r} \lesssim \langle t \rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}}-\frac{1}{r}\right)-\frac{\sigma}{2}-2} \|f\|_{L^{\tilde{r}}}.$$

By these L^p–L^q type estimates, we obtain the following homogeneous Strichartz estimate.

Lemma 2.2. (Homogeneous Strichartz estimate for low frequency part) *Let* $\sigma \ge 0$. *Let* $1 \le \tilde{r} \le r \le \infty$ *and* $1 \le q \le \infty$. *Assume that they satisfy*

$$\frac{d}{2}\left(\frac{1}{\tilde{r}} - \frac{1}{r}\right) > \frac{1}{q},$$

or

$$\frac{d}{2}\left(\frac{1}{\tilde{r}} - \frac{1}{r}\right) = \frac{1}{q} \text{ and } q > \tilde{r} > 1.$$

Then, for any $f \in L^{\tilde{r}}(\mathbb{R}^d)$,

$$\|\langle \nabla \rangle^\sigma \mathcal{D}_l(t) f\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \|f\|_{L^{\tilde{r}}},$$

where $I \subset [0,\infty)$ is a time interval and the implicit constant is independent of I. Moreover, we also have

$$\|\langle\nabla\rangle^{\sigma}\,\partial_t\mathcal{D}_l(t)f\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \|f\|_{L^{\tilde{r}}},$$
$$\left\|\langle\nabla\rangle^{\sigma}\,\partial_t^2\mathcal{D}_l(t)f\right\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \|f\|_{L^{\tilde{r}}}.$$

Proof. These Strichartz estimates are same as those of the heat equation. Thus, the same proof does work. However, we give the proof for reader's convenience.

We first consider the case of $\frac{d}{2}(1/\tilde{r}-1/r) > 1/q$. By the L^r–$L^{\tilde{r}}$ estimate (Lemma 2.1),

$$\begin{aligned}\|\langle\nabla\rangle^{\sigma}\,\mathcal{D}_l(t)f\|_{L^r} &\lesssim \|\mathcal{D}_l(t)f\|_{L^r} + \||\nabla|^{\sigma}\mathcal{D}_l(t)f\|_{L^r}\\ &\lesssim \langle t\rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}}-\frac{1}{r}\right)}\|f\|_{L^{\tilde{r}}} + \langle t\rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}}-\frac{1}{r}\right)-\frac{\sigma}{2}}\|f\|_{L^{\tilde{r}}}\\ &\lesssim \langle t\rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}}-\frac{1}{r}\right)}\|f\|_{L^{\tilde{r}}}.\end{aligned}$$

Then, we obtain

$$\|\langle\nabla\rangle^{\sigma}\,\mathcal{D}_l(t)f\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \left\|\langle t\rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}}-\frac{1}{r}\right)}\|f\|_{L^{\tilde{r}}}\right\|_{L^q([0,\infty))} \lesssim \|f\|_{L^{\tilde{r}}}.$$

Next, we consider the second case. We set $Tf := \|\langle\nabla\rangle^{\sigma}\,\mathcal{D}_l(t)f\|_{L^r(\mathbb{R}^d)}$ and $(q_1,r_1) := (\infty,r)$ and $(q_2,r_2) = (\rho,\gamma)$, where (ρ,γ) satisfies $\frac{d}{2}\left(1/\gamma - 1/r\right) = 1/\rho$ and $\rho,\gamma > 1$. Then T is sub-additive and we have $T : L^{r_j}(\mathbb{R}^d) \to L^{q_j,\infty}([0,\infty))$ for $j=1,2$. Indeed, we have

$$\|Tf(t)\|_{L^\infty(I)} \lesssim \|\langle\nabla\rangle^{\sigma}\,\mathcal{D}_l(t)f\|_{L^\infty(I:L^r(\mathbb{R}^d))} \lesssim \|f\|_{L^r},$$
$$\|Tf(t)\|_{L^{\rho,\infty}(I)} \lesssim \|\langle\nabla\rangle^{\sigma}\,\mathcal{D}_l(t)f\|_{L^{\rho,\infty}(I:L^r(\mathbb{R}^d))} \lesssim \|f\|_{L^\gamma}.$$

If $\rho \geq \gamma$, we can use the Marcinkiewicz interpolation theorem so that we have

$$\|\mathcal{D}_l(t)f\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \|f\|_{L^{\tilde{r}}},$$

for $(q,\tilde{r})$ satisfying $q > \tilde{r} > 1$ and

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \frac{1}{\tilde{r}} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}, \quad 0<\theta<1.$$

This means that the desired inequality holds for (q,r) such that $\frac{d}{2}(1/\tilde{r}-1/r) = 1/q$ and $q > \tilde{r} > 1$. See also [2,31]. In the same way, we get the second and the third inequalities. □

Remark 2.1. We exclude the end-point case in Lemma 2.2 since it is not clear whether the end-point Strichartz estimate holds or not for $q = \tilde{r}$ and $\tilde{r} \neq 2$. We will show the heat end-point Strichartz estimate for $\tilde{r} = 2$ (see Lemma 2.11) as stated in Remark 1.1.

Lemma 2.3. (Inhomogeneous Strichartz estimate for low frequency part) *Let $\sigma \geq 0$. Let $1 \leq \tilde{r}' \leq r \leq \infty$ and $1 \leq q,\tilde{q} \leq \infty$. Assume that they satisfy*

$$\frac{d}{2}\left(\frac{1}{2}-\frac{1}{r}\right) + \frac{d}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right) > \frac{1}{q} + \frac{1}{\tilde{q}},$$
$$\frac{d}{2}\left(\frac{1}{2}-\frac{1}{r}\right) + \frac{d}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right) = \frac{1}{q} + \frac{1}{\tilde{q}} \text{ and } 1 < \tilde{q}' < q < \infty,$$

or

$$(q, r) = (\tilde{q}, \tilde{r}) = (\infty, 2).$$

Then it holds that

$$\left\| \langle \nabla \rangle^{\sigma} \int_0^t \mathcal{D}_l(t-s) F(s) ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \|F\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))},$$
$$\left\| \int_0^t \partial_t \mathcal{D}_l(t-s) F(s) ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \|F\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))},$$

where $I \subset [0, \infty)$ is a time interval such that $0 \in \overline{I}$ and the implicit constant is independent of I.

Proof. We only show the first estimate since the second can be proved similarly. Applying the L^r–$L^{\tilde{r}}$ estimate (Lemma 2.1), we obtain

$$\begin{aligned}
&\left\| \langle \nabla \rangle^{\sigma} \int_0^t \mathcal{D}_l(t-s) F(s) ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \\
&\lesssim \left\| \int_0^t \|\langle \nabla \rangle^{\sigma} \mathcal{D}_l(t-s) F(s)\|_{L^r} ds \right\|_{L^q(I)} \\
&\lesssim \left\| \int_0^t \langle t-s \rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right)} \|F(s)\|_{L^{\tilde{r}'}} ds \right\|_{L^q(I)}.
\end{aligned}$$

When $\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) > \frac{1}{q} + \frac{1}{\tilde{q}}$, by the Young inequality, we obtain

$$\begin{aligned}
&\left\| \int_0^t \langle t-s \rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right)} \|F(s)\|_{L^{\tilde{r}'}} ds \right\|_{L^q(I)} \\
&\lesssim \left\| \langle t \rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right)} \right\|_{L^{\frac{q\tilde{q}}{\tilde{q}+q}}} \|F\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))} \\
&\lesssim \|F\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))}.
\end{aligned}$$

On the other hand, when $\frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) = \frac{1}{q} + \frac{1}{\tilde{q}}$ and $1 < \tilde{q}' < q < \infty$, applying the Hardy–Littlewood–Sobolev inequality, we obtain

$$\left\| \int_0^t \langle t-s \rangle^{-\frac{d}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right)} \|F(s)\|_{L^{\tilde{r}'}} ds \right\|_{L^q(I)} \lesssim \|F\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))}.$$

When $(q, r) = (\tilde{q}, \tilde{r}) = (\infty, 2)$, the inequality is trivial. This completes the proof. □

2.2. The Strichartz estimates for high frequency part

Since we have

$$\mathcal{D}_h(t) = e^{-\frac{t}{2}} \mathcal{F}^{-1} \frac{e^{it\sqrt{|\xi|^2 - 1/4}} - e^{-it\sqrt{|\xi|^2 - 1/4}}}{2i\sqrt{|\xi|^2 - 1/4}} \chi_{>1}(\xi) \mathcal{F},$$

it is enough to estimate

$$e^{-t/2} e^{\pm it\sqrt{-\Delta - 1/4}} P_{>1}.$$

Lemma 2.4. (Homogeneous Strichartz estimate for high frequency part) *Let $d \geq 2$. Let $2 \leq r < \infty$ and $2 \leq q \leq \infty$. Then, we have*

$$\left\| e^{-t/2} e^{\pm it\sqrt{-\Delta-1/4}} P_{>1} f \right\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \| |\nabla|^\gamma f \|_{L^2}$$

where $I \subset [0,\infty)$ is a time interval and the implicit constant is independent of I. In particular, we have

$$\begin{aligned} \|\mathcal{D}_h(t) f\|_{L^q(I:L^r(\mathbb{R}^d))} &\lesssim \left\| |\nabla|^\gamma \langle \nabla \rangle^{-1} f \right\|_{L^2}, \\ \|\partial_t \mathcal{D}_h(t) f\|_{L^q(I:L^r(\mathbb{R}^d))} &\lesssim \| |\nabla|^\gamma f \|_{L^2}, \\ \left\|\partial_t^2 \mathcal{D}_h(t) f\right\|_{L^q(I:L^r(\mathbb{R}^d))} &\lesssim \left\| |\nabla|^{\gamma+1} f \right\|_{L^2}. \end{aligned}$$

Proof. First, we consider $e^{it\sqrt{-\Delta-1/4}}$. We note that

$$e^{it\sqrt{-\Delta-1/4}} = e^{it|\nabla|} e^{it\left(\sqrt{-\Delta-1/4}-|\nabla|\right)}.$$

Since we have

$$\left| \sqrt{|\xi|^2 - 1/4} - |\xi| \right| = \frac{1}{4(\sqrt{|\xi|^2-1/4} + |\xi|)} \approx |\xi|^{-1},$$

a simple calculation shows

$$\left| \partial_\xi^\alpha e^{it\left(\sqrt{|\xi|^2-1/4}-|\xi|\right)} \right| \lesssim \langle t \rangle^{|\alpha|} |\xi|^{-|\alpha|}$$

for $\xi \neq 0$ and $\alpha \in \mathbb{Z}_{\geq 0}^n$. Thus, the Mihlin–Hörmander multiplier theorem (see [5, Theorem 6.2.7]) gives

$$\left\| e^{it\sqrt{-\Delta-1/4}} P_{>1} f \right\|_{L^r} \lesssim \langle t \rangle^{\delta_r} \left\| e^{it|\nabla|} f \right\|_{L^r}$$

for some $\delta_r > 0$. Therefore, we obtain

$$\begin{aligned} \left\| e^{-t/2} e^{it\sqrt{-\Delta-1/4}} P_{>1} f \right\|_{L^q(I:L^r(\mathbb{R}^d))} &= \left\| e^{-t/2} \left\| e^{it\sqrt{-\Delta-1/4}} P_{>1} f \right\|_{L^r} \right\|_{L^q(I)} \\ &\lesssim \left\| e^{-t/2} \langle t \rangle^{\delta_r} \left\| e^{it|\nabla|} f \right\|_{L^r} \right\|_{L^q(I)} \\ &\lesssim \left\| e^{it|\nabla|} f \right\|_{L^{\tilde{q}}(I:L^r(\mathbb{R}^d))}, \end{aligned}$$

where we have used the Hölder inequality in the last inequality and we take $\tilde{q}$ such that

$$\tilde{q} = \begin{cases} q & \text{if } \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \geq \frac{1}{q}, \\ \left\{ \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \right\}^{-1} & \text{if } \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) < \frac{1}{q}. \end{cases}$$

Then, $(\tilde{q}, r)$ is a wave admissible pair. Namely, it satisfies

$$\frac{1}{\tilde{q}} + \frac{d-1}{2r} \leq \frac{d-1}{4}, \ \tilde{q}, r, d \geq 2, \text{ and } (q,r,d) \neq (2,\infty,3)$$

and

$$\frac{1}{\tilde{q}} + \frac{d}{r} = \frac{d}{2} - \gamma,$$

where we note that $\gamma \geq 0$. Therefore, by the Strichartz estimate for the free wave equation (see [4] or [18, Corollary 2.5 in p.233]), we get

$$\begin{aligned}\left\| e^{-t/2} e^{it\sqrt{-\Delta-1/4}} P_{>1} f \right\|_{L^q(I:L^r(\mathbb{R}^d))} &\lesssim \left\| e^{it|\nabla|} f \right\|_{L^{\tilde{q}}(I:L^r(\mathbb{R}^d))} \\ &\lesssim \| |\nabla|^\gamma f \|_{L^2}.\end{aligned}$$

Similarly, we also have

$$\left\| e^{-t/2} e^{-it\sqrt{-\Delta-1/4}} P_{>1} f \right\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \| |\nabla|^\gamma f \|_{L^2}.$$

Combining them with the formula of $\mathcal{D}_h$, we obtain

$$\| \mathcal{D}_h(t) f \|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \left\| |\nabla|^\gamma \langle \nabla \rangle^{-1} f \right\|_{L^2},$$

where we use $\sqrt{|\xi|^2 - 1/4} \approx \langle \xi \rangle$ for $|\xi| \geq 1$. Moreover, we also get the estimates related to $\partial_t \mathcal{D}_h(t)$ and $\partial_t^2 \mathcal{D}_h(t)$. □

Remark 2.2. We can also obtain the homogeneous Strichartz estimates for high frequency part when $1 \leq q < 2$. Indeed, taking

$$\tilde{q} = \begin{cases} 2 & \text{if } \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \geq \frac{1}{2}, \\ \left\{ \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \right\}^{-1} & \text{if } \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) < \frac{1}{2}, \end{cases}$$

$(\tilde{q}, r)$ is a wave admissible pair and thus the above argument does work. We note that, in this case, we need to redefine γ such that

$$\gamma := \max\left\{ \frac{d+1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), d\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{2} \right\} \geq 0.$$

To prove inhomogeneous Strichartz estimates for high frequency part, we show the L^p–L^q type estimate.

Lemma 2.5. (L^r–$L^{r'}$ estimate for high frequency part) *Let $d \geq 1$. Let $2 \leq r < \infty$. Then, it holds that*

$$\left\| e^{\pm it\sqrt{-\Delta-1/4}} P_{>1} P_N f \right\|_{L^r} \lesssim \langle t \rangle^{\delta_r} (1 + |t|N)^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} N^{d\left(1-\frac{2}{r}\right)} \| P_N f \|_{L^{r'}}$$

for any $t > 0$ and $N \in 2^{\mathbb{Z}}$, where δ_r is a positive constant.

Proof. Combining the L^p–L^q type estimate for free wave equation (see [1] or [18, Lemma 2.1 in p.230]) and the Mihlin–Hörmander multiplier theorem, we get the statement. □

Lemma 2.6. (Inhomogeneous Strichartz estimate for high frequency part) *Let $d \geq 2$. Let $2 \leq r < \infty$ and $2 \leq q \leq \infty$. We exclude the wave end-point case, that is, we assume that $(q, r) \neq (2, 2(d-1)/(d-3))$ when $d \geq 4$. Then, we have*

$$\begin{aligned}&\left\| \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1} P_N F(s) ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \\ &\quad \lesssim N^{2\gamma} \| P_N F \|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))},\end{aligned}$$

where $I \subset [0,\infty)$ is a time interval such that $0 \in \overline{I}$ and the implicit constant is independent of I.

Proof. By the L^r–$L^{r'}$ estimate for high frequency part, Lemma 2.5, we get

$$\begin{aligned}
&\left\| \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1} P_N F(s) ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \\
&\quad \lesssim N^{d\left(1-\frac{2}{r}\right)} \left\| \int_0^t e^{-\frac{t-s}{4}} (1+|t-s|N)^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \|P_N F(s)\|_{L^{r'}} ds \right\|_{L^q(I)}.
\end{aligned} \tag{2.1}$$

Here, by the Young inequality, we obtain

$$\begin{aligned}
&N^{d\left(1-\frac{2}{r}\right)} \left\| \int_0^t e^{-\frac{t-s}{4}} (1+|t-s|N)^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \|P_N F(s)\|_{L^{r'}} ds \right\|_{L^q(I)} \\
&\quad \lesssim N^{d\left(1-\frac{2}{r}\right)} \left\| e^{-\frac{\cdot}{4}} (1+|\cdot|N)^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \right\|_{L^{q/2}([0,\infty))} \|P_N F\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))}.
\end{aligned} \tag{2.2}$$

In the case of $\frac{d-1}{2}(1-2/r) > 2/q$, since we have

$$\begin{aligned}
&\left\| e^{-\frac{\cdot}{4}} (1+|\cdot|N)^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \right\|_{L^{q/2}([0,\infty))}^{q/2} \\
&\quad \le \int_0^\infty (1+|t|N)^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)\frac{q}{2}} dt \lesssim N^{-1},
\end{aligned}$$

we obtain, from (2.1) and (2.2),

$$\begin{aligned}
\text{(L.H.S. of (2.1))} &\lesssim N^{2\left\{d\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{q}\right\}} \|P_N F\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))} \\
&= N^{2\gamma} \|P_N F\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))}.
\end{aligned}$$

On the other hand, in the case of $\frac{d-1}{2}(1-2/r) < 2/q$, we have

$$\begin{aligned}
&\left\| e^{-\frac{\cdot}{4}} (1+|\cdot|N)^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \right\|_{L^{q/2}([0,\infty))}^{q/2} \\
&\quad = \int_0^\infty e^{-\frac{q}{8}t} (1+|t|N)^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)\frac{q}{2}} dt \\
&\quad \le N^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)\frac{q}{2}} \int_0^\infty e^{-\frac{q}{8}t} t^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)\frac{q}{2}} dt \\
&\quad \le N^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)\frac{q}{2}} \left(\int_0^1 t^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)\frac{q}{2}} dt + \int_1^\infty e^{-\frac{q}{8}t} dt \right) \\
&\quad \lesssim N^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)\frac{q}{2}}.
\end{aligned}$$

Therefore, we obtain, from (2.1) and (2.2),

$$\begin{aligned}
\text{(L.H.S. of (2.1))} &\lesssim N^{2\left\{\frac{d+1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)\right\}} \|P_N F\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))} \\
&= N^{2\gamma} \|P_N F\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))}.
\end{aligned}$$

At last, we consider the case of $\frac{d-1}{2}(1-2/r) = 2/q$. Then, we have

$$
\begin{aligned}
&N^{d\left(1-\frac{2}{r}\right)} \left\| \int_0^t e^{-\frac{t-s}{4}} (1+|t-s|N)^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \|P_N F(s)\|_{L^{r'}} \, ds \right\|_{L^q(I)} \\
&\lesssim N^{d\left(1-\frac{2}{r}\right)} N^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \left\| \int_0^t |t-s|^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \|P_N F(s)\|_{L^{r'}} \, ds \right\|_{L^q(I)} \\
&= N^{2\gamma} \left\| \int_0^t |t-s|^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \|P_N F(s)\|_{L^{r'}} \, ds \right\|_{L^q(I)} \qquad (2.3)
\end{aligned}
$$

and it follows from the Hardy–Littlewood–Sobolev inequality that

$$
\left\| \int_0^t |t-s|^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} \|P_N F(s)\|_{L^{r'}} \, ds \right\|_{L^q(I)} \lesssim \|P_N F(s)\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))}, \quad (2.4)
$$

since (q, r) is not the end-point. Combining (2.1), (2.3), and (2.4), we get the desired inequality. □

Remark 2.3. In the previous lemma, we exclude the end-point case. However, we can obtain the Strichartz estimate in the end-point case. See the sequel paper [11].

Lemma 2.7. ($L_t^\infty L_x^2$–$L_t^{q'} L_x^{r'}$ estimate for high frequency part) *Let $d \geq 2$. Let $2 \leq r < \infty$ and $2 \leq q \leq \infty$. We assume that $(q, r) \neq (2, 2(d-1)/(d-3))$ when $d \geq 4$. Then, we have*

$$
\left\| \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1} P_N F(s) ds \right\|_{L^\infty(I:L^2(\mathbb{R}^d))} \lesssim N^\gamma \|P_N F\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))},
$$

where $I \subset [0, \infty)$ is a time interval such that $0 \in \overline{I}$ and the implicit constant is independent of I.

Proof. We set $\mathcal{W}_N^\pm(t-s) := e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1} P_N$ for simplicity. Now, we have

$$
\begin{aligned}
&\left\| \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1} P_N F(s) ds \right\|_{L^2}^2 \\
&= \left\langle \int_0^t e^{-\frac{t-s}{2}} \mathcal{W}_N^\pm(t-s) F(s) ds, \int_0^t e^{-\frac{t-\tau}{2}} \mathcal{W}_N^\pm(t-\tau) F(\tau) d\tau \right\rangle_{L^2} \\
&= \int_0^t \int_0^s \left\langle e^{-\frac{t-s}{2}} \mathcal{W}_N^\pm(t-s) F(s), e^{-\frac{t-\tau}{2}} \mathcal{W}_N^\pm(t-\tau) F(\tau) \right\rangle_{L^2} d\tau ds \\
&\quad + \int_0^t \int_0^\tau \left\langle e^{-\frac{t-s}{2}} \mathcal{W}_N^\pm(t-s) F(s), e^{-\frac{t-\tau}{2}} \mathcal{W}_N^\pm(t-\tau) F(\tau) \right\rangle_{L^2} ds d\tau \\
&= I + \mathit{II}.
\end{aligned}
$$

By the symmetry, it is enough to estimate I. By the Hölder inequality, $e^{-\frac{t-s}{2}} e^{-\frac{t-\tau}{2}} = e^{-(t-s)}e^{-\frac{s-\tau}{2}}$, and $e^{-(t-s)} \leq 1$ for $s \in [0,t]$ we obtain

$$\begin{aligned} I &= \int_0^t \left\langle e^{-\frac{t-s}{2}}\mathcal{W}_N^{\pm}(t-s)F(s), \int_0^s e^{-\frac{t-\tau}{2}}\mathcal{W}_N^{\pm}(t-\tau)F(\tau)d\tau \right\rangle_{L^2} ds \\ &\leq \int_0^t e^{-(t-s)} \left\langle |P_N F(s)|, \left|\int_0^s e^{-\frac{s-\tau}{2}}\mathcal{W}_N^{\pm}(s-\tau)P_{>1}F(\tau)d\tau\right| \right\rangle_{L^2} ds \\ &\leq \|P_N F\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))} \left\| \int_0^s e^{-\frac{s-\tau}{2}}\mathcal{W}_N^{\pm}(s-\tau)P_{>1}F(\tau)d\tau \right\|_{L_s^q((0,t):L^r(\mathbb{R}^d))}. \end{aligned}$$

By Lemma 2.6, we obtain

$$I \leq N^{2\gamma} \|P_N F\|^2_{L^{q'}(I:L^{r'}(\mathbb{R}^d))}.$$

Thus, it follows that

$$\left\| \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1}P_N F(s)ds \right\|^2_{L^2} \lesssim N^{2\gamma} \|P_N F\|^2_{L^{q'}(I:L^{r'}(\mathbb{R}^d))}.$$

This finishes the proof. □

Remark 2.4. Let $T>0$, $2 \leq r < \infty$ and $2 \leq q \leq \infty$, $\gamma := \max\{d(1/2-1/r) - 1/q, \frac{d+1}{2}(1/2-1/r)\}$, and $(q,r) \neq (2, 2(d-1)/(d-3))$ when $d \geq 4$. Then, we have the following inequality by the same argument as in Lemma 2.6.

$$\begin{aligned} &\left\| \int_s^t e^{-(\tau-s)}e^{-\frac{t-\tau}{2}} e^{\mp i(t-\tau)\sqrt{-\Delta-1/4}} P_{>1}P_N F(\tau)d\tau \right\|_{L_t^q((s,T):L^r(\mathbb{R}^d))} \\ &\quad\lesssim N^{2\gamma} \|P_N F\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))}, \end{aligned} \tag{2.5}$$

where $s<T$ is a parameter. Moreover, we also have the following estimate from (2.5) and the similar argument to Lemma 2.7.

$$\begin{aligned} &\left\| \int_s^T e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1}P_N F(t)dt \right\|_{L_s^\infty([0,T):L^2(\mathbb{R}^d))} \\ &\quad\lesssim N^{\gamma} \|P_N F\|_{L^{q'}([0,T):L^{r'}(\mathbb{R}^d))}. \end{aligned} \tag{2.6}$$

Lemma 2.8. ($L_t^q L_x^r$–$L_t^1 L_x^2$ estimate for high frequency part) *Let $2 \leq r < \infty$ and $2 \leq q \leq \infty$. We assume that $(q,r) \neq (2, 2(d-1)/(d-3))$ when $d \geq 4$. Then, we have*

$$\begin{aligned} &\left\| \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1}P_N F(s)ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \\ &\quad\lesssim N^{\gamma} \|P_N F\|_{L^1(I:L^2(\mathbb{R}^d))}, \end{aligned}$$

where $I \subset [0,\infty)$ is a time interval such that $0 \in \overline{I}$ and the implicit constant is independent of I.

Proof. We may write $I=[0,T)$. We use a standard duality argument. Let $G \in C_0^\infty(I \times \mathbb{R}^d)$ and $\tilde{P}_N := P_{N/2} + P_N + P_{2N}$. Since we have $\tilde{P}_N P_N = P_N$, it follows from the Fubini theorem and Hölder inequality that

$$\begin{aligned}
&\int_0^T \left\langle \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1} P_N F(s) ds, G(t) \right\rangle dt \\
&\quad = \int_0^T \int_0^t e^{-\frac{t-s}{2}} \left\langle e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1} P_N F(s), G(t) \right\rangle dsdt \\
&\quad = \int_0^T \int_s^T e^{-\frac{t-s}{2}} \left\langle P_N F(s), e^{\mp i(t-s)\sqrt{-\Delta-1/4}} P_{>1} \tilde{P}_N G(t) \right\rangle dtds \\
&\quad = \int_0^T \left\langle P_N F(s), \int_s^T e^{-\frac{t-s}{2}} e^{\mp i(t-s)\sqrt{-\Delta-1/4}} P_{>1} \tilde{P}_N G(t) dt \right\rangle ds \\
&\quad \leq \|P_N F\|_{L^1(I:L^2(\mathbb{R}^d))} \left\| \int_s^T e^{-\frac{t-s}{2}} e^{\mp i(t-s)\sqrt{-\Delta-1/4}} P_{>1} \tilde{P}_N G(t) dt \right\|_{L^\infty(I:L^2(\mathbb{R}^d))}
\end{aligned} \tag{2.7}$$

By (2.6) in Remark 2.4, we get

$$\begin{aligned}
&\left\| \int_s^T e^{-\frac{t-s}{2}} e^{\mp i(t-s)\sqrt{-\Delta-1/4}} P_{>1} \tilde{P}_N G(t) dt \right\|_{L^\infty(I:L^2(\mathbb{R}^d))} \\
&\quad \leq \sum_{j=N/2,N,2N} \left\| \int_s^T e^{-\frac{t-s}{2}} e^{\mp i(t-s)\sqrt{-\Delta-1/4}} P_{>1} P_j G(t) dt \right\|_{L^\infty(I:L^2(\mathbb{R}^d))} \\
&\quad \lesssim N^\gamma \sum_{j=N/2,N,2N} \|P_j G\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))} \\
&\quad \lesssim N^\gamma \|G\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))} .
\end{aligned} \tag{2.8}$$

Since we have the duality

$$\begin{aligned}
&\|F\|_{L^q(I:L^r(\mathbb{R}^d))} \\
&\quad = \sup \left\{ \int_I \langle F(t), G(t) \rangle \, dt : G \in C_0^\infty(I \times \mathbb{R}^d), \|G\|_{L^{q'}(I:L^{r'}(\mathbb{R}^d))} = 1 \right\},
\end{aligned}$$

the desired estimate follows from (2.7) and (2.8). □

Combining these estimates, we obtain the following Strichartz estimates when $(1/q, 1/r)$ and $(1/\tilde{q}, 1/\tilde{r})$ are on a same line.

Lemma 2.9. *Let $2 \leq r, \tilde{r} < \infty$ and $2 \leq q, \tilde{q} \leq \infty$. Assume that*

$$\frac{1}{\tilde{q}} \left(\frac{1}{2} - \frac{1}{r} \right) = \frac{1}{q} \left(\frac{1}{2} - \frac{1}{\tilde{r}} \right).$$

We also assume that $(q, r) \neq (2, 2(d-1)/(d-3))$ and $(\tilde{q}, \tilde{r}) \neq (2, 2(d-1)/(d-3))$ when $d \geq 4$. Then, we have

$$\begin{aligned}
&\left\| \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1} F(s) ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \\
&\quad \lesssim \left\| |\nabla|^{\gamma+\tilde{\gamma}} F \right\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))} ,
\end{aligned}$$

where $I \subset [0, \infty)$ is a time interval such that $0 \in \overline{I}$ and the implicit constant is independent of I.

Proof. We set

$$\Psi[F](t,x) := \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1}F(s)ds.$$

First, we consider the case of $2 \le r \le \tilde{r}$. Then, $\tilde{q} \le q$ and thus there exists $\theta \in [0,1]$ such that

$$\frac{1}{q} = \frac{\theta}{\tilde{q}} + \frac{1-\theta}{\infty}, \quad \frac{1}{r} = \frac{\theta}{\tilde{r}} + \frac{1-\theta}{2}.$$

By this formula, we have $\theta\tilde{\gamma} = \gamma$. Therefore, by the Hölder inequality, Lemmas 2.6 and 2.7, we obtain

$$\begin{aligned} &\|\Psi[F]\|_{L^q(I:L^r(\mathbb{R}^d))} \\ &\lesssim \|\Psi[F]\|^{\theta}_{L^{\tilde{q}}(I:L^{\tilde{r}}(\mathbb{R}^d))} \|\Psi[F]\|^{1-\theta}_{L^\infty(I:L^2(\mathbb{R}^d))} \\ &\lesssim \left(N^{2\tilde{\gamma}} \|P_N F\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))}\right)^{\theta} \left(N^{\tilde{\gamma}} \|P_N F\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))}\right)^{1-\theta} \\ &\approx N^{\gamma+\tilde{\gamma}} \|P_N F\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))}, \end{aligned}$$

where we use $\theta\tilde{\gamma} = \gamma$.

At second, we consider the case of $2 \le \tilde{r} \le r$. Then, we have $\tilde{q} \ge q$. Let $\eta \in [0,1]$ satisfy

$$\frac{1}{\tilde{q}'} = \frac{1-\eta}{1} + \frac{\eta}{q'}, \quad \frac{1}{\tilde{r}'} = \frac{1-\eta}{2} + \frac{\eta}{r'}.$$

Then, we have $\eta\gamma = \tilde{\gamma}$. By the interpolation, Lemmas 2.6, and 2.8, we get the desired inequality, where we note that $N^{(1-\eta)\gamma}N^{\eta 2\gamma} = N^{\gamma+\tilde{\gamma}}$. Taking summation for dyadic number N gives the statement. □

We can get Strichartz estimates even when $(1/q, 1/r)$ and $(1/\tilde{q}, 1/\tilde{r})$ are not on a same line by permitting more derivative loss.

Lemma 2.10. *Let $d \ge 2$. Let $2 \le r, \tilde{r} < \infty$ and $2 \le q, \tilde{q} \le \infty$. Assume that*

$$\frac{1}{\tilde{q}}\left(\frac{1}{2} - \frac{1}{r}\right) \neq \frac{1}{q}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right).$$

We also assume that $(q,r) \neq (2, 2(d-1)/(d-3))$ and $(\tilde{q},\tilde{r}) \neq (2, 2(d-1)/(d-3))$ when $d \ge 4$. Then, we have

$$\begin{aligned} &\left\| \int_0^t e^{-\frac{t-s}{2}} e^{\pm i(t-s)\sqrt{-\Delta-1/4}} P_{>1}F(s)ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \\ &\quad \lesssim \left\| |\nabla|^{\gamma+\tilde{\gamma}+\delta} F \right\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))}, \end{aligned}$$

where $\delta \ge 0$ is defined in Table 1 (see Proposition 1.2). Moreover, we have

$$\left\| \int_0^t \mathcal{D}_h(t-s)F(s)ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \left\| |\nabla|^{\gamma+\tilde{\gamma}+\delta} \langle\nabla\rangle^{-1} F \right\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))},$$

$$\left\| \int_0^t (\partial_t\mathcal{D}_h)(t-s)F(s)ds \right\|_{L^q(I:L^r(\mathbb{R}^d))} \lesssim \left\| |\nabla|^{\gamma+\tilde{\gamma}+\delta} F \right\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))}.$$

Proof. We consider the following cases respectively.

1. $\frac{1}{\tilde{q}}\left(\frac{1}{2}-\frac{1}{r}\right)<\frac{1}{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)$
2. $\frac{1}{\tilde{q}}\left(\frac{1}{2}-\frac{1}{r}\right)>\frac{1}{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)$

a. $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)\geq\frac{1}{q}$ and $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)\geq\frac{1}{\tilde{q}}$
b. $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)\geq\frac{1}{q}$ and $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)<\frac{1}{\tilde{q}}$
c. $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)<\frac{1}{q}$ and $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)\geq\frac{1}{\tilde{q}}$
d. $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)<\frac{1}{q}$ and $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)<\frac{1}{\tilde{q}}$

It is easy to show that Cases (1)-(b) and (2)-(c) do not occur.

Case(1). We treat the case of $\frac{1}{\tilde{q}}\left(\frac{1}{2}-\frac{1}{r}\right)<\frac{1}{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)$. Since $\frac{1}{\tilde{q}}\left(\frac{1}{2}-\frac{1}{r}\right)<\frac{1}{q}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)$, there exists $r_1\in[2,\tilde{r})$ such that

$$\frac{1}{\tilde{q}}\left(\frac{1}{2}-\frac{1}{r}\right)=\frac{1}{q}\left(\frac{1}{2}-\frac{1}{r_1}\right).$$

Let γ_1 be the derivative loss for the pair $(\tilde{q},r_1)$. Then, by Lemma 2.9 and the Bernstein inequality, we get

$$\begin{aligned}\|\Psi[F]\|_{L_t^qL_x^r}&\lesssim N^{\gamma+\gamma_1}\|P_NF\|_{L_t^{\tilde{q}'}L_x^{r_1'}}\\&\lesssim N^{\gamma+\gamma_1}N^{d\left(\frac{1}{\tilde{r}'}-\frac{1}{r_1'}\right)}\|P_NF\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}.\end{aligned}$$

Case(1)-(a). If $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)\geq\frac{1}{q}$, which also gives $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r_1}\right)\geq\frac{1}{\tilde{q}}$, we have $\gamma_1=d(1/2-1/r_1)-1/\tilde{q}$. Thus, we obtain

$$\begin{aligned}\gamma+\gamma_1+d\left(\frac{1}{\tilde{r}'}-\frac{1}{r_1'}\right)&=\gamma+d\left(\frac{1}{2}-\frac{1}{r_1}\right)-\frac{1}{q_1}+d\left(\frac{1}{\tilde{r}'}-\frac{1}{r_1'}\right)\\&=\gamma+d\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)-\frac{1}{\tilde{q}}\\&=\gamma+\tilde{\gamma}.\end{aligned}$$

Case(1)-(c). $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)<\frac{1}{q}$ gives $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r_1}\right)<\frac{1}{\tilde{q}}$. Then, we have $\gamma_1=\frac{d+1}{2}(1/2-1/r_1)$. Moreover, since $\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)\geq\frac{1}{\tilde{q}}$, we have $\tilde{\gamma}=d(1/2-1/\tilde{r})-1/\tilde{q}$. Therefore, we obtain

$$\begin{aligned}\gamma+\gamma_1+d\left(\frac{1}{\tilde{r}'}-\frac{1}{r_1'}\right)&=\gamma+\tilde{\gamma}+\gamma_1-\tilde{\gamma}+d\left(\frac{1}{\tilde{r}'}-\frac{1}{r_1'}\right)\\&=\gamma+\tilde{\gamma}+\frac{q}{\tilde{q}}\left\{\frac{1}{q}-\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{r}\right)\right\},\end{aligned}$$

where we use $q\left(\frac{1}{2}-\frac{1}{r}\right)=\tilde{q}\left(\frac{1}{2}-\frac{1}{r_1}\right)$ in the last equality.

Case(1)-(d). We have $\gamma_1 = \frac{d+1}{2}(1/2 - 1/r_1)$ since $\frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) < \frac{1}{q}$. Since $\frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) < \frac{1}{\tilde{q}}$, we have $\tilde{\gamma} = \frac{d+1}{2}(1/2 - 1/\tilde{r})$ and thus we obtain

$$\begin{aligned}\gamma + \gamma_1 + d\left(\frac{1}{\tilde{r}'} - \frac{1}{r_1'}\right) &= \gamma + \tilde{\gamma} + \gamma_1 - \tilde{\gamma} + d\left(\frac{1}{\tilde{r}'} - \frac{1}{r_1'}\right) \\ &= \gamma + \tilde{\gamma} + \frac{1}{\tilde{q}}\frac{d-1}{2}\left\{\tilde{q}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right) - q\left(\frac{1}{2} - \frac{1}{r}\right)\right\},\end{aligned}$$

where we use $\frac{1}{\tilde{q}}\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{1}{q}\left(\frac{1}{2} - \frac{1}{r_1}\right)$ in the last equality.

Case(2). We treat the case of $\frac{1}{\tilde{q}}\left(\frac{1}{2} - \frac{1}{r}\right) > \frac{1}{q}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right)$. Since $\frac{1}{\tilde{q}}\left(\frac{1}{2} - \frac{1}{r}\right) > \frac{1}{q}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right)$, there exists $r_2 \in [2, r)$ such that

$$\frac{1}{\tilde{q}}\left(\frac{1}{2} - \frac{1}{r_2}\right) = \frac{1}{q}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right).$$

Let γ_2 be the derivative loss for the pair (q, r_2). Then, by the Bernstein inequality and Lemma 2.9, we get

$$\begin{aligned}\|\Psi[F]\|_{L_t^q L_x^r} &\lesssim N^{d\left(\frac{1}{r_2} - \frac{1}{r}\right)} \|\Psi[F]\|_{L_t^q L_x^{r_2}} \\ &\lesssim N^{d\left(\frac{1}{r_2} - \frac{1}{r}\right)} N^{\gamma_2 + \tilde{\gamma}} \|P_N F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.\end{aligned}$$

By the symmetric argument, we get the desired statements. □

2.3. Proof of the Strichartz estimates

Proof of Proposition 1.1. We only show the inequality for $\mathcal{D}$ since the similar argument works for $\partial_t \mathcal{D}$ and $\partial_t^2 \mathcal{D}$. We have

$$\|\mathcal{D}(t)f\|_{L^q(I:L^r(\mathbb{R}^d))} \leq \|\mathcal{D}_l(t)f\|_{L^q(I:L^r(\mathbb{R}^d))} + \|\mathcal{D}_h(t)f\|_{L^q(I:L^r(\mathbb{R}^d))}.$$

Let (q, r) satisfy the assumption of Proposition 1.1 and $(q, r) \neq (2, 2d/(d-2))$ when $d \geq 3$. By the assumption of (q, r), we can apply Lemma 2.2 to the first term as $\tilde{r} = 2$ and $\sigma = 1$ and Lemma 2.4 to the second term. Then it follows that

$$\begin{aligned}\|\mathcal{D}(t)f\|_{L^q(I:L^r(\mathbb{R}^d))} &\leq \|\mathcal{D}_l(t)f\|_{L^q(I:L^r(\mathbb{R}^d))} + \|\mathcal{D}_h(t)f\|_{L^q(I:L^r(\mathbb{R}^d))} \\ &\lesssim \left\|\langle\nabla\rangle^{-1} f\right\|_{L^2} + \left\||\nabla|^{\gamma}\langle\nabla\rangle^{-1} f\right\|_{L^2} \\ &\approx \|f\|_{H^{\gamma-1}}\end{aligned}$$

This finishes the proof except for the heat end-point case. Next, we show the heat end-point estimate $(q, r) = (2, 2d/(d-2))$ for $d \geq 3$. Watanabe [30] obtained the following heat end-point estimate.

Lemma 2.11. [Homogeneous Strichartz estimate in the heat end-point case (see [30, Lemma 2.8])] *Let* $d \geq 3$. *Then, we have*

$$\|\mathcal{D}(t)f\|_{L_t^2(I:L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))} \lesssim \|f\|_{L^2},$$
$$\|\partial_t\mathcal{D}(t)f\|_{L_t^2(I:L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))} \lesssim \|\langle\nabla\rangle f\|_{L^2},$$
$$\left\|\partial_t^2\mathcal{D}(t)f\right\|_{L_t^2(I:L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))} \lesssim \left\|\langle\nabla\rangle^2 f\right\|_{L^2}.$$

By the first estimate in Lemma 2.11, we have

$$\|\langle\nabla\rangle^\sigma \mathcal{D}(t)P_{\leq 1}f\|_{L_t^2(I:L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))} \lesssim \|\langle\nabla\rangle^\sigma P_{\leq 1}f\|_{L^2} \lesssim \|f\|_{L^2},$$

for $\sigma \geq 0$. Therefore, it follows from this inequality and Lemma 2.4 that

$$\begin{aligned}\|\mathcal{D}(t)f\|_{L_t^2(I:L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))} &\leq \|\mathcal{D}_l(t)f\|_{L_t^2(I:L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))} + \|\mathcal{D}_h(t)f\|_{L_t^2(I:L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))}\\ &\lesssim \left\|\langle\nabla\rangle^{-1} f\right\|_{L^2} + \left\||\nabla|^\gamma \langle\nabla\rangle^{-1} f\right\|_{L^2}\\ &\approx \|f\|_{H^{\gamma-1}}.\end{aligned}$$

This completes the proof of the heat end-point homogeneous Strichartz estimate. □

Proof of Proposition 1.2. We only show the inequality for $\mathcal{D}$ since the similar argument works for $\partial_t\mathcal{D}$. By the integral inequality, we get

$$\begin{aligned}&\left\|\int_0^t \mathcal{D}(t-s)F(s)ds\right\|_{L^q(I:L^r(\mathbb{R}^d))}\\ &\quad\leq \left\|\int_0^t \mathcal{D}_l(t-s)F(s)ds\right\|_{L^q(I:L^r(\mathbb{R}^d))} + \left\|\int_0^t \mathcal{D}_h(t-s)F(s)ds\right\|_{L^q(I:L^r(\mathbb{R}^d))}\end{aligned}$$

By the assumption of (q, r), we can apply Lemma 2.3 to the first term as $\tilde{r} = 2$ and $\sigma = 1$ and Lemmas 2.9, 2.10 to the second term. Then it follows that

$$\begin{aligned}&\left\|\int_0^t \mathcal{D}(t-s)F(s)ds\right\|_{L^q(I:L^r(\mathbb{R}^d))}\\ &\quad\leq \left\|\int_0^t \mathcal{D}_l(t-s)F(s)ds\right\|_{L^q(I:L^r(\mathbb{R}^d))} + \left\|\int_0^t \mathcal{D}_h(t-s)F(s)ds\right\|_{L^q(I:L^r(\mathbb{R}^d))}\\ &\quad\lesssim \left\|\langle\nabla\rangle^{-1} F\right\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))} + \left\||\nabla|^{\gamma+\tilde{\gamma}+\delta} \langle\nabla\rangle^{-1} F\right\|_{L^{\tilde{q}'}(I:L^{\tilde{r}'}(\mathbb{R}^d))}\\ &\quad\approx \|F\|_{L^{\tilde{q}'}(I:W^{\gamma+\tilde{\gamma}+\delta-1,\tilde{r}'}(\mathbb{R}^d))}\end{aligned}$$

This is the desired estimate. □

3. Well-posedness for the energy critical nonlinear damped wave equation

In this section, we prove local well-posedness for (NLDW), Theorem 1.3, by contraction mapping principle. We define the complete metric space

$$X(T,L,M) := \left\{ v \text{ on } [0,T)\times\mathbb{R}^d : \left\| \langle\nabla\rangle^{\frac{1}{2}} v \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))} \le L, \|v\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))} \le M \right\}.$$

Remark 3.1. $(q,r) = (2(d+1)/(d-1), 2(d+1)/(d-1))$ and $(2(d+1)/(d-2), 2(d+1)/(d-2))$ satisfy the assumptions of the Strichartz estimates in Propositions 1.1 and 1.2. Moreover, $\gamma = 1/2$ when $(q,r) = (2(d+1)/(d-1), 2(d+1)/(d-1))$ and $\gamma = 1$ when $(q,r) = (2(d+1)/(d-2), 2(d+1)/(d-2))$. We note that these exponents are same as in the local well-posedness for the critical nonlinear wave equation.

We define

$$\Phi[u](t) = \Phi_{u_0,u_1}[u](t) := \mathcal{D}(t)(u_0+u_1) + \partial_t\mathcal{D}(t)u_0 + \int_0^t \mathcal{D}(t-s)\mathcal{N}(u(s))ds.$$

Proof of Theorem 1.3. As stated in Remark 3.1, the exponents are same as in the argument for the energy critical nonlinear wave equation. Thus, the proof if similar so that we only give sketch of the proof. See [3,16,26,27] for details. Since $(u_0,u_1) \in H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$, by the Strichartz estimates in Proposition 1.1, we obtain

$$\begin{aligned}
&\|\mathcal{D}(t)(u_0+u_1) + \partial_t\mathcal{D}(t)u_0\|_{X(T)} \\
&\quad\le \left\| \langle\nabla\rangle^{\frac{1}{2}} \mathcal{D}(t)(u_0+u_1) \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))} + \left\| \langle\nabla\rangle^{\frac{1}{2}} \partial_t\mathcal{D}(t)u_0 \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}} \\
&\qquad + \|\mathcal{D}(t)(u_0+u_1)\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))} + \|\partial_t\mathcal{D}(t)u_0\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))} \\
&\quad\lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2} < A < \infty.
\end{aligned} \tag{3.1}$$

We estimate the nonlinear term as follows. By the Strichartz estimates in Proposition 1.2 and the fractional Leibnitz rule (see [16, Lemma 2.5] and references therein), we get

$$\begin{aligned}
&\left\| \langle\nabla\rangle^{\frac{1}{2}} \int_0^t \mathcal{D}(t-s)\mathcal{N}(u(s))ds \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))} \\
&\quad\lesssim \left\| \langle\nabla\rangle^{\frac{1}{2}} \mathcal{N}(u) \right\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([0,T))} \\
&\quad\lesssim \|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}^{\frac{4}{d-2}} \left\| \langle\nabla\rangle^{\frac{1}{2}} u \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))}
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\left\|\int_0^t \mathcal{D}(t-s)\mathcal{N}(u(s))ds\right\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}\\ &\lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}\mathcal{N}(u)\right\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([0,T))}\\ &\lesssim \|u\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}\left\|\langle\nabla\rangle^{\frac{1}{2}}u\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))}. \end{aligned}\tag{3.3}$$

Combining (3.1) and (3.2), we obtain

$$\begin{aligned}\left\|\langle\nabla\rangle^{\frac{1}{2}}\Phi[u]\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))} \leq& \left\|\langle\nabla\rangle^{\frac{1}{2}}\mathcal{D}(t)(u_0+u_1)+\partial_t\mathcal{D}(t)u_0\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))}\\ &+\left\|\langle\nabla\rangle^{\frac{1}{2}}\int_0^t \mathcal{D}(t-s)\mathcal{N}(u(s))ds\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))}\\ \leq& CA+CLM^{\frac{4}{d-2}}\\ \leq& L\end{aligned}$$

if we choose $L=2CA$ and M such that $CM^{4/(d-2)}\leq 1/2$. By (3.1) and (3.3), we get

$$\begin{aligned}\|\Phi[u]\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))} \leq& \|\mathcal{D}(t)(u_0+u_1)+\partial_t\mathcal{D}(t)u_0\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}\\ &+\left\|\int_0^t \mathcal{D}(t-s)\mathcal{N}(u(s))ds\right\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}\\ \leq& \delta+CLM^{\frac{4}{d-2}}\\ \leq& M\end{aligned}$$

if we choose $\delta = M/2$ and $L\leq (2C)^{-1}M^{(d-6)/(d-2)}$ (which is possible if $3\leq d\leq 5$). Thus, Φ is a mapping on $X(T,L,M)$.

$$\begin{aligned}&\left\|\langle\nabla\rangle^{\frac{1}{2}}(\Phi[u]-\Phi[v])\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))}+\|\Phi[u]-\Phi[v]\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}\\ &\lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}(\mathcal{N}(u)-\mathcal{N}(v))\right\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([0,T))}\\ &\lesssim \left(\|u\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}+\|v\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}\right)\left\|\langle\nabla\rangle^{\frac{1}{2}}(u-v)\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))}\\ &\quad+\left(\|u\|^{\frac{6-d}{d-2}}_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}+\|v\|^{\frac{6-d}{d-2}}_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}\right)\\ &\quad\times\left(\left\|\langle\nabla\rangle^{\frac{1}{2}}u\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))}+\left\|\langle\nabla\rangle^{\frac{1}{2}}v\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))}\right)\end{aligned}$$

$$
\begin{aligned}
&\times \|u-v\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))} \\
&\le CM^{\frac{4}{d-2}} \left\| \langle \nabla \rangle^{\frac{1}{2}} (u-v) \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([0,T))} + CM^{\frac{6-d}{d-2}} L \|u-v\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T))}.
\end{aligned}
$$

Taking L and M sufficiently small, Φ is a contraction mapping on $X(T,L,M)$. By the Banach fixed point theorem, we obtain the solution such that $u = \Phi[u]$. Then, $(u, \partial_t u)$ belongs to $C([0,T); H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ because of the Strichartz estimates (Proposition 1.1 and 1.2) and the nonlinear estimates (for example $\langle \nabla \rangle^{\frac{1}{2}} \mathcal{N}(u) \in L_{t,x}^{\frac{2(d+1)}{d+3}}$). We give a proof of the standard blow-up criterion. We suppose that $T_+ = T_+(u_0,u_1) < \infty$ and $\|u\|_{L_{t,x}^{2(d+1)/d-2}([0,T_+))} < \infty$. Take τ and T arbitrary such that $0 < \tau < T < T_+$. By the Duhamel formula, we have

$$
u(t) = \mathcal{D}(t-\tau)(u(\tau) + \partial_t u(\tau)) + \partial_t \mathcal{D}(t-\tau) u(\tau) + \int_\tau^t \mathcal{D}(t-s)\mathcal{N}(u(s))ds,
$$

for $t > \tau$. By the Strichartz estimates, we obtain

$$
\begin{aligned}
&\left\| \langle \nabla \rangle^{\frac{1}{2}} u \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}((\tau,T))} \\
&\lesssim \|(u(\tau), \partial_t u(\tau))\|_{H^1 \times L^2} + \left\| \int_\tau^t \langle \nabla \rangle^{\frac{1}{2}} \mathcal{D}(t-s)\mathcal{N}(u(s))ds \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}((\tau,T))} \\
&\lesssim \|(u(\tau), \partial_t u(\tau))\|_{H^1 \times L^2} + \left\| \langle \nabla \rangle^{\frac{1}{2}} \mathcal{N}(u(s)) \right\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}((\tau,T))} \\
&\lesssim \|(u(\tau), \partial_t u(\tau))\|_{H^1 \times L^2} + \|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}((\tau,T))}^{\frac{4}{d-2}} \left\| \langle \nabla \rangle^{\frac{1}{2}} u \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}((\tau,T))}.
\end{aligned}
$$

Since $\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}((\tau,T))} \ll 1$ for τ close to T_+, we obtain

$$
\left\| \langle \nabla \rangle^{\frac{1}{2}} u \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}((\tau,T))} \lesssim \|(u(\tau), \partial_t u(\tau))\|_{H^1 \times L^2}.
$$

Fix such τ. Since T is arbitrary, we get

$$
\left\| \langle \nabla \rangle^{\frac{1}{2}} u \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}((\tau,T_+))} \lesssim \|(u(\tau), \partial_t u(\tau))\|_{H^1 \times L^2}. \tag{3.4}
$$

Take a sequence $\{t_n\}$ such that $t_n \to T_+$ and $t_n > \tau$. Then, by the integral formula, the Strichartz estimates the assumption, and 3.4, we have

$$
\begin{aligned}
&\|\mathcal{D}(t-t_n)(u(t_n) + \partial_t u(t_n)) + \partial_t \mathcal{D}(t-t_n)u(t_n)\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([t_n,T_+))} \\
&\lesssim \|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([t_n,T_+))} + \left\| \int_{t_n}^t \mathcal{D}(t-s)\mathcal{N}(u(s))ds \right\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([t_n,T_+))} \\
&\lesssim \|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([t_n,T_+))} + \|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([t_n,T_+))}^{\frac{4}{d-2}} \left\| \langle \nabla \rangle^{\frac{1}{2}} u \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}([t_n,T_+))} \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}
$$

Thus, $\|\mathcal{D}(t-t_n)(u(t_n)+\partial_t u(t_n))+\partial_t\mathcal{D}(t-t_n)u(t_n)\|_{L^{2(d+1)/(d-2)}_{t,x}([t_n,T_+))}$ $<\delta/2$ is true for large n. Then, for some $\varepsilon>0$, we get

$$\|\mathcal{D}(t-t_n)(u(t_n)+\partial_t u(t_n))+\partial_t\mathcal{D}(t-t_n)u(t_n)\|_{L^{2(d+1)/(d-2)}_{t,x}([t_n,T_++\varepsilon))}<\delta.$$

The local well-posedness derives a contradiction. □

Proof of Theorem 1.4. By the Strichartz estimate (Proposition 1.1), we have

$$\|\mathcal{D}(t)(u_0+u_1)+\partial_t\mathcal{D}(t)u_0\|_{L^{\frac{2(d+1)}{d-2}}_{t,x}([0,\infty))}\le C\,\|(u_0,u_1)\|_{H^1\times L^2}.$$

Thus, if we take δ_0 satisfying $C\delta_0<\delta$, where δ is in Theorem 1.3, then we get a global solution from Theorem 1.3. Moreover, the solution u satisfies

$$\|u\|_{L^{\frac{2(d+1)}{d-2}}_{t,x}([0,\infty))}\lesssim\|\mathcal{D}(t)(u_0+u_1)+\partial_t\mathcal{D}(t)u_0\|_{L^{\frac{2(d+1)}{d-2}}_{t,x}([0,\infty))}\lesssim\delta_0.$$

□

4. Decay of global solution with finite Strichartz norm

In this section, we give a proof of Theorem 1.5.

Lemma 4.1. *If u is a global solution of* (NLDW) *with $\|u\|_{L^{\frac{2(d+1)}{d-2}}_{t,x}([0,\infty))}<\infty$, then u satisfies*

$$\left\|\langle\nabla\rangle^{\frac{1}{2}}u\right\|_{L^{\frac{2(d+1)}{d-1}}_{t,x}([0,\infty))}<\infty$$

Proof. The proof is very similar to the proof of the standard blow-up criterion. Take $0<\tau<T<\infty$ arbitrary. We know that the global solution belongs to $L^{\frac{2(d+1)}{d-1}}_{t,x}(K)$ for any compact interval $K\subset[0,\infty)$. It follows from the Duhamel's formula and the Strichartz estimates that

$$\begin{aligned}\left\|\langle\nabla\rangle^{\frac{1}{2}}u\right\|_{L^{\frac{2(d+1)}{d-1}}_{t,x}((\tau,T))}&\lesssim\|(u(\tau),\partial_t u(\tau))\|_{H^1\times L^2}\\&\quad+\|u\|^{\frac{4}{d-2}}_{L^{\frac{2(d+1)}{d-2}}_{t,x}((\tau,T))}\left\|\langle\nabla\rangle^{\frac{1}{2}}u\right\|_{L^{\frac{2(d+1)}{d-1}}_{t,x}((\tau,T))}.\end{aligned}$$

Since $\|u\|_{L^{\frac{2(d+1)}{d-2}}_{t,x}((\tau,T))}\ll 1$ for large τ, we obtain

$$\left\|\langle\nabla\rangle^{\frac{1}{2}}u\right\|_{L^{\frac{2(d+1)}{d-1}}_{t,x}((\tau,T))}\lesssim\|(u(\tau),\partial_t u(\tau))\|_{H^1\times L^2}$$

for large $\tau>0$. Fix such τ. Since T is arbitrary, we obtain

$$\left\|\langle\nabla\rangle^{\frac{1}{2}}u\right\|_{L^{\frac{2(d+1)}{d-1}}_{t,x}((\tau,\infty))}\lesssim\|(u(\tau),\partial_t u(\tau))\|_{H^1\times L^2}.$$

Thus, we obtain $\left\|\langle\nabla\rangle^{\frac{1}{2}}u\right\|_{L^{\frac{2(d+1)}{d-1}}_{t,x}([0,\infty))}<\infty.$ □

Proof of Theorem 1.5. We have

$$\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \int_0^t \mathcal{A}(t-s) \begin{pmatrix} 0 \\ \mathcal{N}(u(s)) \end{pmatrix} ds,$$

where

$$\mathcal{A}(t) = \begin{pmatrix} \mathcal{D}(t) + \partial_t \mathcal{D}(t) & \mathcal{D}(t) \\ \partial_t \mathcal{D}(t) + \partial_t^2 \mathcal{D}(t) & \partial_t \mathcal{D}(t) \end{pmatrix}.$$

We set

$$\begin{aligned} I &:= \mathcal{A}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\ I\!I &:= \int_0^\tau \mathcal{A}(t-s) \begin{pmatrix} 0 \\ \mathcal{N}(u(s)) \end{pmatrix} ds, \\ I\!I\!I &:= \int_\tau^t \mathcal{A}(t-s) \begin{pmatrix} 0 \\ \mathcal{N}(u(s)) \end{pmatrix} ds. \end{aligned}$$

We begin with the estimate of I. Approximating (u_0, u_1) by $(\psi_0, \psi_1) \in (C_0^\infty(\mathbb{R}^d))^2$ in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, we obtain

$$\begin{aligned} \|I\|_{H^1 \times L^2} &= \left\|\mathcal{A}(t)(u_0, u_1)^T\right\|_{H^1 \times L^2} \\ &\leq \left\|\mathcal{A}(t)\{(u_0, u_1) - (\psi_0, \psi_1)\}^T\right\|_{H^1 \times L^2} + \left\|\mathcal{A}(t)(\psi_0, \psi_1)^T\right\|_{H^1 \times L^2}, \end{aligned}$$

where ^{T} denotes transposition. By [9, Theorem 1.1], we have the following L^p–L^q type estimates:

$$\begin{aligned} \|\mathcal{D}(t) f\|_{H^1} &\lesssim \langle t \rangle^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{2}\right)} \|f\|_{L^q} + e^{-\frac{t}{2}} \langle t \rangle^\delta \|f\|_{L^2}, \\ \|\partial_t \mathcal{D}(t) f\|_{L^2} &\lesssim \langle t \rangle^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{2}\right)-1} \|f\|_{L^q} + e^{-\frac{t}{2}} \langle t \rangle^\delta \|f\|_{L^2}, \\ \|\partial_t \mathcal{D}(t) f\|_{H^1} &\lesssim \langle t \rangle^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{2}\right)-1} \|f\|_{W^{1,q}} + e^{-\frac{t}{2}} \langle t \rangle^\delta \|f\|_{H^1}, \\ \left\|\partial_t^2 \mathcal{D}(t) f\right\|_{L^2} &\lesssim \langle t \rangle^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{2}\right)-2} \|f\|_{L^q} + e^{-\frac{t}{2}} \langle t \rangle^\delta \|\langle \nabla \rangle f\|_{L^2}, \end{aligned}$$

for any $q \in [1, 2]$ and some $\delta > 0$. Therefore, applying these as $q = 2$, we get

$$\left\|\mathcal{A}(t)\{(u_0, u_1) - (\psi_0, \psi_1)\}^T\right\|_{H^1 \times L^2} \lesssim \|(u_0, u_1) - (\psi_0, \psi_1)\|_{H^1 \times L^2}.$$

Thus, this can be made arbitrary small by the approximation. Applying the above L^p–L^q type estimates as $q = 1$, we obtain

$$\begin{aligned} &\left\|\mathcal{A}(t)(\psi_0, \psi_1)^T\right\|_{H^1 \times L^2} \\ &\quad \leq \|\mathcal{D}(t)(\psi_0 + \psi_1)\|_{H^1} + \|\partial_t \mathcal{D}(t) \psi_0\|_{H^1} \\ &\qquad + \|\partial_t \mathcal{D}(t)(\psi_0 + \psi_1)\|_{L^2} + \left\|\partial_t^2 \mathcal{D}(t) \psi_0\right\|_{L^2} \\ &\quad \lesssim \langle t \rangle^{-\frac{d}{4}} (\|\psi_0\|_{W^{1,1}} + \|\psi_1\|_{L^1}) + e^{-\frac{t}{4}} (\|\psi_0\|_{H^1} + \|\psi_1\|_{L^2}) \\ &\quad \to 0 \text{ as } t \to \infty. \end{aligned}$$

Next, we consider the estimate of $I\!I\!I$. By the Strichartz estimates, we have

$$\begin{aligned}\|I\!I\!I\|_{H^1\times L^2} &= \left\|\langle\nabla\rangle\int_\tau^t \mathcal{D}(t-s)\mathcal{N}(u(s))ds\right\|_{L^2} \\ &\quad + \left\|\int_\tau^t \partial_t\mathcal{D}(t-s)\mathcal{N}(u(s))ds\right\|_{L^2} \\ &\lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}\mathcal{N}(u)\right\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}((\tau,t))} \\ &\lesssim \|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}((\tau,t))}^{\frac{4}{d-2}} \left\|\langle\nabla\rangle^{\frac{1}{2}} u\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}((\tau,t))}. \end{aligned} \tag{4.1}$$

Therefore, the term is arbitrary small taking τ sufficiently close to t. At last, we calculate $I\!I$. We note that

$$I\!I = \int_0^\tau \mathcal{A}(t-s)\begin{pmatrix}0\\ \mathcal{N}(u(s))\end{pmatrix}ds = \mathcal{A}(t-\tau)\int_0^\tau \mathcal{A}(\tau-s)\begin{pmatrix}0\\ \mathcal{N}(u(s))\end{pmatrix}ds.$$

Since by (4.1) we know

$$\int_0^\tau \mathcal{A}(\tau-s)\begin{pmatrix}0\\ \mathcal{N}(u(s))\end{pmatrix} \in H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d),$$

approximating it by $\vec{\psi}\in (C_0^\infty(\mathbb{R}^d)^2)$, we obtain

$$\begin{aligned}\|I\!I\|_{H^1\times L^2} &\le \left\|\mathcal{A}(t-\tau)\left\{\int_0^\tau \mathcal{A}(\tau-s)\begin{pmatrix}0\\ \mathcal{N}(u(s))\end{pmatrix}ds - \vec{\psi}\right\}\right\|_{H^1\times L^2} \\ &\quad + \left\|\mathcal{A}(t-\tau)\vec{\psi}\right\|_{H^1\times L^2}.\end{aligned}$$

In the smae way as I, the first term is arbitrary small by the approximation and the second term tends to 0 as $t\to\infty$. Combining the estimates of I, $I\!I$, and $I\!I\!I$, we get the decay. □

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Takahisa Inui
Department of Mathematics, Graduate School of Science
Osaka University
Toyonaka Osaka560-0043
Japan
e-mail: inui@math.sci.osaka-u.ac.jp

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