



Formation of singularities of spherically symmetric solutions to the 3D compressible Euler equations and Euler–Poisson equations

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Abstract. By introducing a new averaged quantity with a fast decay weight to perform Sideris’s argument (Commun Math Phys 101:475–485, 1985) developed for the Euler equations, we extend the formation of singularities of classical solution to the 3D Euler equations established in Makino et al. (Jpn J Appl Math 3:249–257, 1986) and Sideris (1985) for the initial data with compactly supported disturbances to the spherically symmetric solution with general initial data in Sobolev space. Moreover, we also prove the formation of singularities of the spherically symmetric solutions to the 3D Euler–Poisson equations, but remove the compact support assumptions on the initial data in Makino and Perthame (Jpn J Appl Math 7:165–170, 1990) and Perthame (Jpn J Appl Math 7:363–367, 1990). Our proof also simplifies that of Lei et al. (Math Res Lett 20:41–50, 2013) for the Euler equations and is undifferentiated in dimensions.

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1. Introduction

It is well-known that Sideris [15] first proved the important fact that the C^1 solution to the three-dimensional Euler equations for compressible fluids must develop singularity in finite time. His proof consists of two critical ingredients, one is the finite propagation of compactly supported disturbances of the solution, the other is the evolution of certain averaged quantities formed out of the solution. In the proof, he assumed that the initial density ρ_0 is positive on the entire space ($\inf \rho_0 > 0$) and has a positive background state outside a ball ($\rho_0 = \bar{\rho}, B_R^c$), and the initial velocity u_0 is compactly supported on the same ball. This design makes sure that, on the one hand, the Euler equations may be written as a positive definite, symmetric hyperbolic system by the

usual symmetrization, and thus possess a unique, local C^1 solution $(\rho - \bar{\rho}, u)$ via the theory [4, 9], on the other hand, by the local energy estimates [14], the solution $(\rho - \bar{\rho}, u)$ is supported on a ball, whose radius is determined by the sound speed corresponding to the background state $\bar{\rho}$, and thus has linear spreading rate. To prove the first main result [15], Sideris introduced an averaged quantity, the radial momentum $\int \rho u x \, dx$, whose evolution connects with some other averaged quantities, for example, the mass, energy and moment of inertia $\int \rho |x|^2 \, dx$ (refer to [17] for an exhaustive account). The crucial idea is to estimate the upper bound of the moment of inertia which heavily relies on the finite propagation speed. The sign of relative total pressure mainly depends on the convexity of the pressure but may be independent of the finite propagation speed (see [18]). Finally, the radial momentum obeys a Riccati type inequality which forces the lifespan of the solution to be finite if the initial velocity is supersonic in some region. By studying other alternative averaged quantities, in the third main result, Sideris established the formation of singularities without any condition of largeness on the initial velocity, which also provides an upper bound of the lifespan in exponential type $\exp(C/\varepsilon^2)$, if the initial data has a perturbation of order ε from the positive background state. In another paper [16], Sideris refined the upper bound $\exp(C/\varepsilon^2)$ of the lifespan to the order $\exp(C/\varepsilon)$ for irrotational initial velocity, which is based on a key observation that the linear operator of acoustics is invariant under the Lorentz transformations for irrotational velocity fields.

As the initial density is surrounded by vacuum, it does not seem yet to be straightforward to apply the theory [4, 9] directly to the Euler equations via the usual symmetrization for lack of positive definiteness. Makino et al. [10] made use of the sound speed as an independent variable in place of the density which together with the original velocity symmetrized the Euler equations to a positive definite hyperbolic system, and thus the well-posedness of classical solution follows. Then, under the further compact support assumptions on the initial density and velocity, the authors showed that the support of the solution does not change with respect to time as long as the solution is tame. This fact gives rise to the moment of inertia an upper bound independent of the time, however, whose lower bound spreads quadratically with respect to time. This obvious contradiction implies that the blowup must occur in finite time for the tame solution. Makino and Ukai [11] continually applied the same symmetrization [10] to establish the local solvability of three-dimensional Euler–Poisson equations with the gravitational forces by a fixed point argument, the idea in which is also effective in the presence of the electrostatic forces.

If the initial data are spherically symmetric and have compact support, for the three-dimensional Euler–Poisson equations with gravitational forces, Makino and Perthame [12] proved that the support of the tame solution does not spread over time with the aid of the sign of the gravitational force, which presents a similar blowup result of classical to [10]. When the external forces are the electrostatic force, Perthame [13] showed that the support of spherically symmetric tame solution grows quadratically. This means that the moment of inertia is sandwiched between a pair of quadratic functions whose coefficients

consist of the total energy, mass and so on. By comparing the coefficients of the two quadratic functions, the blowup of classical solution follows if the initial energy is large in comparison with the initial mass.

In [15], the maximum propagation speed of a front into a constant state is a priori determined by the positive background state. In [10, 12, 13], the maximum propagation speed of compact support is also a priori determined by the initial data. However it is not clear how the support of the general spherically symmetric solution in Sobolev space propagates, especially, as the presence of non-local terms in the compressible Euler–Poisson equations. To overcome the difficulty caused by the lack of the finite speed propagation, one needs to consider some new averaged quantities which may localize the far field for applying Sideris’s argument [15]. Starting from the formulation that, as observed in [15], the Euler equations may be written as a quasi-linear wave-type equation in terms of the density with inhomogeneous terms involving the velocity, Lei et al. [7] studied the evolutions of some new averaged quantities with fast decay weights which lead to some formation of singularities of spherically symmetric solutions to the Euler equations in two and three spatial dimensions, in which the authors removed the compactly supported disturbances assumptions on the initial data. The proofs present delicate differences between the two and three spatial dimensions due to the different weights.

There are extensive studies on the formation of singularities of the one-dimensional Euler equations and Euler–Poisson equations which are most closely related to the method of characteristics, also developed at the background of hyperbolic systems of conservation laws in one spatial dimension. However we cannot exhaust all of them here and only list some of them for the reader’s reference convenience [2, 6, 8, 20], one may also refer to [1, 3] for a fairly complete list of the references.

This work in the present paper is devoted to the investigation of the formation of singularities of spherically symmetric solutions with general initial data in Sobolev space to the three-dimensional Euler equations and Euler–Poisson equations with gravitational force or electrostatic force. However it seems that the method of [7] doesn’t apply directly to the Euler–Poisson equations. Different from [7], we study the equation of the velocity and work with spherical coordinates instead of Eulerian coordinates, which leads us to find a very simple weight $\exp(-r)$ that is effective to both the Euler equations and Euler–Poisson equations. On the other hand, our proof simplifies that of [7] and is also undifferentiated between the two- and three-dimensional Euler equations. Specifically, we introduce a simple averaged quantity $F(t)$ with the new weight $\exp(-r)$, which is efficient in performing Sideris’s argument [15] to the Euler–Poisson equations, as well as the Euler equations. Due to the good sign of the gravitational forces, we may handle the Euler equations and Euler–Poisson equations with gravitational forces at the same time. Following Sideris’s argument [15], we study the evolution of $F(t)$ that yields a middle term, the weighted kinetic energy $Q(t)$, which may bound $F(t)$ from above, but independent of the time. On the occasions of the Euler–Poisson equations with electrostatic forces, no good sign to use, we have to employ the weighted

Hardy inequality to treat with the high singularity near $r = 0$. However one still can control the other middle term coming from the electrostatic forces, but in terms of the initial mass and physical energy instead, and all the estimates are independent of the time. Finally $F(t)$ satisfies a Riccati type inequality and thus the singularity forms in finite time. The above formation of singularities of the Euler equations and Euler–Poisson equations with gravitational forces provide an upper bound for the lifespan to a time of the order C/ε , this together with the lower bound C/ε from [10, 11], yields a by-product, the sharp bound of the lifespan in these two situations.

Section 2 contains the reformulation of the problem, as well as some necessary preliminaries. In Sect. 3 we revisit the local well-posedness of classical solutions to the three-dimensional Euler equations and Euler–Poisson equations with gravitational forces or electrostatic forces. We establish the formation of singularities for spherically symmetric solutions in Sobolev space in Sect. 4. Section 5 presents some examples to justify the conditions in the formation of singularities.

2. Reformulation of the problem

We consider here the Euler equations for an isentropic ideal fluid in three-dimension which are governed by

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0. \end{cases} \tag{2.1}$$

Here $\rho, u = (u_1, u_2, u_3)$ and $p = p(\rho)$ denote the density, velocity and pressure, respectively. For the ideal polytropic gas, the equation of state is given by $p = A\rho^\gamma$ with the adiabatic index $\gamma > 1$ and a constant $A > 0$. When considering gas motion under self-gravitation or describing the dynamics of a plasma, these physical phenomenon may be modeled by the Euler–Poisson system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \pm \rho \nabla \phi, \\ \Delta \phi = 4\pi \mathcal{G} \rho, \end{cases} \tag{2.2}$$

where we use the signs “ \pm ” to distinguish electrostatic forces “ $+$ ” and self-gravitational forces “ $-$ ”. We set both the constant A and universal gravitational constant \mathcal{G} to be unit for simplicity. One can solve the Newtonian potential ϕ from the elliptic equation in (2.2) in terms of ρ to reduce (2.2) to a much simpler system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \pm \rho \nabla G * \rho, \end{cases} \tag{2.3}$$

in which G stands for the three-dimensional Green’s function (factoring out a constant) which is given by $G(x) = -1/|x|$. We denote by $s(\rho)$ the local sound speed which is determined by the formula $s(\rho) = \sqrt{p_\rho(\rho)}$. Denote $\zeta = \frac{2}{\gamma-1}s(\rho)$, as an independent variable instead of ρ together with velocity, which may

symmetrize the Euler equations (2.1) and Euler–Poisson equations (2.3) to

$$\partial_t V + \sum_{i=1}^n A_i(V) \partial_{x_i} V + \delta B(V) = 0. \quad (2.4)$$

Here δ takes the values 0, 1, -1 which are corresponding to the Euler equations, Euler–Poisson equations with electrostatic forces and with gravitational forces, respectively. The notations in the system (2.4) are specifically as follows

$$V = \begin{pmatrix} \zeta \\ u \end{pmatrix}, \quad B(V) = \begin{pmatrix} 0 \\ -\left[\frac{(\gamma-1)^2}{4\gamma}\right]^{\frac{1}{\gamma-1}} \nabla G * \zeta^{\frac{2}{\gamma-1}} \end{pmatrix},$$

$$A_1(V) = \begin{pmatrix} u_1 & \frac{(\gamma-1)\zeta}{2} & 0 & 0 \\ \frac{(\gamma-1)\zeta}{2} & u_1 & 0 & 0 \\ 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & u_1 \end{pmatrix},$$

$$A_2(V) = \begin{pmatrix} u_2 & 0 & \frac{(\gamma-1)\zeta}{2} & 0 \\ 0 & u_2 & 0 & 0 \\ \frac{(\gamma-1)\zeta}{2} & 0 & u_2 & 0 \\ 0 & 0 & 0 & u_2 \end{pmatrix},$$

$$A_3(V) = \begin{pmatrix} u_3 & 0 & 0 & \frac{(\gamma-1)\zeta}{2} \\ 0 & u_3 & 0 & 0 \\ 0 & 0 & u_3 & 0 \\ \frac{(\gamma-1)\zeta}{2} & 0 & 0 & u_3 \end{pmatrix}.$$

For notational convenience, we define the partial and total energy functionals as

$$\mathcal{E}^{(k)}(t) = \|V^{(k)}(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2$$

and

$$\mathcal{E}(t) = \sum_{k=0}^3 \mathcal{E}^{(k)}(t),$$

respectively. Here the maximum index in the above summation is to make sure that the solutions constructed under these energy functionals are classical via Sobolev embedding. We denote by \mathcal{X}_T the Sobolev space

$$C([0, T]; H^3(\mathbb{R}^3)) \cap C^1([0, T]; H^2(\mathbb{R}^3))$$

for simplicity which will be frequently used in the well-posedness and singularity formation theories.

We are interested in studying formation of singularities of classical solutions with conservative mass and physical energy to the three-dimensional Euler equations and Euler–Poisson equations with gravitational forces or electrostatic forces. We use $M(t)$ and $E_\delta(t)$ to denote the mass and physical energy respectively which are defined by

$$M(t) = \int_{\mathbb{R}^3} \rho \, dx$$

and

$$E_\delta(t) = \int_{\mathbb{R}^3} \left(\frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \rho^\gamma - \frac{\delta}{2} \rho G * \rho \right) dx.$$

Here δ has the same meaning as (2.4). We should mention that $E_1(t)$ is always nonnegative due to the negative sign of $G(x)$. However $E_{-1}(t)$ may be nonnegative or negative, whose sign depends on initial setup.

The notation C always denotes a nonnegative universal constant which may be different from line to line but is independent of the parameters involved. Otherwise, we will specify it by the notation $C(a, b, \dots)$. We write $f \lesssim g$ ($f \gtrsim g$) when $f \leq Cg$ ($f \geq Cg$), and $f \approx g$ when $f \lesssim g \lesssim f$. The notation ε always stands for a sufficiently small positive number throughout the paper.

3. Local well-posedness

In this section we will revisit the well-posedness of classical solutions to the Euler equations and Euler–Poisson equations with electrostatic forces or gravitational forces established in [10, 11]. To avoid the trivial cases we always assume in the present paper that the initial data satisfy

$$\rho_0 \geq 0, \quad 0 < \mathcal{E}(0), M(0), E_\delta(0) < \infty.$$

We first state the well-posedness of the Euler equations ($\delta = 0$).

Proposition 3.1. *If $1 < \gamma \leq \frac{5}{3}$. Then there exist a positive number*

$$T \gtrsim \mathcal{E}^{-\frac{1}{2}}(0) \tag{3.1}$$

and a unique solution $(\rho, u) \in \mathcal{X}_T$ of the Euler equations (2.1) with $(\rho, u)|_{t=0} = (\rho_0, u_0)$ satisfying $\rho \geq 0$, which possesses the conservations of total mass and energy.

Proof. Since the system (2.4) is a positive definite hyperbolic system, the standard theory [4, 9] applies to show that the system (2.4) is locally well-posed in \mathcal{X}_T for some positive time T . Moreover $\mathcal{E}(t)$ satisfies the energy estimate

$$\frac{d}{dt} \mathcal{E}(t) \lesssim \mathcal{E}^{\frac{3}{2}}(t),$$

which provides a lower bound $T \gtrsim \mathcal{E}^{-\frac{1}{2}}(0)$. One solves the first equation of (2.4) to deduce $\zeta \geq 0$.

Since ζ belongs to \mathcal{X}_T , so does ρ if $1 < \gamma \leq \frac{5}{3}$. This is a consequence of $\rho = C(\gamma)\zeta^{\frac{2}{\gamma-1}}$ (the definition of the sound speed). Then Sobolev embedding yields $(\rho, u) \in C^1([0, T] \times \mathbb{R}^3)$, which implies that the inverse map of $(\rho, u) \mapsto (\zeta, u)$ is continuously differentiable. So (ρ, u) solves the Cauchy problem of (2.1). Since we have shown $(\rho, u) \in \mathcal{X}_T$, which guarantees the integration by parts in proving the conservations of mass. \square

The well-posedness of the Euler–Poisson equations ($\delta = \pm 1$) may be stated as follows.

Proposition 3.2. *If $1 \leq \gamma \leq \frac{5}{3}$. Then there exist a positive number*

$$T \gtrsim \max \left\{ \mathcal{E}^{-\frac{1}{2}}(0), \mathcal{E}^{\frac{\gamma-3}{2(\gamma-1)}}(0) \right\} \tag{3.2}$$

and a unique solution $(\rho, u) \in \mathcal{X}_T$ of the Euler–Poisson equations (2.3) with $(\rho, u)|_{t=0} = (\zeta_0, u_0)$ satisfying $\rho \geq 0$, which possesses the conservations of total mass and energy.

Proof. We here give only an a priori estimate. Applying D^k , $0 \leq k \leq 3$ to (2.4), then multiplying it by $D^k V$ and using integration by parts to get

$$\frac{d}{dt} \mathcal{E}(t) \lesssim \mathcal{E}^{\frac{3}{2}}(t) + \sum_{k=0}^3 \|D^k(\nabla G * \zeta^{\frac{2}{\gamma-1}})\|_{L^2(\mathbb{R}^3)} \|D^k U\|_{L^2(\mathbb{R}^3)}. \tag{3.3}$$

Note that $|\nabla G| = 1/|x|^2$, we employ the Hardy–Littlewood–Sobolev inequality from ‘‘Appendix’’ to deduce

$$\begin{aligned} \|D^k(\nabla G * \zeta^{\frac{2}{\gamma-1}})\|_{L^2(\mathbb{R}^3)} &= \|\nabla G * D^k(\zeta^{\frac{2}{\gamma-1}})\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|D^k(\zeta^{\frac{2}{\gamma-1}})\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \\ &\lesssim \|D^k(\zeta^{\frac{2}{\gamma-1}})\|_{L^1(\mathbb{R}^3)}^{\frac{2}{3}} \|D^k(\zeta^{\frac{2}{\gamma-1}})\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}}. \end{aligned} \tag{3.4}$$

The Gagliardo–Nirenberg inequality shows

$$\|D^k(\zeta^{\frac{2}{\gamma-1}})\|_{L^1(\mathbb{R}^3)} \lesssim \|\zeta^{\frac{2}{\gamma-1}}\|_{L^1(\mathbb{R}^3)}^{\frac{3}{3+2k}} \|D^k(\zeta^{\frac{2}{\gamma-1}})\|_{L^2(\mathbb{R}^3)}^{\frac{2k}{3+2k}}. \tag{3.5}$$

Then substituting (3.5) to (3.4) yields

$$\|D^k(\nabla G * \zeta^{\frac{2}{\gamma-1}})\|_{L^2(\mathbb{R}^3)} \lesssim \|\zeta^{\frac{2}{\gamma-1}}\|_{L^1(\mathbb{R}^3)}^{\frac{2}{3+2k}} \|D^k(\zeta^{\frac{2}{\gamma-1}})\|_{L^2(\mathbb{R}^3)}^{\frac{1+2k}{3+2k}}. \tag{3.6}$$

Note that $1 < \gamma \leq \frac{5}{3}$, each term $D^k(\zeta^{\frac{2}{\gamma-1}})$, $k = 0, 1, 2, 3$ may be bounded by the norm $H^3(\mathbb{R}^3)$ of ζ via Sobolev embedding. Thus one can estimate (3.6) continually to obtain

$$\|D^k(\nabla G * \zeta^{\frac{2}{\gamma-1}})\|_{L^2(\mathbb{R}^3)} \lesssim \|\zeta^{\frac{4-2\gamma}{\gamma-1}}\|_{L^\infty(\mathbb{R}^3)}^{\frac{2}{3+2k}} \|\zeta^2\|_{L^1(\mathbb{R}^3)}^{\frac{2}{3+2k}} \|\zeta\|_{H^3(\mathbb{R}^3)}^{\frac{2(1+2k)}{(\gamma-1)(3+2k)}} \lesssim \|\zeta\|_{H^3(\mathbb{R}^3)}^{\frac{2}{\gamma-1}}. \tag{3.7}$$

Inserting (3.7) into (3.3) gives

$$\frac{d}{dt} \mathcal{E}(t) \lesssim \mathcal{E}^{\frac{3}{2}}(t) + \mathcal{E}^{\frac{\gamma+1}{2(\gamma-1)}}(t),$$

which in view of the Grönwall’s inequality yields (3.2). The rest is akin to the corresponding part in the proof of Proposition 3.1. □

4. Formation of singularities

In this section we study the formation singularities of spherically symmetric solutions to the three-dimensional Euler equations and Euler–Poisson equations with gravitational forces or electrostatic forces. For spherically symmetric motions, the solution to (2.1) or (2.3) established in Sect. 3, emanating from spherically symmetric initial data

$$\rho_0(x) = \rho_0(r), \quad u_0(x) = \frac{x}{r}v_0(r), \quad r = |x|, \tag{4.1}$$

possess the spherically symmetric form

$$\rho(t, x) = \rho(t, r), \quad u(t, x) = \frac{x}{r}v(t, r), \tag{4.2}$$

which also satisfies

$$\begin{cases} \partial_t \rho + \frac{1}{r^2} \partial_r (r^2 \rho v) = 0, \\ \rho(\partial_t v + v \partial_r v) + \partial_r p = \delta \frac{4\pi \rho}{r^2} \int_0^r \rho(t, l) l^2 dl. \end{cases} \tag{4.3}$$

Here δ has the same meaning with (2.4) and $r > 0$. Let $(\rho, u) \in \mathcal{X}_T$. As observed in [7], we see from (4.2) that each component of u is an odd function, so the velocity u should satisfy $u(t, 0) = 0$ which in turn implies $v(t, 0) = 0$. Since the density ρ is radial, the characteristic curve always maps the center $r = 0$ to itself as long as the solution is smooth, or there must be at least two characteristic curves intersecting at $r = 0$. It follows from the mass equation of (2.1) or (2.3) that $\rho(t, 0) = 0$ if the initial density is imposed on $\rho_0(0) = 0$. In the following we will study formation singularities of spherically symmetric solutions with the initial data $\rho_0(0) = 0$. For convenience, we formulate the above observation as

$$\rho(t, 0) = 0, \quad v(t, 0) = 0. \tag{4.4}$$

The mass and physical energy defined in Sect. 3 may be transferred into the following corresponding to spherically symmetric solutions to (4.3) as

$$M(t) = 4\pi \int_0^\infty \rho(t, r) r^2 dr$$

and

$$\begin{aligned} E_\delta(t) &= 4\pi \int_0^\infty \left(\frac{1}{2} \rho v^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) (t, r) r^2 dr \\ &\quad + 8\pi^2 \delta \int_0^\infty \int_0^\infty K(l, r) \rho(t, l) \rho(t, r) l^2 r^2 dl dr, \end{aligned}$$

where the kernel $K(l, r)$ is defined by

$$K(l, r) = \begin{cases} 1/l & \text{if } r \leq l, \\ 1/r & \text{if } r \geq l. \end{cases}$$

We should emphasize that the mass and physical energy are only used in the formation of singularities of Euler–Poisson equations with electrostatic forces ($\delta = 1$).

Apart from the mass and physical energy, we further introduce the weighted total radial velocity

$$F(t) = - \int_0^\infty v(t, r) \exp(-r) dr,$$

which will play a crucial role in studying formation of singularities afterwards. It is clear that the functional $F(t)$ is well defined as long as the solution (ρ, u) stays in \mathcal{X}_T via Sobolev embedding.

Let $(\rho, u) \in \mathcal{X}_T$ be the solution to (2.1) or (2.3) with the spherically symmetric initial data (4.1).

We first state singularity formation of the Euler equations ($\delta = 0$).

Theorem 4.1. *If the initial data (ρ_0, u_0) satisfy*

$$\rho_0(0) = 0, \quad F(0) > 0.$$

Then, the lifespan T^ of the classical solution (ρ, u) to (2.1) with (4.1) is bounded above by*

$$T^* \leq \frac{2}{F(0)}. \tag{4.5}$$

Proof. We will prove that the lifespan T^* is finite by contradiction. Assume $T^* = \infty$. It follows from the momentum equation of (4.3) that

$$\frac{d}{dt}F(t) = \int_0^\infty \left[v\partial_r v + \frac{\gamma}{\gamma-1}\partial_r(\rho^{\gamma-1}) \right] \exp(-r) \, dr. \tag{4.6}$$

Integrating by parts in view of (4.4), one has

$$\int_0^\infty v\partial_r v \exp(-r) \, dr = \frac{1}{2} \int_0^\infty v^2 \exp(-r) \, dr \tag{4.7}$$

and

$$\int_0^\infty \frac{\gamma}{\gamma-1}\partial_r(\rho^{\gamma-1}) \exp(-r) \, dr = \frac{\gamma}{\gamma-1} \int_0^\infty \rho^{\gamma-1} \exp(-r) \, dr. \tag{4.8}$$

Inserting (4.7) and (4.8) into (4.6) yields

$$\begin{aligned} \frac{d}{dt}F(t) &= \frac{1}{2} \int_0^\infty v^2 \exp(-r) \, dr + \frac{\gamma}{\gamma-1} \int_0^\infty \rho^{\gamma-1} \exp(-r) \, dr \\ &\geq \frac{1}{2} \int_0^\infty v^2 \exp(-r) \, dr =: Q(t), \end{aligned} \tag{4.9}$$

in which we have used the fact that ρ is non-negative in the last inequality.

To control $F(t)$ in terms of $Q(t)$, we employ the Cauchy-Schwartz inequality to deduce

$$F^2(t) \leq \int_0^\infty v^2 \exp(-r) \, dr \int_0^\infty \exp(-r) \, dr = 2Q(t). \tag{4.10}$$

We conclude from (4.9) and (4.10) that

$$\frac{d}{dt}F(t) \geq \frac{1}{2}F^2(t).$$

It follows that

$$F(t) \geq \frac{F(0)}{1 - \frac{F(0)}{2}t}.$$

This implies that $F(t)$ will develop singularity no later than the time $\frac{2}{F(0)}$. This contradiction ends the proof. \square

Remark 4.1. The Euler equations in general dimension may be written in spherical coordinates as

$$\begin{cases} \partial_t \rho + \frac{1}{r^{n-1}} \partial_r (r^{n-1} \rho v) = 0, \\ \rho (\partial_t v + v \partial_r v) + \partial_r p = 0. \end{cases}$$

The proof of Theorem 4.1 can be applied uniformly here and the same result follows. Our proof also simplifies that of [7].

The following is the formation singularity result for the Euler–Poisson equations in the presence of gravitational forces ($\delta = -1$).

Theorem 4.2. *If the initial data (ρ_0, u_0) satisfy*

$$\rho_0(0) = 0, \quad F(0) > 0.$$

Then, the lifespan T^ of (ρ, u) to (2.3) ($\delta = -1$) with (4.1) is bounded above by*

$$T^* \leq \frac{2}{F(0)}.$$

Proof. One calculates from the momentum equation of (4.3) that

$$\begin{aligned} \frac{d}{dt} F(t) &= \int_0^\infty \left[v \partial_r v + \frac{\gamma}{\gamma - 1} \partial_r (\rho^{\gamma-1}) + \frac{4\pi}{r^2} \int_0^r \rho(t, l) l^2 dl \right] \exp(-r) dr \\ &\geq \int_0^\infty \left[v \partial_r v + \frac{\gamma}{\gamma - 1} \partial_r (\rho^{\gamma-1}) \right] \exp(-r) dr. \end{aligned}$$

The rest is same as the proof of Theorem 4.1. □

In contrast to the gravitational forces case, without good sign to use, the blowup formation in the case of the electrostatic forces ($\delta = 1$) is slightly more complicated.

Theorem 4.3. *Let $\sigma \in (0, \frac{1}{2})$ and $\frac{3-\sigma}{2-\sigma} < \gamma \leq \frac{5}{3}$. If (ρ_0, u_0) satisfy*

$$\rho_0(0) = 0$$

and

$$\frac{1}{4} F^2(0) \geq c_1 \left(M(0) + (\gamma - 1) E_1(0) \right) + c_2, \tag{4.11}$$

where

$$c_1 = \frac{2 - \sigma}{3 - \sigma} \left(\frac{3 - \sigma}{3 - 3\sigma + \sigma^2} \right)^{\frac{3-\sigma}{2-\sigma}}, \quad c_2 = \frac{4\pi e^{-1} (1 + \sigma)}{\sigma (3 - \sigma)}.$$

Then the lifespan T^ of (ρ, u) to (2.3) ($\delta = 1$) with (4.1) is bounded above by*

$$T^* \leq \frac{4}{F(0)}.$$

Proof. It follows from the momentum equation of (4.3) that

$$\frac{d}{dt}F(t) = \int_0^\infty \left[v\partial_r v + \frac{\gamma}{\gamma-1}\partial_r(\rho^{\gamma-1}) - \frac{4\pi}{r^2} \int_0^r \rho(t, l)l^2 dl \right] \exp(-r) dr. \tag{4.12}$$

Denote

$$R(t) = 4\pi \int_0^\infty \frac{1}{r^2} \int_0^r \rho(t, l)l^2 dl \exp(-r) dr.$$

There is too much singularity near $r = 0$ in the integrand of $R(t)$, which can not be handled by the weighted Hardy’s inequality directly. To get round this difficulty, by compensating a power of r^σ , via the Young’s inequality and the weighted Hardy’s inequality from “Appendix”, one has

$$\begin{aligned} R(t) &= 4\pi \int_0^\infty \frac{1}{r^2} \int_0^r \rho(t, l)l^{2-\sigma} l^\sigma dl \exp(-r) dr \\ &\leq 4\pi \int_0^\infty \frac{1}{r^{2-\sigma}} \int_0^r \rho(t, l)l^{2-\sigma} dl \exp(-r) dr \\ &\leq 4\pi \int_0^\infty \frac{1}{r^{1-\sigma}} \left[\frac{2-\sigma}{3-\sigma} \left(\frac{1}{r} \int_0^r \rho(t, l)l^{2-\sigma} dl \right)^{\frac{3-\sigma}{2-\sigma}} + \frac{1}{3-\sigma} \right] \exp(-r) dr \\ &\leq \frac{4\pi(2-\sigma)}{3-\sigma} \int_0^\infty \frac{1}{r^{1-\sigma+\frac{3-\sigma}{2-\sigma}}} \left(\int_0^r \rho(t, l)l^{2-\sigma} dl \right)^{\frac{3-\sigma}{2-\sigma}} dr + \frac{4\pi}{3-\sigma} \int_0^\infty \frac{\exp(-r)}{r^{1-\sigma}} dr \\ &\leq \frac{4\pi(2-\sigma)}{3-\sigma} \left(\frac{3-\sigma}{3-3\sigma+\sigma^2} \right)^{\frac{3-\sigma}{2-\sigma}} \int_0^\infty \rho^{\frac{3-\sigma}{2-\sigma}}(t, r)r^2 dr + \frac{4\pi e^{-1}(1+\sigma)}{\sigma(3-\sigma)}, \end{aligned} \tag{4.13}$$

and the first term in (4.13) may be continuously estimated as

$$\begin{aligned} &\int_0^\infty \rho^{\frac{3-\sigma}{2-\sigma}}(t, r)r^2 dr \\ &= \int_{(0, \infty) \cap \{\rho(t, r) \leq 1\}} \rho^{\frac{3-\sigma}{2-\sigma}}(t, r)r^2 dr + \int_{(0, \infty) \cap \{\rho(t, r) > 1\}} \rho^{\frac{3-\sigma}{2-\sigma}}(t, r)r^2 dr \\ &\leq \int_{(0, \infty) \cap \{\rho(t, r) \leq 1\}} \rho(t, r)r^2 dr + \int_{(0, \infty) \cap \{\rho(t, r) > 1\}} \rho^\gamma(t, r)r^2 dr \\ &\leq \int_0^\infty \rho(t, r)r^2 dr + \int_0^\infty \rho^\gamma(t, r)r^2 dr \\ &\leq \frac{1}{4\pi} [M(0) + (\gamma - 1)E_1(0)], \end{aligned} \tag{4.14}$$

where we have used $\frac{3-\sigma}{2-\sigma} < \gamma \leq \frac{5}{3}$. Substituting (4.14) into (4.13) gives

$$R_3(t) \leq c_1 \left(M(0) + (\gamma - 1)E_1(0) \right) + c_2, \tag{4.15}$$

in which $c_1 = \frac{2-\sigma}{3-\sigma} \left(\frac{3-\sigma}{3-3\sigma+\sigma^2} \right)^{\frac{3-\sigma}{2-\sigma}}$ and $c_2 = \frac{4\pi e^{-1}(1+\sigma)}{\sigma(3-\sigma)}$. It follows from (4.12) and (4.15) that

$$\frac{d}{dt}F(t) \geq Q(t) - c_1 \left(M(0) + (\gamma - 1)E_1(0) \right) - c_2. \tag{4.16}$$

We conclude from (4.10) and (4.16) that

$$\frac{d}{dt}F(t) \geq \frac{1}{2}F^2(t) - c_1 \left(M(0) + (\gamma - 1)E_1(0) \right) - c_2.$$

In view of (4.11), by a bootstrap argument one shows that $F(t)$ is increasing and thus satisfies

$$F(t) \geq \frac{F(0)}{1 - \frac{F(0)}{4}t},$$

which shows that $F(t)$ will break down before the time $\frac{4}{F(0)}$. This is a contradiction. \square

5. Sharp bound and example

Let $(\rho, u) \in \mathcal{X}_T$ be solution to (2.1) or (2.3) with the spherically symmetric initial data (4.1). Based on the local well-posedness in Sect. 3 and formation of singularities in Sect. 4 we can obtain a sharp bound on the lifespan of the solutions to the Euler equations and Euler–Poisson equations with gravitational forces by choosing a special scale relation between the initial density and initial velocity. We also give an example to justify the conditions of the formation of singularity of the Euler–Poisson equations with electrostatic forces.

We have the following sharp bound on lifespan of the solutions to the Euler equations and Euler–Poisson equations with gravitational forces.

Theorem 5.1. *Assume that the initial density ρ_0 and velocity u_0 satisfy the scaling relation $(\rho_0, u_0) = (\varepsilon^{\frac{2}{\gamma-1}} \varrho, \varepsilon w)$. If the initial data (ρ_0, u_0) also satisfy*

$$\rho_0(0) = 0, \quad F(0) > 0. \tag{5.1}$$

Then, the lifespan T^ of (ρ, u) to (2.1) with (4.1) satisfies*

$$T^* \approx \frac{1}{\varepsilon}. \tag{5.2}$$

Proof. A direct calculation gives

$$\mathcal{E}(0) \approx \varepsilon^2, \quad F(0) \approx \varepsilon.$$

The conclusion (5.2) follows from (3.1) and (4.5). \square

Theorem 5.2. *Assume that the initial density ρ_0 and velocity u_0 satisfy the scale relation $(\rho_0, u_0) = (\varepsilon^{\frac{2}{\gamma-1}} \varrho, \varepsilon w)$. If the initial data (ρ_0, u_0) also satisfy*

$$\rho_0(0) = 0, \quad F(0) > 0.$$

Then, the lifespan T^ of (ρ, u) to (2.3) ($\delta = -1$) with (4.1) satisfies*

$$T^* \approx \frac{1}{\varepsilon}.$$

Remark 5.1. Let

$$\varrho(x) = |x|^2 \exp(-|x|^2), \quad w(x) = -x|x|^2 \exp(-|x|^2). \tag{5.3}$$

It is clear that (ρ_0, u_0) belong to $H^3(\mathbb{R}^3)$ and satisfy (5.1).

We finally give an example to justify (4.11). Taking (ϱ, w) of (5.3) and setting $(\rho_0, u_0) = (\varepsilon\varrho, \varepsilon^{-1}w)$, one calculates that

$$M(0) \approx \varepsilon, \quad F(0) \approx \varepsilon^{-1}$$

and

$$E_1(0) \approx \varepsilon^{-1} + \varepsilon^\gamma - \varepsilon^2 \approx \varepsilon^{-1}.$$

Consequently, for given $\sigma \in (0, \frac{1}{2})$, let ε go to zero, one has

$$\frac{F^2(0)}{c_1 \left(M(0) + (\gamma - 1)E_1(0) \right) + c_2} \approx \varepsilon^{-1} \gg 1.$$

This implies that (4.11) holds.

6. Appendix

We state here the Hardy–Littlewood–Sobolev inequality and weighted Hardy inequality for reference convenience, whose proofs can be found in [5, 19], respectively.

Lemma 6.1. *Given $0 < \lambda < d$ and $1 < q_1 < q_2 < \infty$ with $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\lambda}{d}$. Let I_λ be the Riesz potential of order λ on \mathbb{R}^d which is defined by*

$$I_\lambda f(x) = C(d, \lambda) \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-\lambda}} dy.$$

Then

$$\|I_\lambda f\|_{L^{q_2}(\mathbb{R}^d)} \leq C(d, \lambda, q_1) \|f\|_{L^{q_1}(\mathbb{R}^d)}.$$

Lemma 6.2. *Given $1 < \mu, q < \infty$. Then*

$$\int_0^\infty \frac{1}{r^\mu} \left(\int_0^r f(l) dl \right)^q dr \leq C(\mu, q) \int_0^\infty f^q(r) r^{q-\mu} dr,$$

where the best constant $C(\mu, q) = \left(\frac{q}{\mu-1} \right)^q$. In particular, it is the classical Hardy’s inequality as $\mu = q$.

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Compliance with ethical standards

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