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Energy release rate and stress intensity factors in planar elasticity in presence of smooth cracks

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Abstract. In this work we first analyze the singular behavior of the displacement u of a linearly elastic body in dimension 2 close to the tip of a smooth crack, extending the well-known results for straight fractures to general smooth ones. As conjectured by Griffith (Phys Eng Sci 221:163–198, 1921), u behaves as the sum of an H^2-function and a linear combination of two singular functions whose profile is similar to the square root of the distance from the tip. The coefficients of the linear combination are the so called stress intensity factors. Afterwards, we prove the differentiability of the elastic energy with respect to an infinitesimal fracture elongation and we compute the energy release rate, enlightening its dependence on the stress intensity factors.

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1. Introduction

In this paper we are concerned with the stability properties of cracks in *brittle* materials. We consider a linear elastic, isotropic, and homogeneous body, whose reference configuration is represented by an infinite cylinder of the form $\overline{\Omega} \times \mathbb{R}$, where the cross section $\Omega \subseteq \mathbb{R}^2$ is an open bounded set with Lipschitz boundary $\partial\Omega$. For simplicity of exposition, we also assume that the origin 0 belongs to Ω. In the setting of plane elasticity, we assume that the applied boundary and volume forces produce a horizontal displacement, namely the deformed configuration is described by

$$\overline{\Omega} \times \mathbb{R} \ni (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3) + (u_1(x_1, x_2), u_2(x_1, x_2), 0).$$

This assumption allows us to work in the planar domain Ω.

As usual, the behavior of the elastic material is fully described by the *elasticity tensor* $\mathbb{C}$, which is interpreted as a linear operator on the space $\mathbb{M}^2_{sym}$ of squared symmetric matrices of order 2. Under our assumptions, $\mathbb{C}$ can be expressed in terms of the so called *Lamé coefficients* $\lambda, \mu \in \mathbb{R}$ of the material. Namely, for every $F \in \mathbb{M}^2_{sym}$ we have

$$\mathbb{C}F := \lambda \mathrm{tr}(F)\, \mathbf{I} + 2\mu F,$$

where $\mathbf{I}$ is the identity matrix and $\mathrm{tr}(F)$ denotes the trace of the matrix F. Under the conditions

$$\mu > 0, \qquad \lambda + \mu > 0,$$

we also have that $\mathbb{C}$ is positive definite (see, e.g., [13]).

The physical model of *brittle fracture* we deal with goes back to Griffith [10] and has been formulated in [9] in a variational language: the crack evolution is the result of the interplay between the energy released by the elastic body when the fracture increases and the energy needed to produce such a new crack. Therefore, if we assume that the elastic body Ω is fractured along a sufficiently smooth curve $\Gamma \subset \overline{\Omega}$, that no volume force is applied, and that no force is transmitted across the fracture lips (*traction free* condition), we define the *elastic energy* as

$$\mathcal{E}(u, \Gamma) := \frac{1}{2} \int_{\Omega \backslash \Gamma} \mathbb{C}Eu : Eu \, dx, \tag{1.1}$$

where $u \in H^1(\Omega \backslash \Gamma; \mathbb{R}^2)$ is the displacement field and Eu denotes the symmetric part of the gradient of u, i.e., $Eu := (Du + Du^T)/2$. In what follows, we will assume the crack set Γ to be a simple closed C^∞-curve in Ω with initial point belonging to $\partial\Omega$ and endpoint in the origin.

In this framework, our aim is first to study the regularity of the displacement of the body Ω at the equilibrium for a given crack set Γ and boundary datum $g \in H^1(\Omega \backslash \Gamma; \mathbb{R}^2)$. As usual, the equilibrium condition is expressed by the minimum problem

$$\min \{\mathcal{E}(u, \Gamma) : u \in H^1(\Omega \backslash \Gamma; \mathbb{R}^2),\ u = g \text{ on } \partial\Omega\}. \tag{1.2}$$

We denote by $\mathcal{E}_{min}(\Gamma)$ the value of the minimum. Clearly, a solution $u \in H^1(\Omega \backslash \Gamma; \mathbb{R}^2)$ to (1.2) exists unique and solves, in a variational sense, the PDE system

$$\begin{cases} \mathrm{div}\,(\mathbb{C}Eu) = 0 & \text{in } \Omega \backslash \Gamma, \\ u = g & \text{on } \partial\Omega, \\ (\mathbb{C}Eu)\nu_\Gamma = 0 & \text{on } \Gamma, \end{cases} \tag{1.3}$$

where ν_Γ is the unit normal vector to Γ. Notice that the first equation in (1.3) is intended in the sense of distributions, the second one in the sense of traces, and the third one in the duality $H^{-1/2}(\Gamma) - H^{1/2}(\Gamma)$.

Our main result is then the following: we show that the solution u to (1.2) is of class H^2 far from the crack tip, while close to the origin it exhibits a singularity, approaching the crack tip with a profile similar to the square root of the distance from the tip. Indeed, denoting by (ρ, θ) the usual polar

coordinates, we are able to prove that there exist two regular functions $\phi_1(\theta)$, $\phi_2(\theta)$ and two constants Q_1 and Q_2 such that

$$u - Q_1\rho^{1/2}\phi_1 - Q_2\rho^{1/2}\phi_2 \in H^2(\Omega'\backslash\Gamma; \mathbb{R}^2), \tag{1.4}$$

for every $\Omega' \subset\subset \Omega$. We emphasize that the functions $\rho^{1/2}\phi_i$, $i = 1, 2$, belong to $H^1(\Omega\backslash\Gamma; \mathbb{R}^2)$ but not to $H^2(\Omega\backslash\Gamma; \mathbb{R}^2)$. For this reason, they are referred to as the *singular solutions* of elasticity (see, e.g., [12] and Remark 2.6 for further discussions). Moreover, denoting with $\boldsymbol{\sigma}(u) := \mathbb{C}Eu$ the stress field associated to the deformation u, the decomposition (1.4) implies that $\boldsymbol{\sigma}(u)$ blows up as $\rho^{-1/2}$ in the origin, that is, the elastic body develops an infinite stress close to the crack tip, unless $Q_1 = Q_2 = 0$. Therefore, the constants Q_1 and Q_2, which depend only on the crack set Γ and are uniquely determined, are called *stress intensity factors*. As we discuss in Remark 2.7, Q_1 and Q_2 are related to Mode I and II of crack growth.

The singularity (1.4), which is a consequence of the linearization (1.3) of the elasticity system, was already formally noticed by Griffith in [10] while studying the behavior of the stress field in elastic bodies with elliptic cracks degenerating into lines. After [10], the decomposition (1.4) has been widely studied: in [11,12] Grisvard proved it rigorously for domains with a straight crack. The proof is based on sophisticated tools in differential operator theory and complex analysis. Roughly speaking, problem (1.3) is recast in the variational form

$$\begin{cases} \int_{\Omega\backslash\Gamma} \mathbb{C}Eu : Ev \, dx = 0 & \text{for every } v \in H_0^1(\Omega\backslash\Gamma; \mathbb{R}^2), \\ u - g \in H_0^1(\Omega\backslash\Gamma; \mathbb{R}^2), \end{cases}$$

and, because of the absence of volume forces, it is shown to be equivalent to a biharmonic problem in the fractured planar domain

$$\begin{cases} \Delta^2 w = 0 & \text{in } \Omega\backslash\Gamma, \\ w = 0 & \text{on } \Gamma, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma, \end{cases} \tag{1.5}$$

where w is the so called *Airy function* satisfying

$$D^2 w = \boldsymbol{\sigma}(u)^{\perp} := \begin{pmatrix} \boldsymbol{\sigma}_{22}(u) & -\boldsymbol{\sigma}_{12}(u) \\ -\boldsymbol{\sigma}_{12}(u) & \boldsymbol{\sigma}_{11}(u) \end{pmatrix}. \tag{1.6}$$

The behavior of w close to the crack tip is then investigated following the Kondrat'ev method in suitable weighted Sobolev spaces (see [15–17] for more details). These techniques make strong use of the particular geometry of the crack set near its tip. Indeed, being Γ a straight line, with the help of a suitable change of variable and of a Fourier transform, system (1.5) reduces to the study of the singularities of an ODE with constant coefficients and whose solutions are explicit. The same strategy in the case of a curved crack (even if smooth) would lead to the study of a more complicated ODE with coefficients depending on the fracture and whose solutions could not be easily computed. Therefore, another approach is in order.

In the context of in-plane linearized elasticity, we also mention the work [3], in which the authors find a characterization of the form (1.4), by studying the asymptotic behavior of the displacement near the crack tip, under very weak assumptions on the fracture: Γ is assumed to be a closed set whose density with respect to the Hausdorff $\mathcal{H}^1$-measure at the tip equals $1/2$. The idea of add-crack of density $1/2$ at the tip, and the corresponding generalized notion of energy release rate, have been introduced in [5], in the case of a straight initial crack. The strategy in [3] is to reduce the equilibrium problem (1.2) (or (1.3)) to the biharmonic system (1.5) and study the blow-up limit of a proper rescaling of w near the origin. Such a blow-up approach was successfully proposed in [6] for the anti-plane case, that is, when the elasticity system (1.3) reduces to the Poisson equation. However, in [3] the lack of regularity of the crack does not allow one to determine in a unique way the blow-up limit and, hence, the stress intensity factors Q_i.

In this paper we follow the lines of [11,18], in which a similar problem is tackled in the simplified setting of anti-plane linearized elasticity. Thanks to classical results, we reduce the elasticity system to the biharmonic problem (1.5) for the associated Airy function $w \in H^2(\Omega\backslash\Gamma)$ in the fractured domain $\Omega\backslash\Gamma$, where the crack set Γ, as mentioned above, is assumed to be of class C^∞. As in [18], we perform a change of variables that straightens Γ close to the tip and, clearly, perturbs the coefficients of the biharmonic equation. The C^∞-regularity of the fracture allows us to apply the regularity and pencil operator theories in weighted Sobolev spaces (see Sect. 3 for a short discussion and [16,17] for full details) in order to show that w behaves, close to the origin, as $\rho^{3/2}$. The splitting (1.4) can then be deduced by the relation (1.6). We stress that, similar to [18], with our approach the stress intensity factors Q_1 and Q_2 are uniquely determined (see Theorem 2.5).

Going back to the comparison with Griffith's theory [10], we have already discussed the postulated relation between the elastic energy released during the crack extension and the energy dissipated in order to create a new fracture surface. In the mathematical language, the above interaction is expressed in terms of the so called *energy release rate*, i.e., the opposite of the derivative of the elastic energy (1.1) with respect to the crack elongation. More precisely, if we assume that we are given an increasing family Γ_s of sufficiently regular crack sets in Ω parametrized by their arc-length s, the energy release rate $\mathcal{G}(\Gamma_s)$ is formally defined as

$$\mathcal{G}(\Gamma_s) := -\frac{\mathrm{d}\mathcal{E}_{min}(\Gamma_s)}{\mathrm{d}s}.$$

In other words, $\mathcal{G}(\Gamma_s)$ represents the amount of elastic energy released as a consequence of an infinitesimal elongation of the fracture. Griffith's original model is therefore based on the comparison between $\mathcal{G}(\Gamma_s)$ and the *toughness* of the material, a positive parameter that depends on the physical properties of the material and represents the energy needed to produce a new fracture of length one.

Exploiting the decomposition result (1.4), we show the differentiability of the energy $s \mapsto \mathcal{E}_{min}(\Gamma_s)$ and we explicitly compute its derivative. This allows us to show the dependence of the energy release rate $\mathcal{G}(\Gamma_s)$ on the stress intensity factors $Q_1(\Gamma_s)$ and $Q_2(\Gamma_s)$ at the configuration Γ_s, and on the Lamé coefficients λ, μ of the material. Namely, we get (see Theorem 4.1)

$$\mathcal{G}(\Gamma_s) = 2\pi(Q_1^2(\Gamma_s) + Q_2^2(\Gamma_s))\mu(\lambda + \mu)(\lambda + 2\mu). \tag{1.7}$$

In particular, we remark that $\mathcal{G}(\Gamma_s)$ depends only on the actual crack set Γ_s. Moreover, in the proof of Theorem 4.1 we actually show that the energy release rate admits an integral representation involving the displacement at the equilibrium and on the geometry of the crack Γ_s. As a consequence, $\mathcal{G}$ turns out to be continuous with respect to the Hausdorff convergence of the fracture set, under suitable regularity constraints. We refer to Remark 4.2 and Corollary 4.3 for further details.

We mention that in the anti-plane setting a similar result was obtained in [18] with weaker regularity of the crack set, and then applied in [19] to the study of the vanishing viscosity approach to the quasi-static fracture evolution problem. In [7], instead, the authors computed, always in the anti-plane context, higher order derivatives of the elastic energy for a straight crack set.

1.1. Plan of the paper

In Sect. 2 we present the mechanical problem, we show the relationship with the biharmonic problem in the plane (1.6), and we state the decomposition result (1.4) (see Theorem 2.5), together with the corresponding property for the Airy function (Theorem 2.4). Section 3 is devoted to the proof of the decomposition result: in Theorems 3.11 and 3.12 we show the singular behavior of the Airy function exploiting the already mentioned regularity theory of PDEs in polygonal domains. Finally, in Sect. 4 we prove the differentiability of the energy with respect to an infinitesimal increment of fracture length and, taking advantage of the decomposition (1.4) of the displacement, we compute the energy release rate, enlightening its dependence on the stress intensity factors.

2. The planar elasticity system

In this section we specify the standing assumptions, we show the relation between the elasticity system (1.3) and the biharmonic equation (1.5), and we prove the decomposition result (1.4) for the displacement.

Let us consider an open bounded subset Ω of $\mathbb{R}^2$ containing the origin 0 and with Lipschitz boundary $\partial\Omega$. The set Ω represents the reference configuration of the horizontal cross section of a cylindrical vertical elastic body. As already pointed out in the Introduction, we assume that the applied boundary and volume forces produce a horizontal displacement, thus we are allowed to work on the planar cross section, without mentioning the vertical component, which is unchanged during the deformation. The material under consideration is assumed to be isotropic, homogeneous, and linearly elastic, so that its

behavior is completely described by the elasticity tensor $\mathbb{C} \colon \mathbb{M}^2_{sym} \to \mathbb{M}^2_{sym}$, where $\mathbb{M}^2_{sym}$ denotes the space of symmetric matrices of order 2. In particular, $\mathbb{C}$ is independent of the position $x \in \Omega$ and can be expressed in terms of the Lamé coefficients of the material $\lambda, \mu \in \mathbb{R}$ by

$$\mathbb{C}F := \lambda \operatorname{tr}(F)\, \mathbf{I} + 2\mu F \quad \text{for every } F \in \mathbb{M}^2_{sym}, \tag{2.1}$$

where $\operatorname{tr}(F)$ denotes the trace of the matrix F. We work under the typical hypotheses that $\mu > 0$ and $\lambda + \mu > 0$, which ensure the positive definiteness of the elasticity tensor $\mathbb{C}$.

Furthermore, we suppose that the elastic body Ω presents a crack along a smooth set Γ, which is a simple and closed C^∞-curve, having one endpoint on $\partial\Omega$ and the other one, the crack tip, in the interior of Ω. Without loss of generality, we assume that the crack tip is at the origin 0 and that, if we choose the arc length parametrization of the crack set which starts on $\partial\Omega$ and ends at the origin, the unit tangent vector to Γ at 0 is $-e_1 = (-1, 0)$. These assumptions are summarized in Fig. 1.
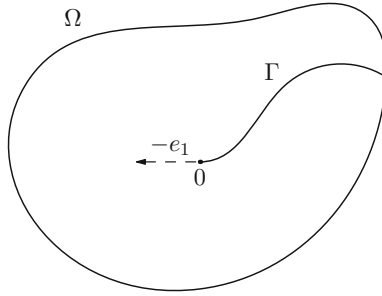

FIGURE 1. *The crack tip is assumed to be at the origin* 0, *and the tangent vector to* Γ *at* 0 *is assumed to be* $-e_1$

In this framework, the energy of Ω subject to a displacement $u \in H^1(\Omega\backslash\Gamma; \mathbb{R}^2)$ is given by the functional

$$\mathcal{E}(u, \Gamma) := \frac{1}{2} \int_{\Omega\backslash\Gamma} \mathbb{C}Eu : Eu \, dx, \tag{2.2}$$

where the colon denotes the scalar product between matrices and Eu stands for the symmetric part of the gradient of u, namely $Eu := (Du + Du^T)/2$.

Given a Dirichlet boundary datum $g \in H^1(\Omega\backslash\Gamma; \mathbb{R}^2)$, we consider the usual elasticity equilibrium problem

$$\min\{\mathcal{E}(u, \Gamma) : u \in H^1(\Omega\backslash\Gamma; \mathbb{R}^2),\, u = g \text{ on } \partial\Omega\}. \tag{2.3}$$

Clearly, a solution to (2.3) exists and is unique, by strict convexity of the integral functional $\mathcal{E}(\cdot, \Gamma)$, ensured by the positive definiteness of $\mathbb{C}$. Moreover, (2.3) is equivalent to the PDE system

$$\begin{cases} \operatorname{div}\boldsymbol{\sigma}(u) = 0 & \text{in } \Omega\backslash\Gamma, \\ u = g & \text{on } \partial\Omega, \\ \boldsymbol{\sigma}(u)\nu_\Gamma = 0 & \text{on } \Gamma, \end{cases} \tag{2.4}$$

where $\boldsymbol{\sigma}(u) := \mathbb{C}Eu$ denotes the stress tensor induced by the displacement u and ν_Γ is the unit normal vector to Γ.

In order to describe the regularity of u, we first reduce (2.4) to the biharmonic problem. To this aim we make use of the so called *Airy function* (see, e.g., [3]), whose construction is possible thanks to the absence of external volume forces.

Lemma 2.1. *Let* $u \in H^1(\Omega\backslash\Gamma; \mathbb{R}^2)$ *be the solution of the minimum problem* (2.3). *Then there exists a function* $w_u \in H^2(\Omega\backslash\Gamma)$ *solution of the system*

$$\begin{cases} \Delta^2 w = 0 & in\ \Omega\backslash\Gamma, \\ w = 0 & on\ \Gamma, \\ \dfrac{\partial w}{\partial \nu} = 0 & on\ \Gamma, \end{cases} \tag{2.5}$$

and such that

$$\boldsymbol{\sigma}(u) = \begin{pmatrix} \mathrm{D}_{22}w_u & -\mathrm{D}_{12}w_u \\ -\mathrm{D}_{12}w_u & \mathrm{D}_{11}w_u \end{pmatrix}. \tag{2.6}$$

Proof. Even if the statement is well-known, for the benefit of the reader, we recall the proof. Since $\Omega\backslash\Gamma$ is simply connected, the divergence free condition on $\boldsymbol{\sigma}(u)$ in (2.4) implies that

$$\boldsymbol{\sigma}(u) = \begin{pmatrix} -\mathrm{D}_2\phi & \mathrm{D}_1\phi \\ -\mathrm{D}_2\psi & \mathrm{D}_1\psi \end{pmatrix},$$

for some $\phi, \psi \in H^1(\Omega\backslash\Gamma)$. The same trick can be applied a second time: since $\boldsymbol{\sigma}(u)$ is symmetric, the vector field (ϕ, ψ) is divergence free in $\Omega\backslash\Gamma$, thus there exists $w \in H^2(\Omega\backslash\Gamma)$ such that $\phi = -\mathrm{D}_2 w$ and $\psi = \mathrm{D}_1 w$, namely (2.6) holds true. In particular, w is biharmonic: given a test function $\varphi \in \mathcal{D}(\Omega\backslash\Gamma)$, we have

$$\langle \Delta^2 w, \varphi \rangle = \int_{\Omega\backslash\Gamma} \nabla^2 w : \nabla^2\varphi \,\mathrm{d}x = \int_{\Omega\backslash\Gamma} \boldsymbol{\sigma}(u) : (\nabla^2\varphi)^\perp \,\mathrm{d}x = \int_{\Omega\backslash\Gamma} \boldsymbol{\sigma}(u) : \nabla V \,\mathrm{d}x$$
$$= -\langle \mathrm{div}\boldsymbol{\sigma}(u), V \rangle = 0,$$

where V is a suitable vector field in $\mathcal{D}(\Omega\backslash\Gamma; \mathbb{R}^2)$, the brackets denote the duality product in the distributional sense, and

$$M^\perp := \begin{pmatrix} M_{22} & -M_{12} \\ -M_{12} & M_{11} \end{pmatrix}, \quad \text{for } M \in \mathbb{M}^2.$$

Here we have exploited again the fact that $\Omega\backslash\Gamma$ is simply connected: since the rows of $(\nabla^2\varphi)^\perp$ are irrotational, there exists a potential V such that $(\nabla^2\varphi)^\perp = \nabla V$.

As it is clear from its construction, because of three constants of integration, the function w is not unique. In particular, in view of the boundary condition $\boldsymbol{\sigma}(u)\nu_\Gamma = 0$, we infer that ∇w is constant on Γ; therefore, by choosing such a constant to be zero, we may impose w and its normal derivative to be zero on the crack. To sum up, w solves (2.5). $\qquad\square$

As an immediate consequence, we have the following result.

Proposition 2.2. *Let $u \in H^1(\Omega\backslash\Gamma;\mathbb{R}^2)$ be the solution of* (2.3) *and let $w_u \in H^2_0(\Omega\backslash\Gamma)$ be the associated Airy function found in Lemma* 2.1. *Then, for every open subset $\Omega' \subseteq \Omega$, $u \in H^2(\Omega'\backslash\Gamma;\mathbb{R}^2)$ if and only if $w_u \in H^3(\Omega'\backslash\Gamma)$.*

Proof. Let Ω' be an open subset of Ω. The statement follows immediately from the identity (2.6) and from the invertibility of the elasticity tensor $\mathbb{C}$. Indeed, the latter allows us to express the elements $\mathrm{E}_{ij}u$ of the symmetric gradient $\mathrm{E}u$ in terms of the stress components $\boldsymbol{\sigma}_{ij}(u)$. Therefore, taking the derivatives $\mathrm{D}_k\mathrm{E}_{ij}(u)$ and combining them, we obtain that all the third derivatives of u belong to $L^2(\Omega'\backslash\Gamma)$ whenever $w_u \in H^3(\Omega'\backslash\Gamma)$. The converse implication is obvious. $\qquad\square$

The regularity of the Airy function far from the crack tip is described in the following proposition, and follows from standard regularity theory in fourth order systems (see [4]).

Proposition 2.3. *Let $w \in H^2(\Omega\backslash\Gamma)$ be a solution of* (2.5). *Let Ω' and Ω'' be two open subsets of Ω such that $0 \in \Omega' \subset \Omega'' \subset\subset \Omega$. Then, for every $\varphi \in C^\infty_c(\Omega''\backslash\overline{\Omega'})$ we have $\varphi w \in H^3(\Omega\backslash\Gamma)$.*

By combining Propositions 2.2 and 2.3, we infer that the displacement u belongs to H^2 far from the tip. In order to complete the regularity analysis, we shall describe the behavior of w_u close to the origin. To do this, we introduce the polar coordinates system (ρ, ϑ), where ρ stands for the usual distance from the origin, and ϑ is the determination of the angle between $x - 0$ and the vector e_1 continuous in $\Omega\backslash\Gamma$. Furthermore, we set

$$\phi_1(\rho,\vartheta) := \rho^{3/2}\left(\frac{2}{3}\sin\frac{3\vartheta}{2} - 2\sin\frac{\vartheta}{2}\right) \quad \text{and} \quad \phi_2(\rho,\vartheta) := \rho^{3/2}\left(\cos\frac{3\vartheta}{2} - \cos\frac{\vartheta}{2}\right). \tag{2.7}$$

With this notation, we have the following result, whose proof is postponed to Sect. 3.

Theorem 2.4. *Let Γ be a C^∞-crack set with a vertex in the origin, and let $w_u \in H^2(\Omega\backslash\Gamma)$ be the Airy function determined in Lemma* 2.1. *Then, there exist unique two constants C_1 and C_2 such that*

$$w_u - C_1\phi_1 - C_2\phi_2 \in H^3(\Omega'\backslash\Gamma) \tag{2.8}$$

for every $\Omega' \subset\subset \Omega$.

Thanks to Proposition 2.3 and Theorem 2.4, we deduce the decomposition result for the displacement u, solution to (2.3).

Theorem 2.5. *Let $\Gamma \subseteq \Omega$ be a C^∞-crack set with a vertex in the origin and let $u \in H^1(\Omega\backslash\Gamma;\mathbb{R}^2)$ be the solution of* (2.3). *Let us set*

$$\psi_1(\vartheta) := \begin{pmatrix} \dfrac{\lambda+\mu}{2}\cos\dfrac{3\vartheta}{2} - \dfrac{5\lambda+9\mu}{2}\cos\dfrac{\vartheta}{2} \\[12pt] \dfrac{\lambda+\mu}{2}\sin\dfrac{3\vartheta}{2} + \dfrac{\lambda-3\mu}{2}\sin\dfrac{\vartheta}{2} \end{pmatrix},$$

$$\psi_2(\vartheta) := \begin{pmatrix} -\dfrac{\lambda+\mu}{2}\sin\dfrac{3\vartheta}{2} - \dfrac{\lambda+5\mu}{2}\sin\dfrac{\vartheta}{2} \\[2ex] \dfrac{\lambda+\mu}{2}\cos\dfrac{3\vartheta}{2} + \dfrac{3\lambda+7\mu}{2}\cos\dfrac{\vartheta}{2} \end{pmatrix}, \tag{2.9}$$

and

$$\Phi_1(\rho,\vartheta) := \rho^{1/2}\psi_1(\vartheta), \quad \Phi_2(\rho,\vartheta) := \rho^{1/2}\psi_2(\vartheta). \tag{2.10}$$

Then, there exist unique two constants Q_1 *and* Q_2 *such that*

$$u - Q_1\Phi_1 - Q_2\Phi_2 \in H^2(\Omega'\backslash\Gamma;\mathbb{R}^2) \tag{2.11}$$

for every $\Omega' \subset\subset \Omega$.

Remark 2.6. In the language of Grisvard [12], if Γ is straight, say $\Gamma \subseteq \{x_2 = 0, x_1 \geq 0\}$, the functions Φ_1 and Φ_2 are the *singular solutions* of the elasticity system. Indeed, for every $R > 0$, Φ_1 and Φ_2 belong to $H^1(\mathrm{B}_R\backslash\{x_2 = 0, x_1 \geq 0\};\mathbb{R}^2)\backslash H^2(\mathrm{B}_R\backslash\{x_2 = 0, x_1 \geq 0\};\mathbb{R}^2)$ and solve

$$\begin{cases} \operatorname{div}\boldsymbol{\sigma}(\Phi) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2\backslash\{x_2 = 0\}), \\ \boldsymbol{\sigma}(\Phi)e_2 = 0 & \text{on } \{x_2 = 0\}, \end{cases}$$

where $e_2 := (0,1)$.

Remark 2.7. In literature the stress intensity factors Q_1 and Q_2 are related to Mode-I and Mode-II crack growth, respectively: indeed, as it can be seen from formulas (2.9)–(2.10), the function Φ_1 corresponds to a pure opening of the fracture, while Φ_2 describes a sliding of the fracture lips in the plane of Ω (see, for instance, [21]).

We now prove Theorem 2.5.

Proof of Theorem 2.5. By a direct computation it can be shown that

$$\mu(\lambda+\mu)\mathrm{D}^2\phi_1 = -\boldsymbol{\sigma}(\Phi_2)^\perp \quad \text{and} \quad 2\mu(\lambda+\mu)\mathrm{D}^2\phi_2 = \boldsymbol{\sigma}(\Phi_1)^\perp, \tag{2.12}$$

where Φ_1 and Φ_2 have been defined in (2.9) and (2.10), respectively.

Let us fix $\Omega' \subset\subset \Omega$. By Theorem 2.4 we know that there exist two constants C_1 and C_2 such that

$$w_u - C_1\phi_1 - C_2\phi_2 \in H^3(\Omega'\backslash\Gamma). \tag{2.13}$$

Exploiting the equalities (2.12) and Proposition 2.2, we easily deduce (2.11) for

$$Q_1 := \frac{C_2}{2\mu(\lambda+\mu)} \quad \text{and} \quad Q_2 := -\frac{C_1}{\mu(\lambda+\mu)}.$$

The uniqueness follows by the uniqueness of the decomposition (2.13). $\qquad\square$

Remark 2.8. The hypothesis of constant Lamé coefficients is here important to determine the functions ψ_1 and ψ_2 in (2.9) and to prove the relation (2.12), involving the functions ϕ_i and Φ_i, $i = 1, 2$. Indeed, if λ and μ were not constant, in (2.12) we would have some extra terms depending on their derivatives, so that the splitting (2.11) would not follow.

We conclude the section by briefly describing the strategy of the proof of Theorem 2.4. First, we localize the biharmonic system (2.5) around the crack tip by multiplying the Airy function w_u by a cut-off function $\eta \in C_c^\infty(\Omega)$, which is identically 1 in a neighborhood of 0 and has compact support in Ω. The product $w_{u,\eta} := \eta w_u$ solves

$$\begin{cases} \Delta^2 w = f & \text{in } \Omega \backslash \Gamma, \\ w \in H_0^2(\Omega \backslash \Gamma), \end{cases} \tag{2.14}$$

where $f \in H^{-1}(\Omega \backslash \Gamma)$ is such that $0 \notin \operatorname{supp}(f)$. System (2.14) is obtained by taking into account that w_u solves (2.5) and by splitting its bilaplacian as $\Delta^2 w_{u,\eta} = \eta \Delta^2 w_u + [\Delta^2, \eta] w_u$, where $[\Delta^2, \eta]$ is a differential operator of order 3 whose coefficients contain the derivatives of the cut-off function η. Note that, by construction, the forcing term f depends on the cut-off function chosen.

As a second step, we introduce a change of variables that straightens the crack close to the origin and keeps the domain Ω unchanged far from it. Without loss of generality, we may assume that in the open ball B_δ of center 0 and radius $\delta > 0$ the crack set Γ can be extended to a C^∞-curve Λ in such a way that Λ is the graph of a C^∞-function $\zeta = \zeta(x_1)$ in B_δ with $\zeta'(0) = 0$. We consider $S \colon B_\delta \to \mathbb{R}^2$ given by

$$S(x) := (l(x_1, \zeta(x_1)), x_2 - \zeta(x_1)), \tag{2.15}$$

being $l(x_1, \zeta(x_1)) := \int_0^{x_1} \sqrt{1 + \zeta'(t)^2} \, dt$ the signed length of the portion of Γ between the origin and the point $(x_1, \zeta(x_1))$. It is clear that S is a C^∞-diffeomorphism such that $S(\Gamma \cap B_\delta) \subseteq \{x_2 = 0, \, x_1 \geq 0\}$, that is, S straightens the crack close to the tip. Moreover, we can extend S to a C^∞-diffeomorphism on the whole domain Ω in such a way that S is the identity in a neighborhood of $\partial \Omega$. Note that, thanks to the regularity of ζ and to the conditions $S(0) = 0$ and $\nabla S(0) = \mathbf{I}$, we have $\|\nabla S - \mathbf{I}\|_{L^\infty(\Omega)} < C\delta$, for some constant $C > 0$. We recall that $\mathbf{I}$ denotes the identity matrix.

By applying the change of variables S, system (2.14) becomes a fourth order problem set in the fractured domain $\Omega \backslash \widehat{\Gamma}$, with $\widehat{\Gamma} := S(\Gamma)$ horizontal near the origin, of the form

$$\begin{cases} (\Delta^2 + \mathcal{B})w = \widehat{f} & \text{in } \Omega \backslash \widehat{\Gamma}, \\ w \in H_0^2(\Omega \backslash \widehat{\Gamma}), \end{cases} \tag{2.16}$$

where $\mathcal{B}$ is a fourth order differential operator with C^∞-coefficients and $\widehat{f}$ is a suitable element of $H^{-1}(\Omega \backslash \widehat{\Gamma})$ such that $0 \notin \operatorname{supp}(\widehat{f})$.

Remark 2.9. For the sake of completeness, we show here the variational formulation of (2.16), which can be directly deduced by the variational formulation of (2.14): for every $v \in H_0^2(\Omega \backslash \widehat{\Gamma})$,

$$\int_{\Omega \backslash \widehat{\Gamma}} \left[M^T \nabla^2 w \, M + Q \, \nabla w \right] : \left[M^T \nabla^2 v \, M + Q \, \nabla v \right] \quad \det \nabla S(x)^{-1} \, dx$$
$$= \langle f, v \circ S \rangle, \tag{2.17}$$

where the brackets in the right-hand denote the duality product in H^{-1}-H_0^1 over $\Omega\backslash\Gamma$, and M, $Q\nabla v$ (and, similarly, $Q\nabla w$) are the matrices

$$M(x) := \nabla\mathrm{S}(\mathrm{S}^{-1}(x)),$$

$$(Q\nabla v)_{ij}(x) := \sum_k (\nabla^2\mathrm{S}_k)_{ij}(\mathrm{S}^{-1}(x))\mathrm{D}_k v(x) \quad \text{for every } x \in \Omega\backslash\widehat{\Gamma}. \tag{2.18}$$

In view of (2.17) and (2.18) we could also deduce the explicit formula of the differential operator $\mathcal{B}$, but we limit ourselves to notice that its coefficients of order 4 are δ-close to 0 in $\Omega\backslash\widehat{\Gamma}$ and that $\mathcal{B}(0)$ contains only terms of order less than or equal to 3.

Remark 2.10. We mention that the results of Theorems 2.4 and 2.5 do not depend on the choice of the extension Λ, which is only needed to define in a rigorous way the change of variable (2.15).

3. The Airy function close to the tip

This section is devoted to the proof of Theorem 2.4. In view of the discussion in Sect. 2, we will deduce the decomposition (2.8) by investigating the behavior of the solution of the auxiliary "perturbed" biharmonic problem (2.16), related to the Airy function w_u through the localized system (2.14) and the change of variables S defined in (2.15). We refer to Theorem 3.12 below for the precise statement.

We start by recalling some notation and results concerning differential operators with smooth coefficients in domains with conical points. Since we are interested in an asymptotic expansion close to the tip 0 of the straightened crack $\widehat{\Gamma}$ in the domain $\Omega\backslash\widehat{\Gamma}$, we will adapt all the results to our particular setting. For further details and more general assumptions we refer to [16, Section 6].

Since S is such that $\mathrm{S}(\Gamma \cap B_\delta) \subseteq \{x_2 = 0,\, x_1 \geq 0\}$, it is convenient to represent, in a neighborhood of the origin, the domain $\Omega\backslash\widehat{\Gamma}$ in the polar coordinates $(\rho, \theta) \in [0, +\infty) \times (0, 2\pi)$, where ρ denotes the distance of $x \in \mathbb{R}^2$ from the origin and θ is the usual angle between the vector $x-0$ and e_1, having its discontinuity line on the set $\{x_2 = 0,\, x_1 \geq 0\}$. For simplicity of notation, we denote by $\mathcal{K}_{\mathfrak{r}}$ a neighborhood of the crack tip in $\Omega\backslash\widehat{\Gamma}$ described by (ρ, θ), namely,

$$\mathcal{K}_{\mathfrak{r}} := \{(\rho, \theta) : 0 \leq \rho \leq \mathfrak{r},\, \theta \in (0, 2\pi)\}$$

for a suitable $\mathfrak{r} > 0$. We also set $\mathcal{K}_\infty := \mathbb{R}^2\backslash\{x_2 = 0,\, x_1 \geq 0\}$.

Remark 3.1. We notice that the angle θ used above and the angle ϑ introduced in (2.7) do not coincide, since they have two different lines of discontinuity ($\{x_2 = 0,\, x_1 \geq 0\}$ and Γ, respectively).

We now define a family of weighted Sobolev spaces in $\mathcal{K}_{\mathfrak{r}}$. The same definition can be given in $\mathcal{K}_\infty$.

Definition 3.2. For every integer $\ell \geq 0$ and every $\beta \in \mathbb{R}$ we set

$$V_\beta^\ell(\mathcal{K}_\mathfrak{r}) := \left\{ v \colon \mathcal{K} \to \mathbb{R} \colon \int_{\mathcal{K}_\mathfrak{r}} \rho^{2(\beta-\ell+|\alpha|)} |\mathrm{D}^\alpha v|^2 \, \mathrm{d}x < +\infty \text{ for every multi-index } \alpha, \, |\alpha| \leq \ell \right\}.$$

The space $V_\beta^\ell(\mathcal{K}_\mathfrak{r})$ is a Hilbert space endowed with the norm

$$\|v\|_{V_\beta^\ell(\mathcal{K}_\mathfrak{r})}^2 := \sum_{|\alpha| \leq \ell} \int_{\mathcal{K}_\mathfrak{r}} \rho^{2(\beta-\ell+|\alpha|)} |\mathrm{D}^\alpha v|^2 \, \mathrm{d}x.$$

For $\ell < 0$, the space $V_\beta^\ell(\mathcal{K}_\mathfrak{r})$ is defined as the dual space of $V_{-\beta}^{-\ell}(\mathcal{K}_\mathfrak{r})$.

Remark 3.3. We notice that, in view of [12, Theorem 1.2.16], we have that $H_0^2(\mathcal{K}_\mathfrak{r}) \subset V_0^2(\mathcal{K}_\mathfrak{r})$. Moreover, every element of $V_0^{-1}(\mathcal{K}_\mathfrak{r})$ with compact support belongs to $H^{-1}(\mathcal{K}_\mathfrak{r})$.

The following embeddings hold true.

Proposition 3.4. *Let* $\beta_1, \beta_2 \in \mathbb{R}$ *and* $\ell_1 \geq \ell_2 \geq 0$ *be such that* $\beta_1 - \ell_1 \leq \beta_2 - \ell_2$. *Then* $V_{\beta_1}^{\ell_1}(\mathcal{K}_\mathfrak{r})$ *is continuously embedded in* $V_{\beta_2}^{\ell_2}(\mathcal{K}_\mathfrak{r})$.

The following two definitions are given for a general differential operator of order k

$$\mathrm{P}(x, \partial_x) := \sum_{|\alpha| \leq k} p_\alpha(x) \partial_x^\alpha \tag{3.1}$$

with coefficients $p_\alpha \in C^\infty(\Omega)$ (see, e.g., [17, Chapter 6]). We will later on specify them in our setting.

Definition 3.5. Given a differential operator $\mathrm{P}(x, \partial_x)$ as in (3.1), we define the *leading part* of P in the origin 0 as

$$\mathrm{P}^\circ(\partial_x) := \sum_{|\alpha| \leq k} \bar{p}_\alpha(0, \theta) \, \partial_x^\alpha,$$

where the coefficients $\bar{p}_\alpha \in C^\infty((0, +\infty) \times (0, 2\pi))$ are such that

$$p_\alpha(x) = \rho^{|\alpha|-k} \bar{p}_\alpha(\rho, \theta),$$

$$(\rho\partial_\rho)^j \partial_\theta^\gamma (\bar{p}_\alpha(\rho, \theta) - \bar{p}_\alpha(0, \theta)) \to 0 \qquad \text{as } \rho \to 0, \text{ for every } j, \gamma \in \mathbb{N}.$$

Definition 3.6. We say that a differential operator $\mathrm{P}(x, \partial_x)$ of the form (3.1) is δ-*admissible* in the origin if the following conditions are satisfied:

(a) the coefficients $p_\alpha(x)$ are of class C^∞ in $\overline{\Omega} \setminus \{0\}$;

(b) there exist $p_{\alpha,0} \in C^\infty([0, 2\pi])$ and $p_{\alpha,1} \in C^\infty((0, +\infty) \times [0, 2\pi])$ such that, in a neighborhood of the origin, we can write

$$p_\alpha(x) = \rho^{|\alpha|-k} \left(p_{\alpha,0}(\theta) + \rho^\delta p_{\alpha,1}(\rho, \theta) \right) ;$$

(c) for every pair of non negative integers j and γ, there exists a positive constant C such that

$$|(\rho\partial_\rho)^j \, \partial_\theta^\gamma \, p_{\alpha,1}(\rho, \theta)| \leq C \quad \text{for every } (\rho, \theta) \in (0, +\infty) \times [0, 2\pi].$$

Remark 3.7. If the coefficients p_α belong to $C^\infty(\overline{\Omega})$, then the operator $P(x, \partial_x)$ in (3.1) is δ-admissible with $\delta = 1$ and its leading part reads

$$P^\circ(\partial_x) := \sum_{|\alpha|=k} p_\alpha(0) \, \partial_x^\alpha.$$

The last definition we give here involves a general boundary condition problem

$$\begin{cases} L(x, \partial_x) \, u = f & \text{in } \Omega \backslash \widehat{\Gamma}, \\ C_k(x, \partial_x) \, u = g_k & \text{on } \partial\Omega \cup \widehat{\Gamma}, \ k = 1, \ldots, m, \end{cases} \tag{3.2}$$

where $L(x, \partial_x)$ is a differential operator of order $2m$ with coefficients in $C^\infty(\overline{\Omega})$ and $C_k(x, \partial_x)$ are differential operators of order $\mu_k < 2m$ with smooth coefficients. We rewrite the leading parts $L^\circ(\partial_x)$ and $C_k^\circ(\partial_x)$ of L and C_k in the origin 0, respectively, in the polar coordinates $(\rho, \theta) \in \mathcal{K}_{\mathfrak{r}}$ as follows:

$$L^\circ(\partial_x) = \rho^{-2m} \mathcal{L}^\circ(\theta, \partial_\theta, \rho\partial_\rho),$$
$$C_k^\circ(\partial_x) = \rho^{-\mu_k} \mathcal{C}_k^\circ(\theta, \partial_\theta, \rho\partial_\rho).$$

In order to study the behavior of solutions of a boundary value problem of the form (3.2) close to the origin, it is customary to introduce the so called *pencil operator*.

Definition 3.8. Given $\lambda \in \mathbb{C}$, we define the *pencil operator* $\mathfrak{U}(\lambda)$ of (3.2) in 0 as the map $\mathfrak{U}(\lambda)\colon H^{2m}(0, 2\pi) \to L^2(0, 2\pi) \times \mathbb{C}^{2m}$ such that

$$\mathfrak{U}(\lambda)\, v := \Big(\mathcal{L}^\circ(\theta, \partial_\theta, \lambda)v, \mathcal{C}_1^\circ(\theta, \partial_\theta, \lambda)v|_{\theta=0}, \mathcal{C}_1^\circ(\theta, \partial_\theta, \lambda)v|_{\theta=2\pi}, \ldots,$$

$$\mathcal{C}_m^\circ(\theta, \partial_\theta, \lambda)v|_{\theta=0}, \mathcal{C}_m^\circ(\theta, \partial_\theta, \lambda)v|_{\theta=2\pi} \Big).$$

We say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of the pencil operator $\mathfrak{U}$ if $\ker \mathfrak{U}(\lambda) \neq \{0\}$.

According to the definitions above and to Remark 3.7, the differential operators defining the boundary value problems (2.16) are 1-admissible in the origin 0. From the properties of $\mathcal{B}$ (see Remark 2.9), we have that the leading part of $\Delta^2 + \mathcal{B}$ at the origin coincides with Δ^2 and the corresponding pencil operator $\mathfrak{U}(\lambda)$ reads

$$\mathfrak{U}(\lambda)v := \Big((\partial_\theta^2 + (\lambda-2)^2)(\partial_\theta^2 + \lambda^2)\, v, v(0), v(2\pi), \partial_\theta\, v(0), \partial_\theta\, v(2\pi) \Big), \quad (3.3)$$

for every $v \in H^4(0, 2\pi)$. As it can be checked in [17, Chapter 7], the set of eigenvalues of (3.3) is

$$\mathcal{S} = \left\{ 1 + \frac{k}{2} \, : \, k \in \mathbb{Z}\backslash\{0\} \right\}. \tag{3.4}$$

We now state two results that relate the operator pencil $\mathfrak{U}(\lambda)$ with the solvability of the system (2.16), and that will be exploited in the proof of Theorem 3.11. The first proposition asserts that the system

$$\begin{cases} \Delta^2\phi = 0 & \text{in } \mathcal{D}'(\mathcal{K}_\infty), \\ \phi(\rho, 0) = \phi(\rho, 2\pi) = 0, \quad \partial_\theta\phi(\rho, 0) = \partial_\theta\phi(\rho, 2\pi) = 0, \end{cases} \tag{3.5}$$

admits a unique solution $\phi \in V_\beta^\ell(\mathcal{K}_\infty)$ for every $l \in \mathbb{Z}$ and every $\beta \in \mathbb{R}$ out of a countable set. The proof can be found, for instance, in [16, Chapter 6].

Proposition 3.9. *Let $\ell \in \mathbb{Z}$ and $\beta \in \mathbb{R}$ be such that $-\beta + \ell - 1 \notin \mathcal{S}$. Then, the system* (3.5) *admits only $\phi = 0$ as a solution in $V_\beta^\ell(\mathcal{K}_\infty)$.*

For the following theorem, it is useful to introduce some notation: for every $\lambda \in \mathcal{S}$, we denote by $m(\lambda)$ the algebraic multiplicity of the eigenvalue λ and by $u_\lambda^{(i)}$, $1 \leq i \leq m(\lambda)$, the family of linearly independent functions solving $\mathfrak{U}(\lambda) u_\lambda^{(i)} = 0$. Notice that all the eigenvalues of the operator pencil (3.3) have multiplicity 2, except for $\lambda = 0$ and $\lambda = 2$, which are simple.

Theorem 3.10. *Let $\ell_1, \ell_2 \in \mathbb{Z}$ and $\beta, \gamma \in \mathbb{R}$ be such that $0 < (\ell_2 - \gamma) - (\ell_1 - \beta) < 1$ and that*

$$-\beta + \ell_1 - 1 \notin \mathcal{S} \qquad and \qquad -\gamma + \ell_2 - 1 \notin \mathcal{S}.$$

Moreover, assume that

$$\mathcal{S} \cap (-\beta + \ell_1 - 1, -\gamma + \ell_2 - 1) = \{\lambda_1, \ldots, \lambda_N\}.$$

If $w \in V_\beta^{\ell_1}(\mathcal{K}_\mathfrak{r})$ is a solution of (2.16) *with $\widehat{f} \in V_\gamma^{\ell_2 - 4}(\mathcal{K}_\mathfrak{r})$, then there exist a function $w_R \in V_\gamma^{\ell_2}(\mathcal{K}_\mathfrak{r})$ and constants c_j^i, $i = 1, \ldots, m(\lambda_j)$, $j = 1, \ldots, N$ such that*

$$w = w_R + \sum_{j=1}^N \sum_{i=1}^{m(\lambda_j)} c_j^i \, \rho^{\lambda_j} u_{\lambda_j}^{(i)}.$$

Proof. We refer to [16, Theorem 6.4.1]. $\qquad\qquad\square$

For the sake of clearness, we introduce two auxiliary functions $\widehat{\phi}_1$ and $\widehat{\phi}_2$, which simply correspond to ϕ_1 and ϕ_2 computed in the usual angular coordinate θ (continuous in $B_\delta \backslash \widehat{\Gamma}$, for δ small enough). Namely, we set

$$\widehat{\phi}_1(\rho, \theta) := \phi_1(\rho, \theta) \qquad \text{and} \qquad \widehat{\phi}_2(\rho, \theta) := \phi_2(\rho, \theta). \qquad (3.6)$$

Although this notation seems to be redundant, it will be very useful in Proposition 3.13. We clarify the difference between the angles θ and ϑ in Fig. 2.

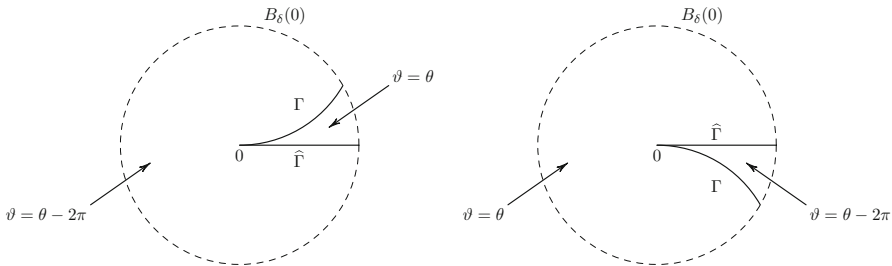We are now in a position to state the main result of the section.

FIGURE 2. *Relation between ϑ and θ in $B_\delta(0)$, when $\Gamma \subset \{x_1 \geq 0, \; x_2 \geq 0\}$ (left) and $\Gamma \subset \{x_1 \geq 0, \; x_2 \leq 0\}$ (right)*

Theorem 3.11. *Let $\widehat{w} \in H_0^2(\Omega \backslash \widehat{\Gamma})$ be the solution of (2.16). Then, there exist unique two constants C_1 and C_2 such that*

$$\widehat{w} - C_1 \widehat{\phi}_1 - C_2 \widehat{\phi}_2 \in H^3(\mathcal{K}_\mathfrak{r}). \tag{3.7}$$

Proof. In view of Remark 3.3, since $\widehat{w} \in H_0^2(\mathcal{K}_\mathfrak{r})$, we have $\widehat{w} \in V_0^2(\mathcal{K}_\mathfrak{r})$. Moreover, since $\widehat{f} \in H^{-1}(\mathcal{K}_\mathfrak{r})$ has support far from the origin, we deduce that $\widehat{f} \in V_{-\varepsilon}^{-1}(\mathcal{K}_\mathfrak{r})$ for every $\varepsilon > 0$.

Choosing $0 < \varepsilon < 1/2$, we are in a position to apply Theorem 3.10 with $\ell_1 = \ell_2 = 2$, $\beta = 0$ and $\gamma = -\varepsilon$. Since the set of eigenvalues $\mathcal{S}$ in (3.4) does not intersect the interval $(1, 1 + \varepsilon)$, we conclude that $\widehat{w} \in V_{-\varepsilon}^2(\mathcal{K}_\mathfrak{r})$.

We now apply a second time Theorem 3.10 with $\beta = -\varepsilon$, $\ell_1 = 2$, $\ell_2 = 3$, and $-\varepsilon < \gamma < 0$. Since

$$\mathcal{S} \cap (1 + \varepsilon, 2 - \gamma) = \left\{ \frac{3}{2}, 2 \right\},$$

there exist $C_1, C_2, C_3 \in \mathbb{R}$ such that

$$\widehat{w} - C_1 \rho^{3/2} u_{\frac{3}{2}}^{(1)}(\theta) - C_2 \rho^{3/2} u_{\frac{3}{2}}^{(2)}(\theta) - C_3 \rho^2 u_2(\theta) \in V_\gamma^3(\mathcal{K}_\mathfrak{r}), \tag{3.8}$$

where $u_{\frac{3}{2}}^{(1)}$, $u_{\frac{3}{2}}^{(2)}$, and u_2 are the eigenfunctions of the pencil operator (3.3) associated to the eigenvalues $\frac{3}{2}$ (with multiplicity 2) and 2 (simple). By Proposition 3.4, we have that $V_\gamma^3(\mathcal{K}_\mathfrak{r})$ embeds in $V_0^3(\mathcal{K}_\mathfrak{r})$ and, recalling Remark 3.3, $V_0^3(\mathcal{K}_\mathfrak{r}) \subseteq H^3(\mathcal{K}_\mathfrak{r})$. Therefore, the difference in (3.8) belongs to $H^3(\mathcal{K}_\mathfrak{r})$.

By standard computations regarding the bilaplacian (see, for instance, [17, Section 7]), we have that

$$u_{\frac{3}{2}}^{(1)}(\theta) = \frac{2}{3} \sin \frac{3\theta}{2} - 2 \sin \frac{\theta}{2}, \quad u_{\frac{3}{2}}^{(2)}(\theta) = \cos \frac{3\theta}{2} - \cos \frac{\theta}{2}, \quad u_2(\theta) = 1 - \cos 2\theta.$$

Clearly, the function $\rho^2 u_2(\theta)$ belongs to $H^3(\mathcal{K}_\mathfrak{r})$. Hence, (3.8) reduces to (3.7).

The uniqueness of the constants C_1 and C_2 follows by Proposition 3.9. Indeed, if $\overline{C}_1, \overline{C}_2 \in \mathbb{R}$ are such that (3.7) is satisfied, we deduce from (3.8) that

$$(C_1 - \overline{C}_1) \widehat{\phi}_1 + (C_2 - \overline{C}_2) \widehat{\phi}_2 \in V_\gamma^3(\mathcal{K}_\mathfrak{r}). \tag{3.9}$$

By a direct computation, it can be shown that the functions $\widehat{\phi}_1$ and $\widehat{\phi}_2$ extended to the whole of $\mathcal{K}_\infty$ are solutions of the Dirichlet problem (3.5), so that the same holds for the function in (3.9). Since $2 - \gamma \notin \mathcal{S}$, applying Proposition 3.9 we get that

$$(C_1 - \overline{C}_1) \widehat{\phi}_1 + (C_2 - \overline{C}_2) \widehat{\phi}_2 = 0,$$

that is, $C_1 = \overline{C}_1$ and $C_2 = \overline{C}_2$, and the proof is thus concluded. $\square$

Thanks to Theorem 3.11, we can easily deduce the next regularity result regarding the Airy function $w_u \in H^2(\Omega \backslash \Gamma)$ defined in Lemma 2.1.

Theorem 3.12. *Let $\Gamma \subset \Omega$ be a C^∞-crack set with a vertex in the origin, and let $w_u \in H^2(\Omega \backslash \Gamma)$ be the Airy function satisfying (2.5) and (2.6). Then, there exist unique two constants C_1 and C_2 such that*

$$w_u - C_1 \widehat{\phi}_1 \circ \mathrm{S} - C_2 \widehat{\phi}_2 \circ \mathrm{S} \in H^3(\Omega' \backslash \Gamma) \tag{3.10}$$

for every $\Omega' \subset\subset \Omega$.

Proof. We recall that, with the notation introduced in Sect. 2, $w_{u,\eta} = \eta w_u$ (for a suitable cut-off function η) solves the system (2.14) and is related to the solution $\widehat{w} \in H_0^2(\Omega\backslash\widehat{\Gamma})$ of (2.16) by the change of variable $w_{u,\eta} = \widehat{w} \circ S$. Hence, Theorem 3.11 implies that

$$w_{u,\eta} - C_1\widehat{\phi}_1 \circ S - C_2\widehat{\phi}_2 \circ S \in H^3(\mathcal{K}_{\mathfrak{r}}),$$

with the same constants C_1 and C_2 determined in (3.7). Proposition 2.3 allows us to get rid of the cut-off function, since w_u is an H^3-function far from the crack tip. Hence we deduce (3.10). $\qquad\square$

As in [18, Proposition 1.6], the last step of the proof of Theorem 2.4 consists in showing that the singularities (3.10) of the Airy function $w_u \in H^2(\Omega\backslash\Gamma)$ can be expressed by the simpler functions $\phi_1(\rho, \vartheta)$ and $\phi_2(\rho, \vartheta)$ introduced in Theorem 2.4. We stress that the advantage of using ϕ_1, ϕ_2 instead of $\widehat{\phi}_1$, $\widehat{\phi}_2$ is that they do not require the explicit computation of the change of variables S in (2.15). In order to make the presentation simpler, in the following proposition we assume that, at least in a neighborhood of the origin, the crack Γ is contained in the cone $\{x_1 \geq 0,\, x_2 \geq 0\}$. We will briefly specify in Remark 3.14 what happens in the other scenario, when $\Gamma \subseteq \{x_1 \geq 0,\, x_2 \leq 0\}$ (see also Fig. 2).

Proposition 3.13. *Let $\Gamma \subset \Omega$ be a C^∞-crack set with a vertex in the origin and assume that, in a neighborhood of the origin, it is contained in the cone $\{x_1 \geq 0,\, x_2 \geq 0\}$. Then $\phi_i + \widehat{\phi}_i \circ S \in H^3(\Omega'\backslash\Gamma)$, for $i = 1, 2$, and for every $\Omega' \subset\subset \Omega$.*

Proof. Let us consider $i = 1$. Since both the functions ϕ_1 and $\widehat{\phi}_1 \circ S$ belong to $H^2(\Omega\backslash\Gamma)$ and are regular far from the origin, we only have to estimate the sum of their third derivatives $D_{jkl}\phi_1 + D_{jkl}(\widehat{\phi}_1 \circ S)$ close to 0. By a direct computation, we have that

$$\begin{aligned}
&|D_{jkl}\phi_1 + D_{jkl}(\widehat{\phi}_1 \circ S)| \\
&\leq \left|D_{jkl}\phi_1 + \sum_{n,m,r} D_{nmr}\widehat{\phi}_1(S(x))D_jS_nD_kS_mD_lS_r\right| \\
&\quad + \left|\sum_{m,r} D_{mr}\widehat{\phi}_1(S(x))D_lS_rD_{jk}S_m\right| + \left|\sum_{m,r} D_{mr}\widehat{\phi}_1(S(x))D_kS_mD_{jl}S_r\right| \\
&\quad + \left|\sum_{m,r} D_{mr}\widehat{\phi}_1(S(x))D_jS_mD_{kl}S_r\right| + \left|\sum_r D_r\widehat{\phi}_1(S(x))D_{jkl}S_r\right|.
\end{aligned}$$

$$(3.11)$$

Adding and subtracting in (3.11) the term $\sum_{n,m,r} D_{nmr}\phi_1 D_jS_nD_kS_mD_lS_r$ and taking into account the smoothness of S, we deduce that there exists a positive constant C such that

$$
\begin{aligned}
|D_{jkl}\phi_1 + &\, D_{jkl}(\widehat{\phi}_1 \circ S)| \\
\leq &\, C\big(|\nabla\widehat{\phi}_1| + |\nabla^2\widehat{\phi}_1|\big) \\
&+ \sum_{n,m,r} |D_{nmr}\phi_1||\delta_j^n \delta_k^m \delta_l^r - D_j S_n D_k S_m D_l S_r| \\
&+ \sum_{n,m,r} |D_j S_n D_k S_m D_l S_r||D_{nmr}\widehat{\phi}_1(S(x)) + D_{nmr}\phi_1(x)|, \quad (3.12)
\end{aligned}
$$

where δ_i^j denotes the usual Kronecker delta. In view of the 3/2-homogeneity of $\widehat{\phi}_1$ (see (3.6)), of the regularity of S, and of the fact that $\nabla S(0) = \mathbf{I}$, we easily see that, at least in a neighborhood of the origin,

$$
\sum_{n,m,r} |D_{nmr}\phi_1||\delta_j^n \delta_k^m \delta_l^r - D_j S_n D_k S_m D_l S_r| \leq C|x|^{-\frac{3}{2}}|x| = C|x|^{-\frac{1}{2}}, \quad (3.13)
$$

for some positive constant C independent of x. In particular, we notice that the function $x \mapsto |x|^{-\frac{1}{2}}$ is square integrable in $\Omega\backslash\Gamma$.

We now estimate the last term in the right-hand side of (3.12). Of course, $|D_j S_n D_k S_m D_l S_r|$ is uniformly bounded by definition of S (2.15). Therefore, it remains to estimate the sum $|D_{nmr}\widehat{\phi}_1(S(x)) + D_{nmr}\phi_1(x)|$ for $x \in \Omega\backslash\Gamma$. To this end, we notice that, by definition of the angles θ and ϑ and of S in (2.15), we have either $\vartheta(x) = \theta(x) - 2\pi$ or $\theta(S(x)) = \vartheta(S(x)) + 2\pi$. In view of these relations, we can always find a new function $\overline{\phi}_1$ continuous on the segment $[x, S(x)]$ and such that

$$
\overline{\phi}_1(x) = \phi_1(x) \iff \overline{\phi}_1(S(x)) = -\widehat{\phi}_1(S(x)). \quad (3.14)
$$

Indeed, when the segment $[x, S(x)]$ meets Γ, we set $\overline{\phi}_1(y) := -\widehat{\phi}_1(\theta(y))$, while we define $\overline{\phi}_1(y) := \widehat{\phi}_1(\vartheta(y))$ whenever $[x, S(x)] \cap \Gamma = \emptyset$. By direct comparison, it is easy to check that (3.14) holds true. Therefore, the sum $|D_{nmr}\widehat{\phi}_1(S(x)) + D_{nmr}\phi_1(x)|$ takes the form $|D_{nmr}\overline{\phi}_1(S(x)) - D_{nmr}\overline{\phi}_1(x)|$ and, in view of the above discussion, the angle used to define $\overline{\phi}_1$ is continuous along the segment $[x, S(x)]$.

Applying the mean value theorem to $\overline{\phi}_1$, we find a point $\bar{x} \in [x, S(x)]$ such that

$$
\begin{aligned}
|D_{nmr}\widehat{\phi}_1(S(x)) + D_{nmr}\phi_1(x)| &= |D_{nmr}\overline{\phi}_1(S(x)) - D_{nmr}\overline{\phi}_1(x)| \\
&\leq |\nabla D_{nmr}\overline{\phi}_1(\bar{x})||x - S(x)|. \quad (3.15)
\end{aligned}
$$

Since $S \in C^\infty$ with $S(0) = 0$ and $\nabla S(0) = \mathbf{I}$, we have that, close to the origin,

$$
|x - S(x)| \leq L|x|^2, \quad (3.16)
$$

for $L := \mathrm{Lip}(\nabla^2 S)/2$, $\mathrm{Lip}(\cdot)$ denoting the Lipschitz constant of a function. If we indicate with d the distance between the origin 0 and the segment $[x, S(x)]$, by [18, Lemma 1.7] we have that, at least for $|x|$ small enough, $d \geq \frac{1}{2}|x|$. Hence, we deduce that

$$
|\bar{x}|^{-\frac{5}{2}} \leq d^{-\frac{5}{2}} \leq 2^{\frac{5}{2}}|x|^{-\frac{5}{2}}. \quad (3.17)
$$

Combining (3.16) and (3.17) with the behavior of the fourth order derivatives of $\overline{\phi}_1$ and the estimate (3.15), we get that

$$|\mathrm{D}_{nmr}\widehat{\phi}_1(\mathrm{S}(x)) + \mathrm{D}_{nmr}\phi_1(x)| \leq C|x|^{-\frac{1}{2}}, \tag{3.18}$$

for some positive constant C.

Finally, inserting (3.14) and (3.18) in (3.12), we deduce that the sum $|\mathrm{D}_{jkl}\phi_1 - \mathrm{D}_{jkl}(\widehat{\phi}_1 \circ \mathrm{S})|$ is square integrable close to the origin, and this concludes the proof of the proposition. $\qquad\square$

Remark 3.14. If the crack set Γ is contained, at least close to its tip, in the cone $\{x_1 \geq 0, x_2 \leq 0\}$, the statement of Proposition 3.13 needs to be slightly modified: we can prove that $\phi_i - \widehat{\phi}_i \circ \mathrm{S} \in H^3(\Omega'\backslash\Gamma)$, for $i = 1, 2$ and for every $\Omega' \subset\subset \Omega$. The strategy of the proof remains the same, apart from (3.14). In this case, we can find a function $\overline{\phi}_1$ such that

$$\overline{\phi}_1(x) = \phi_1(x) \iff \overline{\phi}_1(\mathrm{S}(x)) = \widehat{\phi}_1(\mathrm{S}(x)).$$

The immediate consequence of this fact is that the constants appearing in the decomposition (2.8) of Theorem 2.4 could not coincide with the ones found in Theorem 3.12. Precisely, as shown in the proof of Proposition 3.13, the definition of the angle ϑ could result in a change of sign in the definition of the constants C_1, C_2 when passing from $\widehat{\phi}_i \circ \mathrm{S}$ (Theorem 3.12) to ϕ_i (Theorem 2.4).

Nevertheless, we stress that this possible "uncertainty" of the sign of the constants C_1 and C_2 does not affect the results in Sect. 4. Indeed, in Theorem 4.1, only the squares of the stress intensity factors Q_1 and Q_2 appear. Hence, their sign, which depend on C_1, C_2, is not of particular interest for us.

Proof of Theorem 2.4. The splitting (2.8) follows from Theorem 3.12, Proposition 3.13, and Remark 3.14. $\qquad\square$

4. The energy release rate

In this section we are concerned with the computation of the so called energy release rate, i.e., the opposite of the derivative of the elastic energy with respect to the crack elongation. In order to define it rigorously, we first need to prove that the elastic energy at the equilibrium is in some suitable sense differentiable with respect to the crack length parameter. To do this, we start by considering an increasing family of C^∞-curves Γ_s parametrized by their arc-length $s \in [0, L]$, $L > 0$, and having an endpoint in $\partial\Omega$. It will be useful to write the curves Γ_s in the form

$$\Gamma_s := \{\gamma(\sigma) : 0 \leq \sigma \leq s\}, \tag{4.1}$$

where γ is the arc-length parametrization of Γ_s, with $\gamma(0) \in \partial\Omega$. Given $s \in [0, L]$, we denote by $u_s \in H^1(\Omega\backslash\Gamma_s; \mathbb{R}^2)$ the solution to the minimum problem

$$\min\{\mathcal{E}(u, \Gamma_s) : u \in H^1(\Omega\backslash\Gamma_s; \mathbb{R}^2), u = g \text{ on } \partial\Omega\},$$

where $g \in H^1(\Omega\backslash\Gamma_L; \mathbb{R}^2)$ is the usual boundary datum. For what follows, it is not restrictive to assume that $0 \notin \mathrm{supp}(g)$. Moreover, we set $\mathcal{E}_{min}(\Gamma_s) :=$

$\mathcal{E}(u_s, \Gamma_s)$ and we notice that the functions $s \mapsto \mathcal{E}_{min}(\Gamma_s)$ and $s \mapsto u_s$ are continuous from $\mathbb{R}$ to $\mathbb{R}$ and from $\mathbb{R}$ to H^1, respectively.

Let us fix $s_0 \in (0, L)$. Without loss of generality, we may assume that $\gamma(s_0) = 0$ and $\gamma'(s_0) = -e_1$. Therefore, Theorem 2.5 ensures that there exist two constants Q_1 and Q_2, which depend on Γ_{s_0}, such that

$$u_{s_0} = u_R + Q_1\Phi_1 + Q_2\Phi_2, \tag{4.2}$$

where $u_R \in H^2(\Omega' \backslash \Gamma_{s_0}; \mathbb{R}^2)$ for every $\Omega' \subset\subset \Omega$ and Φ_1, Φ_2 are defined in (2.9)–(2.10). With this notation, the following result holds.

Theorem 4.1. *The energy $\mathcal{E}_{min}$ is differentiable at s_0 and*

$$\frac{d\mathcal{E}_{min}(\Gamma_s)}{ds}\Big|_{s=s_0} = -2\pi\big(Q_1^2(\Gamma_{s_0}) + Q_2^2(\Gamma_{s_0})\big)\mu(\lambda + \mu)(\lambda + 2\mu), \tag{4.3}$$

where λ and μ are the Lamé coefficients.

Proof. In order to prove the differentiability of the energy $s \mapsto \mathcal{E}_{min}(\Gamma_s)$ in s_0 we adapt the argument of [18, Theorem 2.1]. For simplicity, we adopt here the Einstein notation of summation over repeated indices.

In order to make explicit computations, for $r > 0$ small enough we may assume that the curve $\Gamma_s \cap B_r$, $s \geq s_0$, is the graph of a C^∞-function ζ in a neighborhood of $\gamma(s_0)$ such that $\zeta'(\gamma_1(s_0)) = 0$. For $\delta > 0$ small, we introduce a C^∞-diffeomorphism $F_{s_0,\delta}$ that maps Γ_{s_0} in $\Gamma_{s_0+\delta}$ and coincides with the identity of $\mathbb{R}^2$ far from the origin (see, e.g., [1,14]). Therefore, for $x \in B_{r/2}$ we set

$$F_{s_0,\delta}(x) := x + \begin{pmatrix} (\gamma_1(s_0 + \delta) - \gamma_1(s_0))\varphi(x) \\ \zeta(x_1 + (\gamma_1(s_0 + \delta) - \gamma_1(s_0))\varphi(x)) - \zeta(x_1) \end{pmatrix}, \tag{4.4}$$

where $\varphi \in C_c^\infty(B_{r/2})$ is a suitable cut-off function equal to 1 close to the origin and such that

$$\text{supp}(\varphi) \cap \text{supp}(g) = \emptyset. \tag{4.5}$$

We extend $F_{s_0,\delta}$ with the identity out of $B_{r/2}$.

By the regularity of ζ, we deduce that $F_{s_0,\delta}$ is a C^∞-diffeomorphism of $\mathbb{R}^2$ such that $F_{s_0,\delta}(\Gamma_{s_0}) = \Gamma_{s_0+\delta}$. Moreover, the following equalities hold:

$$\rho_{s_0}(x) := \partial_\delta(F_{s_0,\delta}(x))|_{\delta=0} = \gamma_1'(s_0)\varphi(x)\begin{pmatrix} 1 \\ \zeta'(x_1) \end{pmatrix} = -\varphi(x)\begin{pmatrix} 1 \\ \zeta'(x_1) \end{pmatrix}, \tag{4.6}$$

$$\partial_\delta(\det\nabla F_{s_0,\delta})|_{\delta=0} = \text{div}\rho_{s_0},$$
$$\partial_\delta(\nabla F_{s_0,\delta})|_{\delta=0} = -\partial_\delta(\nabla F_{s_0,\delta})^{-1}|_{\delta=0} = \nabla\rho_{s_0}. \tag{4.7}$$

For what follows, let us denote by $U_{s_0+\delta} := u_{s_0+\delta} \circ F_{s_0,\delta} \in H^1(\Omega \backslash \Gamma_{s_0}; \mathbb{R}^2)$. Since $F_{s_0,\delta} = id$ on $\partial\Omega$, we clearly have $U_{s_0+\delta} = g$ on $\partial\Omega$. Moreover, it is easy to check that

$$U_{s_0+\delta} = \arg\min \left\{ \frac{1}{2} \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}(\nabla v(\nabla F_{s_0,\delta})^{-1}) : \nabla v(\nabla F_{s_0,\delta})^{-1} \det \nabla F_{s_0,\delta} \, dx : \right.$$

$$\left. v \in H^1(\Omega\backslash\Gamma_{s_0}; \mathbb{R}^2), \ v = g \text{ on } \partial\Omega \right\}. \tag{4.8}$$

In particular,

$$\mathcal{E}_{min}(\Gamma_{s_0+\delta}) = \mathcal{E}(u_{s_0+\delta}, \Gamma_{s_0+\delta})$$

$$= \frac{1}{2} \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}(\nabla U_{s_0+\delta}(\nabla F_{s_0,\delta})^{-1}) : \nabla U_{s_0+\delta}(\nabla F_{s_0,\delta})^{-1} \det \nabla F_{s_0,\delta} \, dx.$$

We also notice that, since $u_{s_0+\delta} \to u_{s_0}$ in H^1 as $\delta \to 0$, $U_{s_0+\delta} \to u_{s_0}$ in $H^1(\Omega\backslash\Gamma_{s_0}; \mathbb{R}^2)$.

We now show that the function $\delta \mapsto U_{s_0+\delta}$ is differentiable in $\delta = 0$. Indeed, by definition of u_{s_0}, the finite difference $(U_{s_0+\delta} - u_{s_0})/\delta$ satisfies, for every $v \in H^1(\Omega\backslash\Gamma_{s_0}; \mathbb{R}^2)$ with $v = 0$ on $\partial\Omega$,

$$\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C} \left(\nabla \frac{U_{s_0+\delta} - u_{s_0}}{\delta} \right) : \nabla v \, dx$$

$$= \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C} \nabla \frac{U_{s_0+\delta}}{\delta} : \nabla v \, dx = \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C} \left(\nabla U_{s_0+\delta} \frac{(\mathbf{I} - (\nabla F_{s_0,\delta})^{-1})}{\delta} \right) : \nabla v \, dx$$

$$+ \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C} \left(\nabla U_{s_0+\delta}(\nabla F_{s_0,\delta})^{-1} \right) : \nabla v \frac{(\mathbf{I} - (\nabla F_{s_0,\delta})^{-1})}{\delta} \, dx$$

$$+ \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C} \left(\nabla U_{s_0+\delta}(\nabla F_{s_0,\delta})^{-1} \right) : \nabla v(\nabla F_{s_0,\delta})^{-1} \frac{(1 - \det \nabla F_{s_0,\delta})}{\delta} \, dx.$$

The previous equality implies that $(U_{s_0+\delta} - u_{s_0})/\delta$ is the solution of

$$\min \left\{ \frac{1}{2} \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C} \nabla v : \nabla v \, dx - \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C} \left(\nabla U_{s_0+\delta} \frac{(\mathbf{I} - (\nabla F_{s_0,\delta})^{-1})}{\delta} \right) : \nabla v \, dx \right.$$

$$- \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C} \left(\nabla U_{s_0+\delta}(\nabla F_{s_0,\delta})^{-1} \right) : \nabla v \frac{(\mathbf{I} - (\nabla F_{s_0,\delta}))^{-1})}{\delta} \, dx$$

$$- \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C} \left(\nabla U_{s_0+\delta}(\nabla F_{s_0,\delta})^{-1} \right) : \nabla v(\nabla F_{s_0,\delta})^{-1} \frac{(1 - \det \nabla F_{s_0,\delta})}{\delta} \, dx :$$

$$v \in H^1(\Omega\backslash\Gamma_{s_0}; \mathbb{R}^2), \ v = 0 \text{ on } \partial\Omega \right\}.$$

Thus, we deduce that the function $\delta \mapsto \frac{U_{s_0+\delta}-u_{s_0}}{\delta}$ is bounded in $H^1(\Omega\backslash\Gamma_{s_0};\mathbb{R}^2)$ and, up to a subsequence, it converges weakly to some function $\dot{U} \in H^1(\Omega\backslash\Gamma_{s_0};\mathbb{R}^2)$ as $\delta \to 0$. Since it turns out that $\dot{U}$ is such that

$$
\begin{aligned}
\dot{U} = \arg\min \Bigg\{ &\frac{1}{2} \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}\nabla v : \nabla v \, dx - \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}\nabla u_{s_0}\nabla\rho_{s_0} : \nabla v \, dx \\
&- \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}\nabla u_{s_0} : \nabla v\nabla\rho_{s_0} \, dx \\
&+ \int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}\nabla u_{s_0} : \nabla v \operatorname{div}\rho_{s_0} \, dx : v \in H^1(\Omega\backslash\Gamma_{s_0};\mathbb{R}^2),\ v = 0 \text{ on } \partial\Omega \Bigg\},
\end{aligned}
\tag{4.9}
$$

we also deduce that

$$
\lim_{\delta\to 0} \frac{U_{s_0+\delta} - u_{s_0}}{\delta} = \dot{U} \qquad \text{strongly in } H^1(\Omega\backslash\Gamma_{s_0};\mathbb{R}^2). \tag{4.10}
$$

We now estimate the finite difference quotient

$$
\frac{\mathcal{E}_{min}(\Gamma_{s_0+\delta}) - \mathcal{E}_{min}(\Gamma_{s_0})}{\delta} = \frac{\mathcal{E}(u_{s_0+\delta},\Gamma_{s_0+\delta}) - \mathcal{E}(u_{s_0},\Gamma_{s_0})}{\delta}.
$$

By the minimality of u_{s_0}, we have

$$
\mathcal{E}_{min}(\Gamma_{s_0}) = \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}Eu_{s_0} : Eu_{s_0} \, dx = \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}Eu_{s_0} : Eg \, dx. \tag{4.11}
$$

In a similar way, recalling (4.4) and (4.5), we deduce that

$$
\mathcal{E}_{min}(\Gamma_{s_0+\delta}) = \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}Eu_{s_0+\delta} : Eg \, dx = \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}EU_{s_0+\delta} : Eg \, dx. \tag{4.12}
$$

Combining (4.11) and (4.12) we get that

$$
\frac{\mathcal{E}_{min}(\Gamma_{s_0+\delta}) - \mathcal{E}_{min}(\Gamma_{s_0})}{\delta} = \frac{1}{2\delta}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}E(U_{s_0+\delta} - u_{s_0}) : Eg \, dx.
$$

Passing to the limit as $\delta \to 0$ in the previous equality and taking into account the convergence (4.10), we obtain that $\delta \mapsto \mathcal{E}_{min}(\Gamma_{s_0+\delta})$ is differentiable in $\delta = 0$ and

$$
\frac{d\mathcal{E}_{min}(\Gamma_s)}{ds}\bigg|_{s=s_0} = \frac{d\mathcal{E}_{min}(\Gamma_{s_0+\delta})}{d\delta}\bigg|_{\delta=0} = \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}E\dot{U} : Eg \, dx. \tag{4.13}
$$

Considering the minimum problem (4.9) and the Euler-Lagrange equation associated to it and recalling that $\dot{U} = 0$ on $\partial\Omega$, we can rewrite (4.13) as

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{E}_{min}(\Gamma_s)}{\mathrm{d}s}\Big|_{s=s_0} &= \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}\mathrm{E}\dot{U} : \mathrm{E}(g - u_{s_0})\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}(\nabla u_{s_0}\nabla\rho_{s_0}) : \nabla(g - u_{s_0})\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}\nabla u_{s_0} : \nabla(g - u_{s_0})\nabla\rho_{s_0}\,\mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}\nabla u_{s_0} : \nabla(g - u_{s_0})\mathrm{div}\rho_{s_0}\,\mathrm{d}x.
\end{aligned}
\tag{4.14}
$$

In view of (4.5) we have that $\mathrm{supp}(\rho_{s_0}) \cap \mathrm{supp}(g) = \emptyset$, so that (4.14) can be rewritten as

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{E}_{min}(\Gamma_s)}{\mathrm{d}s}\Big|_{s=s_0} &= -\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}\mathrm{E}u_{s_0} : \nabla u_{s_0}\nabla\rho_{s_0}\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \mathbb{C}\mathrm{E}u_{s_0} : \nabla u_{s_0}\mathrm{div}\rho_{s_0}\,\mathrm{d}x \\
&= -\int_{\Omega\backslash\Gamma_{s_0}} \boldsymbol{\sigma}(u_{s_0}) : \nabla u_{s_0}\nabla\rho_{s_0}\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\Omega\backslash\Gamma_{s_0}} \boldsymbol{\sigma}(u_{s_0}) : \nabla u_{s_0}\,\mathrm{div}\rho_{s_0}\,\mathrm{d}x,
\end{aligned}
\tag{4.15}
$$

where we have used the notation $\boldsymbol{\sigma}(u) := \mathbb{C}\mathrm{E}u$.

In order to compute explicitly the right-hand side of (4.15), we first integrate out of the closure of the ball B_ε centered in the origin and of radius ε and then pass to the limit as $\varepsilon \to 0$. By integration by parts and writing the integrands in components, we get

$$
\begin{aligned}
&-\int_{(\Omega\backslash\Gamma_{s_0})\backslash\overline{\mathrm{B}}_\varepsilon} \boldsymbol{\sigma}(u_{s_0}) : \nabla u_{s_0}\nabla\rho_{s_0}\,\mathrm{d}x + \frac{1}{2}\int_{(\Omega\backslash\Gamma_{s_0})\backslash\overline{\mathrm{B}}_\varepsilon} \boldsymbol{\sigma}(u_{s_0}) : \nabla u_{s_0}\mathrm{div}\rho_{s_0}\,\mathrm{d}x \\
&= \int_{(\Omega\backslash\Gamma_{s_0})\backslash\overline{\mathrm{B}}_\varepsilon} \mathrm{D}_j\boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_k u_{s_0,i}\rho_{s_0,k}\,\mathrm{d}x \\
&\quad + \int_{(\Omega\backslash\Gamma_{s_0})\backslash\overline{\mathrm{B}}_\varepsilon} \boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_j\mathrm{D}_k u_{s_0,i}\,\rho_{s_0,k}\,\mathrm{d}x \\
&\quad + \int_{\partial\mathrm{B}_\varepsilon} \boldsymbol{\sigma}_{ij}(u_{s_0})\nu_{\partial\mathrm{B}_\varepsilon,j}\mathrm{D}_k u_{s_0,i}\,\rho_{s_0,k}\,\mathrm{d}\mathcal{H}^1 \\
&\quad + \int_{\Gamma_{s_0}\backslash\overline{\mathrm{B}}_\varepsilon} \boldsymbol{\sigma}_{ij}(u_{s_0})\nu_{\Gamma_{s_0},j}\mathrm{D}_k u_{s_0,i}\,\rho_{s_0,k}\,\mathrm{d}\mathcal{H}^1 \\
&\quad - \frac{1}{2}\int_{(\Omega\backslash\Gamma_{s_0})\backslash\overline{\mathrm{B}}_\varepsilon} \mathrm{D}_k\boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_j u_{s_0,i}\,\rho_{s_0,k}\,\mathrm{d}x
\end{aligned}
\tag{4.16}
$$

$$-\frac{1}{2}\int_{(\Omega\backslash\Gamma_{s_0})\backslash\overline{\mathrm{B}}_\varepsilon} \boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_k\mathrm{D}_j u_{s_0,i}\, \rho_{s_0,k}\, \mathrm{d}x$$

$$-\frac{1}{2}\int_{\partial\mathrm{B}_\varepsilon} \boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_j u_{s_0,i}\, \rho_{s_0,k}\nu_{\partial\mathrm{B}_\varepsilon,k}\, \mathrm{d}\mathcal{H}^1$$

$$-\frac{1}{2}\int_{\Gamma_{s_0}\backslash\overline{\mathrm{B}}_\varepsilon} \boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_j u_{s_0,i}\, \rho_{s_0,k}\nu_{\Gamma_{s_0},k}\, \mathrm{d}\mathcal{H}^1,$$

where $\nu_{\partial\mathrm{B}_\varepsilon}$ denotes the outer unit normal to the ball B_ε and $\nu_{\Gamma_{s_0}}$ is the unit normal vector to Γ_{s_0}. In the right-hand side of (4.16) we have that, by definition of u_{s_0}, $\mathrm{div}\boldsymbol{\sigma}(u_{s_0}) = 0$, so that

$$\int_{\Omega\backslash\overline{\mathrm{B}}_\varepsilon} \mathrm{D}_j\boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_k u_{s_0,i}\rho_{s_0,k}\, \mathrm{d}x = 0.$$

The remaining volume integrals sum up to

$$\frac{1}{2}\int_{\Omega\backslash\overline{\mathrm{B}}_\varepsilon} \boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_j\mathrm{D}_k u_{s_0,i}\, \rho_{s_0,k}\, \mathrm{d}x - \frac{1}{2}\int_{\Omega\backslash\overline{\mathrm{B}}_\varepsilon} \mathrm{D}_k\boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_j u_{s_0,i}\, \rho_{s_0,k}\, \mathrm{d}x.$$

Recalling the definitions of the stress $\boldsymbol{\sigma}(u_{s_0}) = \mathbb{C}\mathrm{E}u_{s_0}$ and of the elasticity tensor in terms of the Lamé coefficients (2.1), we can show that

$$\boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_j\mathrm{D}_k u_{s_0,i}\, \rho_{s_0,k} - \mathrm{D}_k\boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_j u_{s_0,i}\, \rho_{s_0,k} = 0 \qquad \text{in } \Omega\backslash\overline{\mathrm{B}}_\varepsilon.$$

Hence, the sum of the volume terms in (4.16) is zero.

As for the surface integrals, due to the stress free condition $\boldsymbol{\sigma}(u_{s_0})\nu_{\Gamma_{s_0}} = 0$ on Γ_{s_0}, we obtain that

$$\int_{\Gamma_{s_0}} \boldsymbol{\sigma}_{ij}(u_{s_0})\nu_{\Gamma_{s_0},j}\mathrm{D}_k u_{s_0,i}\, \rho_{s_0,k}\, \mathrm{d}\mathcal{H}^1 = 0.$$

Moreover, since Γ_{s_0} is the graph of the function ζ close to the crack tip, we have that $\nu_{\Gamma_{s_0}} = \sqrt{1 + \zeta'^2}(\zeta', -1)$. Hence, recalling the definition (4.6) of ρ_{s_0}, $\rho_{s_0}\cdot\nu_{\Gamma_{s_0}} = 0$ and

$$\int_{\Gamma_{s_0}} \boldsymbol{\sigma}_{ij}(u_{s_0})\mathrm{D}_j u_{s_0,i}\, \rho_{s_0,k}\nu_{\Gamma_{s_0},k}\, \mathrm{d}\mathcal{H}^1 = 0.$$

Collecting all the previous equalities, (4.16) reduces to

$$-\int_{\Omega\backslash\overline{\mathrm{B}}_\varepsilon} \boldsymbol{\sigma}(u_{s_0}):\nabla u_{s_0}\nabla\rho_{s_0}\, \mathrm{d}x + \frac{1}{2}\int_{\Omega\backslash\overline{\mathrm{B}}_\varepsilon} \boldsymbol{\sigma}(u_{s_0}):\nabla u_{s_0}\mathrm{div}\rho_{s_0}\, \mathrm{d}x$$

$$= \int_{\partial\mathrm{B}_\varepsilon} \boldsymbol{\sigma}(u_{s_0})\nu_{\partial\mathrm{B}_\varepsilon}\cdot\nabla u_{s_0}\rho_{s_0}\, \mathrm{d}\mathcal{H}^1 - \frac{1}{2}\int_{\partial\mathrm{B}_\varepsilon} \boldsymbol{\sigma}(u_{s_0}):\nabla u_{s_0}\, \rho_{s_0}\cdot\nu_{\partial\mathrm{B}_\varepsilon}\, \mathrm{d}\mathcal{H}^1.$$

$$(4.17)$$

To compute the limit as $\varepsilon \to 0$ of the right-hand side of (4.17), we recall the splitting (4.2), which, combined with (4.15) and (4.17), allows us to write

$$\frac{\mathrm{d}\mathcal{E}_{min}(\Gamma_s)}{\mathrm{d}s}\bigg|_{s=s_0} = \lim_{\varepsilon\to 0} a_\varepsilon + b_\varepsilon + c_\varepsilon + d_\varepsilon, \qquad (4.18)$$

where we have set

$$
\begin{aligned}
a_\varepsilon := {}& \mathrm{Q}_1^2 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_1)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla \Phi_1 \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 - \frac{\mathrm{Q}_1^2}{2} \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_1) : \mathrm{E}\Phi_1 \rho_{s_0} \cdot \nu_{\partial \mathrm{B}_\varepsilon}\, \mathrm{d}\mathcal{H}^1 \\
& + \mathrm{Q}_2^2 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_2)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla \Phi_2 \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 - \frac{\mathrm{Q}_2^2}{2} \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_2) : \mathrm{E}\Phi_2 \rho_{s_0} \cdot \nu_{\partial \mathrm{B}_\varepsilon}\, \mathrm{d}\mathcal{H}^1,
\end{aligned}
$$

$$
\begin{aligned}
b_\varepsilon := {}& \mathrm{Q}_1 \mathrm{Q}_2 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_1)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla \Phi_2 \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 + \mathrm{Q}_1 \mathrm{Q}_2 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_2)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla \Phi_1 \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 \\
& - \mathrm{Q}_1 \mathrm{Q}_2 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_1) : \mathrm{E}\Phi_2 \rho_{s_0} \cdot \nu_{\partial \mathrm{B}_\varepsilon}\, \mathrm{d}\mathcal{H}^1,
\end{aligned}
$$

$$
\begin{aligned}
c_\varepsilon := {}& \mathrm{Q}_1 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_1)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla u_R \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 + \mathrm{Q}_1 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(u_R)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla \Phi_1 \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 \\
& - \mathrm{Q}_1 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_1) : \mathrm{E}u_R \rho_{s_0} \cdot \nu_{\partial \mathrm{B}_\varepsilon}\, \mathrm{d}\mathcal{H}^1 + \mathrm{Q}_2 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_2)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla u_R \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 \\
& + \mathrm{Q}_2 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(u_R)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla \Phi_2 \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 - \mathrm{Q}_2 \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_2) : \mathrm{E}u_R \rho_{s_0} \cdot \nu_{\partial \mathrm{B}_\varepsilon}\, \mathrm{d}\mathcal{H}^1,
\end{aligned}
$$

$$
d_\varepsilon := \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(u_R)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla u_R \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 - \frac{1}{2} \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(u_R) : \mathrm{E}u_R \rho_{s_0} \cdot \nu_{\partial \mathrm{B}_\varepsilon}\, \mathrm{d}\mathcal{H}^1.
$$

Here, for brevity, we have omitted the dependence of Q_i on Γ_{s_0}.

In view of the regularity of $F_{s_0,\delta}$ and of ρ_{s_0} (see (4.6)), we have that $\rho_{s_0}(x) \to (-1,0)$ uniformly as $x \to 0$. Therefore, by a long but elementary computation, we can show that

$$
\begin{aligned}
& \lim_{\varepsilon \to 0} a_\varepsilon = -2\pi(\mathrm{Q}_1^2 + \mathrm{Q}_2^2)\mu(\lambda + \mu)(\lambda + 2\mu), \\
& \lim_{\varepsilon \to 0} b_\varepsilon = 0.
\end{aligned}
\tag{4.19}
$$

It remains to prove that $c_\varepsilon, d_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us consider the first term in c_ε. By Hölder inequality, by the regularity of u_R, and by definition (2.10) of Φ_1, we have that

$$
\begin{aligned}
\Big| \int_{\partial \mathrm{B}_\varepsilon} \boldsymbol{\sigma}(\Phi_1)\nu_{\partial \mathrm{B}_\varepsilon} \cdot \nabla u_R \rho_{s_0}\, \mathrm{d}\mathcal{H}^1 \Big| & \le \frac{C}{\varepsilon^{1/2}} \|\nabla u_R\|_{L^1(\partial \mathrm{B}_\varepsilon)} \\
& \le \frac{C}{\varepsilon^{1/2}} \mathcal{H}^1(\partial \mathrm{B}_\varepsilon)^{1/2} \|\nabla u_R\|_{L^2(\partial \mathrm{B}_\varepsilon)} \\
& = C \|\nabla u_R\|_{L^2(\partial \mathrm{B}_\varepsilon)}
\end{aligned}
$$

for some positive constant C independent of ε. The same estimate can be obtained for the remaining terms of c_ε, so that

$$
|c_\varepsilon| \le C \|\nabla u_R\|_{L^2(\partial \mathrm{B}_\varepsilon)}.
$$

As for d_ε, again by Hölder inequality we simply get

$$
|d_\varepsilon| \le C \|\nabla u_R\|_{L^2(\partial \mathrm{B}_\varepsilon)}^2.
$$

Therefore, we are led to prove that $\|\nabla u_R\|_{L^2(\partial B_\varepsilon)} \to 0$ as $\varepsilon \to 0$. Employing the change of variable $y = \frac{x}{\varepsilon}$ and defining $v(y) := u_R(\varepsilon y)$, we can write

$$\|\nabla u_R\|_{L^2(\partial B_\varepsilon)}^2 = \frac{1}{\varepsilon} \int_{\partial B_1} |\nabla v|^2 \, d\mathcal{H}^1.$$

By the continuity of the trace operator (recall that $v \in H^2(B_1; \mathbb{R}^2)$), there exists a positive contant C such that

$$\|\nabla u_R\|_{L^2(\partial B_\varepsilon)}^2 \leq \frac{C}{\varepsilon} \int_{B_1} |\nabla^2 v|^2 \, dy + \frac{C}{\varepsilon} \int_{B_1} |\nabla v|^2 \, dy$$

$$= C\varepsilon \int_{B_\varepsilon} |\nabla^2 u_R|^2 \, dx + \frac{C}{\varepsilon} \int_{B_\varepsilon} |\nabla u_R|^2 \, dx. \qquad (4.20)$$

By Sobolev embedding and Hölder inequality, we may further estimate (4.20) with

$$\|\nabla u_R\|_{L^2(\partial B_\varepsilon)}^2 \leq C\varepsilon \|\nabla^2 u_R\|_{L^2(\partial B_\varepsilon)}^2 + C\|\nabla u_R\|_{L^4(B_\varepsilon)}^2.$$

By the absolute continuity of the integral and the H^2-regularity of u_R, passing to the limit in the previous inequality we finally get $\|\nabla u_R\|_{L^2(\partial B_\varepsilon)} \to 0$ as $\varepsilon \to 0$. This concludes the proof of the theorem. $\qquad \square$

Remark 4.2. Although the theory of pencil operators exploited in Sect. 3 has been developed for PDE systems with smooth coefficients, and, as a consequence, our analysis only applies to the class of C^∞-crack sets, we notice that the sole proof of Theorem 4.1 works even if we consider a $C^{1,1}$-extension of a C^∞-crack set Γ_{s_0}. Indeed, repeating all the computation above, we would obtain exactly the same result, since the splitting (4.2) of u_{s_0} only depends on Γ_{s_0} and not on Γ_s for $s \in (s_0, L)$, and since the derivatives in (4.6)–(4.8) are defined for almost every x and are uniformly bounded. This allows us to give a definition of energy release rate as the opposite of the derivative of the elastic energy $\mathcal{E}_{min}$ with respect to the crack elongation:

$$\mathcal{G}(\Gamma_{s_0}) := -\frac{d\mathcal{E}_{min}(\Gamma_s)}{ds}\bigg|_{s=s_0} = 2\pi\big(Q_1^2(\Gamma_{s_0}) + Q_2^2(\Gamma_{s_0})\big)\mu(\lambda + \mu)(\lambda + 2\mu).$$

From formula (4.15) in the proof of Theorem 4.1 we also deduce an integral formula for $\mathcal{G}$. Namely,

$$\mathcal{G}(\Gamma_{s_0}) = \int_{\Omega \backslash \Gamma_{s_0}} \boldsymbol{\sigma}(u_{s_0}) : \nabla u_{s_0} \nabla \rho_{s_0} \, dx - \frac{1}{2} \int_{\Omega \backslash \Gamma_{s_0}} \boldsymbol{\sigma}(u_{s_0}) : \nabla u_{s_0} \, \mathrm{div}\rho_{s_0} \, dx, \quad (4.21)$$

where ρ_{s_0} has been defined in (4.6). Moreover, we notice that (4.3) and (4.21) are actually independent of the choice of the cut-off function φ made in (4.6).

We conclude the paper by proving the continuity of the energy release rate $\mathcal{G}$ with respect to the Hausdorff convergence of the fractures, always under the assumptions that the crack sets are smooth. Let us fix $g \in H^1(\Omega \backslash \Gamma_0; \mathbb{R}^2)$, $M > 0$, and Γ_0 a closed C^∞-curve without self-intersection, contained in Ω except for its initial point (belonging to $\partial\Omega$), and such that $\Omega \backslash \Gamma_0$ is the union of two Lipschitz sets. We define $\mathcal{R}_M$ as the set of closed and simple C^∞-curves in Ω containing Γ_0 and such that

- $\Gamma\backslash\Gamma_0 \subset\subset \Omega$;
- denoted by $\gamma\colon [0, \mathcal{H}^1(\Gamma)] \to \mathbb{R}^2$ the arc-length parametrization of Γ, $\|\gamma^{(k)}\|_\infty \leq M$ for every $k \in \{0, 1, 2\}$,

where the superscript (k) denotes the k-th derivative. It is clear that $\mathcal{R}_M$ is compact with respect to the Hausdorff convergence of sets.

Given a sequence $\Gamma^n \in \mathcal{R}_M$ converging in the Hausdorff metric of sets to some $\Gamma^\infty \in \mathcal{R}_M$ as $n \to +\infty$, we denote by $u^n \in H^1(\Omega\backslash\Gamma^n; \mathbb{R}^2)$ the solution of

$$\min\{\mathcal{E}(u, \Gamma^n) : u \in H^1(\Omega\backslash\Gamma^n; \mathbb{R}^2), \, u = g \text{ on } \partial\Omega\}. \qquad (4.22)$$

Similarly, we indicate with $u^\infty \in H^1(\Omega\backslash\Gamma^\infty; \mathbb{R}^2)$ the solution of the same minimum problem for $n = +\infty$.

Moreover, if $s^n := \mathcal{H}^1(\Gamma^n)$ and $\gamma^n\colon [0, s^n] \to \Omega$ is the arc-length parametrization of Γ^n and, similar to (4.6), we assume that Γ^n, in a neighborhood $\mathrm{B}_r(\gamma^n(s^n))$ of its tip, is the graph of a suitable C^∞-function ζ^n, and we set

$$\rho^n = (\gamma_1^n)'(s^n)\varphi^n \begin{pmatrix} 1 \\ (\zeta^n)' \end{pmatrix} \qquad (4.23)$$

for a suitable cut-off function $\varphi^n \in C_c^\infty(\mathrm{B}_{r/2}(\gamma^n(s^n)))$. Recalling (4.21), we can write

$$\mathcal{G}(\Gamma^n) = \int_{\Omega\backslash\Gamma^n} \boldsymbol{\sigma}(u^n) : \nabla u^n \nabla\rho^n \, \mathrm{d}x - \frac{1}{2}\int_{\Omega\backslash\Gamma^n} \boldsymbol{\sigma}(u^n) : \nabla u^n \, \mathrm{div}\rho^n \, \mathrm{d}x. \qquad (4.24)$$

Formulas similar to (4.23) and (4.24) hold also for Γ^∞, setting $s^\infty := \mathcal{H}^1(\Gamma^\infty)$, denoting by $\gamma^\infty\colon [0, s^\infty] \to \mathbb{R}^2$ its arc-length parametrization, and assuming that Γ^∞ is the graph of a C^∞-function ζ^∞ in $\mathrm{B}_r(\gamma^\infty(s^\infty))$. In view of the convergence of Γ^n to Γ^∞ in the Hausdorff metric, we notice that, at least for n sufficiently large, in the definition (4.23) of ρ^n and of ρ^∞ we could fix a common cut-off function φ, independent of n.

We are now ready to prove the continuity of the energy release rate $\mathcal{G}$ with respect to the Hausdorff convergence of sets in the class $\mathcal{R}_M$.

Corollary 4.3. *Let $M > 0$ and $\Gamma^n, \Gamma^\infty \in \mathcal{R}_M$ be such that $\Gamma^n \to \Gamma^\infty$ as $n \to +\infty$ with respect to the Hausdorff metric of sets. Then, $\mathcal{G}(\Gamma^n) \to \mathcal{G}(\Gamma^\infty)$.*

Proof. Following the lines of [20, Lemma 3.7] and of [2, Lemma 5.5], it is possible to construct a family of C^2-diffeomorphisms $\Psi^n : \overline{\Omega} \to \overline{\Omega}$ which, for every n, map Γ^n into Γ^∞, keep the boundary $\partial\Omega$ fixed, and such that Ψ^n and $(\Psi^n)^{-1}$ converge to the identity function in $C^1(\overline{\Omega}; \mathbb{R}^2)$. With the notation introduced in (4.22), setting $u_\infty^n := u^\infty \circ \Psi^n$ we clearly have that $u_\infty^n \in H^1(\Omega\backslash\Gamma^n; \mathbb{R}^2)$, $u_\infty^n \to u^\infty$ in $L^2(\Omega; \mathbb{R}^2)$, and $\nabla u_\infty^n \to \nabla u^\infty$ in $L^2(\Omega; \mathbb{M}^2)$ as $n \to +\infty$.

We now prove that $\nabla u^n \to \nabla u^\infty$ in $L^2(\Omega; \mathbb{M}^2)$. In view of (4.22) and of Korn inequality (see, e.g., [8, Theorem 4.2]), we have that, up to a subsequence, $\nabla u^n \rightharpoonup \nabla v$ weakly in $L^2(\Omega; \mathbb{M}^2)$ for some $v \in H^1(\Omega\backslash\Gamma^\infty; \mathbb{R}^2)$. Again by (4.22) and by lower semicontinuity, we have that

$$\mathcal{E}(v, \Gamma^\infty) \leq \liminf_n \mathcal{E}(u^n, \Gamma^n) \leq \limsup_n \mathcal{E}(u^n, \Gamma^n) \leq \limsup_n \mathcal{E}(u_\infty^n, \Gamma^n)$$
$$= \mathcal{E}(u^\infty, \Gamma^\infty),$$

from which we deduce that $v = u^\infty$ and that $\nabla u^n \to \nabla u^\infty$ in $L^2(\Omega; \mathbb{M}^2)$ as $n \to +\infty$.

In view of the Hausdorff convergence of Γ^n to Γ^∞ and of the regularity of the family $\mathcal{R}_M$, we have that, up to a reparametrization, γ^n converges to γ^∞ weakly* in $W^{2,\infty}([0, s^\infty]; \mathbb{R}^2)$ and $(\zeta^n)' \to (\zeta^\infty)'$ uniformly in $\mathrm{B}_r(\gamma^\infty(s^\infty))$. Therefore, $\rho^n \to \rho^\infty$ in $W^{1,\infty}(\Omega; \mathbb{R}^2)$ and, passing to the limit in the expression (4.24) of $\mathcal{G}(\Gamma^n)$, we deduce that $\mathcal{G}(\Gamma^n) \to \mathcal{G}(\Gamma^\infty)$ as $n \to +\infty$. $\qquad\square$

Acknowledgements

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