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Stability of semi-wavefronts for delayed reaction–diffusion equations

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Abstract. This paper deals with the asymptotic behavior of solutions to the delayed monostable equation: $(*)\; u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x))$, $x \in \mathbb{R}$, $t > 0$; here $h > 0$ and the reaction term $g : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous and has exactly two fixed points (zero and $\kappa > 0$). Under certain condition on the derivative of g at κ (without assuming classic KPP condition for g) the global stability of fast semi-wavefronts is proved. Also, when the Lipschitz constant L_g is equal to $g'(0)$ the stability of all semi-wavefronts (e.g., critical, non-critical and asymptotically periodic semi-wavefronts) on each interval in the form $(-\infty, N], N \in \mathbb{R}$, to $(*)$ is established, which includes classic equations such as the Nicholson's model.

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1. Main results and discussion

In this work, the main object of study is the equation:

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)), \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

where $h > 0$ and the nonlinear reaction term $g : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous with Lipschitz constant L_g which satisfies the monostability condition $L_g \geq g'(0) > 1$ and has exactly two fixed points: 0 and $\kappa > 0$. The Eq. (1) is frequently considered to model problems of population dynamics. In this case g stands for the birth rate function, h is the age when the individual reaches the sexual maturity and $u(t,x)$ is the adult population at location x and time t. The diffusion and death rates have been normalized. In this framework is relevant the stability properties of the positive equilibrium κ and the existence of colonization waves so-called *wavefronts* (see [2,12,21,28,34,35] and references therein). Wavefronts with speed c are non-negative entire bounded solutions $u(t,x) = \psi_c(x+ct)$ such that the profile $\psi_c : \mathbb{R} \to \mathbb{R}$ satisfies $\psi_c(-\infty) = 0$ and $\psi_c(+\infty) = \kappa$. It is well known that when g is monotone and $h \geq 0$ then there exists a positive number $c_* = c_*(h)$ so-called *critical speed* or *minimal speed* such that (1) has wavefronts if and only if $c \geq c_*$ [18,40,43]; wavefronts with speed c_* are called *critical wavefronts*. Moreover, these wavefronts are monotone and unique modulo translation. The main tool to obtain the stability and existence of wavefronts is to construct sub and super- solutions by using monotony arguments.

However, when g is non-monotone the associated semi-flow is non-monotone in general and wavefronts are replaced by positive bounded solutions $u(t,x) = \psi_c(x+ct)$ such that $\psi_c(-\infty) = 0$ and $\liminf_{x\to+\infty} \psi_c(x) > 0$ which are called *semi-wavefronts*. For g satisfying the *subtangency condition* $g(u) \leq g'(0)u$, for all $u \geq 0$, has been demonstrated the existence of a minimal speed $c_* = c_*(h)$ for the existence of semi-wavefronts to (1) for all $h \geq 0$ (see [37, Theorems 4.5 and 5.4] and [13,43]). Under the Diekmann- Kaper (D-K, for short) condition $L_g = g'(0)$ (see [11, Theorem 6.4]) Aguerrea, Gomez and Trofimchuk demonstrated the uniqueness modulo translation of all semi-wavefronts of (1). In the general case $L_g \geq g'(0)$ it is necessary to consider the following characteristic equation

$$E_c(\lambda) := \lambda^2 - c\lambda - 1 + L_g\, e^{-\lambda ch} = 0, \tag{2}$$

for which it has been showed that there exits a speed $\mathfrak{c}_* = \mathfrak{c}_*(L_g)$ defined as

$$\mathfrak{c}_* = \mathfrak{c}_*(L_g) := \inf\{c > 0 : E_c(\lambda) \text{ has a positive root}\}, \tag{3}$$

such that E_c has exactly two positive zeros $\lambda_1(c) \leq \lambda_2(c)$, also $\lambda_1(c) = \lambda_2(c)$ if and only if $c = \mathfrak{c}_*$ (for a more detailed study of (2) see [16, Lemma 22]). Thus, the authors in [1, Theorem 4] showed that for $c \geq \mathfrak{c}_*$ semi-wavefronts have the following representation

$$\psi_c(z) = A_{\psi_c}(-z)^{j_c} e^{\lambda_1(c)z} + e^{(\lambda_1(c)+\epsilon)z}\, r(z), \tag{4}$$

where $A_{\psi_c}, \epsilon \in \mathbb{R}_+$, $r \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $j_c = 0, 1$ with $j_c = 1$ if and only if $L_g = g'(0)$, moreover, semi-wavefronts are unique (modulo translation) for all $c > \mathfrak{c}_*$ and $h \geq 0$ [1, Theorem 8]. We should mention that when L_g in (2) is replaced by $g'(0)$ the speed $\mathfrak{c}_*$ in (3) coincides with the definition of the so-called *linear speed* $c_\#$ and

$$c_\# \leq c_* \leq \mathfrak{c}_*, \tag{5}$$

(see [37, Theorem 4.5 and Theorem 5.4]), also when g is subtangential then $c_\#$ coincides with the critical speed c_* for the existence of semi-wavefronts (see [43, Theorem 4.4]). In particular, if g satisfies the D-K condition $L_g = g'(0)$ then g is subtangential and $\mathfrak{c}_* = c_* = c_\#$. One of the main results of this work is show the stability of semi-wavefronts (with unbounded exponential weights) for $c \geq \mathfrak{c}_*(h)$ for all delay $h \geq 0$ (the same conditions for c and h to establish the uniqueness of semi-wavefronts in [1]).

In order to overcome the non-monotony of some reaction–diffusion equations with delay a *quasi-monotonicity* condition is assumed which usually requires the monotony of the delayed argument. Indeed, in a pioneering work, Schaaf [29] considered the following parabolic functional differential equation

$$u_t(t,x) = u_{xx}(t,x) + f(u(t,x), u(t-h,x)), \quad x \in \mathbb{R}, \, t > 0. \tag{6}$$

Schaaf proved that for a concave nonlinearity f with exactly two equilibria (0 and 1) satisfying a certain positivity condition (see [29, Section 2.1]) and

$$\partial_2 f(u,v) \geq 0 \quad \text{for all}\, u, v \in \mathbb{R}_+ \qquad \text{(quasi-monotonicity condition)} \tag{7}$$

wavefronts are *linearly* stable for small delay h [29, Theorem 4.13].

After the pioneering work by Schaaf, a series of other studies appeared where the KPP condition $|g'(u)| \leq g'(0)$, for all $u \geq 0$, or the concave condition $g''(u) \leq 0$, for all $u \geq 0$, was instrumental for the stability analysis. Among these studies, we would like to distinguish an important contribution [24] by Mei, Ou and Zhao where the authors proved the *global* stability of monotone wavefronts (critical as well as non-critical ones, see [24, Theorem 2.2]) of the following non-local equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + \int_{\mathbb{R}} K(y) g(u(t-h, x-y)) dy \quad x \in \mathbb{R}, \; t > 0, \tag{8}$$

for monotone and concave g and K a heat kernel; here the perturbations are taken in weighted Sobolev spaces. At the same time, Lv and Wang [20] proved the global stability of non-critical wavefronts of (6) for monostable f (with exactly two equilibria: 0 and $\kappa > 0$) satisfying (7) and the concavity condition: $\partial_{ij} f(u,v) \leq 0$ (i, j $= 1, 2$), for all $u, v \in [0, \kappa]$. The authors in [20] also study (6) with non-local reaction term (which includes (8), for monotone g) and demonstrated the stability of non-critical monotone wavefronts in Sobolev spaces with exponential weights; this result can also be obtained by our approach to (1) even to non-monotone wavefronts, see Remark 17 (for the non-local equation (8) see our recent work [31]).

With respect to non-monotone wavefronts, we should mention a work of Wu et al. [41, Theorem 2.4] where the authors take some type non-monotone $g \in C^2([0,\kappa], \mathbb{R})$ ('crossing monostable' nonlinearity) satisfying $|g'(\kappa)| < 1$ and prove the *local* stability of wavefronts with speed c for $c > 2\sqrt{2(L_g - 1)}$ and for all $h \geq 0$. Additionally, by assuming the KPP condition and $|g'(\kappa)|$ sufficiently small they prove the local stability of wavefronts with speed c for all $c > c_*$ and $h \geq 0$ [41, Theorem 2.6]; here the existence of non-monotone wavefronts can be deduced, e.g., from [36] and [15]. Our second result for non-linear stability of wavefronts generalizes these results (see Remark 10). In this regard, for unimodal g (i.e., g has exactly one critical point which is the absolute maximum point) satisfying the KPP condition and $|g'(\kappa)| < 1$, Lin et al. [19] proved the local stability of non-critical wavefronts for all $h \geq 0$ (monotone or non-monotone) which includes well-known models such as Nicholson's model (see [15,19,33] and references therein) described by

$$u_t(t,x) = u_{xx}(t,x) - \delta u(t,x) + \rho u(t-h,x)e^{-u(t-h,x)}, \quad t>0, x\in\mathbb{R}, \quad (9)$$

where $\rho, \delta > 0$, or the Mackey–Glass model [1,3,19,23,24] given by

$$u_t(t,x) = u_{xx}(t,x) - \tau\, u(t,x) + \frac{ab^n u(t-h,x)}{b^n + u^n(t-h,x)}, \quad t>0, x\in\mathbb{R}. \quad (10)$$

where $\tau, a, b > 0$ and $n \in \mathbb{Z}_+$. These stability results were stablished in weighted Sobolev spaces to initial data with a suitable convergence to κ at $x = +\infty$.

When $\rho/\delta \in (1, e]$ in (9) wavefronts are monotone and by [24] they (critical as well as non-critical ones) are globally stable. The authors in [19] proved the local stability of (monotone and non-monotone) non-critical wavefronts to (9) when $\rho/\delta \in (e, e^2)$ for all $h \geq 0$ and for small delay h when $\rho/\delta \in (1, +\infty)$. Then, assuming $|g'(\kappa)| < 1$, Chern et al. [8, Theorem 2.3] have demonstrated the local stability of critical wavefronts (monotone or non-monotone) in the same Sobolev spaces.

Next, for Lipschitz continuous function g satisfying the D-K condition $L_g = g'(0)$ and $|g'(u)| < 1$ in some neighborhood of κ, Solar and Trofimchuk have established the global stability of (monotone or non-monotone) non-critical wavefronts [33, Corollary 3]. In particular, they obtained the global stability of non-critical wavefronts for (9) when $\rho/\delta \in (1, e^2)$ for all $h \geq 0$. Here initial data are not required to convergence to κ at $x = +\infty$ as above mentioned works. Then, in a recent work, for unimodal $g \in C^2[0, +\infty)$ satisfying the KPP condition, Mei et al. [22] have generalized the results in [8,19] for a global perturbation in the same Sobolev spaces.

On the other hand, non-subtangential models have recently attracted a lot of interest because of their connection to the so-called *Allee effect* in population dynamics [6,7,10,26]. More precisely, if we only consider as benefit to species a greater availability of resources then the per capita growth rate $g(u)/u$ attains its maximum at $u = 0$, however if animal behavior is cooperative then individuals obtain benefits for intermediate densities $u > 0$ (individual fitness) which are not generated for low densities ($u = 0$), so that the per capita growth rate $g(u)/u$ attains its maximum at some $u_0 > 0$. In this case model is said to have an Allee effect [9, Chapter 1] (since in our case the per capita growth birth rate $g(u)/u$ is non-decreasing in a neighborhood of $u = 0$ model is said to have a *weak Allee effect*). In contrast to subtangential case, for a model with Allee effect it could occur $c_* > c_\#$, critical wavefronts with speed $c_* > c_\#$ are called *pushed wavefronts*. In this direction, for monotone g (necessarily non-subtangential), it has been possible to establish the stability of pushed wavefronts (see [32] and [42]) as well as that of non-critical wavefronts [33, Theorem 1]. These results show that pushed wavefronts are more attractive than critical wavefronts with speed $c_* = c_\#$, for instance pushed wavefronts attract (orbitally) to the solution of (1) generated by the Heaviside step function while a critical wavefront (which is not a pushed wavefront) requires a logarithmic correction to attract this solution (see e.g. [12,39] for

$h = 0$ and [4] for $h > 0$). It is important to mention that the problem of the existence of semi-wavefronts for non-subtangential models is not completely solved (e.g., see [36, Corollary 4]: of course, in the available literature there are some partial results on the existence of semi-wavefronts for certain subclasses of equations, e.g., see [37, Theorem 2.4]).

Hence, in the above mentioned works, we can find stability results for Eq. (1) only when g either is monotone or meets the sub-tangency condition. In this work we study the stability of semi-wavefronts without assuming the quasi-monotonicity nor the sub-tangency condition on g. Our approach uses ideas from [33] and a suitable Fourier analysis for partial functional differential equations. In the particular case when g is unimodal and satisfies the KPP condition, i.e. $L_g = g'(0)$ and $g \in C^1[0, +\infty)$, our estimates (in different spaces) are similar than [8,19,22,24] for perturbations of wavefronts, but our Fourier analysis (for wavefronts and proper semi-wavefronts) is different in many aspects, for instance by our approach Fourier transforms are estimated by means of a Halanay inequality on *Banach* spaces (see Lemma 12) instead of finite-dimensional spaces while the non-critical case $c > c_*$ (for wavefronts and proper semi-wavefronts) does not require Fourier analysis as in [19] (see Corollary 20 and Remark 17 below). However, approach used in [8,19,22,24] allows us to obtain stability results of wavefronts on the real line when $|g'(\kappa)| > 1$ for small h whenever the initial datum $u_0(s, x)$ converges to κ at $x = +\infty$. In this regard, we obtain a general stability result for semi-wavefronts on each semi-infinite interval $(-\infty, N]$, $N \in \mathbb{R}$, without assume the restriction $|g'(\kappa)| < 1$ for all $h \geq 0$ (Theorem 3) which also includes critical semi-wavefronts and *asymptotically periodic* semi-wavefronts (see, e.g [36, Theorem 3]). This kind of stability seems to be transversal to another models, indeed in a recent work [5] Benguria and Solar have stablished the stability of a class of non-monotone semi-wavefronts for the Hutchinson diffusive equation

$$u_t(t, x) = u_{xx}(t, x) + u(t, x)(1 - u(t - h, x)), \qquad t > 0,\, x \in \mathbb{R}, \tag{11}$$

on each semi-infinite interval $(-\infty, N]$.

More precisely, by a suitable Fourier analysis we can show that for $c \geq c_*$, an initial perturbation

$$u_0(s, x + cs) - \psi_c(x + cs) = e^{\lambda(x+cs)}\, r(s, x), \quad \text{for all} \quad (s, x) \in [-h, 0] \times \mathbb{R},$$

with $r \in C([-h, 0], L^1(\mathbb{R}))$ and λ satisfying $E_c(\lambda) \leq 0$ [according to definition (2)], evolves as

$$u(t, x + ct) - \psi_c(x + ct) = O(t^{-1/2} e^{\gamma t}), \quad \text{for all} \quad t \geq -h,$$

uniformly for $x + ct \in (-\infty, N]$, $N \in \mathbb{R}$ and some $\gamma = \gamma(\lambda) \leq 0$. Moreover, $\gamma = 0$ if and only if $E(\lambda) = 0$, i.e. $\lambda = \lambda_1(c)$ or $\lambda = \lambda_2(c)$.

On the other hand, in our second main result we study the convergence of perturbations of wavefronts on the remaining domain $(N, +\infty)$, $N \in \mathbb{R}$. In this case it is necessary to assume the stability condition $|g'(\kappa)| < 1$ in order to establish (without assuming monotonicity or sub-tangency condition on g) the local stability of wavefronts with $c > \mathfrak{c}_*$ on whole the real line $(-\infty, +\infty)$ (Corollary 20). Additionally, assuming $|g'(u)| < 1$ for u in a suitable neighborhood of κ we obtain the global stability of wavefronts with $c > \mathfrak{c}_*$ on whole real line $(-\infty, +\infty)$ (see Theorem 9 below).

In order to obtain these stability results we study the decay of solutions of the constant coefficient linear equation with delay,

$$u_t(t,x) = u_{xx}(t,x) + mu_x(t,x) + pu(t,x) + qu(t-h, x+d), \quad x \in \mathbb{R}, t > 0, \tag{12}$$

where the parameters m, p, q y d are real numbers.

For an initial datum $u_0 \in C([-h,0], L^1(\mathbb{R}))$, let us denote $C_{u_0} := \sup_{s\in[-h,0]} ||u_0(s,\cdot)||_{L^1}$.

Theorem 1. *Suppose that $-p \geq q \geq 0$ and $m, d \in \mathbb{R}$. Let $\gamma \leq 0$ be the only real solution of the following equation:*

$$\gamma - p = qe^{-h\gamma}. \tag{13}$$

If the initial datum u_0 belongs to $C([-h,0]; L^1(\mathbb{R}))$ then the solution $u(t,x)$ of (12) satisfies the estimate:

$$\sup_{x\in\mathbb{R}} |u(t,x)| < A_0 \frac{e^{\gamma t}}{\sqrt{t}}, \qquad \text{for all} \quad t > h, \tag{14}$$

where $A_0 = C_{u_0}/2\sqrt{1 + h(\gamma - p)}$.

We note that in the special case $-p = q$ (which implies $\gamma = 0$) an exponential estimate is no longer available. In some cases, it can be established that the decay is not faster than that given by (14). For instance, if $d = 0$ for the evolution equation (12), the behavior of the solutions in the $L^1(\mathbb{R})$ phase space with an appropriate weight can be specified. In fact, we obtain the exact behavior which is embodied in Theorem below

Theorem 2. (Asymptotic behavior) *Let us consider (12) with $m, p \in \mathbb{R}$, $q \geq 0$ and $d = 0$. Let $u(t,x)$ be the solution generated by the initial data $u(s,\cdot) = e^{\sigma s} u_0$ where u_0 is such that $e^{\frac{m}{2}\cdot} u_0 \in L^1(\mathbb{R})$ and σ is the only real solution of*

$$qe^{-\sigma h} = \sigma + \frac{m^2}{4} - p, \tag{15}$$

then

$$\lim_{t\to\infty} \sqrt{t} e^{-\sigma t} u(t, x + o(\sqrt{t})) = \frac{\sqrt{1 + hqe^{-\sigma h}}}{2\sqrt{\pi}} e^{-\frac{m}{2}x} \int_{\mathbb{R}} e^{\frac{m}{2}y} u_0(y) dy, \tag{16}$$

for all $x \in \mathbb{R}$.

Now, for the study of the stability of semi-wavefronts with speed c , the following equation should be considered

$$v_t(t,z) = v_{zz}(t,z) - cv_z(t,z) - v(t,z) + g(v(t-h, z-ch)), \quad t > 0, z \in \mathbb{R}. \tag{17}$$

For $c \geq \mathfrak{c}_*$ let us fix $\lambda_c \in [\lambda_1(c), \lambda_2(c)]$ and let us denote by $\xi_c(z) := e^{-\lambda_c z}$. Now, the first main result of this article can be set out.

Theorem 3. (Stability with weight) *Assume that* $c \geq \mathfrak{c}_*$. *Let* $v_0(s,z)$ *and* $\psi_0(s,z)$ *be two initial data to* (17) *such that* $v_0, \psi_0 \in C([-h,0]; L^\infty(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}))$, *some* $\alpha \in (0,1]$, *and*

$$u_0(s,z) := \xi_c(z)|v_0(s,z) - \psi_0(s,z)| \in C([-h,0], L^1(\mathbb{R})), \tag{18}$$

then there are unique solutions $v(t,z)$ *and* $\psi(t,z)$ *of* (17) *with initial data* v_0 *and* ψ_0, *respectively, and these solutions satisfies* $v(\cdot + kh, \cdot), \psi(\cdot + kh, \cdot) \in C([-h,0]; L^\infty(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}))$ *for all* $k \in \mathbb{Z}_+$. *Moreover, if* $u(t,z)$ *satisfies* (12) *with initial data* $u_0(s,z)$ *and with parameters* $m = m(\lambda_c) = 2\lambda_c - c$, $p = p(\lambda_c) = \lambda_c^2 - c\lambda_c - 1$, $q = q(\lambda_c) = L_g e^{-\lambda_c ch}$ *and* $d = -ch$, *then*

$$\xi_c(z)|v(t,z) - \psi(t,z)| \leq u(t,z), \quad \textit{for all} \quad t \geq -h, z \in \mathbb{R}, \tag{19}$$

in particular

$$|v(t,z) - \psi(t,z)| \leq A_0\, \xi_c(-z)\, \frac{e^{\gamma t}}{\sqrt{t}}, \quad \textit{for all} \quad t > h, z \in \mathbb{R} \tag{20}$$

where $\gamma = \gamma(\lambda_c)$ *is defined by* (13) *with* $p = p(\lambda_c)$ *and* $q = q(\lambda_c)$.

Corollary 4. (Uniqueness) *If* $\psi_c(z)$ *and* $\phi_c(z)$ *are two semi-wavefronts with speed* $c \geq \mathfrak{c}_*$ *satisfying* (18) *then there exists* $z_* \in \mathbb{R}$ *such that* $\psi_c(z+z_*) = \phi_c(z)$ *for all* $z \in \mathbb{R}$.

Remark 5. If $h = 0$ in (1) then semi-wavefronts are monotone wavefronts and by taking a wavefront $\psi(t,z) = \psi_c(z)$ in Theorem 3 we get the stability of the wavefront on the sets $(-\infty, N], N \in \mathbb{R}$, which is comparable to a result obtained by Uchiyama [39, Theorem 4.1].

It has recently been showed that the estimation $u(t,x) = O(t^{-1/2})$ in (19)—(20) for critical semi-wavefronts, in the D-K case, is actually $u(t,x) = o(t^{-1/2})$ for all $h \geq 0$ (see [4, Corollary 1.2]). Also, since Theorem 3 does not assume some stability condition on κ then semi-wavefronts could be asymptotically periodic at $+\infty$ [36, Theorem 3] and oscillations around κ can be approximated by the solution $v(t,z)$ on each interval in the form $(-\infty, N]$ with $N \in \mathbb{R}$. The Corollary 4 refers essentially to the fact that semi-wavefronts are equal (up to translation) if they have the same one-order asymptotic terms at $z = -\infty$, i.e., the condition (18).

By the change of variable $t' := \delta^{-1} t$ and $x' := \delta^{-1/2} x$ Eq. (9) can be reduced to (1) with delay $h' := h\delta$ so that by Theorem 3 we obtain the stability of semi-wavefronts with speed c for the Nicholson's model,

Corollary 6. (Nicholson Model) *Let* $\rho/\delta \in [1, +\infty)$ *be in* (9). *Consider* $N \in \mathbb{R}$ *and the initial datum satisfying the conditions of Theorem* 3, *if* $c \geq c_*$ *and* $\lambda_c \in [\lambda_1(c), \lambda_2(c)]$ *then*

$$\sup_{z \in (-\infty, N]} |v(t,z) - \psi_c(z)| = O(t^{-1/2} e^{\gamma t}),$$

where $\gamma = \gamma(\lambda_c) \leq 0$ *is determined by* (13).

It is well known that if $\rho/\delta \in (e, e^2)$ then there are non-monotone wavefronts (see [15, Theorem 2.3]). Moreover, for some critical value $\nu_0 = 2.808\ldots$ and some delay h_0 if $\rho/\delta \in [\nu_0, +\infty)$ then each minimal wavefront has oscillations around κ at $+\infty$ and there exist a critical value c^* (a extended real number) such that each semi-wavefront with speed $c > c_*$ has non-decaying slow oscillations [36, Theorem 3].

Nevertheless, the semi-wavefronts of Theorem 3 could exhibit a type of *convective instability* due to the positive equilibrium (e.g., see [27]), however by controlling the size of the slope of g at the positive equilibrium, the stability of the semi-wavefront on the remaining domain $[N, +\infty)$, $N \in \mathbb{R}$, can be obtained. In this framework, it is necessary to assume some additional hypotheses in order to establish the existence of semi-wavefronts, such as the following condition.

(M) The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is such that the equation $g(x) = x$ has exactly two solutions on $[0, +\infty)$: 0 and $\kappa > 0$. Moreover, g is C^1-smooth in some δ_0-neighborhood of the equilibria where $g'(0) > 1 > g'(\kappa)$. In addition, there are $C > 0$, $\theta \in (0, 1]$, such that $|g'(u) - g'(0)| + |g'(\kappa) - g'(\kappa - u)| \le Cu^\theta$ for $u \in (0, \delta_0]$.

We note that for g satisfying **(M)**, there are real numbers $0 < \zeta_1 \le \zeta_2$ such that

(B1) $g([\zeta_1, \zeta_2]) \subset [\zeta_1, \zeta_2]$ and $g([0, \zeta_1]) \subset [0, \zeta_2]$;
(B2) $\min_{\zeta \in [\zeta_1, \zeta_2]} g(\zeta) = g(\zeta_1)$;
(B3) $g(x) > x$ for $x \in [0, \zeta_1]$ and $1 < g'(0) \le g_+^* := \sup_{s \ge 0} g(s)/s < \infty$;
(B4) In $[0, \zeta_2]$, the equation $g(x) = x$ has exactly two solutions 0 and κ.

Thus, from [37, Theorem 4.5] we obtain the following result to the existence of semi-wavefronts.

Proposition 7. (Existence of semi-wavefronts) *Let g satisfy* **(M)**. *Then, for each $c > \mathfrak{c}_*(g_+^*)$ (according to definition (3)) Eq, (1) has semi-wavefronts with speed c. Moreover, if $0 < \zeta_1 \le \zeta_2$ meet (B1)–(B4) then each semi-wavefront ψ_c satisfies:*

$$\zeta_1 \le \liminf_{z \to -\infty} \psi_c(z) \le \limsup_{z \to +\infty} \psi_c(z) \le \zeta_2, \quad \textit{for all } z \in \mathbb{R}.$$

Remark 8. (Minimal speed for semi-wavefronts) Due [16, Theorem 18], in the case that $g_+^* = g'(0)$ the number $\mathfrak{c}_*(g_+^*)$ is actually the minimal speed for the existence of semi-wavefronts.

Now, let us introduce some notation. If $I \subset \mathbb{R}_+ = Dom(g)$, let us denote by

$$L_g(I) := \sup_{x \neq y;\, x, y \in I} \frac{|g(x) - g(y)|}{|x - y|},$$

and for $b \in \mathbb{R}$, let us denote by $\eta_b(z) = \min\{1, e^{\lambda_c(z-b)}\}$, with $\lambda_c \in [\lambda_1(c), \lambda_2(c)]$. With these notations, the second main result of this paper can be established

Theorem 9. (Global stability) *Let $c > \mathfrak{c}_*$ and $\bar{g}$ be a non decreasing function satisfying* **(M)** *with equilibrium K such that $\bar{g}(u) \ge g(u)$ for all $u \in \mathbb{R}_+$ such* *that $L_{\bar{g}} \leq L_g$. We denote by $m_K = \min_{u\in[\kappa,K]} g(u)$ and $\mathcal{I}_K := [m_K, K]$ and we suppose that $L_g(\mathcal{I}_K) < 1$. If for some $q_0 > 0$ and $z_0 \in \mathbb{R}$ the initial datum satisfies*

$$v_0(s,z) \geq q_0 \quad \text{for all} \quad (s,z) \in [-h,0]\times[z_0,+\infty) \tag{21}$$

and for some wavefront ψ_c, $b \in \mathbb{R}$ and $q > 0$

$$|v_0(s,z) - \psi_c(z)| \leq q\eta_b(z) \quad \text{for all} \quad (s,z) \in [-h,0]\times\mathbb{R}, \tag{22}$$

then there exists $C = C(\bar{g}, m_K, b) > 0$ and $\gamma_0 \geq 0$ satisfying

$$-\lambda_c^2 + c\lambda_c + 1 \geq \gamma_0 + L_g e^{\gamma_0 h} e^{-\lambda_c ch} \quad \text{and} \quad L_g(\mathcal{I}_K) \leq e^{-\gamma_0 h}(1-\gamma_0), \tag{23}$$

such that

$$|v(t,z) - \psi_c(z)| \leq Cqe^{-\gamma_0 t}, \quad \text{for all} \quad (t,z) \in [-h,\infty)\times\mathbb{R}. \tag{24}$$

Remark 10. (Crossing-monostable case) In [41] Wu et al. established the local stability for sufficiently fast wavefronts of the so-called crossing-monostable case. Theorem 9 generalizes those results by including global perturbations of wavefronts. More precisely, we suppose that for some positive number $K \geq \kappa$, the birth function g is such that

- (C_1) g satisfies **(M)**
- (C_2) $g(u) \leq \bar{g}(u) := \max\{g'(0)u, K\}$ for all $u \geq 0$ and
- (C_3) $L_g(\mathcal{I}_K) < 1$,

then the non decreasing function $\bar{g}(u)$ clearly satisfies $L_{\bar{g}} = g'(0) \leq L_g$, therefore g satisfies the conditions of Theorem 9. Hence if g satisfies (C_1)-(C_3) then (22) implies (24). Note that the condition (A_3) in [41, Theorem 2.4] is essentially our condition (C_3). Moreover, by (2) and the definition of $\mathfrak{c}_*(h)$ we have $\mathfrak{c}_*(h)$ is a non-increasing function of h, therefore we have $\mathfrak{c}_*(h) \leq \mathfrak{c}_*(0) = 2\sqrt{L_g - 1}$ for all $h \geq 0$, so that we have improved the minimal speed $\tilde{c} := 2\sqrt{2(L_g-1)}$ given in [41] for the local stability of wavefronts with speed $c > \tilde{c}$.

Now, if we take $\bar{g}(u) = \max_{s\in[0,u]} g(s)$ then we have that $K = M_g := \max_{s\in[0,\kappa]} g(s)$ and by writing $m_g = \min_{u\in[\kappa,M_g]} g(u)$ and $\mathcal{I}_K = I_g := [m_g, M_g]$ the following global stability result is obtained

Corollary 11. *Let g satisfy* **(M)** *such that $L_g(I_g) < 1$. If ψ_c is a semi-wavefront with speed $c > \mathfrak{c}_*$, then ψ_c is globally stable in the sense of Theorem 9.*

Corollary 11 generalizes results for wavefronts which assume the D-K condition (see, e.g. [33]). In the Allee case with monotone g, Corollary 11 is an improvement, in terms of the globality of the disturbance, of [33, Theorem 2] for wavefronts with a speed greater than $\mathfrak{c}_*$ and it also gives us an exponential convergence rate for these waves. In this regard, exponential (in the time) stability as in (24) for pushed wavefronts was not studied in [32] but a recent work [42] by Wu, Niu and Hsu, has given a positive answer to this problem.

This paper is organized as follows. The linear theorems (Theorems 1 and 2) are proven in Sect. 2. Finally, results on the stability of semi-wavefronts are proven in Sect. 3.

2. Proof of Linear Theorems

In order to demonstrate both Theorems 1 and 2, the following two lemmas will be needed. The first one is an abstract version of the Halanay type inequalities [17]

Lemma 12. (Halanay Type Inequality) *Let X be a complex Banach space. Suppose that $\sigma, k \in \mathbb{C}$ and $h > 0$. If $r \in C([-h, \infty), X)$ is a function satisfying:*

$$r_t(t) = \sigma r(t) + kr(t-h), \quad a.e.,$$

then

$$|r(t)|_X \leq \sup_{s\in[-h,0]} |r(s)|_X \; e^{\max\{0,\lambda\}h}\, e^{\lambda t}, \quad \text{for all} \quad t > -h, \tag{25}$$

where λ is the only real root of the equation :

$$\lambda = Re(\sigma) + |k|e^{-\lambda h}. \tag{26}$$

Moreover

(i) $\lambda \leq 0 \iff -Re(\sigma) \geq |k|$.
(ii) $\lambda = 0 \iff -Re(\sigma) = |k|$.

Proof. It is clear that:

$$\frac{d}{dt}(r(t)e^{-\sigma t}) = ke^{-\sigma t} r(t-h) \quad a.e.$$

and from here, it is obtained that $|r(t)|_X$ meets the following inequality:

$$x(t) \leq |k| \int_0^t e^{Re(\sigma)(t-s)} x(s-h)ds + x(0)e^{Re(\sigma)t} \quad \text{for all} \quad t > 0 \tag{27}$$

We note that for $A \in \mathbb{R}$ the function $e_A(t) = Ae^{\lambda t}$ meets (27) with equality. Now, for $A := \sup_{s\in[-h,0]} |r(s)|_X \, e^{\max\{0,\lambda\}h}$ the function $\delta(t) = |r(t)|_X - e_A$ satisfies (27) for $t \in [0, h]$ and therefore $\delta(t) \leq 0$ for all $t \in [0, h]$. Similarly, it is concluded that $\delta(t) \leq 0$ for the intervals $[h, 2h], [2h, 3h] \ldots$ This proves (25).

Let us prove (i). If $-Re(\sigma) \geq |k|$ then: $\lambda \leq |k|(e^{-h\lambda} - 1)$ which necessarily implies that $\lambda \leq 0$. Otherwise, if $\lambda \leq 0$ let us suppose that $-Re(\sigma) < b$, then $\lambda > |k|(e^{-h\lambda} - 1)$ which is a contradiction.

In order to prove (ii) let us note that since the derivative of $f(\lambda) := \lambda - Re(\sigma) - |k|e^{-h\lambda}$ is always positive then $f(\lambda)$ has at most one zero. So, if $Re(\sigma) = |k|$ then $\lambda = 0$ is the only solution of (26), this proves (ii). □

Now, let us consider the function $\lambda : \mathbb{R} \to \mathbb{R}$ defined by

$$\lambda(\zeta) = -\zeta^2 + p + qe^{-h\lambda(\zeta)}, \tag{28}$$

where $q \geq 0$. Next, we proceed to estimate the even function $\lambda(\zeta)$.

For $\epsilon_h = \frac{1}{1+h(\gamma-p)}$ we define the function

$$\alpha_h(\zeta) := -\frac{1}{h}\log(1 + h\epsilon_h\zeta^2).$$

Here $\gamma \in \mathbb{R}$ is defined by (13) for any $p \in \mathbb{R}$ and $q \geq 0$.

Lemma 13. *If λ is defined by (28) then*

$$-\epsilon_h\zeta^2+\gamma\le\lambda(\zeta)\le\alpha_h(\zeta)+\gamma\quad \textit{for all } \zeta\in\mathbb{R}. \tag{29}$$

Moreover, if $q>0$ then

$$\lim_{|\zeta|\to\infty} q^{-1}\zeta^2\, e^{h\lambda(\zeta)}=1. \tag{30}$$

Remark 14. The function α_h is a generalization of the function $\alpha_0(\zeta):=-\zeta^2=\lim_{h\to 0}\alpha_h(\zeta)$ for each $\zeta\in\mathbb{R}$. Also, when $h=0$ then $\gamma=p+q$ [according to definition (13)] therefore $\lambda(\zeta)=-\zeta^2+\gamma$ in (28). Thus, by passing the limit $h\to 0$ in (29) we have the equality $-\zeta^2=\lambda(\zeta)-\gamma=\alpha_0(\zeta)$. In this regard, the estimates in (29) are sharp.

Proof. Let us denote $\beta(\zeta)=\lambda(\zeta)-\alpha(\zeta)-\gamma$. Then $\beta(\zeta)$ satisfies the following equation

$$\beta(\zeta)=-\zeta^2+\frac{1}{h}\log(1+h\epsilon_h\zeta^2)-\gamma+p+qe^{-h\gamma}(1+h\epsilon_h\zeta^2)e^{-h\beta(\zeta)}.$$

From Lemma 12 we have that $\beta(\zeta)\le 0$ if and only if:

$$\zeta^2-\frac{1}{h}\log(1+h\epsilon_h\zeta^2)+\gamma-p\ge qe^{-h\gamma}(1+h\epsilon_h\zeta^2). \tag{31}$$

Now, using $\log(1+x)\le x$, fo rall $x\ge 0$, in order to obtain (31) it is enough to have

$$\begin{aligned}&\zeta^2-\epsilon_h\zeta^2+\gamma-p\ge qe^{-h\gamma}(1+h\epsilon_h\zeta^2)\quad \text{for all}\quad \zeta\in\mathbb{R}\\ &\iff (1-\epsilon_h-qh\epsilon_h e^{-h\gamma})\zeta^2+\gamma-p-qe^{-h\gamma}\ge 0\quad \text{for all}\quad \zeta\in\mathbb{R},\end{aligned}$$

which is a consequence of definition of γ and ϵ_h. So that, this proves the upper estimate in (29)

To complete left hand side of (29) we note that due to (28), (13) and upper estimate in (29)

$$\begin{aligned}\lambda(\zeta)&\ge -\zeta^2+\gamma-q^{-\gamma h}+qe^{-h\gamma}[1+h\epsilon_h\zeta^2]\\ &=-\epsilon_h\zeta^2+\gamma.\end{aligned}$$

Next, by multiplying (28) by $e^{h\lambda(\zeta)}$ and by using that $\lambda(\zeta)\to-\infty$ as $|\zeta|\to+\infty$ [which is obtained from upper estimation in (29)] we conclude

$$\lim_{\zeta\to\pm\infty} e^{h\lambda(\zeta)}\zeta^2=p$$

which implies (30). □

Consider the following equation

$$u_t(t,z)=u_{zz}(t,z)+d_1u_z(t,z)+d_2u(t,z)+e^{-\lambda z}g(e^{\lambda(z-ch)}u(t-h,z-ch)) \tag{32}$$

where $d_1,d_2,\lambda\in\mathbb{R}$

Proposition 15. *If* $u_0 \in C([-h,0];L^\infty(\mathbb{R})\cap C^{0,\alpha}(\mathbb{R}))$*, some* $\alpha \in (0,1]$*, then there is a unique solution* $u(t,z)$ *of* (32) *with initial data* u_0 *and this solution satisfies* $u(\cdot + kh,\cdot) \in C([-h,0];L^\infty(\mathbb{R})\cap C^{0,\alpha}(\mathbb{R}))$ *for all* $k \in \mathbb{Z}_+$*. Moreover, if* $u_0 \in C([-h,0];L^1(\mathbb{R}))$ *and* $\lambda = 0$ *in* (32) *then* $u(t,\cdot), u_z(t,\cdot) \in L^1(\mathbb{R})$ *for all* $t \geq 0$ *and* $u_{zz}(t,\cdot) \in L^1(\mathbb{R})$ *for all* $t > h$*.*

Proof. By defining

$$d_3(t,z) := e^{-\lambda ch} g(e^{\lambda(z-ch)}u(t-h,z-ch))/e^{\lambda(z-ch)}u(t-h,z-ch)$$

we have $d_3 \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ and the function u satisfies

$$u_t(t,z) = u_{zz}(t,z) + d_1 u_z(t,z) + d_2 u(t,z) + d_3(t,z)u(t-h,z-ch) \quad (33)$$

By making the change of variables $\bar{u}(t,z) := u(t,z-d_1 t)e^{-d_2 t}$ the equation (33) is reduced to an inhomogeneous heat equation

$$\bar{u}_t(t,z) = \bar{u}_{zz}(t,z) + f(t,z), \quad (34)$$

where

$$f(t,z) = e^{-d_2 h} d_3(t,z-d_1 t)\bar{u}(t-h,z-h(c+d_1)), \quad (35)$$

Now, note that for $1 \leq p \leq \infty$

$$|f(t,\cdot)|_{L^p(\mathbb{R})} \leq e^{-d_2 h}|d_3|_{L^\infty} \max_{s\in[-h,0]} |\bar{u}_0(s,\cdot)|_{L^p(\mathbb{R})} \quad \text{for all } t \in [0,h]. \quad (36)$$

Similarly, by using the definition of d_3, we get

$$|f(t,\cdot)|_{C^{0,\alpha}(\mathbb{R})} \leq L_g\, e^{-d_2 h} \max_{s\in[-h,0]} |\bar{u}_0(s,\cdot)|_{C^{0,\alpha}(\mathbb{R})} \quad \text{for all } t \in [0,h]. \quad (37)$$

So that, by [14, Chapter 1, Theorems 12 and 16] there exist a unique solution to (32) and this solution satisfies

$$\bar{u}(t) := \Gamma_t * \bar{u}(0) + \int_0^t \Gamma_{t-s} * f(s)ds, \quad (38)$$

where Γ_t is the one-dimensional heat kernel.

Now, we take $1 \leq p \leq \infty$. Then, for $t \in [0,h]$ and $t_n \to t$ we have

$$|\bar{u}(t) - \bar{u}(t_n)|_{L^p} \leq |\Gamma_t - \Gamma_{t_n}|_{L^1}\, |\bar{u}(0)|_{L^p} + \int_0^t |\Gamma_{t-s} - \Gamma_{t_n-s}|_{L^1}|f(s)|_{L^p}ds$$
$$+ \int_t^{t_n} |\Gamma_{t_n-s}|_{L^1}|f(s)|_{L^p}ds, \quad (39)$$

and by using (36),

$$|\bar{u}(t) - \bar{u}(t_n)|_{L^p} \leq (|\Gamma_t - \Gamma_{t_n}|_{L^1}$$
$$+ \int_0^t |\Gamma_{t-s} - \Gamma_{t_n-s}|_{L^1}ds + |t-t_n|)R \max_{s\in[-h,0]} |\bar{u}_0(s,\cdot)|_{L^p(\mathbb{R})}, \quad (40)$$

where $R = \max\{1, e^{-d_2 h}|d_3|_{L^\infty}\}$. Since $|\Gamma_{t_n}|_{L^1} = |\Gamma_t|_{L^1} = 1$ the last inequality implies $|\bar{u}(t) - \bar{u}(t_n)| \to 0$ as $t_n \to t$, therefore if $u_0(\cdot,\cdot) \in C([-h,0];L^p(\mathbb{R}))$ then $u(\cdot + h,\cdot) \in C([-h,0];L^p(\mathbb{R}))$. Similarly, we get $u(\cdot + h,\cdot) \in C([-h,0]; C^{0,\alpha}(\mathbb{R}))$ whenever $u_0(\cdot,\cdot) \in C([-h,0];C^{0,\alpha}(\mathbb{R}))$

Analogously, by using the initial data $u(t+h,\cdot), u(t+2h,\cdot)\dots$ we obtain $u(\cdot+kh,\cdot)\in C([-h,0];L^p(\mathbb{R})\cap C^{0,\alpha}(\mathbb{R}))$ for $k=2,3\dots$ Therefore, with $p=\infty$ we obtain the first assertion of the Proposition 15.

Otherwise, if $u_0\in C([-h,0],L^1(\mathbb{R}))$ then with $p=1$ we get $u(\cdot+kh,\cdot)\in C([-h,0];L^1(\mathbb{R}))$ for all $k\in\mathbb{Z}_+$. Then, note that by (38) for $t>0$ we get

$$\begin{aligned}\bar u_z(t,z)=&\int_{\mathbb{R}}\frac{(y-z)e^{-(z-y)^2/4t}}{4t^{3/2}\sqrt{\pi}}u_0(0,y)dy\\&+\int_0^t\int_{\mathbb{R}}\frac{(y-z)e^{-(z-y)^2/4(t-s)}}{4(t-s)^{3/2}\sqrt{\pi}}f(s,y)dyds,\end{aligned}\tag{41}$$

and using (36) with $p=1$ for $t\in(0,h]$ we obtain

$$\begin{aligned}|\bar u_z(t,\cdot)|_{L^1(\mathbb{R})}\le&\frac{|u_0(0,\cdot)|_{L^1(\mathbb{R})}}{\sqrt{\pi t}}\int_{\mathbb{R}}|y|e^{-y^2}dy\\&+2\sqrt{\frac{t}{\pi}}\int_{\mathbb{R}}|y|e^{-y^2}dy\max_{s\in[-h,0]}|f(s,\cdot)|_{L^1(\mathbb{R})}\\&\le(\frac{1}{\sqrt{t}}+2\sqrt{t}e^{-d_2h}|d_3|_{L^\infty(\mathbb{R}_+\times\mathbb{R})})\frac{1}{\sqrt{\pi}}\max_{s\in[-h,0]}|\bar u_0(s,\cdot)|_{L^1(\mathbb{R})}\end{aligned}\tag{42}$$

and by using the initial data $\bar u(t+h,\cdot),\bar u(t+2h,\cdot)\dots$, with $t\in(0,h]$, we obtain $\bar u_z(t+kh,\cdot)\in L^1(\mathbb{R})$ for $k\in\mathbb{Z}_+$ and $t\in(0,h]$. Moreover, if we differentiate in (32) and proceed as in (39) and (40) then we have $|\bar u_z(t,\cdot)|_{L^1(\mathbb{R})}$ continuously depends on $t\in\mathbb{R}_+$.

Finally, if $T>h$ then $\bar u(T+\cdot,\cdot)\in C([-h,0];L^1(\mathbb{R}))$, by taking $\lambda=0$, we obtain $\bar u_z(t,z)$ satisfies (33) with $d_3(t,z)=g'(\bar u(t-h,z-ch))$ and taking as initial datum the function $\bar u(T+s,z)$ and using (42) (replacing $\bar u_z$ by $\bar u_{zz}$) we obtain $\bar u_{zz}(T+t,\cdot)\in L^1(\mathbb{R})$ for all $t\in(0,h]$. Similarly, by using the initial data $\bar u_z(t+h,\cdot),\bar u_z(t+2h,\cdot)\dots$, with $t\in(0,h]$, we obtain $\bar u_{zz}(t+T+kh,\cdot)\in L^1(\mathbb{R})$ for $k\in\mathbb{Z}_+$ and $t\in(0,h]$, which completes the proof. □

Remark 16. Since by Proposition 15 $u(\cdot+hk,\cdot)\in C([-h,0],L^\infty(\mathbb{R}))$ for all $k\in\mathbb{Z}$ then for each $t>0$ we have $f(\cdot,\cdot)\in C([0,t],L^\infty(\mathbb{R}))$, therefore

$$\left|\int_{\mathbb{R}}\frac{(y-z)e^{-(z-y)^2/4(t-s)}}{4(t-s)^{3/2}\sqrt{\pi}}f(s,y)dy\right|\le\frac{C}{\sqrt{t-s}}\quad\text{for all}\quad t>s,$$

for some constant $C>0$ (which does not depend on (z,t,s)) so that from (41) we conclude $u(t,\cdot)\in C^1(\mathbb{R})$ for all $t>0$.

Proof of Theorem 1. By using Proposition 15 with $d_1=m$, $d_2=p$, $\lambda=0$ and $g(u)=qu$ we get $u(t,\cdot),u_z(t,\cdot),u_{zz}(t,\cdot)\in L^1(\mathbb{R})$ for all $t>h$. Next, by applying the Fourier transform, here

$$\hat u(z):=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-izy}u(y)dy,$$

to Eq. (12) we have

$$\hat u_t(t,\zeta)=\sigma(\zeta)\hat u(t,\zeta)+k(\zeta)\hat u(t-h,\zeta)\quad\text{for all}\quad t>h,$$

where $\sigma(\zeta) = -\zeta^2 + im\zeta + p$ and $k(\zeta) = qe^{-id\zeta}$.

Since $-Re(\sigma(\zeta)) \geq |k(\zeta)|$, by Lemma 12 we obtain $\lambda(\zeta) \leq 0$ for all $\zeta \in \mathbb{R}$ and :

$$|\hat{u}(t,\zeta)| \leq C_{u_0} e^{\lambda(\zeta)t} \quad \text{for all } \zeta \in \mathbb{R}.$$

If $t > h$ then by the Fourier's inversion formula (since by Remark 16 $u(t,\cdot) \in C^1(\mathbb{R})$ for $t > 0$) and Lemma 13, we have

$$|u(t,x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{u}(t,\zeta)| d\zeta \leq \frac{C_{u_0}}{2\pi}\int_{\mathbb{R}} e^{\lambda(\zeta)t} d\zeta \leq \frac{C_{u_0}}{2\pi} e^{\gamma t}\int_{\mathbb{R}} \frac{d\zeta}{(1+\epsilon\zeta^2)^{\frac{t}{h}}}.$$

Moreover, by Bernoulli's inequality, we conclude that

$$\int_{\mathbb{R}} \frac{d\zeta}{(1+\epsilon\zeta^2)^{\frac{t}{h}}} \leq \int_{\mathbb{R}} \frac{d\zeta}{1+\frac{t\epsilon}{h}\zeta^2} = \frac{1}{\sqrt{t}}\left[\sqrt{\frac{h}{\epsilon}}\int_{\mathbb{R}} \frac{d\zeta}{1+\zeta^2}\right] = \frac{1}{\sqrt{t}}\sqrt{\frac{h}{\epsilon}}\pi.$$

□

Proof of Theorem 2. If we make the change of variable $v(t,x) = e^{\frac{m}{2}x}u(t,x)$, then $v(t,x)$ solves

$$v_t(t,x) = v_{xx}(t,x) + \left(p - \frac{m^2}{4}\right)v(t,x) + qv(t-h,x). \tag{43}$$

By applying the Fourier transform to (43) we get

$$\hat{v}_t(t,z) = \left(-z^2 + p - \frac{m^2}{4}\right)\hat{v}(t,z) + q\hat{v}(t-h,z) \quad \text{for all} \quad t > 0 \tag{44}$$

Let us note that due to $q \geq 0$, we have that (44) satisfies the Comparison Principle; that is, if for each $z \in \mathbb{R}$ we consider two solutions $v(s)$ and $w(s)$ of (44) defined on $[-h,+\infty)$ then, by denoting $\Re(\hat{u}(t)) = u_1(t), \Im(\hat{u}(t)) = u_2(t)$, the inequality

$$v_i(s) \leq w_i(s) \quad \text{for all } s \in [-h,0] \text{ and } \quad i = 1,2.$$

implies

$$v_i(s) \leq w_i(s) \quad \text{for all } s \in [-h,+\infty) \text{ and } \quad i = 1,2$$

Let us denote by $e_A(t,z) = Ae^{\lambda(z)t}$, where $\lambda(z)$ satisfies

$$\lambda(z) = -z^2 + p - \frac{m^2}{4} + qe^{-\lambda(z)h}. \tag{45}$$

Let us note that $e_A(t,z)$ satisfies (44) for all $A \in \mathbb{C}$. Also, let us denote that

$$m_i(z) = \min_{s\in[-h,0]}(v_i(s,z)e^{-\lambda(z)s}) \quad \text{and}$$
$$M_i(z) = \max_{s\in[-h,0]}(v_i(s,z)e^{-\lambda(z)s}) \quad i = 1,2$$

then we have that

$$e_{m_i}(s,z) \leq v_i(s,z) \leq e_{M_i}(s,z) \quad \text{for all } (s,z) \in [-h,0]\times\mathbb{R}; i = 1,2.$$

By the comparison principle applied to real and imaginary part in (44), we have that

$$e_{m_i}(t,z) \le v_i(t,z) \le e_{M_i}(t,z) \quad \text{for all} \quad (t,z) \in [-h,\infty) \times \mathbb{R}; i = 1,2 \tag{46}$$

or

$$m_i(z)e^{\lambda(z)t} \le v_i(t,z) \le M_i(z)e^{\lambda(z)t} \quad \text{for all} \quad (t,z) \in [-h,\infty) \times \mathbb{R}; i = 1,2 \tag{47}$$

Now, by the Fourier inversion formula, we have that

$$v(t,x) = \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{\frac{xy}{\sqrt{t}}} \hat{v}(t, y/\sqrt{t}) dy. \tag{48}$$

However, if we apply Lemma 13 to (45) with $\gamma = \sigma$ we have

$$\lim_{t\to\infty} t[\lambda(y/\sqrt{t}) - \sigma] = -\frac{y^2}{1 + hqe^{-\sigma h}}. \tag{49}$$

and due to $v(s,\cdot) \in L^1(\mathbb{R})$ by the Lebesgue's dominated convergence theorem

$$\lim_{t\to\infty} M_1(y/\sqrt{t}) = \lim_{t\to\infty} m_1(y/\sqrt{t}) = \int_{\mathbb{R}} e^{\frac{m}{2}x} v(s,x)dx$$

and

$$\lim_{t\to\infty} M_2(y/\sqrt{t}) = \lim_{t\to\infty} m_2(y/\sqrt{t}) = 0.$$

Therefore by (47)

$$\lim_{t\to\infty} \hat{v}(t, y/\sqrt{t}) = e^{-\frac{y^2}{1+hqe^{-\sigma h}}} \int_{\mathbb{R}} e^{\frac{m}{2}x} v(s,x)dx \tag{50}$$

However, by (25) there exists $C(p,q,m) > 0$ such that

$$|\hat{v}(t, y/\sqrt{t})| \le C \sup_{s\in[-h,0]} |\hat{v}(s, y/\sqrt{t})| e^{\lambda(y/\sqrt{t})t}$$

but by (29) and Bernoulli's Inequality

$$|\hat{v}(t, y/\sqrt{t})| \le \frac{Ce^{|\sigma|h} ||e^{\frac{m}{2}\cdot} u_0(\cdot)||_{L^1(\mathbb{R})}}{1 + \epsilon_h y^2} \quad \text{for all} \quad t > 0. \tag{51}$$

Finally, by (48), (51), Lebesgue's dominated convergence theorem and (50), the result obtained. □

3. Proof of results of stability of semi-wavefronts

Proof of Theorem 3. The first assertion follows from Proposition 15 with $\lambda = 0$. Next, for a solution $w(t,z)$ of (17), let us denote the function $\tilde{w}(t,z) = \xi_c(z)w(t,z)$ which satisfies

$$\begin{aligned}\tilde{w}_t(t,z) &= \tilde{w}_{zz}(t,z) + m\tilde{w}_z(t,z) + p\tilde{w}(t,z) \\ &\quad + \xi_c(z)g(\xi_c(-z+ch)\tilde{w}(t-h, z-ch)).\end{aligned}$$

We consider the linear operator

$$\mathcal{L}\delta(t,z) := \delta_{zz}(t,z) + m\delta_z(t,z) + p\delta(t,z) - \delta_t(t,z).$$

If $\delta_\pm(t,z) := \pm[\tilde{v}(t,z) - \tilde{\psi}(t,z)] - u(t,z)$, then by (18): $\delta_\pm(s,z) \le 0$ for $(s,z) \in [-h,0]\times\mathbb{R}$. For $(t,z) \in [0,h]\times\mathbb{R}$ by (17) and (18) we have

$$\begin{aligned}\mathcal{L}\delta_\pm(t,z) &= \mp\xi(z)[g(\xi(-z+ch)\tilde{\psi}(t-h,z-ch))\\ &\qquad -g(\xi(-z+ch)\tilde{v}(t-h,z-ch))] - \mathcal{L}u(t,z)\\ &\ge -L_g e^{-\lambda ch}|\tilde{v}(t-h,z-ch) - \tilde{\psi}(t-h,z-ch)| - \mathcal{L}u(t,z)\\ &\ge -L_g e^{-\lambda ch}u(t-h,z-ch) - \mathcal{L}u(t,z) = 0.\end{aligned}$$

Now, by Proposition 15, $\tilde{w}(\cdot + kh,\cdot) \in C([-h,0];L^\infty(\mathbb{R}))$ for all $k \in \mathbb{Z}_+$ therefore by using the Phragmèn–Lindelöf principle from [25, Chapter 3, Theorem 1], we have $\delta_\pm(t,z) \le 0$ for $(t,z) \in [0,h]\times\mathbb{R}$. The argument is repeated for intervals $[h,2h],[2h,3h]\ldots$ to conclude (19). Finally, the estimate in (20) is obtained using Theorem 1. □

Remark 17. Note that in Proof of Theorem 3 it was only necessary to have an initial datum u_0 exponentially bounded to apply the Phragmèn–Lindelöf principle in order to obtain estimate (20). So, we could use the elementary exponential solutions of (12) of the form $u(t,z) = Be^{\gamma t + r\,z}$, with r and γ satisfying

$$q(\lambda_c)e^{-rch}\,e^{-\gamma h} = -r^2 - (2\lambda_c - c)r - p(\lambda_c) + \gamma. \tag{52}$$

Here, $\gamma \le 0$ if and only if

$$-r^2 - (2\lambda_c - c)r - p(\lambda_c) \ge q(\lambda_c)\,e^{-rch}, \tag{53}$$

with $\gamma = 0$ if and only if (53) holds with equality. Thus, for $c > \mathfrak{c}_*$ and $\lambda_c \in (\lambda_1(c),\lambda_2(c))$ we have $-p(\lambda_c) > q(\lambda_c)$ and therefore by taking $r = 0$ in (53) we obtain $\gamma < 0$ in (52) and therefore the asymptotic stability of non-critical is obtained. However, when $c = \mathfrak{c}_*$ we have $\lambda_{\mathfrak{c}_*} = \lambda_1(\mathfrak{c}_*) = \lambda_2(\mathfrak{c}_*)$ and $-p(\lambda_{\mathfrak{c}_*}) = q(\lambda_{\mathfrak{c}_*})$ in (53), also due to the curves $-\lambda^2 + c\lambda + 1$ and $L_g\,e^{-\lambda ch}$ in (2) are tangent at $\lambda = \lambda_{\mathfrak{c}_*}$ the function $\Theta(r) := q(\lambda_{\mathfrak{c}_*})\,e^{-r\mathfrak{c}_* h} + r^2 + (2\lambda_{\mathfrak{c}_*} - \mathfrak{c}_*)r + p(\lambda_{\mathfrak{c}_*})$ holds $\Theta'(0) = 0$. Consequently, since Θ is strictly convex and $\Theta(0) = 0$ we conclude $r = 0$ is the only solution in (53) and therefore $-p(\lambda_{\mathfrak{c}_*}) = q(\lambda_{\mathfrak{c}_*})$ implies $\gamma = 0$ in (52). Thus, this approach does not allow us to obtain the asymptotic stability of critical semi-wavefronts.

Theorem 18. *Let $v(t,z)$ and $\psi(t,z)$ be solutions of equation (17) for $c \ge \mathfrak{c}_*$. Assume that for some compact interval $I \subset \mathbb{R}$, such that $L_g(I) < 1$, and $b \in \mathbb{R}$ we have*

$$\psi(t,z), v(t,z) \in I \quad \text{for all } (t,z) \in [-h,\infty)\times[b-ch,\infty), \tag{54}$$

and for some $q > 0$ and $\lambda_c \in [\lambda_1(c),\lambda_2(c)]$

$$|v_0(s,z) - \psi_0(s,z)| \le q\eta_b(z) \quad \text{for all } (s,z) \in [-h,0]\times\mathbb{R}. \tag{55}$$

If $\gamma_0 \ge 0$ satisfies

$$-\lambda_c^2 + c\lambda_c + 1 \ge \gamma_0 + L_g e^{\gamma_0 h}e^{-\lambda_c ch} \quad \text{and} \quad L_g(I) \le e^{-\gamma_0 h}(1-\gamma_0), \tag{56}$$

then

$$|v(t,z) - \psi(t,z)| \le q e^{-\gamma_0 t}\eta_b(z) \quad \textit{for all } (t,z) \in [-h,\infty) \times \mathbb{R}. \tag{57}$$

Proof. We define $\eta(t,z) = qe^{-\gamma_0 t}\eta_b(z)$ and write the operator

$$\mathcal{L}_0\delta(t,z) := \delta_{zz}(t,z) - c\delta_z(t,z) - \delta(t,z) - \delta_t(t,z).$$

Note that by (55) if $\delta_\pm(t,z) := \pm[v(t,z) - \psi(t,z)] - \eta(t,z)$ then $\delta_\pm(s,z) \le 0$ for $(s,z) \in [-h,0] \times \mathbb{R}$. Now, for $(t,z) \in [0,h] \times (-\infty, b]$ due to (17), (22) and (56) we have that

$$\begin{aligned}\mathcal{L}_0\delta_\pm(t,z) &= \pm[-g(v(t-h,z-ch)) + g(\psi(t-h,z-ch))] - \mathcal{L}_0\eta(t,z)\\ &\ge qe^{-\gamma_0 t + \lambda_c(z-b)}[-L_g e^{\gamma_0 h} e^{-\lambda ch} - (\lambda_c^2 - c\lambda_c - 1 + \gamma_0)] \ge 0.\end{aligned}$$

Similarly, if $(t,z) \in [0,h] \times [b,\infty)$ we obtain:

$$\begin{aligned}\mathcal{L}_0\delta_\pm(t,z) &= \pm[-g(v(t-h,z-ch)) + g(\psi(t-h,z-ch))] - \mathcal{L}_0\eta(t,z)\\ &\ge qe^{-\gamma_0 t}[-L_g(I)e^{\gamma_0 h}\eta(z-ch) - (-1+\gamma_0)]\\ &\ge qe^{-\gamma_0 t}[-L_g(I)e^{\gamma_0 h} + 1 - \gamma_0] \ge 0.\end{aligned}$$

Now, as in the proof of the [33, Lemma 1], due to

$$\frac{\partial\delta_\pm(t,b+)}{\partial z} - \frac{\partial\delta_\pm(t,b-)}{\partial z} > 0, \tag{58}$$

we have that $\delta_\pm(t,z) \le 0$ for all $t \in [0,h],\ z \in \mathbb{R}$. Indeed, otherwise there exists $r_0 > 0$ such that $\delta(t,z)$ restricted to any rectangle $\Pi_r = [-r,r] \times [0,h]$ with $r > r_0$, reaches its maximum positive value $M_r > 0$ at some point $(t',z') \in \Pi_r$.

We claim that (t',z') belongs to the parabolic boundary $\partial\Pi_r$ of Π_r. Indeed, suppose on the contrary, that $\delta(t,z)$ reaches its maximum positive value at some point (t',z') of $\Pi_r \setminus \partial\Pi_r$. Then clearly $z' \neq z_*$ because of (58). Suppose, for instance that $z' > z_*$. Then $\delta(t,z)$ considered on the subrectangle $\Pi = [z_*, r] \times [0,h]$ reaches its maximum positive value M_r at the point $(t',z') \in \Pi \setminus \partial\Pi$. Then the classical results [25, Chapter 3, Theorems 5,7] show that $\delta(t,z) \equiv M_r > 0$ in Π, a contradiction.

Hence, the usual maximum principle holds for each $\Pi_r,\ r \ge r_0$, so that we can appeal to the proof of the Phragmèn–Lindelöf principle from [25] (see Theorem 10 in Chapter 3 of this book), in order to conclude that $\delta(t,z) \le 0$ for all $t \in [0,h],\ z \in \mathbb{R}$.

We can again repeat the above argument on the intervals $[h,2h]$, $[2h,3h]$, … establishing that the inequality $w_-(t,z) \le w(t,z) \le w_+(t,z)$, $z \in \mathbb{R}$, holds for all $t \ge -h$. □

Remark 19. We can generalize the function $\eta_b(z)$ for $b = +\infty$ and, thus, have $\eta_\infty(z) = \xi_c(-z)$. In this proof, it was not necessary to use the condition (54) for $z \le b$ so by replacing $\xi_c(-z)$ by $\eta_b(z)$ it can be concluded that (55) implies (57).

Corollary 20. (Local stability) *Suppose that there exist $M, b \in \mathbb{R}$ and $l_0 > 0$, such that:*

$$\psi(t,z) \in [M - l_0, M + l_0] \quad \textit{for all} \quad (t,z) \in [-h,\infty) \times [b - ch, \infty), \tag{59}$$

and that for some $l_1 > l_0$ the initial data satisfy

$$|v_0(s,z) - \psi_0(s,z)| < (l_1 - l_0)e^{-\gamma_0 s}\eta_b(z) \quad \textit{for all} \quad (s,z) \in [-h,0] \times \mathbb{R}. \tag{60}$$

where $\gamma_0 \geq 0$ is defined by (56). If $L_g(\mathcal{I}_1) < 1$, where $\mathcal{I}_1 := [M - l_1, M + l_1]$, then

$$|v(t,z) - \psi(t,z)| \leq (l_1 - l_0)e^{-\gamma_0 t}\eta_b(z) \quad \textit{for all } (t,z) \in [-h,\infty) \times \mathbb{R}. \tag{61}$$

Proof. Clearly, $\psi(t,z) \in \mathcal{I}_1$ for all $(t,z) \in [-h,\infty) \times [b - ch, \infty)$. Now if we suppose that the inequality in (60) is satisfied for $v_0(s,z) = v(hk + s, z)$ and $\psi_0(s,z) = \psi(hk + s, z)$, with $k \in \mathbb{Z}_+$, then $v(hk + t - h, z) \in \mathcal{I}_1$ for all $(t,z) \in [0,h] \times \mathbb{R}$ and, arguing as in the proof of Theorem 18, we get

$$\mathcal{L}_0\, \delta_\pm^k(t,z) \leq 0 \quad \text{for all} \quad (t,z) \in [0,h] \times \mathbb{R},$$

where $\delta_\pm^k(t,z) = \pm[v(hk + t, z) - \psi(hk + t, z)] - (l_1 - l_0)e^{-\gamma_0(hk+t)}\eta_b(z)$ and from [33, Lemma 1] we conclude

$$\delta_\pm^k(t,z) \leq 0 \quad \text{for all } (t,z) \in [0,h] \times \mathbb{R}. \tag{62}$$

But (62) implies $v((k+1)h + t - h, z) \in \mathcal{I}_1$ for all $(t,z) \in [0,h] \times \mathbb{R}$ and, arguing as above, by using (62) we obtain $\delta_\pm^{k+1}(t,z) \leq 0$ for all $(t,z) \in [0,h] \times \mathbb{R}$. Therefore, it is sufficient to suppose (60) in order to conclude (61) for $(t,z) \in [0,h] \times \mathbb{R}$ and then we procede inductively to obtain (61) for all $(t,z) \in [-h,\infty) \times \mathbb{R}$. □

To prove Theorem 9, we will use the following lemma

Lemma 21. *Suppose that functions $g_1, g_2 : D \subset \mathbb{R}_+ \to \mathbb{R}_+$ satisfy: $g_1(u) \leq g_2(u)$ for all $u \in D$. Let $v_1(t,z), v_2(t,z) : [-h,\infty) \times \mathbb{R} \to D$ be solutions to (17), with $g = g_1$ and $g = g_2$, respectively, such that: $v_1(s,z) \leq v_2(s,z)$ for $(s,z) \in [h,0] \times \mathbb{R}$. If g_1 or g_2 is a non decreasing function, then we have : $v_1(t,z) \leq v_2(t,z)$ for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$.*

Proof. We take $\delta(t,z) = v_1(t,z) - v_2(t,z)$. Let us note that if $(t,z) \in [0,h] \times \mathbb{R}$ then

$$\mathcal{L}_0\delta(t,z) = g_2(v_2(t-h, z-ch)) - g_1(v_1(t-h, z-ch)) \geq 0,$$

because if g_2 is a non decreasing function we have that

$$\begin{aligned} &g_2(v_2(t-h, z-ch)) - g_1(v_1(t-h, z-ch)) \\ &\quad \geq g_2(v_1(t-h, z-ch)) - g_1(v_1(t-h, z-ch)) \geq 0, \end{aligned}$$

or if g_1 is a non decreasing function, we have

$$\begin{aligned} &g_2(v_2(t-h, z-ch)) - g_1(v_1(t-h, z-ch)) \\ &\quad \geq g_2(v_2(t-h, z-ch)) - g_1(v_2(t-h, z-ch)) \geq 0 \end{aligned}$$

Now, as $\delta(t,z) \leq 0$ for all $(t,z) \in [-h,0] \times \mathbb{R}$ the Phragmèn–Lindelöf principle from [25][Chapter 3, Theorem 10] implies that $\delta(t,z) \leq 0$ for $(t,z) \in [0,h] \times \mathbb{R}$. The argument is repeated for intervals $[h,2h], [2h,3h] \ldots$ □

Proof Theorem 9. Let us take $\epsilon > 0$ such that $L_g(\mathcal{I}_\epsilon) < 1$, where $\mathcal{I}_\epsilon := [m_K - \epsilon, K + \epsilon] \subset \mathbb{R}_+$.Then, there is an increasing function $\bar{g}_\epsilon$ satisfying **(M)** with positive equilibrium $\kappa_+ \in (K, K+\epsilon)$, $\mathfrak{c}_*(L_{\bar{g}_\epsilon}) \le \mathfrak{c}_*(L_g)$ and $g \le \bar{g}_\epsilon$. Furthermore, there is also an increasing $\underline{g}_\epsilon$ function meeting **(M)** with positive equilibrium $\kappa_- \in (m_K - \epsilon, m_K)$ and $\mathfrak{c}_*(L_{\underline{g}_\epsilon}) \le \mathfrak{c}_*(L_g)$ such that: $\underline{g}_\epsilon(x) \le g(x)$ for $x \in [0, K+\epsilon]$.

Now, if $\bar{v}(t)$ is the homogenous solution of (17) replacing g by $\bar{g}_\epsilon$ with initial datum $v_0(s) = q, s \in [-h, 0]$, and $c > \mathfrak{c}_*(L_g)$ then by Lemma 21 and the global stability of κ_+ there is a number $T > 0$ such that

$$v(t,z) \le \bar{v}(t) \le K + \epsilon \quad \text{for all} \quad (t,z) \in [T, +\infty) \times \mathbb{R}. \tag{63}$$

Next, by (63)

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} v(t,z) =: v_\infty < \infty,$$

and by denoting $\kappa_* := \min_{u\in[\kappa,u_\infty]} g(u)$ we take an increasing function g_∞ satisfying **(M)** with equilibrium $\kappa_\infty \in (\kappa_* - \epsilon, \kappa_*)$, $\mathfrak{c}_*(L_{g_\infty}) \le \mathfrak{c}_*(L_g)$ and $g_\infty(u) \le g(u)$ for all $u \in [0, u_\infty]$. Next, without loss of generality we take $q_\infty := \kappa_\infty - q_0 > 0$. Then, by [1, Theorem 3], (21) and (22) there exist a monotone wavefront ϕ_c^∞ to (17) (with nonlinearity g_∞) such that

$$\phi_c^\infty(z) - q\eta_b(z) \le v_0(s,z) \quad (s,z) \in [-h, 0] \times \mathbb{R} \tag{64}$$

thus by [32, Lemma 2.1] there are $C_0 > 0$ and $\gamma \ge 0$ such that

$$\phi_c^\infty(z - C_0 q) - q^{-\gamma t}\eta_b(z) \le v_\infty(t,z) \quad (t,z) \times [-h, \infty) \times \mathbb{R} \tag{65}$$

where $v_\infty(t,z)$ is the solution generated by the initial datum $v(s,z)$ to Eq. (17). Now, by applying Lemma 21 with $D = [0, u_\infty]$

$$\phi_c^\infty(z - C_0 q) - q^{-\gamma t}\eta_b(z) \le v(t,z) \quad (t,z) \times [-h, \infty) \times \mathbb{R}.$$

So, there are z_0' and $q_0' > 0$ such that

$$v(t,z) \ge q_0' > 0 \quad (t,z) \in [-h, \infty) \times [z_0', \infty) \tag{66}$$

Otherwise, denoting $\underline{v}(t,z)$ the solution of (17) replacing g by $\underline{g}_\epsilon$ with initial data $\underline{v}_0(s,z) = v(s + T + h, z)$. Due to (66) and Remark 19 the initial datum $\underline{v}_0$ satisfies (21) and (22). Next, if we denote by $\underline{v}(t)$ the homogenous solution of (17) replacing g by $\underline{g}_\epsilon$ with initial datum $v_0(s) = K + \epsilon, s \in [-h, 0]$, then by [30, Corollary 2.2,p.82] $\underline{v}(t)$ converges monotonically to κ_-, therefore

$$\underline{v}(t,z) \le \underline{v}(t) \le K + \epsilon \quad \text{for all } (t,z) \in [-h, +\infty) \times \mathbb{R}$$

So, for $c > c(L_g)$ by Lemma 21 (with $D = [0, K+\epsilon]$), Proposition 7 and [33, Theorem 1] there is a wavefront $\underline{\phi}_c$ and $T_0 > 0$ such that

$$m_K - \epsilon \le \underline{\phi}_c(z) + \epsilon/2 \le \underline{v}(t,z) \le v(t,z) \quad \text{for all} \quad (t,z) \in [T_0, \infty)^2. \tag{67}$$

Thus there is T_v such that the function $\tilde{v}(t,z) := v(t + T_v + h, z)$ satisfies (54) with $b = t_v + ch$ and $I = \mathcal{I}_\epsilon$. Analogously, for some T_{ψ_c} we have $\psi_c(z) \in \mathcal{I}_\epsilon$ for all $z \ge T_{\psi_c}$. Finally, by applying Theorem 18 we conclude (24) with $C := \max_{z\in\mathbb{R}} \eta_b(z)/\eta_{t_0+ch}(z)$ where $t_0 := \max\{T_v, T_{\psi_c}\}$. □

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