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Lipschitz regularity for viscosity solutions to parabolic $p(x,t)$-Laplacian equations on Riemannian manifolds

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Abstract. We study viscosity solutions to parabolic $p(x,t)$-Laplacian equations on Riemannian manifolds under the assumption that a continuous exponent function p is Lipschitz continuous with respect to spatial variables, and satisfies $1 < p_- \le p(x,t) \le p_+ < \infty$ for some constants $1 < p_- \le p_+ < \infty$. Using Ishii–Lions' method, a Lipschitz estimate of viscosity solutions is established on Riemannian manifolds with sectional curvature bounded from below.

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1. Introduction

In this paper, we consider viscosity solutions to the parabolic $p(x,t)$-Laplacian equation

$$\Delta_{p(x,t)}u + |\nabla u|^{p(x,t)-2+q}\langle V, \nabla u\rangle - \partial_t u = f \tag{1.1}$$

over a complete Riemannian manifold M of dimension n. Here an exponent function $p : M \times [0,\infty) \to \mathbb{R}$ satisfies that $1 < p_- \le p(x,t) \le p_+ < \infty$ for some constants $1 < p_- \le p_+ < \infty$, $q \in [0,1)$ is a constant, and V is a bounded vector field on $M \times (0,\infty)$. The $p(x,t)$-Laplacian operator defined by

$$\Delta_{p(x,t)}u := \operatorname{div}\left(|\nabla u|^{p(x,t)-2}\nabla u\right)$$

appears in the study of motions of the non-Newtonian fluids, particularly electro-rheological fluids. There is an extensive literature on elliptic/parabolic problems with a variable exponent of nonlinearity in the context of divergent equations (in the Euclidean space); for instance, we refer to [1–4, 42, 43] for parabolic problems. With regard to regularities of weak solutions for the parabolic $p(x,t)$-Laplacian equations, a local boundedness and a Hölder estimate have been established in [19,44] in the framework of Orlicz–Sobolev spaces, provided that p is logarithmic Hölder continuous in space-time variables with $1 < p_- \le p_+ < \infty$. A Hölder estimate for the spatial gradient of weak solutions was proved in [2,15,45] under the assumption that p is Hölder continuous with $\max\left(1, \frac{2n}{n+2}\right) < p_- \le p_+ < \infty$ (see also [25] for partial regularity). Lastly, we mention [5,9,14,26,46] and references therein for Calderón–Zygmund estimates and higher integrability results.

The $p(x,t)$-Laplacian operator can be expressed in non-divergence form as follows:

$$\begin{aligned}\Delta_{p(x,t)} u = |\nabla u|^{p(x,t)-2} \operatorname{tr}\left\{\left(\mathbf{I} + \Big(p(x,t)-2\Big)\frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\right) D^2 u\right\} \\ + |\nabla u|^{p(x,t)-2} \log|\nabla u| \left\langle \nabla p, \nabla u \right\rangle,\end{aligned} \tag{1.2}$$

provided that p is differentiable with respect to spatial variables. The $p(x,t)$-Laplacian operator becomes degenerate at a point (x,t) where the gradient of u vanishes with $p(x,t) > 2$ while it is singular at a point (y,s) with zero gradient of u and $1 < p(y,s) < 2$. In this paper, we are concerned with a Lipschitz estimate of viscosity solutions to the parabolic $p(x,t)$-Laplacian equation (1.1) on $M \times (0,\infty)$ with a non-homogenous right-hand side, under the assumption that a continuous exponent function p is Lipschitz continuous in spatial variables; see the condition (1.3).

In the Euclidean space, Ishii and Lions in [31] proved a Lipschitz regularity of viscosity solutions to fully nonlinear elliptic/parabolic equations, which are continuous with respect to all variables including ∇u and D^2u, and possibly degenerate. Ishii–Lions' method based on a doubling variable technique was adapted by Imbert–Jin–Silvestre in [30] to prove a Lipschitz estimate for viscosity solutions to the singular/degenerate parabolic p-Laplacian type equation $\partial_t u = |\nabla u|^\gamma \Delta_p u$ with the constants $p > 1$ and $\gamma > -p$. Moreover, a Hölder estimate for the spatial gradient of viscosity solutions was also obtained in [30] provided with the constants $p > 1$ and $\gamma > 1 - p$. We also mention [35] for the result on parabolic normalized p-Laplacian operators (when $\gamma = 2 - p$), and refer to [12,13,22,28–30] and references therein for relevant results in the context of non-divergent p-Laplacian equations.

To show a Lipschitz estimate of viscosity solutions to (1.1) on $M \times (0,\infty)$, we employ the approach in [30] which relies on Ishii–Lions' method [31], with the help of Jensen's sup- and inf-convolutions. When we deal with viscosity solutions on Riemannian manifolds, there is a smoothness issue of test functions such that the squared distance function from a point is not smooth on the cut locus of the point, as well as the non-smoothness of viscosity solutions. In order to overcome difficulties from the non-smoothness of viscosity solutions and test functions in the Riemannian case, an appropriate regularization would be required, and Jensen's regularization by sup- and inf-convolutions plays a role in this paper. Jensen's regularization is a standard approximation in the context of viscosity solutions (see [17,33,34] for the Euclidean case) and was studied in the setting of Riemannian manifolds in order to establish regularities of viscosity solutions in [39], where a geometric property of sup- and inf-convolutions: semi-convexity and semi-concavity respectively, was useful in dealing with the non-smoothness of test functions due to the cut locus. We mention [16,38–40] for Krylov–Safonov type Harnack inequalities and Hölder estimates on Riemannian manifolds, where certain non-divergent techniques in the Euclidean space have been adapted for the operators including p-Laplacian elliptic operators and uniform parabolic operators. It is worth noting that a lower bound of Ricci curvature of the underlying manifold is a natural assumption for Harnack inequalities of the p-Laplacian operator (with a constant exponent $p > 1$) in [40] and the heat operator in [38], whereas sectional curvature bounded from below is taken into account for the analysis of $p(x,t)$-Laplacian operators owing to nonlinearity of a variable exponent.

Compared to Jensen's regularization used in this paper, the authors in [30] considered a smooth solution u_δ ($\delta > 0$) of the approximating equation

$$\partial_t u_\delta = \left(|\nabla u_\delta|^2 + \delta^2\right)^{\frac{\gamma+p-2}{2}} \operatorname{tr}\left\{\left(\mathbf{I} + (p-2)\frac{\nabla u_\delta \otimes \nabla u_\delta}{|\nabla u_\delta|^2 + \delta^2}\right) D^2 u_\delta\right\}$$

in $\Omega \times (0,T) \subset \mathbb{R}^n \times [0,\infty)$ with the Dirichlet boundary data $u_\delta = u$ on the parabolic boundary $\partial_p\big(\Omega \times (0,T]\big)$. Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain, and u is a viscosity solution of $\partial_t u = |\nabla u|^\gamma \Delta_p u$. They established uniform gradient estimates for u_δ with respect to $\delta > 0$, and then such gradient estimates hold for the solution u by utilizing results on existence, uniqueness and stability of approximating solutions u_δ (see also [41]). On an arbitrary Riemannian manifold which may be compact, it is not always possible to construct smooth solutions u_δ as above which approximate a viscosity solution to (1.1), so we apply Ishii–Lions' method to regularized solutions by Jensen's sup- and inf-convolutions in the Riemannian case.

Regarding the notion of viscosity solutions to the parabolic $p(x,t)$-Laplacian equation (1.1) (see Definition 2.1), we adapt the definition by Demengel in [23]. A main difficulty in defining viscosity solutions for singular operators lies in the fact that one can not test functions at a point where the gradient of a solution is zero, which has been dealt with by many authors in [10,18,27,32,37,41]. In particular, Demengel in [23] obtained existence and regularity results on viscosity solutions of the parabolic p-Laplacian equations with non-homogenous right-hand sides in the Euclidean space (for a constant exponent $p \in (1,\infty)$). Moreover, she proved equivalence between a viscosity solution introduced in [23] and a viscosity solution defined by Ohnuma–Sato [41] in the case of the homogeneous parabolic p-Laplacian equation ($f \equiv 0$ and $V \equiv 0$ in (1.1)). We remark that the authors in [41] defined viscosity solutions utilizing a certain class of admissible test functions when the gradient of a solution vanishes, which was employed in [30].

We end the introduction by stating our main result. Below and hereafter, let (M,g) be a smooth, complete Riemannian manifold of dimension n, where

g is the Riemannian metric. We denote $\langle X, Y\rangle := g(X, Y)$ and $|X|^2 := \langle X, X\rangle$ for $X, Y \in T_xM$, where T_xM is the tangent space at $x \in M$. Let $d(\cdot,\cdot)$ be the distance function on M. For a given point $y \in M$, $d_y(\cdot)$ denotes the distance function from y, i.e., $d_y(x) = d(x, y)$. Throughout this paper, a continuous function p is assumed to be differentiable with respect to spatial variables, and to satisfy

$$|\, p(x,t) - p(y,s)\,| \le \theta\, d(x,y) + \omega_0\,(|t-s|) \tag{1.3}$$

for $x, y \in B_{2R}(z_0) \subset M$ and $t, s \in (0, T]$, where θ is a positive constant and $\omega_0 : [0,\infty) \to [0,\infty)$ is a non-negative function with $\omega_0(0+) = 0$, and

$$1 < p_- \ \le\ p(x,t)\ \le\ p_+ < \infty \tag{1.4}$$

for any $(x,t) \in B_{2R}(z_0) \times (0, T]$ with some constants p_- and p_+.

Theorem 1.1. *Assume that sectional curvature is bounded from below by $-\kappa$ for $\kappa \ge 0$, i.e.,* Sec $\ge -\kappa$. *Let $0 < R \le R_0 < \infty$, $T > 0$, and $p : B_{2R}(z_0) \times (0,T] \subset M \times [0,\infty) \to \mathbb{R}$ satisfy (1.3) and (1.4) on $B_{2R}(z_0) \times (0,T]$. Let $u \in C\,(B_{2R}(z_0) \times (0,T])$ be a viscosity solution of*

$$\Delta_{p(x,t)} u + |\nabla u|^{p(x,t)-2+q}\langle V, \nabla u\rangle - \partial_t u = R^{-p(x,t)}\, f \tag{1.5}$$

in $B_{2R}(z_0) \times (0,T]$ for a constant $q \in [0,1)$ and a vector field V with $\|V\|_{L^\infty(B_{2R}(z_0)\times(0,T])} \le \beta$. Then we have that for any $x, y \in B_R(z_0)$ and $t, s \in (\,T/2,\, T]$,

$$|u(x,t) - u(y,s)| \le C\left(\frac{d(x,y)}{R} + \frac{|t-s|^{1/2}}{T^{1/2}}\right)$$

where a constant $C > 0$ depends only on n, p_-, p_+, q, $\sqrt{\kappa}R_0$, θ $\max(e, R_0 \log R_0)$, βR_0^{1-q}, $T^{-1}\max\,(R^{p_-}, R^{p_+})$, $T\max\,(R^{-p_-}, R^{-p_+})$, $\|u\|_{L^\infty(B_{2R}(z_0)\times(0,T])}$, and $\|f\|_{L^\infty(B_{2R}(z_0)\times(0,T])}$.

Consider a modulus of continuity ω_0 in time for a variable exponent p in (1.3) given by

$$\omega_0(\tau) = \begin{cases} \dfrac{c_0}{1 + |\log \tau|} & \forall\, 0 < \tau < 1; \\ c_0 & \forall\, \tau \ge 1; \end{cases} \tag{1.6}$$

with a nonnegative constant c_0. Letting $p_0 := p(z_0, 0)$, we observe that for any $(x,t), (y,s) \in B_{2R}(z_0) \times [0, R^{p_0}]$,

$$R^{p(x,t)-p(y,s)} \le \max\left(1,\ e^{4\theta\ \max(e, R_0\log R_0) + c_0\max(1,\,\log R_0)}\right)$$

in light of (1.3) and (1.6) since

$$\begin{aligned} |p(x,t) - p(y,s)| \cdot |\log R| &\le 4\theta R|\log R| + \omega_0(|t-s|)|\log R| \\ &\le 4\theta\ \max\,(e, R_0\log R_0) + c_0\max\,(1, \log R_0)\,. \end{aligned}$$

Thus it holds that

$$T^{-1}R^{p(x,t)} = R^{p(x,t)-p_0} \le C \quad \text{and} \quad TR^{-p(x,t)} = R^{p_0 - p(x,t)} \le C$$

for a constant $C > 0$ depending on $\theta\ \max\,(e, R_0\log R_0)$ and $c_0\max\,(1, \log R_0)$ and hence a locally uniform estimate follows:

Corollary 1.2. *Assume that* $\mathrm{Sec} \geq -\kappa$ *for* $\kappa \geq 0$. *Let* $0 < R \leq R_0 < \infty$, *and* $p : B_{2R}(z_0) \times [0, R^{p_0}] \to \mathbb{R}$ *satisfy* (1.3) *with* (1.6) *and* (1.4) *on* $B_{2R}(z_0) \times [0, R^{p_0}]$, *where* $p_0 := p(z_0, 0)$. *Let* $u \in C\left(B_{2R}(z_0) \times (0, R^{p_0}]\right)$ *be a viscosity solution of* (1.5) *in* $B_{2R}(z_0) \times (0, R^{p_0}]$ *for a constant* $q \in [0, 1)$ *and a vector field* V *with* $\|V\|_{L^\infty(B_{2R}(z_0)\times(0,R^{p_0}])} \leq \beta$. *Then we have that for any* $x, y \in B_R(z_0)$ *and* $t, s \in (\, R^{p_0}/2,\, R^{p_0}]$,

$$|u(x,t) - u(y,s)| \leq C\left(\frac{d(x,y)}{R} + \frac{|t-s|^{1/2}}{R^{p_0/2}}\right)$$

where a constant $C > 0$ *depends only on* n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta \max(e, R_0 \log R_0)$, $c_0 \max(1, \log R_0)$, βR_0^{1-q}, $\|u\|_{L^\infty(B_{2R}(z_0)\times(0,R^{p_0}])}$, *and* $\|f\|_{L^\infty(B_{2R}(z_0)\times(0,R^{p_0}])}$.

This result can be seen as a generalization of a scaling-invariant Lipschitz estimate for parabolic p-Laplacian equations with a constant $p > 1$ in the Euclidean space. Moreover, considering the parabolic p-Laplacian equation with a constant $p > 1$ on M with nonnegative sectional curvature, we have a Liouville type theorem for eternal solutions.

Corollary 1.3. *Assume that* M *has nonnegative sectional curvature:* $\mathrm{Sec} \geq 0$. *If* u *is a bounded viscosity solution to the parabolic* p*-Laplacian equation with a constant* $p > 1$*:*

$$\Delta_p u - \partial_t u = 0 \qquad \text{on } M \times \mathbb{R},$$

then u *is a constant.*

The rest of the paper is organized as follows. In Sect. 2, we give a notion of viscosity solutions and some known results on Jensen's regularization that are used in the paper. Section 3 is devoted to proving a logarithmic-type Lipschitz estimate in spatial variables to show a Lipschitz estimate of Theorem 1.1. In Sect. 4, we prove an 1/2-Hölder estimate in time variable.

2. Viscosity solutions

We present a refined definition of viscosity solutions for singular parabolic operators, which is adapted from [23].

Definition 2.1. (*Viscosity solution*) Let Ω be an open set in M and $T > 0$. Define

$$G\left(x, t, \nabla u, D^2 u\right) := \Delta_{p(x,t)} u + |\nabla u|^{p(x,t)-2+q} \langle V, \nabla u \rangle$$

with the $p(x,t)$-Laplacian operator $\Delta_{p(x,t)} u$ as in (1.2) in non-divergence form. For a function $f : \Omega \times (0, T] \to \mathbb{R}$, we say that $u \in C\left(\Omega \times (0, T]\right)$ is a viscosity supersolution (respectively subsolution) of the equation

$$G\left(x, t, \nabla u, D^2 u\right) - \partial_t u = f \quad \text{in } \Omega \times (0, T]$$

if the following holds: for any $(\bar{x}, \bar{t}) \in \Omega \times (0, T]$,

(i) either if any function $\varphi \in C^{2,1}\left(\Omega \times (0,T]\right)$ satisfies that $u-\varphi$ has a local minimum (respectively maximum) at $(\bar{x},\bar{t})$ in $\Omega\times(0,\bar{t}\,]$ with $\nabla\varphi(\bar{x},\bar{t}) \neq 0$, then it holds that

$$G\left(\bar{x},\bar{t},\nabla\varphi(\bar{x},\bar{t}\,),D^2\varphi(\bar{x},\bar{t}\,)\right) - \partial_t\varphi(\bar{x},\bar{t}\,) \leq f(\bar{x},\bar{t}\,) \quad (\text{respectively} \geq);$$

(ii) or, if there exist $\epsilon > 0$ and $h \in C^1\big((\bar{t}-\epsilon,\bar{t}\,]\big)$ such that

$$\begin{cases} \inf_{t\in(\bar{t}-\epsilon,\,\bar{t}\,]}\left\{u(\bar{x},t)-h(t)\right\} = u(\bar{x},\bar{t})-h(\bar{t}) \quad (\text{respectively sup}); \qquad \text{and} \\ \inf_{t\in(\bar{t}-\epsilon,\,\bar{t}\,]}\left\{u(x,t)-h(t)\right\} \text{ is locally constant in a neighborhood of } \bar{x} \quad (\text{respectively sup}), \end{cases}$$

then $-h'(\bar{t}\,) \leq f(\bar{x},\bar{t}\,)$ (respectively $\geq$).

We say that u is a viscosity solution if u is both a viscosity subsolution and a viscosity supersolution.

Remark 2.2. (a) When a viscosity solution is independent of time variable, the definition above is equivalent to the one for singular elliptic operators given in [11,40].

(b) Assuming that either $p > 2$ in $\Omega \times (0,T]$ or $p \equiv 2$, the operator $G\left(x,t,\nabla u,D^2u\right)$ is considered a continuous operator with respect to all variables x, t, ∇u, and D^2u by setting

$$G\big(x,t,0,Q\big) = \begin{cases} 0 & \text{when } p > 2 \text{ in } \Omega\times(0,T]; \\ \operatorname{tr} Q & \text{when } p \equiv 2. \end{cases} \tag{2.1}$$

Then u is a usual parabolic viscosity supersolution of (1.1) if and only if u is a viscosity supersolution of (1.1) in the sense of Definition 2.1, provided that f is continuous.

When $M = \mathbb{R}^n$, the above equivalence can be proved by a similar argument to the proof of Lemma 2 in Appendix 2 of [23]. Indeed, in light of (2.1), it is clear that a usual parabolic viscosity supersolution is a viscosity supersolution in the sense of Definition 2.1. For the proof of the converse, the condition (ii) of Definition 2.1 is useful as well as the continuity of the operator G and the Lipschitz continuity of p with respect to x. In the Riemannian setting, one can modify a proof of the Euclidean case employing some arguments for the elliptic operators on manifolds in [40, Lemma 3.2].

Remark 2.3. The notion of parabolic viscosity solutions in Definition 2.1 is slightly different from the one introduced in Definition 1 of [23] (cf. [7,30,41]). The difference with [23, Definition 1] is that for our refined parabolic viscosity solutions, we do not impose any condition on what is to happen after $t = \bar{t}$ for admissible test functions; refer to [36,39].

Considering continuous parabolic operators (with respect to all variables x, t, ∇u and D^2u) on $M = \mathbb{R}^n$, Juutinen in [36] called a viscosity solution ignoring what happens after time $\bar{t}$, *a parabolic viscosity solution*. The equivalence between a parabolic viscosity solution and the one in the usual sense (as for elliptic operators as in Definition 1 of [23]) was proved in [36, Theorem 1] for continuous parabolic operators, where a crucial ingredient of the proof is the comparison principle for viscosity solutions in the usual sense. Due to the (possible) singularity of the operator G (the condition (ii) of Definition 2.1) and a variable exponent p, the approach of [36] seems not directly applicable to the proof of the equivalence between Definition 2.1 and [23, Definition 1], which we believe is worth further investigation.

The following technical lemma will be used in the proof of a Hölder estimate in time (Lemma 4.1). The proof is similar to the one of Lemma 1 of [23]; we omit it.

Lemma 2.4. *Let $u \in C(\Omega \times (0,T])$ be a subsolution of*

$$\Delta_{p(x,t)} u + \beta |\nabla u|^{p(x,t)-1+q} - \partial_t u \ \geq \ f \quad \text{in } \Omega \times (0,T]$$

for constants $\beta \in \mathbb{R}$ and $q \geq 0$, where f is a continuous function on $\Omega \times (0,T]$. Suppose that for a point $(\bar{x}, \bar{t}) \in \Omega \times (0,T]$ and a function $h \in C^1((0,\bar{t}])$, it holds that

$$\sup_{B_\epsilon(\bar{x}) \times (\bar{t}-\epsilon,\, \bar{t}\,]} \left\{ u(x,t) - h(t) - C\, d^{\,k}(x,\bar{x}) \right\} = u(\bar{x},\bar{t}) - h(\bar{t}\,)$$

with some constants $\epsilon > 0$, $C \in \mathbb{R}$, and $k > \max\left(2,\, 1 + \dfrac{1}{p_- - 1}\right)$. Then, we have that $-h'(\bar{t}\,) \geq f(\bar{x}, \bar{t}\,)$.

As mentioned in the introduction, we make use of sup- and inf-convolutions by Jensen [33,34] in order to approximate viscosity solutions (see also [17, Chapter 5]). Recall the definition of sup- and inf-convolutions from [39]. For a bounded open set $\Omega \subset M$ and $T_2 > T_0$, let u be a continuous function on $\overline{\Omega} \times [T_0, T_2]$. For $\varepsilon > 0$, the inf-convolution of u (with respect to $\Omega \times (T_0, T_2]$), denoted by u_ε, is defined as follows: for $(x,t) \in \overline{\Omega} \times [T_0, T_2]$,

$$u_\varepsilon(x,t) := \inf_{(y,s) \in \overline{\Omega} \times [T_0, T_2]} \left\{ u(y,s) + \frac{1}{2\varepsilon} \left(d^2(y,x) + |s-t|^2 \right) \right\}.$$

In a similar way, we define the sup-convolution u^ε of u by

$$u^\varepsilon(x,t) := \sup_{(y,s) \in \overline{\Omega} \times [T_0, T_2]} \left\{ u(y,s) - \frac{1}{2\varepsilon} \left(d^2(y,x) + |s-t|^2 \right) \right\}.$$

For properties of sup- and inf-convolutions such as uniform convergence to u as ε tends to 0, we refer to Section 3 of [39]. Below and hereafter, the same notations as in [39] will be used. For example, $\operatorname{Sym} TM$ denotes the bundle of symmetric 2-tensors over M, and $\mathcal{P}^{2,\pm} u$ stand for the second order parabolic super- and sub-jets of u.

In the following lemma, we are concerned with a link between a viscosity solution and its sup- and inf-convolutions. Before stating the lemma, limiting parabolic super- and sub-jets $\overline{\mathcal{P}}^{2,\pm} u$ of u at $(x,t) \in \Omega \times (T_0, T_2]$ are defined as follows:

$$\begin{aligned}\overline{\mathcal{P}}^{2,\pm}u(x,t) := \Big\{ &(a,\zeta,A) \in \mathbb{R}\times T_xM \times \operatorname{Sym} TM_x : \exists\,(x_k,t_k,a_k,\zeta_k,A_k)\\ &\in \Omega\times(T_0,T_2]\times\mathbb{R}\times T_{x_k}M\times \operatorname{Sym} TM_{x_k}\\ &\text{s. t. } (a_k,\zeta_k,A_k)\in \mathcal{P}^{2,\pm}u(x_k,t_k), \text{ and}\\ &\lim_{k\to\infty}(x_k,t_k,a_k,\zeta_k,A_k) = (x,t,a,\zeta,A)\Big\};\end{aligned}$$

see [6,7] for slightly stronger definitions of $\mathcal{P}^{2,\pm}u$ and $\overline{\mathcal{P}}^{2,\pm}u$ for a function u defined on $\Omega\times(T_0,T_2)$. The following lemma can be deduced from the proof of [39, Proposition 3.3] considering the limiting sub-jet $\overline{\mathcal{P}}^{2,-}u$ instead of the sub-jet $\mathcal{P}^{2,-}u$. Similar properties for the limiting super-jet $\overline{\mathcal{P}}^{2,+}u$ can be obtained by using the facts that $u^\varepsilon = -(-u)_\varepsilon$ and $\overline{\mathcal{P}}^{2,+}u^\varepsilon = -\overline{\mathcal{P}}^{2,-}(-u^\varepsilon) = -\overline{\mathcal{P}}^{2,-}(-u)_\varepsilon$.

Lemma 2.5. *Assume that* $\operatorname{Sec}\ge -\kappa$ *on* M *for* $\kappa\ge 0$. *Let* H *and* Ω *be bounded open sets in* M *such that* $\overline{H}\subset\Omega$, *and* $T_0<T_1<T_2$. *Let* $u\in C\left(\overline{\Omega}\times[T_0,T_2]\right)$, *and let* ω *be a modulus of continuity of* u *on* $\overline{\Omega}\times[T_0,T_2]$. *Then there exists* $\varepsilon_0>0$ *depending only on* $||u||_{L^\infty\left(\overline{\Omega}\times[T_0,T_2]\right)}, H, \Omega, T_0,$ *and* T_1, *such that if* $0<\varepsilon<\varepsilon_0$, *then the following statements hold. Let* $(x_0,t_0)\in\overline{H}\times[T_1,T_2]$ *and let* $(a,\zeta,A)\in\overline{\mathcal{P}}^{2,-}u_\varepsilon(x_0,t_0)$.

(a) There exists a point $(y_0,s_0)\in\Omega\times(T_0,T_2]$ *such that*

$$y_0=\exp_{x_0}(-\varepsilon\zeta),\quad s_0\in\left[t_0-2\sqrt{\varepsilon m},\,t_0+2\sqrt{\varepsilon m}\right]\cap(T_0,T_2],$$

and

$$u_\varepsilon(x_0,t_0)=u(y_0,s_0)+\frac{1}{2\varepsilon}\left\{d^2(y_0,x_0)+|s_0-t_0|^2\right\}.$$

Here u_ε *denotes the inf-convolution of* u *with respect to* $\Omega\times(T_0,T_2]$ *for* $\varepsilon>0$, *and* $m:=||u||_{L^\infty\left(\overline{\Omega}\times[T_0,T_2]\right)}$.

(b) $y_0\notin\operatorname{Cut}(x_0)$, *and there is a unique minimizing geodesic joining* x_0 *to* y_0 *which is contained in* Ω. *Moreover,*

$$d^2(y_0,x_0)=\varepsilon^2|\zeta|^2\le 2\varepsilon\,\omega\left(2\sqrt{\varepsilon m}\right).$$

(c)

$$\left(a,\,L_{x_0,y_0}\zeta,\,L_{x_0,y_0}A-\kappa\min\left\{\varepsilon|\zeta|^2,2\omega\left(2\sqrt{\varepsilon m}\right)\right\}\mathbf{I}\right)\in\overline{\mathcal{P}}^{2,-}u(y_0,s_0).$$

Here L_{x_0,y_0} *denotes the parallel transport along the unique minimizing geodesic joining* x_0 *to* y_0, *and* $L_{x_0,y_0}A$ *is a symmetric bilinear form on* $T_{y_0}M$ *defined by*

$$\langle(L_{x_0,y_0}A)\cdot\xi,\,\xi\rangle_{y_0}:=\langle A\cdot(L_{y_0,x_0}\xi),\,L_{y_0,x_0}\xi\rangle_{x_0}\quad\forall\xi\in T_{y_0}M.$$

3. Lipschitz estimate

In order to prove a Lipschitz estimate for viscosity solutions to the parabolic $p(x,t)$-Laplacian equation (1.5) on a Riemannian manifold M, we follow the proof of a Lipschitz estimate in [30] (in the Euclidean space) based on Ishii–Lions' method [31] with the use of a regularization of Jensen by sup- and inf-convolutions. Firstly, we establish a logarithmic-type Lipschitz estimate for viscosity solutions.

Proposition 3.1. *Assume that* Sec $\geq -\kappa$ *on* $B_{2R}(z_0)$ *for* $\kappa \geq 0$. *Let* $0 < R \leq R_0 < \infty$, $T > 0$, *and* $u \in C\left(B_{2R}(z_0) \times (0, T]\right)$ *be a viscosity solution of* (1.5) *in* $B_{2R}(z_0) \times (0, T]$ *with* $0 \leq q < 1$,

$$\begin{aligned} &\|u\|_{L^\infty(B_{2R}(z_0)\times(0,T])} \leq 1, \quad \|f\|_{L^\infty(B_{2R}(z_0)\times(0,T])} \leq 1 \quad \textit{and} \\ &\|V\|_{L^\infty(B_{2R}(z_0)\times(0,T])} \leq \beta \end{aligned} \tag{3.1}$$

for $\beta \geq 0$. *Then we have that for any* $x, y \in B_R(z_0)$ *and* $t \in (T/2, T]$,

$$\begin{aligned} |u(x,t) - u(y,t)| \leq{}& A_1 \frac{d(x,y)}{R} \left|\log\left(\frac{d(x,y)}{R}\right)\right|^2 \\ &+ A_2 \left\{\frac{1}{R^2} d_{z_0}^2(x) + \frac{1}{R^2} d_{z_0}^2(y) + \frac{1}{T^2}(t-T)^2\right\}. \end{aligned} \tag{3.2}$$

Here constants $A_1 > 0$ *and* $A_2 > 0$ *depend only on* n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta \max\left(e, R_0 \log R_0\right)$, βR_0^{1-q}, *and* $T^{-1}\max\left(R^{p_-}, R^{p_+}\right)$.

Proof. For $0 < \varepsilon < 1$, we denote by u_ε and u^ε the inf- and sup-convolutions of u with respect to $\overline{B_{3R/2}(z_0)} \times [T/8, T]$, respectively. Define a function Φ by

$$\Phi(x,y,t) := A_1 \phi\left(\frac{d(x,y)}{R}\right) + A_2 \left\{\frac{1}{2R^2} d_{z_0}^2(x) + \frac{1}{2R^2} d_{z_0}^2(y) + \frac{1}{2T^2}(t-T)^2\right\},$$

with ϕ given by

$$\phi(r) = \begin{cases} r\left|\log r\right|^2 & \text{for } r \in \left[0, e^{-2}\right); \\ 4e^{-2} & \text{for } r \in \left[e^{-2}, \infty\right). \end{cases}$$

Here positive constants A_1 and A_2 will be determined later. We consider

$$m_\varepsilon := \sup_{x,y \in B_R(z_0),\, T/2<t<T} \left\{ u^\varepsilon(x,t) - u_\varepsilon(y,t) - \Phi(x,y,t) \right\}. \tag{3.3}$$

Let $\nu \in (0,1)$ be a constant. Once we have proved that $m_\varepsilon \leq \nu$ for sufficiently small $\varepsilon > 0$, we conclude that for any $x, y \in \overline{B_R(z_0)}$ and $T/2 \leq t \leq T$,

$$\begin{aligned} u(x,t) - u(y,t) \leq{}& \nu + A_1 \phi\left(\frac{d(x,y)}{R}\right) \\ &+ A_2 \left\{\frac{1}{2R^2} d_{z_0}^2(x) + \frac{1}{2R^2} d_{z_0}^2(y) + \frac{1}{2T^2}(t-T)^2\right\} \end{aligned}$$

using uniform convergence of u^ε and u_ε to u in $\overline{B_{3R/2}(z_0)} \times [T/8, T]$ as ε tends to 0. This implies (3.2) since $\nu > 0$ is arbitrary. Thus it suffices to show that $m_\varepsilon \leq \nu$ for sufficiently small $\varepsilon > 0$. Here $\varepsilon > 0$ may depend on u, ν and R.

Suppose to the contrary that $m_\varepsilon > \nu$ for small $\varepsilon > 0$. Let $(\bar{x}, \bar{y}, \bar{t}) \in \overline{B_R(z_0)} \times \overline{B_R(z_0)} \times [T/2, T]$ be a point such that

$$m_\varepsilon = u^\varepsilon(\bar{x}, \bar{t}) - u_\varepsilon(\bar{y}, \bar{t}) - \Phi(\bar{x}, \bar{y}, \bar{t}). \tag{3.4}$$

Then we have that

$$\nu + A_1\phi\left(\frac{d(\bar{x},\bar{y})}{R}\right) + A_2\left\{\frac{1}{2R^2}d_{z_0}^2(\bar{x}) + \frac{1}{2R^2}d_{z_0}^2(\bar{y}) + \frac{1}{2T^2}(\bar{t}-T)^2\right\}$$
$$< u^\varepsilon(\bar{x},\bar{t}) - u_\varepsilon(\bar{y},\bar{t}) \le 3 \tag{3.5}$$

for sufficiently small $\varepsilon > 0$ by uniform convergence of u^ε and u_ε to u and the assumption (3.1). Selecting $A_2 > 1$ large enough yields that

$$\frac{d_{z_0}(\bar{x})}{R} + \frac{d_{z_0}(\bar{y})}{R} \le 2\sqrt{\frac{6}{A_2}} < \frac{1}{2} \quad \text{and} \quad |\bar{t}-T| \le \sqrt{\frac{6}{A_2}}\,T < \frac{1}{4}T.$$

Here and below, we fix such a large constant $A_2 > 1$, and hence we see that $(\bar{x},\bar{y},\bar{t}\,) \in B_R(z_0)\times B_R(z_0)\times(T/2,T]$. In light of (3.5), we obtain that

$$\frac{4\delta}{R} \le \phi\left(\frac{\delta}{R}\right) \le \frac{3}{A_1} < e^{-4} \quad \text{with} \quad \delta := d(\bar{x},\bar{y}), \tag{3.6}$$

by choosing $A_1 > 1$ large enough.

Let ω be a modulus of continuity of u on $\overline{B_{3R/2}(z_0)}\times[T/8,\,T]$ such that $\omega:[0,\infty)\to[0,\infty)$ is strictly increasing with $\omega(0+)=0$. Since

$$u^\varepsilon(\bar{x},\bar{t}) - u_\varepsilon(\bar{y},\bar{t}) \le u(\bar{x},\bar{t}) - u(\bar{y},\bar{t}) + \nu/2$$

for sufficient small $\varepsilon > 0$ by uniform convergence of u^ε and u_ε to u in $\overline{B_{3R/2}(z_0)}\times[T/8,\,T]$ as $\varepsilon\to 0$, it follows from (3.5) that

$$\nu < u^\varepsilon(\bar{x},\bar{t}) - u_\varepsilon(\bar{y},\bar{t}) \le u(\bar{x},\bar{t}) - u(\bar{y},\bar{t}) + \nu/2 \le \omega(\delta) + \nu/2. \tag{3.7}$$

In light of (3.7), it holds that for sufficiently small $\varepsilon > 0$ satisfying that $\omega\left(4\sqrt{\varepsilon} + \theta^{-1}\omega_0\left(4\sqrt{\varepsilon}\right)\right) < \nu/2$,

$$4\sqrt{\varepsilon} + \theta^{-1}\omega_0\left(4\sqrt{\varepsilon}\right) < \omega^{-1}\left(\nu/2\right) < \delta. \tag{3.8}$$

In particular, we notice that $\delta > 0$, that is, $\bar{x} \ne \bar{y}$ for sufficiently small $\varepsilon > 0$. $\square$

Claim 3.2. $\bar{x} \notin \{\bar{y}\}\cup\mathrm{Cut}(\bar{y})\cup\mathrm{Cut}(z_0)$, and $\bar{y} \notin \{\bar{x}\}\cup\mathrm{Cut}(\bar{x})\cup\mathrm{Cut}(z_0)$.

We postpone the proof of Claim 3.2 after the proof of this proposition, whose proof uses semi-convexity of u^ε and $-u_\varepsilon$, and the contacting property (3.4) together with (3.3).

Since $u^\varepsilon(\cdot,\,\bar{t}\,)$ and $-u_\varepsilon(\cdot,\,\bar{t}\,)$ are semi-convex in $B_R(z_0)$, it follows from the contacting property (3.4) with (3.3), and Claim 3.2 that $u^\varepsilon(\cdot,\bar{t}\,)$ and $-u_\varepsilon(\cdot,\bar{t}\,)$ are differentiable at $\bar{x}$ and $\bar{y}$, respectively, and

$$\begin{aligned}
\xi\ &:= \nabla u^\varepsilon(\bar{x},\bar{t}\,) = \nabla_x\Phi(\bar{x},\bar{y},\bar{t}\,) = \tfrac{A_1}{R}\phi'\left(\frac{\delta}{R}\right)\nabla d_{\bar{y}}(\bar{x}) + \tfrac{A_2}{2R^2}\nabla d_{z_0}^2(\bar{x});\\
\zeta &:= \nabla u_\varepsilon(\bar{y},\bar{t}\,) = -\nabla_y\Phi(\bar{x},\bar{y},\bar{t}\,) = -\tfrac{A_1}{R}\phi'\left(\frac{\delta}{R}\right)\nabla d_{\bar{x}}(\bar{y}) - \tfrac{A_2}{2R^2}\nabla d_{z_0}^2(\bar{y}).
\end{aligned} \tag{3.9}$$

This implies that

$$\xi = L_{\bar{y},\bar{x}}\,\zeta + \eta \quad \text{with} \quad \eta := \frac{A_2}{2R^2}\nabla d_{z_0}^2(\bar{x}) + \frac{A_2}{2R^2}L_{\bar{y},\bar{x}}\nabla d_{z_0}^2(\bar{y}) \tag{3.10}$$

since $\nabla d_{\bar{y}}(\bar{x}) = -L_{\bar{y},\bar{x}}\nabla d_{\bar{x}}(\bar{y})$. Choosing $A_1 > 1$ large such that $8A_2 \le A_1$, we use (3.6), (3.9) and (3.10) to obtain

$$0 < \frac{A_1}{2R}\phi'\left(\frac{\delta}{R}\right) \le |\xi|, \quad |\zeta| \le \frac{3A_1}{2R}\phi'\left(\frac{\delta}{R}\right) \quad \text{and} \quad |\eta| \le \frac{|\xi|}{2} \tag{3.11}$$

since

$$1 < \frac{1}{2}|\log r|^2 \le \phi'(r) = |\log r|\,(|\log r| - 2) \le |\log r|^2 \quad \forall r \in \left(0, e^{-4}\right). \tag{3.12}$$

Here we also used the fact that $|\nabla d_{\bar{x}}(\bar{y})| = 1 = |\nabla d_{\bar{y}}(\bar{x})|$.

In light of Claim 3.2, the test function Φ is of class $C^{2,1}(\mathcal{O})$ for a small neighborhood $\mathcal{O}$ of $(\bar{x}, \bar{y}, \bar{t})$ and we notice that $m_\varepsilon = \sup_{(x,y,t)\in\mathcal{O},\, t\le T}\left\{u^\varepsilon(x,t) - u_\varepsilon(y,t) - \Phi(x,y,t)\right\}$ from the contacting property (3.4) with (3.3). Utilizing this fact and Theorem 3.8 of [7], the following claim which is a Riemannian version of Jensen–Ishii's lemma holds true; refer to [21] for the Euclidean case. For the reader's convenience, a sketch of the proof of Claim 3.3 is provided after the proof of this proposition.

Claim 3.3. For any $\mu > 0$, there exist

$$(a, \xi, P) \in \overline{\mathcal{P}}^{2,+}u^\varepsilon(\bar{x}, \bar{t}) \quad \text{and} \quad (b, \zeta, Q) \in \overline{\mathcal{P}}^{2,-}u_\varepsilon(\bar{y}, \bar{t}) \tag{3.13}$$

such that

$$-\left(\frac{1}{\mu} + \left\|D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t})\right\|\right)\mathbf{I} \le \begin{pmatrix} P & 0 \\ 0 & -Q \end{pmatrix} \le D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t}) + \mu\left(D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t})\right)^2 \tag{3.14}$$

and

$$a - b \ge \frac{A_2}{T^2}\left(\bar{t} - T\right) \qquad (\text{the equality holds if } \bar{t} < T). \tag{3.15}$$

In light of (3.14), it holds that for any $X \in T_{\bar{x}}M$,

$$\begin{aligned} \langle (L_{\bar{x},\bar{y}}P - Q)\cdot L_{\bar{x},\bar{y}}X,\, L_{\bar{x},\bar{y}}X\rangle &= \left\langle P\cdot X,\, X\right\rangle - \langle Q\cdot L_{\bar{x},\bar{y}}X,\, L_{\bar{x},\bar{y}}X\rangle \\ &\le \left\langle \left\{D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t}) + \mu\left(D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t})\right)^2\right\}\cdot (X, L_{\bar{x},\bar{y}}X),\, (X, L_{\bar{x},\bar{y}}X)\right\rangle. \end{aligned} \tag{3.16}$$

Setting $\psi(s) := A_1\phi\left(\sqrt{s}/R\right)$ for $s \ge 0$, we have that for any $X \in T_{\bar{x}}M$,

$$\begin{aligned} &\left\langle D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t})\cdot (X, L_{\bar{x},\bar{y}}X),\, (X, L_{\bar{x},\bar{y}}X)\right\rangle \\ &\quad = \left\langle D^2_{x,y}\left(\psi\circ d^2\right)(\bar{x},\bar{y})\cdot (X, L_{\bar{x},\bar{y}}X),\, (X, L_{\bar{x},\bar{y}}X)\right\rangle \\ &\qquad + \left\langle D^2\left(\frac{A_2}{2R^2}d^2_{z_0}\right)(\bar{x})\cdot X,\, X\right\rangle + \left\langle D^2\left(\frac{A_2}{2R^2}d^2_{z_0}\right)(\bar{y})\cdot L_{\bar{x},\bar{y}}X,\, L_{\bar{x},\bar{y}}X\right\rangle. \end{aligned}$$

Then it holds that

$$\begin{aligned} &\left\langle D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t})\cdot (X, L_{\bar{x},\bar{y}}X),\, (X, L_{\bar{x},\bar{y}}X)\right\rangle \\ &\quad \le \left\langle D^2_{x,y}\left(\psi\circ d^2\right)(\bar{x},\bar{y})\cdot (X, L_{\bar{x},\bar{y}}X),\, (X, L_{\bar{x},\bar{y}}X)\right\rangle \\ &\qquad + \frac{2A_2}{R^2}\sqrt{\kappa}R_0\coth\left(\sqrt{\kappa}R_0\right)|X|^2 \end{aligned} \tag{3.17}$$

by recalling a Hessian estimate for squared distance functions in [20, Lemma 3.12]:

$$\left\langle D^2\left(\frac{1}{2}d_{z_0}^2\right)(\bar{x})\cdot X,\, X\right\rangle,\quad \left\langle D^2\left(\frac{1}{2}d_{z_0}^2\right)(\bar{y})\cdot L_{\bar{x},\bar{y}}X,\, L_{\bar{x},\bar{y}}X\right\rangle \\ \leq \sqrt{\kappa}R_0\coth\left(\sqrt{\kappa}R_0\right)|X|^2 \tag{3.18}$$

provided that Sec $\geq -\kappa$ on $B_{2R}(z_0)$ for $\kappa \geq 0$. Here we used the fact that $|X| = |L_{\bar{x},\bar{y}}X|$, $\bar{x},\bar{y} \in B_R(z_0) \subset B_{R_0}(z_0)$, and the function $\tau \mapsto \tau\coth\tau$ is nondecreasing in $[0,\infty)$.

Employing the chain rule, we have that for $X \in T_{\bar{x}}M$,

$$\begin{aligned} &\left\langle D_{x,y}^2\left(\psi\circ d^2\right)(\bar{x},\bar{y})\cdot\left(X, L_{\bar{x},\bar{y}}X\right),\left(X, L_{\bar{x},\bar{y}}X\right)\right\rangle \\ &\quad= \frac{A_1}{2R\delta}\,\phi'\left(\frac{\delta}{R}\right)\left\langle D_{x,y}^2 d^2(\bar{x},\bar{y})\cdot\left(X, L_{\bar{x},\bar{y}}X\right),\left(X, L_{\bar{x},\bar{y}}X\right)\right\rangle \\ &\qquad+\psi''\left(\delta^2\right)\left\langle \nabla_{x,y}d^2(\bar{x},\bar{y}),\left(X, L_{\bar{x},\bar{y}}X\right)\right\rangle^2 \end{aligned}$$

since $\psi'(s) = \dfrac{A_1}{2R\sqrt{s}}\,\phi'\left(\sqrt{s}/R\right)$. As in Section 3 of [6], using the first variation formula of length, we can prove that

$$\left.\frac{d}{ds}\right|_{s=0} d\left(\exp_{\bar{x}} sX,\, \exp_{\bar{y}} sL_{\bar{x},\bar{y}}X\right) = 0 \qquad \forall X \in T_{\bar{x}}M$$

since $\bar{x} \notin \mathrm{Cut}(\bar{y})$ by Claim 3.2 and the parallel transport preserves inner products. This equality yields that $\left\langle \nabla_{x,y}d^2(\bar{x},\bar{y}),\left(X, L_{\bar{x},\bar{y}}X\right)\right\rangle^2 = 0$, and hence we deduce

$$\begin{aligned} &\left\langle D_{x,y}^2\left(\psi\circ d^2\right)(\bar{x},\bar{y})\cdot\left(X, L_{\bar{x},\bar{y}}X\right),\left(X, L_{\bar{x},\bar{y}}X\right)\right\rangle \\ &\quad= \frac{A_1}{2R\delta}\,\phi'\left(\frac{\delta}{R}\right)\left\langle D_{x,y}^2 d^2(\bar{x},\bar{y})\cdot\left(X, L_{\bar{x},\bar{y}}X\right),\left(X, L_{\bar{x},\bar{y}}X\right)\right\rangle \\ &\quad\leq \frac{A_1}{2R\delta}\phi'\left(\frac{\delta}{R}\right)\cdot 2\kappa\delta^2\,|X|^2. \end{aligned} \tag{3.19}$$

Here, we used [6, Proposition 3.3] for the last inequality of (3.19). Thus the estimates (3.17) and (3.19) imply that

$$\begin{aligned} \left\langle D_{x,y}^2\Phi(\bar{x},\bar{y},\bar{t})\cdot\left(X, L_{\bar{x},\bar{y}}X\right),\,\left(X, L_{\bar{x},\bar{y}}X\right)\right\rangle &\leq \frac{1}{R^2}\left\{\kappa R_0^2\cdot A_1\phi'\left(\frac{\delta}{R}\right)\cdot\frac{\delta}{R}+C_1\right\}|X|^2 \\ &\leq \frac{C_1}{R^2}\left\{A_1\phi'\left(\frac{\delta}{R}\right)\cdot\frac{\delta}{R}+1\right\}|X|^2 \end{aligned} \tag{3.20}$$

for some constant $C_1 > 0$ depending on $\sqrt{\kappa}R_0$ which may vary from line to line. Hereafter, unless specifically stated, a constant $C_1 > 0$ may depend on n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta\max\left(e, R_0\log R_0\right)$, βR_0^{1-q}, and $T^{-1}\max\left(R^{p_-}, R^{p_+}\right)$, and vary from line to line.

Since

$$\left|L_{\bar{x},\bar{y}}X\right|^2 = |X|^2 \quad\text{and}\quad \left|\left(X, L_{\bar{x},\bar{y}}X\right)\right|^2 = 2|X|^2,$$

it holds from (3.16) and (3.20) that

$$L_{\bar{x},\bar{y}}P - Q \;\le\; \frac{C_1}{R^2}\left\{ A_1\phi'\left(\frac{\delta}{R}\right)\cdot\frac{\delta}{R} + 1\right\}\mathbf{I} + 2\mu\left\| \left(D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t})\right)^2 \right\|\mathbf{I}$$

as symmetric bilinear forms, that is,

$$\begin{aligned}\langle (L_{\bar{x},\bar{y}}P - Q)\cdot Y, Y\rangle &\le \frac{C_1}{R^2}\left\{ A_1\phi'\left(\frac{\delta}{R}\right)\cdot\frac{\delta}{R} + 1\right\}|Y|^2\\ &\quad + 2\mu\left\| \left(D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t})\right)^2 \right\| |Y|^2, \quad \forall Y \in T_{\bar{y}}M.\end{aligned}$$

With the choice of

$$\mu = R^{-2}\cdot\left\{1 + \left\| \left(D^2_{x,y}\Phi(\bar{x},\bar{y},\bar{t})\right)^2 \right\| \right\}^{-1} \tag{3.21}$$

in Claim 3.3, we deduce that

$$L_{\bar{x},\bar{y}}P - Q \le \frac{C_1}{R^2}\left\{ A_1\phi'\left(\frac{\delta}{R}\right)\cdot\frac{\delta}{R} + 1\right\}\mathbf{I} \tag{3.22}$$

for some constant $C_1 > 0$ depending on $\sqrt{\kappa}R_0$.

Since

$$\phi''(r) = -\frac{2|\log r|}{r}\left(1 - \frac{1}{|\log r|}\right) \le -\frac{3|\log r|}{2r} < 0 \qquad \forall r \in \left(0, e^{-4}\right), \tag{3.23}$$

a direct computation with the use of Claim 3.2, (3.12) and a Hessian estimate for squared distance functions in [20, Lemma 3.12] (see also (3.18)) implies that

$$\begin{aligned} D^2_x\Phi(\bar{x},\bar{y},\bar{t}) &= \frac{A_1}{R^2}\phi''\left(\frac{\delta}{R}\right)\nabla d_{\bar{y}}\otimes\nabla d_{\bar{y}}(\bar{x}) + \frac{A_1}{R}\phi'\left(\frac{\delta}{R}\right)\frac{1}{\delta}\\ &\quad\times\left\{\frac{1}{2}D^2 d^2_{\bar{y}}(\bar{x}) - \nabla d_{\bar{y}}\otimes\nabla d_{\bar{y}}(\bar{x})\right\} + \frac{A_2}{2R^2}D^2 d^2_{z_0}(\bar{x})\\ &\le \frac{A_1}{2R^2}\phi'\left(\frac{\delta}{R}\right)\frac{R}{\delta}\,D^2 d^2_{\bar{y}}(\bar{x}) + \frac{A_2}{2R^2}\,D^2 d^2_{z_0}(\bar{x})\\ &\le \frac{C_1}{R^2}\left\{A_1\,\phi'\left(\frac{\delta}{R}\right)\frac{R}{\delta} + 1\right\}\mathbf{I}, \end{aligned} \tag{3.24}$$

where we recall from (3.6) that $d(\bar{x},\bar{y}) = \delta < R \le R_0$. By the choice of μ as (3.21), it follows from (3.14) and (3.24) that

$$P \le \frac{C_1}{R^2}\left\{A_1\,\phi'\left(\frac{\delta}{R}\right)\frac{R}{\delta} + 1\right\}\mathbf{I} \tag{3.25}$$

for some constant $C_1 > 0$ depending on $\sqrt{\kappa}R_0$. Similarly, we have that

$$Q \;\ge\; -\frac{C_1}{R^2}\left\{A_1\,\phi'\left(\frac{\delta}{R}\right)\frac{R}{\delta} + 1\right\}\mathbf{I}. \tag{3.26}$$

In view of (3.13) and Lemma 2.5, we find two points (x_*, t_*), $(y_*, s_*) \in B_{3R/2}(z_0) \times (T/8, T]$ such that

$$\begin{aligned} &x_* = \exp_{\bar{x}}(\varepsilon\,\xi), \quad y_* = \exp_{\bar{y}}(-\varepsilon\,\zeta), \quad t_*,\, s_* \in \left[\bar{t} - 2\sqrt{\varepsilon},\, \bar{t} + 2\sqrt{\varepsilon}\right] \cap (T/8,\, T], \\ &\text{with} \quad d^2(x_*, \bar{x}) = \varepsilon^2|\xi|^2 \le 2\varepsilon\,\omega\left(2\sqrt{\varepsilon}\right), \quad d^2(y_*, \bar{y}) = \varepsilon^2|\zeta|^2 \le 2\varepsilon\,\omega\left(2\sqrt{\varepsilon}\right), \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} &\left(a,\, L_{\bar{x},x_*}\xi,\, L_{\bar{x},x_*}P + 2\kappa\omega\left(2\sqrt{\varepsilon}\right)\mathbf{I}\right) \in \overline{\mathcal{P}}^{2,+}u(x_*, t_*); \\ &\left(b,\, L_{\bar{y},y_*}\zeta,\, L_{\bar{y},y_*}Q - 2\kappa\omega\left(2\sqrt{\varepsilon}\right)\mathbf{I}\right) \in \overline{\mathcal{P}}^{2,-}u(y_*, s_*); \end{aligned} \tag{3.28}$$

for sufficiently small $\varepsilon > 0$. Here we used the facts that $\overline{\mathcal{P}}^{2,+}u^{\varepsilon} = -\overline{\mathcal{P}}^{2,-}(-u)_{\varepsilon}$, and $\|u\|_{L^\infty(B_{2R}(z_0)\times(0,T])} \le 1$. Notice that $x_* \notin \mathrm{Cut}(\bar{x})$ and $y_* \notin \mathrm{Cut}(\bar{y})$ from Lemma 2.5. Since u satisfies (1.5) in $B_{2R}(z_0)\times(0,T]$ in the viscosity sense and the $p(x,t)$-Laplacian operator (1.2) in non-divergence form is continuous with respect to $x, t, \nabla u$ and D^2u except when $\nabla u = 0$, utilizing (3.11) and (3.28) yields that

$$\begin{aligned} &|\xi|^{p(x_*,t_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I} + \left(p(x_*,t_*) - 2\right)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)P\right] - a \\ &\quad\ge -|\xi|^{p(x_*,t_*)-2}\left\{2\kappa\,\omega\left(2\sqrt{\varepsilon}\right)(n + p_+ - 2) + \theta|\xi|\big|\log|\xi|\big| + \beta|\xi|^{1+q}\right\} - R^{-p(x_*,t_*)}; \\ &|\zeta|^{p(y_*,s_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I} + \left(p(y_*,s_*) - 2\right)\frac{\zeta}{|\zeta|}\otimes\frac{\zeta}{|\zeta|}\right)Q\right] - b \\ &\quad\le |\zeta|^{p(y_*,s_*)-2}\left\{2\kappa\,\omega\left(2\sqrt{\varepsilon}\right)(n + p_+ - 2) + \theta|\zeta|\big|\log|\zeta|\big| + \beta|\zeta|^{1+q}\right\} + R^{-p(y_*,s_*)}. \end{aligned} \tag{3.29}$$

Here we used the facts that $|\xi| = |L_{\bar{x},x_*}\xi|$, P and $L_{\bar{x},x_*}P$ have the same eigenvalues, and

$$\begin{aligned} &\left\langle \left(L_{\bar{x},x_*}\xi \otimes L_{\bar{x},x_*}\xi\right) L_{\bar{x},x_*}P\cdot X,\, X\right\rangle_{x_*} \\ &\quad= \left\langle (\xi\otimes\xi)\,P\cdot L_{x_*,\bar{x}}X,\, L_{x_*,\bar{x}}X\right\rangle_{\bar{x}} \quad \forall X \in T_{x_*}M. \end{aligned} \tag{3.30}$$

Now we will estimate $|P|$ with the use of (3.25), (3.26) and (3.29). Employing (3.11) and (3.29), we obtain that

$$\begin{aligned} &\mathrm{tr}\left[\left(\mathbf{I} + \left(p(x_*,t_*) - 2\right)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)P\right] - a\,|\xi|^{2-p(x_*,t_*)} \\ &\quad\ge -\left\{2\kappa\,\omega\left(2\sqrt{\varepsilon}\right)(n + p_+ - 2) + \theta|\xi|\big|\log|\xi|\big| + \beta|\xi|^{1+q}\right\} - R^{-p(x_*,t_*)}\,|\xi|^{2-p(x_*,t_*)} \\ &\quad\ge -\frac{C_1}{R^2}\left\{\kappa R_0^2 + \theta R\,A_1\phi'\left(\frac{\delta}{R}\right)\left[\log 2 + \log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right] + |\log R|\right]\right. \\ &\qquad\left. + \beta R_0^{1-q}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{1+q} + \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{2-p_-}\right\} \\ &\quad\ge -\frac{C_1}{R^2}\,A_1\phi'\left(\frac{\delta}{R}\right)\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right] + \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\} \end{aligned} \tag{3.31}$$

for sufficiently small $\varepsilon > 0$ and sufficiently large $A_1 > 1$, where we used that $\theta R, \theta R|\log R| \leq \theta \max(e, R_0 \log R_0)$, $A_1\phi'\left(\frac{\delta}{R}\right) \gg 1$, $A_1 \gg 1$ and $p_- > 1$. Here and hereafter, a constant $C_1 > 0$ may depend on n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta \max(e, R_0 \log R_0)$, βR_0^{1-q}, and $T^{-1}\max(R^{p_-}, R^{p_+})$, and vary from line to line. Utilizing this estimate together with (3.15) and (3.29) implies that

$$\begin{aligned}
&\operatorname{tr}\left[\left(\mathbf{I} + \Big(p(x_*, t_*) - 2\Big)\frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}\right) P\right] \\
&\quad \geq -\frac{C_1}{R^2} A_1\phi'\left(\frac{\delta}{R}\right)\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]\right. \\
&\qquad \left. + \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\} + (a-b)\,|\xi|^{2-p(x_*,t_*)} + b\,|\xi|^{2-p(x_*,t_*)} \\
&\quad \geq -\frac{C_1}{R^2} A_1\phi'\left(\frac{\delta}{R}\right)\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right] + \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\} \\
&\qquad - \frac{A_2}{T}\,|\xi|^{2-p(x_*,t_*)} + |\xi|^{2-p(x_*,t_*)}|\zeta|^{p(y_*,s_*)-2}\operatorname{tr}\left[\left(\mathbf{I} + \Big(p(y_*, s_*) - 2\Big)\frac{\zeta}{|\zeta|} \otimes \frac{\zeta}{|\zeta|}\right) Q\right] \\
&\qquad - |\xi|^{2-p(x_*,t_*)}|\zeta|^{p(y_*,s_*)-2}\,\Big\{2\kappa\omega\left(2\sqrt{\varepsilon}\right)(n + p_+ - 2) \\
&\qquad + \theta|\zeta|\big|\log|\zeta|\big| + \beta|\zeta|^{1+q} + R^{-p(y_*,s_*)}|\zeta|^{2-p(y_*,s_*)}\Big\}.
\end{aligned}$$

Using (3.26) and arguing similarly as for (3.31), we deduce that

$$\begin{aligned}
&\operatorname{tr}\left[\left(\mathbf{I} + \Big(p(x_*, t_*) - 2\Big)\frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}\right) P\right] \\
&\quad \geq -\frac{C_1}{R^2} A_1\phi'\left(\frac{\delta}{R}\right)\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right] + \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q + T^{-1}R^{p(x_*,t_*)}\right\} \\
&\qquad - R^{p(x_*,t_*)-p(y_*,s_*)}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-p(x_*,t_*)}\frac{C_1}{R^2}\left\{A_1\phi'\left(\frac{\delta}{R}\right)\frac{R}{\delta} + 1\right\} \\
&\qquad - R^{p(x_*,t_*)-p(y_*,s_*)}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{1+p(y_*,s_*)-p(x_*,t_*)}\frac{C_1}{R^2}\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]\right. \\
&\qquad \left. + \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\}
\end{aligned} \tag{3.32}$$

for sufficiently small $\varepsilon > 0$ and sufficiently large $A_1 > 1$.

In order to estimate $|P|$ with the use of (3.32) and (3.25), we will establish some estimates regarding the difference between $p(x_*, t_*)$ and $p(y_*, s_*)$. Using the assumption (1.3) on Lipschitz continuity of p, it holds that

$$R^{p(x_*,t_*)-p(y_*,s_*)},\ R^{p(y_*,s_*)-p(x_*,t_*)} \leq \max\left\{1,\ e^{\theta\,\max(e,\,R_0\log R_0)}\right\}. \tag{3.33}$$

Indeed, we only consider the case when $R^{p(x_*,t_*)-p(y_*,s_*)} > 1$ since the other is similar. In light of (3.27) and (3.8), we have that

$$\begin{aligned}\delta_* :&= d(x_*,y_*) + \theta^{-1}\omega_0\left(|t_*-s_*|\right) \le d(x_*,\bar{x}) + d(\bar{x},\bar{y})\\ &\quad + d(\bar{y},y_*) + \theta^{-1}\omega_0\left(4\sqrt{\varepsilon}\right)\\ &\le \delta + 2\sqrt{2\varepsilon\,\omega\left(2\sqrt{\varepsilon}\right)} + \theta^{-1}\omega_0\left(4\sqrt{\varepsilon}\right)\\ &\le \delta + 4\sqrt{\varepsilon} + \theta^{-1}\omega_0\left(4\sqrt{\varepsilon}\right) < 2\delta\end{aligned} \tag{3.34}$$

for sufficiently small $\varepsilon > 0$. Hence the assumption (1.3) on p together with (3.34) and (3.6) yields that

$$\begin{aligned}\log R^{p(x_*,t_*)-p(y_*,s_*)} &= |p(x_*,t_*) - p(y_*,s_*)| \cdot |\log R| \le \theta\delta_* |\log R|\\ &\le \theta R|\log R| \le \theta\, \max\left(e, R_0\log R_0\right)\end{aligned}$$

for large $A_1 > 1$ and small $\varepsilon > 0$, which proves (3.33).

With the use of (3.34) and (3.6), it holds that

$$|p(x_*,t_*) - p(y_*,s_*)| \le \theta\delta_* \le 2\theta\delta \le 2\theta R\, A_1^{-1} \tag{3.35}$$

for sufficiently small $\varepsilon > 0$ and large $A_1 > 1$. Moreover, we show that

$$\left[\phi'\left(\frac{\delta}{R}\right)\right]^{|p(x_*,t_*)-p(y_*,s_*)|} \le \left[\phi'\left(\frac{\delta}{R}\right)\right]^{\theta\delta_*} \le C_1. \tag{3.36}$$

In fact, by (3.35), it follows that

$$\theta\delta_*\log\phi'\left(\frac{\delta}{R}\right) \le 2\theta R\cdot\frac{\delta}{R}\log\phi'\left(\frac{\delta}{R}\right)$$

for large $A_1 > 1$ and small $\varepsilon > 0$. The right-hand side of the above estimate is uniformly bounded in light of (3.6) and (3.12), which yields (3.36). Here we also used that $\theta R \le \theta\,\max\left(e,\, R_0\log R_0\right)$.

From (3.32), (3.33), (3.35) and (3.36), it follows that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}&\mathrm{tr}\left[\left(\mathbf{I} + (p(x_*,t_*)-2)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)P\right]\\ &\quad\ge -\frac{C_1}{R^2}A_1^{1+\theta\delta_*}\,\phi'\left(\frac{\delta}{R}\right)\left\{\frac{R}{\delta} + \log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right] + \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\}\end{aligned} \tag{3.37}$$

with sufficiently large $A_1 > 1$ depending on n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta\max(e, R_0\log R_0)$, βR_0^{1-q}, and $T^{-1}\max\left(R^{p_-}, R^{p_+}\right)$, since

$$T^{-1}R^{p(x_*,t_*)} \le T^{-1}\max\left(R^{p_-}, R^{p_+}\right). \tag{3.38}$$

Here we used the assumption that $1 < p_- \le p(x_*,t_*) \le p_+ < \infty$. The estimate (3.37) combines with (3.25), (3.12) and (3.6) to obtain

$$\begin{aligned}|P| &\le \frac{C_1}{R^2}A_1^{1+\theta\delta_*}\,\phi'\left(\frac{\delta}{R}\right)\left\{\frac{R}{\delta} + \log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right] + \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\}\\ &\le \frac{C_1}{R^2}A_1^{1+\theta\delta_*}\,\phi'\left(\frac{\delta}{R}\right)\cdot\frac{R}{\delta}.\end{aligned} \tag{3.39}$$

Here we used the assumption that $q < 1$, and the fact that $A_1 \le R/\delta$ from (3.6).

Employing estimates (3.29), (3.11), (3.31) and (3.15), we deduce that

$$\begin{aligned}
&|\xi|^{p(x_*,t_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(x_*,t_*)-2\Big)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)P\right]\\
&\quad-|\zeta|^{p(y_*,s_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(y_*,s_*)-2\Big)\frac{\zeta}{|\zeta|}\otimes\frac{\zeta}{|\zeta|}\right)Q\right]\\
&\quad\geq -C_1R^{2-p(x_*,t_*)}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(x_*,t_*)-2}\cdot\frac{1}{R^2}\,A_1\phi'\left(\frac{\delta}{R}\right)\\
&\qquad\times\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]+\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\}\\
&\qquad-C_1R^{2-p(y_*,s_*)}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-2}\frac{1}{R^2}A_1\phi'\left(\frac{\delta}{R}\right)\\
&\qquad\times\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]+\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\}+a-b\\
&\quad\geq -C_1R^{-p(y_*,s_*)}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-1}\\
&\qquad\cdot\left\{1+R^{p(y_*,s_*)-p(x_*,t_*)}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(x_*,t_*)-p(y_*,s_*)}\right\}\\
&\qquad\cdot\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]+\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\}-A_2T^{-1}
\end{aligned}$$

and hence in light of (3.33), (3.35), (3.36), and (3.38), it follows that

$$\begin{aligned}
&|\xi|^{p(x_*,t_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(x_*,t_*)-2\Big)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)P\right]\\
&\quad-|\zeta|^{p(y_*,s_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(y_*,s_*)-2\Big)\frac{\zeta}{|\zeta|}\otimes\frac{\zeta}{|\zeta|}\right)Q\right]\\
&\geq -C_1R^{-p(y_*,s_*)}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-1}A_1^{\theta\delta_*}\\
&\quad\times\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]+\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^q\right\}
\end{aligned}\tag{3.40}$$

for sufficiently small $\varepsilon>0$ and sufficiently large $A_1>1$. Now we decompose the first line of (3.40) into $\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4$, where $\mathcal{T}_j$ for $j=1,2,3,4$ are given by

$$\mathcal{T}_1:=\left\{|\xi|^{p(x_*,t_*)-2}-|\xi|^{p(y_*,s_*)-2}\right\}\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(x_*,t_*)-2\Big)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)P\right];$$

$$\mathcal{T}_2:=|\xi|^{p(y_*,s_*)-2}\Big(p(x_*,t_*)-p(y_*,s_*)\Big)\,\mathrm{tr}\left(\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}P\right);$$

$$\mathcal{T}_3:=|\xi|^{p(y_*,s_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(y_*,s_*)-2\Big)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)P\right]$$

$$-|L_{\bar{y},\bar{x}}\zeta|^{p(y_*,s_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(y_*,s_*)-2\Big)\frac{L_{\bar{y},\bar{x}}\zeta}{|L_{\bar{y},\bar{x}}\zeta|}\otimes\frac{L_{\bar{y},\bar{x}}\zeta}{|L_{\bar{y},\bar{x}}\zeta|}\right)P\right];$$

$$\mathcal{T}_4 := |\zeta|^{p(y_*,s_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(y_*,s_*)-2\Big)\frac{\zeta}{|\zeta|}\otimes\frac{\zeta}{|\zeta|}\right)\Big(L_{\bar{x},\bar{y}}P-Q\Big)\right]$$

Here we used the facts that $|\zeta|=|L_{\bar{y},\bar{x}}\zeta|$, and P and $L_{\bar{x},\bar{y}}P$ have the same eigenvalues, and the equality (3.30).

Estimate for $\mathcal{T}_1$. For $\mathcal{T}_1$, we use the mean value theorem, (3.11) and (3.39) with a similar argument for (3.33) to deduce that

$$\begin{aligned}
|\mathcal{T}_1| &\le C_1|\xi|^{p(y_*,s_*)-2}\cdot\left|\,|\xi|^{p(x_*,t_*)-p(y_*,s_*)}-1\right|\cdot|P| \\
&\le C_1\,|\xi|^{p(y_*,s_*)-2}\cdot\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{|p(x_*,t_*)-p(y_*,s_*)|} \\
&\qquad \big|\log|\xi|\big|\cdot\big|p(x_*,t_*)-p(y_*,s_*)\big|\cdot\frac{1}{R^2}A_1^{1+\theta\delta_*}\phi'\left(\frac{\delta}{R}\right)\cdot\frac{R}{\delta}.
\end{aligned}$$

Then utilizing (3.11), (3.35), (3.36), and (3.34) shows that

$$\begin{aligned}
|\mathcal{T}_1| &\le C_1\left[\frac{A_1}{R}\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-2}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{\theta\delta_*}\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]+|\log R|\right\} \\
&\quad\cdot\theta\delta_*\cdot\frac{1}{R^2}A_1^{1+\theta\delta_*}\phi'\left(\frac{\delta}{R}\right)\cdot\frac{R}{\delta} \\
&\le \frac{C_1}{R^{p(y_*,s_*)}}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-1}\cdot A_1^{2\theta\delta_*}\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]+|\log R|\right\}\cdot\theta\delta_*\cdot\frac{R}{\delta} \\
&\le \frac{C_1}{R^{p(y_*,s_*)}}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-1}\cdot A_1^{2\theta\delta_*}\left\{\log\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]+|\log R|\right\}\cdot\theta R
\end{aligned}\tag{3.41}$$

for sufficiently small $\varepsilon>0$. Selecting $A_1>1$ sufficiently large, (3.41) together with (3.35) yields that

$$|\mathcal{T}_1|\le\frac{C_1}{R^{p(y_*,s_*)}}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-1}A_1^{q_0}\left[\log A_1+\log\phi'\left(\frac{\delta}{R}\right)\right].\tag{3.42}$$

for $q_0:=\frac{1}{2}(1-q)>0$. Here we used that $\theta R, \theta R|\log R|\le\theta\max(e,R_0\log R_0)$.

Estimate for $\mathcal{T}_2$. As in the estimate for $\mathcal{T}_1$, it holds that

$$\begin{aligned}
|\mathcal{T}_2| &\le C_1|\xi|^{p(y_*,s_*)-2}\big|p(x_*,t_*)-p(y_*,s_*)\big|\cdot|P| \\
&\le C_1\left[\frac{A_1}{R}\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-2}\cdot\theta\delta_*\cdot\frac{1}{R^2}A_1^{1+\theta\delta_*}\phi'\left(\frac{\delta}{R}\right)\cdot\frac{R}{\delta} \\
&\le \frac{C_1}{R^{p(y_*,s_*)}}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-1}A_1^{\theta\delta_*}\cdot\theta\delta_*\cdot\frac{R}{\delta} \\
&\le \frac{C_1}{R^{p(y_*,s_*)}}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-1}A_1^{q_0}
\end{aligned}\tag{3.43}$$

by choosing $A_1 > 1$ sufficiently large and $\varepsilon > 0$ sufficiently small.

Estimate for $\mathcal{T}_3$. Employing the mean value theorem, (3.10), (3.11), and (3.39), we obtain

$$\begin{aligned} |\mathcal{T}_3| &\le C_1 \left[\frac{A_1}{R}\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-3} \cdot |\eta| \cdot |P| \\ &\le C_1 \left[\frac{A_1}{R}\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-3} \cdot \frac{1}{R} \cdot \frac{1}{R^2} A_1^{1+\theta\delta_*} \phi'\left(\frac{\delta}{R}\right) \cdot \frac{R}{\delta} \\ &\le \frac{C_1}{R^{p(y_*,s_*)}} \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-2} A_1^{\theta\delta_*} \cdot \frac{R}{\delta} \end{aligned} \tag{3.44}$$

since $|\eta| \le 2A_2 R^{-1}$ in light of (3.10). Thus we conclude from (3.35) that

$$|\mathcal{T}_3| \le \frac{C_1}{R^{p(y_*,s_*)}} \left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-2} A_1^{q_0} \cdot \frac{R}{\delta} \tag{3.45}$$

for sufficiently large $A_1 > 1$ and sufficiently small $\varepsilon > 0$.

Estimate for $\mathcal{T}_4$. Since $\nabla d_{\bar{y}}(\bar{x})$ is an eigenvector of $\frac{1}{2}D^2 d_{\bar{y}}^2(\bar{x})$ associated with eigenvalue 1, the first line of (3.24) and a Hessian estimate for squared distance functions in Lemma 3.12 of [20] (see (3.18)) imply that

$$\begin{aligned} \left\langle D_x^2\Phi\left(\bar{x},\bar{y},\bar{t}\right) \cdot \nabla d_{\bar{y}}, \nabla d_{\bar{y}}(\bar{x})\right\rangle &= \frac{1}{R^2}\left[A_1\phi''\left(\frac{\delta}{R}\right) + \frac{A_2}{2}\left\langle D^2 d_{z_0}^2 \cdot \nabla d_{\bar{y}}, \nabla d_{\bar{y}}(\bar{x})\right\rangle\right] \\ &\le \frac{1}{R^2}\left[A_1\phi''\left(\frac{\delta}{R}\right) + C_1\right] \end{aligned}$$

which is negative in view of (3.23) and (3.6) by selecting $A_1 > 1$ sufficiently large. This yields from (3.14) that

$$\langle P \cdot \nabla d_{\bar{y}}, \nabla d_{\bar{y}}(\bar{x})\rangle \le \frac{1}{R^2}\left[A_1\phi''\left(\frac{\delta}{R}\right) + C_1\right] \tag{3.46}$$

since we have chosen μ as (3.21). In a similar fashion, it holds that

$$-\left\langle Q \cdot \nabla d_{\bar{x}}, \nabla d_{\bar{x}}(\bar{y})\right\rangle \le \frac{1}{R^2}\left[A_1\phi''\left(\frac{\delta}{R}\right) + C_1\right]. \tag{3.47}$$

Since $\nabla d_{\bar{y}}(\bar{x}) = -L_{\bar{y},\bar{x}}\nabla d_{\bar{x}}(\bar{y})$, estimates (3.46) and (3.47) imply that

$$\left\langle \left(L_{\bar{x},\bar{y}}P - Q\right) \cdot \nabla d_{\bar{x}}(y), \nabla d_{\bar{x}}(\bar{y})\right\rangle \le \frac{2}{R^2}\left[A_1\phi''\left(\frac{\delta}{R}\right) + C_1\right]. \tag{3.48}$$

Therefore by (3.11), (3.22) and (3.48) , we deduce that

$$\begin{aligned} \mathcal{T}_4 &= |\zeta|^{p(y_*,s_*)-2}\,\mathrm{tr}\left[\left(\mathbf{I} + \left(p(y_*,s_*)-2\right)\frac{\zeta}{|\zeta|}\otimes\frac{\zeta}{|\zeta|}\right)\left(L_{\bar{x},\bar{y}}P - Q\right)\right] \\ &\le \frac{C_0}{R^{p(y_*,s_*)}}\left[A_1\phi'\left(\frac{\delta}{R}\right)\right]^{p(y_*,s_*)-2}\left[A_1\phi''\left(\frac{\delta}{R}\right) + C_1A_1\phi'\left(\frac{\delta}{R}\right)\cdot\frac{\delta}{R} + C_1\right], \end{aligned} \tag{3.49}$$

where a constant $C_0 > 0$ depends only on p_- and p_+.

Utilizing (3.35), (3.40), (3.42), (3.43), (3.45) and (3.49), it follows that

$$-A_1\phi''\left(\frac{\delta}{R}\right) \le C_1\left[\log A_1 + \log\phi'\left(\frac{\delta}{R}\right)\right] \cdot\left\{A_1^{1+q_0}\,\phi'\left(\frac{\delta}{R}\right) + A_1^{q_0}\cdot\frac{R}{\delta} + A_1^{1+q+q_0}\left[\phi'\left(\frac{\delta}{R}\right)\right]^{1+q}\right\} \tag{3.50}$$

for sufficiently large $A_1 > 1$ and small $\varepsilon > 0$. Letting $q_1 := 1 + q + q_0$, and recalling (3.12) and (3.23), we conclude from (3.50) and (3.6) that

$$A_1{}^{1+q_1/2}\left(\frac{R}{\delta}\right)^{1-q_1/2} + A_1\left(\log\frac{R}{\delta}\right)\cdot\frac{R}{\delta} \le C_1\left(\log A_1 + \log\log\frac{R}{\delta}\right)\cdot\left\{A_1^{q_1}\left(\log\frac{R}{\delta}\right)^4 + A_1^{q_0}\cdot\frac{R}{\delta}\right\}.$$

This gives a contradiction by choosing $A_1 > 1$ sufficiently large since $0 < q_0 = \frac{1}{2}(1-q) < 1$ and $q_1 = 1 + q + q_0 < 2$. Therefore we have proved that $m_\varepsilon \le \nu$ for sufficiently small $\varepsilon > 0$ provided that positive constants A_1 and A_2 are sufficiently large.

Lastly, we will give the proofs of Claim 3.2 and Claim 3.3

Proof of Claim 3.2. In light of the assumption that $m_\varepsilon > \nu$, we recall that $\bar{x} \neq \bar{y}$. By the contacting property (3.4) with (3.3), we have that for any $x \in B_R(z_0)$,

$$u^\varepsilon(x,\bar{t}) - A_1\phi\left(\frac{d_{\bar{y}}(x)}{R}\right) - \frac{A_2}{2R^2}d_{z_0}^2(x) \;\le\; u^\varepsilon(\bar{x},\bar{t}) - A_1\phi\left(\frac{d_{\bar{y}}(\bar{x})}{R}\right) - \frac{A_2}{2R^2}d_{z_0}^2(\bar{x}). \tag{3.51}$$

If $\bar{x}$ is a cut point of $\bar{y}$, then Corollary 2.8 of [40] implies that there is a unit vector $X \in T_{\bar{x}}M$ such that

$$\liminf_{\tau\to 0}\frac{1}{\tau^2}\left\{\psi\left(d_{\bar{y}}^2\left(\exp_{\bar{x}}\tau X\right)\right) + \psi\left(d_{\bar{y}}^2\left(\exp_{\bar{x}} -\tau X\right)\right) - 2\psi\left(d_{\bar{y}}^2(\bar{x})\right)\right\} = -\infty,$$

where a function $\psi(s) = A_1\phi(\sqrt{s}/R)$ (for $s \ge 0$) is smooth near $s = \delta^2 > 0$ with $\psi'(\delta^2) > 0$ by (3.6). On the other hand, semi-convexity of $u^\varepsilon(\cdot,\bar{t})$ and semi-concavity of $d_{z_0}^2$ in $B_R(z_0)$ (see a upper Hessian bound in Lemma 3.12 of [20]) with (3.51) yield that

$$\begin{aligned}
&\liminf_{\tau\to 0}\frac{1}{\tau^2}\left\{\psi\left(d_{\bar{y}}^2\left(\exp_{\bar{x}}\tau X\right)\right) + \psi\left(d_{\bar{y}}^2\left(\exp_{\bar{x}} -\tau X\right)\right) - 2\psi\left(d_{\bar{y}}^2(\bar{x})\right)\right\}\\
&\quad\ge \liminf_{\tau\to 0}\frac{1}{\tau^2}\left\{u^\varepsilon\left(\exp_{\bar{x}}\tau X,\,\bar{t}\right) + u^\varepsilon\left(\exp_{\bar{x}} -\tau X,\,\bar{t}\right) - 2u^\varepsilon\left(\bar{x},\bar{t}\right)\right\}\\
&\qquad - \frac{A_2}{2R^2}\limsup_{\tau\to 0}\frac{1}{\tau^2}\left\{d_{z_0}^2\left(\exp_{\bar{x}}\tau X\right) + d_{z_0}^2\left(\exp_{\bar{x}} -\tau X\right) - 2d_{z_0}^2(\bar{x})\right\} > -\infty,
\end{aligned}$$

which is a contradiction. Therefore we conclude that $\bar{x} \notin \mathrm{Cut}(\bar{y}) \cup \{\bar{y}\}$. Moreover, a similar argument above using (3.51) and the fact that $\bar{x} \notin \mathrm{Cut}(\bar{y}) \cup \{\bar{y}\}$ gives that $\bar{x}$ is not a cut point of z_0 since $d_{z_0}^2$ fails to be semi-convex at a cut point of z_0; refer to [20, Proposition 2.5]. Similarly, it can be proved that $\bar{y} \notin$

$\{\bar{x}\} \cup \mathrm{Cut}(\bar{x}) \cup \mathrm{Cut}(z_0)$ with the help of semi-convexity of $-u_\varepsilon(\cdot, \bar{t})$ in $B_R(z_0)$ and the contacting property (3.4). This finishes the proof of Claim 3.2. □

Proof of Claim 3.3. If $\bar{t} < T$, Claim 3.3 follows from Theorem 3.8 of [7] since u^ε and u_ε are Lipschitz continuous in $\overline{B_{3R/2}(z_0)} \times [T/8, T]$. Now we assume that $\bar{t} = T$ and we will use a similar argument as in the proof of Theorem 9 of [24] to prove Claim 3.3 in the case. Here we will use Lipschitz continuity of u^ε and u_ε for a simpler proof; refer to the proof of Theorem 9 of [24] for general viscosity sub-/supersolutions.

For a given sequence $\{s_k\}_{k=1}^\infty$ such that $T/2 < s_k < T$ and $\lim_{k\to\infty} s_k = T$, consider

$$m_k := \sup_{x,y\in B_R(z_0),\, T/2<t<T} \Big\{ u^\varepsilon(x,t) - u_\varepsilon(y,t) - \Phi_k(x,y,t) \Big\},$$

where

$$\Phi_k(x,y,t) := \Phi(x,y,t) + d_{\bar{x}}^4(x) + d_{\bar{y}}^4(y) + (t-T)^2 + \frac{(T-s_k)^2}{T-t}.$$

Since $\bar{t} = T$ and

$$m_\varepsilon \geq m_k \geq u^\varepsilon(\bar{x}, s_k) - u_\varepsilon(\bar{y}, s_k) - \Phi(\bar{x}, \bar{y}, s_k) - (s_k - T)^2 - (T - s_k),$$

it follows from (3.4) that

$$\lim_{k\to\infty} m_k = m_\varepsilon\,. \tag{3.52}$$

Let $(x_k, y_k, t_k) \in \overline{B_R(z_0)} \times \overline{B_R(z_0)} \times [T/2, T)$ be a point such that

$$m_k = u^\varepsilon(x_k, t_k) - u_\varepsilon(y_k, t_k) - \Phi_k(x_k, y_k, t_k).$$

Note that $t_k \neq T$. Then utilizing (3.52), (3.3) and (3.4), it can be checked that (x_k, y_k, t_k) converges to $(\bar{x}, \bar{y}, T)$ as k tends to ∞. Note that $(x_k, y_k, t_k) \in B_R(z_0) \times B_R(z_0) \times (T/2, T)$, and Φ_k is smooth near the point (x_k, y_k, t_k) for sufficiently large $k \in \mathbb{N}$. Applying Theorem 3.8 of [7] to the functions u^ε, u_ε and Φ_k near the point (x_k, y_k, t_k) for large $k \in \mathbb{N}$, we find

$$\Big(a_k, \nabla_x \Phi_k(x_k, y_k, t_k), P_k\Big) \in \overline{\mathcal{P}}^{2,+} u^\varepsilon(x_k, t_k) \quad \text{and}$$
$$\Big(b_k, -\nabla_y \Phi_k(x_k, y_k, t_k), Q_k\Big) \in \overline{\mathcal{P}}^{2,-} u_\varepsilon(y_k, t_k)$$

which satisfies (3.14) with (P_k, Q_k, Φ_k) at the point (x_k, y_k, t_k) replacing (P, Q, Φ) at the point $(\bar{x}, \bar{y}, \bar{t})$, and

$$a_k - b_k = \frac{A_2}{T^2}(t_k - T) + 2(t_k - T) + \frac{(T-s_k)^2}{(T-t_k)^2}.$$

Here we note that a_k and b_k are uniformly bounded with respect to k by Lipschitz continuity of u^ε and u_ε, and then $\lim_{k\to\infty} a_k =: a$ and $\lim_{k\to\infty} b_k =: b$ exist up to a subsequence. Therefore this combines with a diagonal argument (in view of the definition of the limiting sub-and super-jets) to conclude the proof of Claim 3.3 since (x_k, y_k, t_k) converges to $(\bar{x}, \bar{y}, T)$ as k tends to ∞. We only remark that

$$a - b \geq \lim_{k\to\infty} \left\{ \frac{A_2}{T^2}(t_k - T) + 2(t_k - T) \right\} = 0$$

which gives (3.15). □

Therefore, the proof of Proposition 3.1 is finished. □

In light of the proof of Proposition 3.1, we have the following corollary. It should be noted that our viscosity solution is not influenced by what is to happen in the future.

Corollary 3.4. *Assume that* $\operatorname{Sec} \geq -\kappa$ *on* $B_{2R}(z_0)$ *for* $\kappa \geq 0$. *Let* $0 < R \leq R_0 < \infty$, $T > 0$, *and* $u \in C\left(B_{2R}(z_0) \times (0, T]\right)$ *be a viscosity solution of* (1.5) *in* $B_{2R}(z_0) \times (0, T]$ *with* $0 \leq q < 1$ *and*

$$\|V\|_{L^\infty(B_{2R}(z_0)\times(0,T])} \leq \beta$$

for $\beta \geq 0$. *Then we have that for any* $x, y \in B_R(z_0)$ *and* $t \in (T/2, T]$,

$$|u(x,t) - u(y,t)| \leq C_0 \frac{d(x,y)}{R} \left|\log\left(\frac{d(x,y)}{R}\right)\right|^2.$$

Here a constant $C_0 > 0$ *depends only on* $\|u\|_{L^\infty(B_{2R}(z_0)\times(0,T])}$, $\|f\|_{L^\infty(B_{2R}(z_0)\times(0,T])}$, n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta \max\left(e, R_0 \log R_0\right)$, βR_0^{1-q}, *and* $T^{-1} \max\left(R^{p_-}, R^{p_+}\right)$.

By uniform convergence of u^ε and u_ε to u as ε tends to 0, we get the following Corollary.

Corollary 3.5. *With the same assumption as Corollary* 3.4, *let* $\nu \in (0,1)$ *be a constant. Then there exists a small constant* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$, *then*

$$|u^\varepsilon(x,t) - u_\varepsilon(y,t)| \leq C_0 \frac{d(x,y)}{R} \left|\log\left(\frac{d(x,y)}{R}\right)\right|^2 + \nu$$

for any $x, y \in \overline{B_R(z_0)}$ *and* $t \in [T/2, T]$.

Now we will prove a Lipschitz estimate using a logarithmic-type Lipschitz estimate in Proposition 3.1.

Lemma 3.6. (Lipschitz estimate) *With the same assumption as Proposition 3.1, we have that for any* $x, y \in B_R(z_0)$ *and* $t \in (T/2, T]$,

$$|u(x,t) - u(y,t)| \leq A_1 \frac{d(x,y)}{R} + A_2 \left\{ \frac{1}{R^2} d_{z_0}^2(x) + \frac{1}{R^2} d_{z_0}^2(y) + \frac{1}{T^2}(t - T)^2 \right\}.$$

Here constants $A_1 > 0$ *and* $A_2 > 0$ *depend only on* n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta \max\left(e, R_0 \log R_0\right)$, βR_0^{1-q}, *and* $T^{-1} \max\left(R^{p_-}, R^{p_+}\right)$.

Proof. We follow the proof of Proposition 3.1 in order to show a Lipschitz estimate. Let

$$q_0 = \frac{1}{4}(1-q) \in (0,\, 1/4], \quad \text{and} \quad \sigma := \frac{3}{2} - \frac{1}{4}(1+q+q_0) \in (1,\, 3/2).$$

Define a function Φ by

$$\Phi(x,y,t) := A_1 \phi\left(\frac{d(x,y)}{R}\right) + A_2 \left\{\frac{1}{2R^2} d_{z_0}^2(x) + \frac{1}{2R^2} d_{z_0}^2(y) + \frac{1}{2T^2}(t-T)^2\right\}$$

with

$$\phi(r) = \begin{cases} r - \dfrac{1}{\sigma} r^\sigma & \text{for } r \in [0,1); \\ 1 - \dfrac{1}{\sigma} & \text{for } r \in [1,\infty); \end{cases} \tag{3.53}$$

for some constants $A_1 > 1$ and $A_2 > 1$ to be chosen later. With the same notation as in the proof of Proposition 3.1, consider

$$m_\varepsilon := \sup_{x,y \in B_R(z_0),\, T/2<t<T} \Big\{ u^\varepsilon(x,t) - u_\varepsilon(y,t) - \Phi(x,y,t) \Big\}.$$

Let $\nu \in (0,1)$ be a constant. Then it suffices to prove that $m_\varepsilon \le \nu$ for sufficiently small $\varepsilon > 0$ by uniform convergence of u^ε and u_ε to u as $\varepsilon \to 0$.

Suppose to the contrary that $m_\varepsilon > \nu$ for sufficiently small $\varepsilon > 0$. Let $(\bar{x}, \bar{y}, \bar{t}) \in \overline{B_R(z_0)} \times \overline{B_R(z_0)} \times [T/2, T]$ be a point such that

$$m_\varepsilon = u^\varepsilon(\bar{x},\bar{t}) - u_\varepsilon(\bar{y},\bar{t}) - \Phi(\bar{x},\bar{y},\bar{t}), \quad \text{and} \quad \delta := d(\bar{x},\bar{y}).$$

Then by the assumption that $m_\varepsilon > \nu$, and Corollary 3.5, it follows that

$$\begin{aligned} &A_1 \phi\left(\frac{\delta}{R}\right) + A_2\left\{\frac{1}{2R^2} d_{z_0}^2(\bar{x}) + \frac{1}{2R^2} d_{z_0}^2(\bar{y}) + \frac{1}{2T^2}(\bar{t}-T)^2\right\} + \nu \\ &< u^\varepsilon(\bar{x},\bar{t}) - u_\varepsilon(\bar{y},\bar{t}) \\ &\le C_0 \frac{\delta}{R}\left(\log\frac{R}{\delta}\right)^2 + \nu. \end{aligned} \tag{3.54}$$

for sufficiently small $0 < \varepsilon < \varepsilon_0$. Here constants $C_0 > 0$ and $\varepsilon_0 > 0$ are the constants appearing in Corollary 3.5. Arguing similarly as in the proof of Proposition 3.1, we see that $(\bar{x},\bar{y},\bar{t}) \in B_R(z_0) \times B_R(z_0) \times (T/2, T]$ and Claim 3.2 holds true for sufficiently large constants $A_1 > 1$ and $A_2 > 1$, and sufficiently small $\varepsilon > 0$. With the use of (3.10) and (3.54), it follows that

$$|\eta| \le \frac{A_2}{R^2} d_{z_0}(\bar{x}) + \frac{A_2}{R^2} d_{z_0}(\bar{y}) \le \frac{C_1}{R}\sqrt{\frac{\delta}{R}} \cdot \log\frac{R}{\delta} \tag{3.55}$$

for some constant $C_1 > 0$ depending on n, p_-, p_+, q, $\sqrt{\kappa} R_0$, $\theta \max(e, R_0 \log R_0)$, βR_0^{1-q}, and $T^{-1}\max(R^{p_-}, R^{p_+})$. Since

$$\frac{1}{2} \le \phi'(r) < 1 \quad \forall r \in \left(0,\, 2^{-1/(\sigma-1)}\right), \tag{3.56}$$

the estimate (3.11) holds with the function ϕ given by (3.53) for sufficiently large $A_1 > 1$.

As in the estimate (3.42), we have

$$|\mathcal{T}_1| \le \frac{C_1}{R^{p(y_*,s_*)}} A_1^{p(y_*,s_*)-1+q_0} \log A_1 \tag{3.57}$$

by using (3.56) and selecting $A_1 > 1$ sufficiently large and $\varepsilon > 0$ sufficiently small. In light of (3.43) and (3.57), we get

$$|\mathcal{T}_2| \le \frac{C_1}{R^{p(y_*,s_*)}} A_1^{p(y_*,s_*)-1+q_0} \tag{3.58}$$

For the estimate of $\mathcal{T}_3$, we use (3.44), (3.39), (3.35) and (3.55) to deduce that

$$\begin{aligned} |\mathcal{T}_3| &\le C_1 \left[\frac{A_1}{R} \phi' \left(\frac{\delta}{R} \right) \right]^{p(y_*,s_*)-3} \cdot |\eta| \cdot |P| \\ &\le \frac{C_1}{R^{p(y_*,s_*)}} A_1^{p(y_*,s_*)-2+q_0} \cdot \sqrt{\frac{R}{\delta}} \cdot \log \frac{R}{\delta} \,. \end{aligned} \tag{3.59}$$

By (3.49), it holds that

$$\mathcal{T}_4 \le \frac{C_0}{R^{p(y_*,s_*)}} {A_1}^{p(y_*,s_*)-2} \left[A_1 \phi'' \left(\frac{\delta}{R} \right) + C_1 A_1 \cdot \frac{\delta}{R} + C_1 \right].$$

This combined with (3.40), (3.35), (3.57), (3.58) and (3.59) implies that

$$-A_1 \phi'' \left(\frac{\delta}{R} \right) \le C_1 \left\{ A_1^{1+q_0} \log A_1 + A_1^{q_0} \sqrt{\frac{R}{\delta}} \cdot \log \frac{R}{\delta} + A_1^{1+q+q_0} \right\} \tag{3.60}$$

for sufficiently large $A_1 > 1$. Let $q_2 := (1 + q + q_0)/2 \in (1/2, 1)$. Since (3.54) implies that

$$\frac{\delta}{2R} \le \phi \left(\frac{\delta}{R} \right) \le \frac{3}{A_1} \ll 1$$

for sufficiently large $A_1 > 1$ and small $\varepsilon > 0$ (see (3.6)), it follows from (3.60) that

$$A_1^{1+q_2} \left(\frac{R}{\delta} \right)^{2-\sigma-q_2} + A_1 \left(\frac{R}{\delta} \right)^{2-\sigma} \le C_1 \left\{ A_1^{2q_2} \log A_1 + A_1^{q_0} \sqrt{\frac{R}{\delta}} \cdot \log \frac{R}{\delta} \right\},$$

where we note that $1 < \sigma < 3/2$. This gives a contradiction for sufficiently large $A_1 > 0$ since $q_2 < 1$, $2 - \sigma - q_2 = (1 - q_2)/2 > 0$, and $q_0 < 1$. Therefore we have shown that $m_\varepsilon \le \nu$ for sufficiently small $\varepsilon > 0$ provided that positive constants A_1 and A_2 are sufficiently large, completing the proof. □

In view of the proof of Lemma 3.6, the following corollary holds.

Corollary 3.7. (Lipschitz estimate in spatial variables) *With the same assumption as Corollary 3.4, we have that for any $x, y \in B_R(z_0)$ and $t \in (T/2, T]$,*

$$|u(x,t) - u(y,t)| \le C_0 \frac{d(x,y)}{R} \,.$$

Here a constant $C_0 > 0$ depends only on $\|u\|_{L^\infty(B_{2R}(z_0)\times(0,T])}$, $\|f\|_{L^\infty(B_{2R}(z_0)\times(0,T])}$, n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta \max(e, R_0 \log R_0)$, βR_0^{1-q}, and $T^{-1} \max(R^{p_-}, R^{p_+})$.

Using Corollary 3.7 and uniform convergence of u^ε and u_ε to u as ε tends to 0, we have the following corollary.

Corollary 3.8. *With the same assumption as Proposition 3.1, let $\nu \in (0,1)$ be a constant. Then there exists a small constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then*

$$|u^\varepsilon(x,t) - u_\varepsilon(y,t)| \le C_0 \frac{d(x,y)}{R} + \nu$$

for any $x, y \in \overline{B_R(z_0)}$ and $t \in [T/2, T]$.

4. 1/2-Hölder estimate in time variable

Using a comparison argument as in [8] with the help of a Lipschitz estimate in spatial variables, we prove an 1/2-Hölder continuity with respect to time variable for viscosity solutions.

Lemma 4.1. *Under the same assumption as Proposition 3.1, we have*

$$|u(x,t) - u(x,s)| \le C\,T^{-1/2}\,|t-s|^{1/2}$$

for any $x \in B_R(z_0)$ and $t, s \in (T/2, T]$. Here a constant $C > 0$ depends only on n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta\max(e, R_0\log R_0)$, βR_0^{1-q}, $T^{-1}\max(R^{p_-}, R^{p_+})$, and $T\max(R^{-p_-}, R^{-p_+})$.

Proof. Fix $(x_0, t_0) \in B_R(z_0) \times (T/2,\, T)$, and let

$$\sigma_0 := \max\left(2,\, 1 + \frac{1}{p_- - 1}\right).$$

Let ν be a constant in $(0,1)$ and let σ be a constant such that $\sigma_0 < \sigma < \sigma_0 + 1$. For each constant $\eta > 0$, consider

$$\varphi(x,t) := \nu + \frac{\eta}{T} + A_1(t - t_0) + A_2\left[\frac{d_{x_0}(x)}{R}\right]^\sigma \qquad \forall (x,t) \in \overline{B_{R/4}(x_0)} \times [t_0, T]$$

with some constants $A_1 > 0$ and $A_2 > 0$ to be determined later.

Firstly, we select A_2 large enough such that

$$A_2 = C_0^{\sigma_0+1}\left\{\left(\frac{\eta}{T}\right)^{1-\sigma} + 1\right\} \tag{4.1}$$

for a large constant $C_0 > 1$, where a constant $C_0 > 1$ will be chosen later depending only on n, p_-, p_+, q, $\sqrt{\kappa}R_0$, $\theta\max(e, R_0\log R_0)$, βR_0^{1-q}, and $T^{-1}\max(R^{p_-}, R^{p_+})$. With the same notation as in the proof of Proposition 3.1, we deduce in light of Corollary 3.8, and (4.1) that

$$u^\varepsilon(x,t_0) - u_\varepsilon(x_0,t_0) \le C_0\frac{d_{x_0}(x)}{R} + \nu \;<\; \varphi(x,t_0) \quad \forall x \in \overline{B_{R/4}(x_0)}$$

for sufficiently small $\varepsilon > 0$ and large $C_0 > 1$. Hence using the assumption (3.1), it follows that

$$u^\varepsilon(x,t) - u_\varepsilon(x_0,t_0) < \varphi(x,t) \qquad \forall (x,t) \in \left\{\overline{B_{R/4}(x_0)} \times \{t_0\}\right\} \cup \left\{\partial B_{R/4}(x_0) \times [t_0, T]\right\} \tag{4.2}$$

for a sufficiently large constant $C_0 > 1$ and any positive constant A_1. If $A_1 > 1$ is sufficiently large, it can be checked that

$$u^\varepsilon(x,t) - u_\varepsilon(x_0,t_0) \le \varphi(x,t) \qquad \forall (x,t) \in \overline{B_{R/4}(x_0)} \times [t_0,T]. \tag{4.3}$$

We choose the minimal constant A_1 satisfying (4.3), and let $(\bar{x},\bar{t}) \in \overline{B_{R/4}(x_0)} \times [t_0,T]$ be a point such that

$$u^\varepsilon(\bar{x},\bar{t}) - u_\varepsilon(x_0,t_0) = \varphi(\bar{x},\bar{t}). \tag{4.4}$$

We claim that for each $\eta > 0$, the minimal constant A_1 has a uniform upper bound which is independent of small constants ε and ν, and a constant $\sigma \in (\sigma_0, \sigma_0 + 1)$. Here a uniform upper bound for A_1 may depend on η. Obviously, for a fixed $\eta > 0$, the minimal constant A_1 is either bounded from above by $2\max\left(R^{-p_+}, R^{-p_-}\right)$ or bigger than $2\max\left(R^{-p_+}, R^{-p_-}\right)$.

Case 1: $A_1 \le 2\max\left(R^{-p_+}, R^{-p_-}\right)$ for a given $\eta > 0$. In the case, it is immediate from (4.3) that

$$u^\varepsilon(x_0,t) - u_\varepsilon(x_0,t_0) \le \nu + \frac{\eta}{T} + 2\max\left(R^{-p_+}, R^{-p_-}\right)(t - t_0) \qquad \forall t \in [t_0,T]. \tag{4.5}$$

Case 2: $A_1 > 2\max\left(R^{-p_+}, R^{-p_-}\right)$ for a given $\eta > 0$. Firstly, (4.2) implies that $(\bar{x},\bar{t}) \in B_{R/4}(x_0) \times (t_0,T]$, and we see that $\bar{x} \notin \mathrm{Cut}(x_0)$ by employing semi-convexity of u^ε, and [40, Corollary 2.8]; refer to the proof of Claim 3.2. In light of the contacting property (4.4) with (4.3), it holds that

$$\left(\partial_t\varphi(\bar{x},\bar{t}), \nabla\varphi(\bar{x},\bar{t}), D^2\varphi(\bar{x},\bar{t})\right) \in \mathcal{P}^{2,+}u^\varepsilon(\bar{x},\bar{t}),$$

and hence by Proposition 3.3 of [39], there exists a point $(\bar{y},\bar{s}) \in B_{3R/2}(z_0) \times (T/8,T]$ such that

$$\bar{y} = \exp_{\bar{x}} \varepsilon\nabla\varphi(\bar{x},\bar{t}) \quad \text{and} \quad \bar{s} \in \left[\bar{t} - 2\sqrt{\varepsilon},\, T\right] \subset (3T/8,T], \tag{4.6}$$

satisfying

$$u^\varepsilon(\bar{x},\bar{t}) = u(\bar{y},\bar{s}) - \frac{1}{2\varepsilon}\left\{d^2(\bar{y},\bar{x}) + |\bar{s} - \bar{t}|^2\right\} \tag{4.7}$$

and

$$\begin{aligned}&\Big(\partial_t\varphi(\bar{x},\bar{t}),\, L_{\bar{x},\bar{y}}\nabla\varphi(\bar{x},\bar{t}),\, L_{\bar{x},\bar{y}}D^2\varphi(\bar{x},\bar{t})\\ &\quad +\kappa\min\left\{\varepsilon|\nabla\varphi(\bar{x},\bar{t})|^2, 2\omega\left(2\sqrt{\varepsilon}\right)\right\}\mathbf{I}\Big) \in \mathcal{P}^{2,+}u(\bar{y},\bar{s})\end{aligned} \tag{4.8}$$

for sufficiently small $\varepsilon > 0$.

We first consider the case when $\bar{x} = x_0$. Since $\sigma > 2$, we have that $|\nabla\varphi(\bar{x},\bar{t})| = |\nabla\varphi(x_0,\bar{t})| = 0$, which combined with (4.6) and (4.7) yields that $\bar{x} = \bar{y}$ and

$$u^\varepsilon(x_0,\bar{t}) = u(x_0,\bar{s}) - \frac{1}{2\varepsilon}|\bar{s} - \bar{t}|^2. \tag{4.9}$$

On the other hand, by (4.3) and the definition of u^ε, we obtain that

$$u(x,\, t + \bar{s} - \bar{t}) - \frac{1}{2\varepsilon}|\bar{s} - \bar{t}|^2 \le u^\varepsilon(x,t) \le \varphi(x,\, t) + u_\varepsilon(x_0,t_0) \tag{4.10}$$

for any $(x,t) \in B_{R/4}(x_0) \times \left(t_0 + (\bar{t} - t_0)/2,\, \bar{t}\right]$ since

$$t + \bar{s} - \bar{t} \in \left(\bar{s} - (\bar{t} - t_0)/2,\, \bar{s}\right] \subset (\bar{s} - T/4,\, \bar{s}] \subset (T/8,\, T].$$

Setting

$$\psi(x,s) := \varphi(x, s-\bar{s}+\bar{t}) + u_\varepsilon(x_0,t_0) + \frac{1}{2\varepsilon}\left|\bar{s}-\bar{t}\right|^2,$$

the contacting property (4.4) with the use of (4.9) and (4.10) implies that

$$\begin{cases} u(x,s) \le \psi(x,s) \quad \forall (x,s) \in B_{R/4}(x_0) \times \big(\bar{s} - (\bar{t}-t_0)/2,\ \bar{s}\big]; \\ u(x_0,\bar{s}) = \psi(x_0,\bar{s}). \end{cases} \tag{4.11}$$

Since u is a viscosity subsolution of (1.5) and $\sigma > \sigma_0$, we deduce in light of (4.11) and Lemma 2.4 that

$$-A_1 \ \ge\ -R^{-p(x_0,\bar{s})},$$

which is a contradiction to the assumption that $A_1 > 2\max\left(R^{-p_+},\ R^{-p_-}\right)$. Thus we have proved that $\bar{x} \ne x_0$ when $A_1 > 2\max\left(R^{-p_+},\ R^{-p_-}\right)$.

Since $\bar{x} \ne x_0$, it follows from Definition 2.1, and (4.8) that

$$\begin{aligned} &|\nabla\varphi(\bar{x},\bar{t})|^{p(\bar{y},\bar{s})-2}\,\mathrm{tr}\left[\left(\mathbf{I} + \Big(p(\bar{y},\bar{s})-2\Big)\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)\right. \\ &\quad\left.\left\{D^2\varphi(\bar{x},\bar{t}) + \kappa\sqrt{2\varepsilon\omega\left(2\sqrt{\varepsilon}\right)}\,\big|\nabla\varphi(\bar{x},\bar{t})\big|\,\mathbf{I}\right\}\right] \\ &\quad\ge A_1 - R^{-p(\bar{y},\bar{s})} - \beta|\nabla\varphi(\bar{x},\bar{t})|^{p(\bar{y},\bar{s})-1+q} \\ &\qquad - \theta|\nabla\varphi(\bar{x},\bar{t})|^{p(\bar{y},\bar{s})-1}\left|\log\big|\nabla\varphi(\bar{x},\bar{t})\big|\right|. \end{aligned}$$

Here we used the facts that $|\nabla\varphi(\bar{x},\bar{t})| = |L_{\bar{x},\bar{y}}\nabla\varphi(\bar{x},\bar{t})| > 0$, and $D^2\varphi(\bar{x},\bar{t})$ and $L_{\bar{x},\bar{y}}D^2\varphi(\bar{x},\bar{t})$ have the same eigenvalues and the equality (3.30). Moreover, using (4.8) and arguing similarly as for the proof of Corollary 3.7, it holds that

$$\frac{\sigma A_2}{R}\left[\frac{d_{x_0}(\bar{x})}{R}\right]^{\sigma-1} = |\nabla\varphi(\bar{x},\bar{t})| = \big|L_{\bar{x},\bar{y}}\nabla\varphi(\bar{x},\bar{t})\big| \le \frac{C_0}{R} \tag{4.12}$$

for some constant $C_0 > 1$ since $\bar{x} \notin \{x_0\} \cup \mathrm{Cut}(x_0)$ and $(\bar{y},\bar{s}) \in B_{3R/2}(z_0) \times (T/8, T]$. Therefore we conclude that

$$\begin{aligned} A_1 &\le R^{-p(\bar{y},\bar{s})} + \beta R_0^{1-q}\,R^{-p(\bar{y},\bar{s})}\,C_0^{p(\bar{y},\bar{s})-1+q} \\ &\quad + \theta R^{-p(\bar{y},\bar{s})+1}\left[C_0^{p(\bar{y},\bar{s})-1}\log C_0 + C_0^{p(\bar{y},\bar{s})-1}|\log R|\right] \\ &\quad + (n+p_+-2)\kappa\sqrt{2\varepsilon\omega\left(2\sqrt{\varepsilon}\right)}\,R^{-p(\bar{y},\bar{s})+1}\,C_0^{p(\bar{y},\bar{s})-1} \\ &\quad + |\nabla\varphi(\bar{x},\bar{t})|^{p(\bar{y},\bar{s})-2}\,\mathrm{tr}\left[\left(\mathbf{I} + \Big(p(\bar{y},\bar{s})-2\Big)\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)D^2\varphi(\bar{x},\bar{t})\right] \\ &\le C_1 R^{-p(\bar{y},\bar{s})} + |\nabla\varphi(\bar{x},\bar{t})|^{p(\bar{y},\bar{s})-2}\,\mathrm{tr}\left[\left(\mathbf{I} + \Big(p(\bar{y},\bar{s})-2\Big)\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)D^2\varphi(\bar{x},\bar{t})\right] \end{aligned} \tag{4.13}$$

(for sufficiently small $\varepsilon > 0$ which depends on ν and R) since $0 \le q < 1$ and $1 < p_- \le p(\bar{y}, \bar{s}) \le p_+ < \infty$. If $p_- \ge 2$, then $p(\bar{y}, \bar{s}) \ge 2$ and thus it holds that

$$\begin{aligned}&|\nabla\varphi(\bar{x},\bar{t})|^{p(\bar{y},\bar{s})-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(\bar{y},\bar{s})-2\Big)\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)D^2\varphi(\bar{x},\bar{t})\right]\\&\quad\le C_1R^{-p(\bar{y},\bar{s})}C_0^{p_+-2}\cdot\sigma_0(\sigma_0+1)A_2\left[\frac{d_{x_0}(\bar{x})}{R}\right]^{\sigma-2}\le C_1R^{-p(\bar{y},\bar{s})}A_2,\end{aligned}$$

since $d(x_0, \bar{x}) \le R/4$ and $\sigma > 2$. Here we used a Hessian estimate for squared distance functions in [20, Lemma 3.12]. When $1 < p_- < 2$, we use (4.12) to obtain that

$$\begin{aligned}&|\nabla\varphi(\bar{x},\bar{t})|^{p(\bar{y},\bar{s})-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(\bar{y},\bar{s})-2\Big)\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)D^2\varphi(\bar{x},\bar{t})\right]\\&\quad\le C_1\sigma_0\,(\sigma_0+1)^{p(\bar{y},\bar{s})-1}\;R^{-p(\bar{y},\bar{s})}A_2^{p(\bar{y},\bar{s})-1}\left[\frac{d_{x_0}(\bar{x})}{R}\right]^{(\sigma-1)(p(\bar{y},\bar{s})-1)-1}\\&\quad\le C_1R^{-p(\bar{y},\bar{s})}A_2^{p(\bar{y},\bar{s})-1}\left[\frac{d_{x_0}(\bar{x})}{R}\right]^{(\sigma-1)\left(p(\bar{y},\bar{s})-p_-\right)}\cdot\left[\frac{d_{x_0}(\bar{x})}{R}\right]^{(\sigma-1)(p_--1)-1}\\&\quad\le C_1R^{-p(\bar{y},\bar{s})}A_2^{p(\bar{y},\bar{s})-1}\left(\frac{C_0}{\sigma_0\,A_2}\right)^{p(\bar{y},\bar{s})-p_-}\le C_1\left(\frac{C_0}{\sigma_0}\right)^{p_+-p_-}\;R^{-p(\bar{y},\bar{s})}\,A_2^{p_--1}\end{aligned}$$

since $(\sigma-1)(p_- - 1) > (\sigma_0 - 1)(p_- - 1) \ge 1$. Therefore we deduce that

$$\begin{aligned}&|\nabla\varphi(\bar{x},\bar{t})|^{p(\bar{y},\bar{s})-2}\,\mathrm{tr}\left[\left(\mathbf{I}+\Big(p(\bar{y},\bar{s})-2\Big)\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)D^2\varphi(\bar{x},\bar{t})\right]\\&\quad\le C_1R^{-p(\bar{y},\bar{s})}\min\Big(A_2,\;A_2^{p_--1}\Big),\end{aligned}$$

from which (4.13) yields that

$$A_1 \le C_1\max\big(R^{-p_+},\,R^{-p_-}\big)\,A_2^{\sigma_1}\quad\text{with } \sigma_1 := \min\big(1,\,p_- - 1\big) \tag{4.14}$$

in the case when $A_1 > 2\max\big(R^{-p_+},\, R^{-p_-}\big)$ for a given $\eta > 0$. Recalling the definition of φ and using (4.3), (4.1), and (4.14), we obtain that for $t \in [t_0, T]$

$$\begin{aligned}u^\varepsilon(x_0,t)-u_\varepsilon(x_0,t_0)&\le\nu+\frac{\eta}{T}\\&\quad+C_1\max\big(R^{-p_+},\,R^{-p_-}\big)\left\{1+\left(\frac{\eta}{T}\right)^{-(\sigma-1)\sigma_1}\right\}(t-t_0)\end{aligned}\tag{4.15}$$

when $A_1 > 2\max\big(R^{-p_+},\, R^{-p_-}\big)$ for a given $\eta > 0$.

With the use of (4.5), the estimate (4.15) holds true for any cases: either $A_1 \le 2\max\big(R^{-p_+},\, R^{-p_-}\big)$ or $A_1 > 2\max\big(R^{-p_+},\, R^{-p_-}\big)$. In order to optimize the above estimate (4.15) with respect to $\eta > 0$, we choose

$$\frac{\eta}{T}=\Big[C_1\max\big(R^{-p_+},\,R^{-p_-}\big)\cdot(t-t_0)\Big]^\gamma\quad\text{with }\gamma=\frac{1}{1+(\sigma-1)\sigma_1}$$

and then it follows that

$$\begin{aligned}u^\varepsilon(x_0,t)-u_\varepsilon(x_0,t_0)&\le\nu+C_1\max\big(R^{-p_+},\,R^{-p_-}\big)(t-t_0)\\&\quad+\Big[C_1\max\big(R^{-p_+},\,R^{-p_-}\big)\,(t-t_0)\Big]^\gamma.\end{aligned}$$

Letting $\varepsilon \to 0$, $\nu \to 0$ and $\sigma \to \sigma_0$, the proof is finished since $(\sigma_0 - 1)\sigma_1 = 1$. □

Proof of Corollary 1.3. For any $x, y \in M$ and $t, s \in \mathbb{R}$, Corollary 1.2 implies that

$$|u(x,t) - u(y,s)| \leq C \left(\frac{d(x,y)}{R} + \frac{|t-s|^{1/2}}{R^{p/2}} \right)$$

for sufficiently large $R > 0$, where a constant $C > 0$ depends only on n, p, and $\|u\|_{L^\infty(M\times\mathbb{R})}$. Letting $R \to \infty$, the result follows. □

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