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Fractional Choquard equation with critical nonlinearities

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Abstract. In this article, we study the Brezis–Nirenberg type problem of nonlinear Choquard equation involving the fractional Laplacian

$$(-\Delta)^s u = \left(\int_\Omega \frac{|u|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{2^*_{\mu,s}-2}u + \lambda u \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n\backslash\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, λ is a real parameter, $s \in (0,1)$, $n > 2s$, $0 < \mu < n$ and $2^*_{\mu,s} = (2n-\mu)/(n-2s)$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality. We obtain some existence, nonexistence and regularity results for weak solution of the above problem using variational methods.

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1. Introduction

In the present paper, we study the existence of weak solutions of the following doubly nonlocal fractional elliptic problem:

$$(P_\lambda) : (-\Delta)^s u = \left(\int_\Omega \frac{|u|^{2^*_{\mu,s}}}{|x-y|^\mu}dy\right)|u|^{2^*_{\mu,s}-2}u + \lambda u \ \text{ in } \Omega, \ \ u = 0 \ \text{ in } \mathbb{R}^n\backslash\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, λ is a real parameter, $s \in (0,1)$, $2^*_{\mu,s} = (2n-\mu)/(n-2s)$, $0 < \mu < n$, $n > 2s$ and $(-\Delta)^s$ is the fractional Laplace operator defined as

$$(-\Delta)^s u(x) = -\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y$$

(up to a normalizing constant), where P.V. denotes the Cauchy principal value. The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. For more details, we refer to [5,15]. Problems of the type (P_λ) are inspired by the Hardy–Littlewood–Sobolev inequality:

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_{\mu,s}}}\leq C^{\frac{1}{2^*_{\mu,s}}}\|u\|^2_{L^{2^*_s}(\Omega)},\text{ for all } u\in H^s(\mathbb{R}^n), \tag{1.1}$$

where $C=C(n,\mu)$ is a positive constant and $2^*_s=\frac{2n}{n-2s}$.

In the local case $s=1$, authors in [10] studied the existence of ground state solutions for the nonlinear Choquard equation

$$-\Delta u+V(x)u=\left(\frac{1}{|x|^{\mu}}*|u|^p\right)|u|^{p-2}u \text{ in } \mathbb{R}^n, \tag{1.2}$$

for the case $p=2$ and $\mu=1$. S. Pekar [29] used this equation to describe the quantum theory of a polaron at rest and modeling of an electron trapped in its own hole. Lieb [20] adopted it as an approximation to Hartree-Fock theory of one component plasma. In [26], authors proved the existence of ground states under the assumptions of Berestycki-Lions type. With conditions on the potential V, problems of type (1.2) are also studied in [1,2].

In [19], Lieb considered the problem of the form

$$-\Delta u+u=(|x|^{\mu}*F(u))f(u) \text{ in } \mathbb{R}^n,$$

where $f(t)$ is critical growth nonlinearity such that $|tf(t)|\leq C||t|^2+|t|^{\frac{2n-\mu}{n-2s}}|$ for $t\in\mathbb{R}$, $\mu>0$, some constant $C>0$ and $F(t)=\int_0^z f(z)\mathrm{d}z$. Under some appropriate structure conditions on the nonlinearity f, author proved the existence and uniqueness (up to translations) of the ground state solutions. The existence of a sequence of radially symmetric solutions was shown by Lions in [21]. The nonlocal counterpart of this problem with fractional Laplacian has been studied in [35]. A class of Schrödinger equations with a generalized Choquard nonlinearity and fractional diffusion has been investigated in [12]. Some existence, nonexistence and regularity results has been studied in [13]. For more details, we refer to [3,4,11,16,36].

In the pioneering work of Brezis–Nirenberg [7], authors studied the critical exponent problem

$$-\Delta u=|u|^{2^*-2}u+\lambda u \text{ in } \Omega,\ u=0 \text{ in } \partial\Omega,$$

where $2^*=\frac{n+2}{n-2}$. They proved the existence of solutions for $\lambda>0, n>4$ by analyzing the local Palais-Smale sequences below the first critical level. In [14], Gao and Yang established some existence results for the Brezis–Nirenberg type problem of the nonlinear Choquard equation

$$-\Delta u=\left(\int_{\Omega}\frac{|u|^{2^*_{\mu}}}{|x-y|^{\mu}}\mathrm{d}y\right)|u|^{2^*_{\mu}-2}u+\lambda u \text{ in } \Omega,\quad u=0 \text{ on } \partial\Omega, \tag{1.3}$$

where λ is a real parameter and $2^*_\mu = (2n-\mu)/(n-2)$, $0<\mu<n$. Here again, authors obtained the existence results using mountain pass structure of the energy functional and carefully analysing the local Palais-Smale sequences below the first critical level as in [7].

Recently, many people studied the Brezis–Nirenberg type results for semilinear equations with fractional Laplacian, for details and recent works we refer to [8,9,17,24,25,28,33,34] and the references therein. In [22,23], authors discussed recent developments in the description of anamolous diffusion via fractional dynamics and several fractional equations are obtained asymptotically from Lévy random walk models, extending Brownian walk models in a natural way. Particularly in [18], a fractional Schrödinger equation with local power type nonlinearity has been studied. We also refer [27] for a detailed study.

In this paper, we consider the nonlocal counterpart of the problem in (1.3) namely (P_λ). Here, we study the existence, multiplicity, regularity and nonexistence results for (P_λ) in the spirit of [14]. We show several estimates while studying the compactness of Palais-Smale sequences using the minimizers of the inequality in (1.1) and show the L^∞ and $C^{0,\alpha}$ regularity for the solutions of (P_λ). A paper concerning Choquard equation and fractional Laplacian with critical and noncritical nonlinearity is [6]. To the best of our knowledge, there is no paper considering the Brezis–Nirenberg type problem for Choquard equation with fractional Laplacian. We aim at studying the existence and multiplicity of Choquard equation with upper critical exponent $2^*_{\mu,s} = (2n-\mu)/(n-2s)$ on bounded domain in $\mathbb{R}^n$, $n>2s$.

The paper is organized as follows: In Sect. 2, we give the functional setting for the problem to use variational approach and state our main results. In Sect. 3, we show that the weak limit of every bounded Palais-Smale sequence gives a weak solution for (P_λ) by analyzing the Palais-Smale sequences below the critical level. In Sect. 4, we give the proof of our first main theorem (when $n\geq 4s$) for the cases $\lambda\in(0,\lambda_1)$ and $\lambda\geq\lambda_1$ separately, where λ_1 is the first eigenvalue of $(-\Delta)^s$ with homogenous Dirichlet datum given in $\mathbb{R}^n\backslash\Omega$. In Sect. 5, we prove the existence result for (P_λ) when $2s<n<4s$, that is we show that there exists $\bar{\lambda}>0$ such that for any $\lambda>\bar{\lambda}$, different from the eigenvalues of $(-\Delta)^s$, (P_λ) has a nontrivial solution. In Sect. 6, we show some regularity result for weak solutions of (P_λ). Finally in Sect. 7, we prove a non-existence result for $\lambda<0$.

2. Functional setting and main results

In [33], Servadei and Valdinoci discussed the Dirichlet boundary value problem for the fractional Laplacian using variational methods. Due to the nonlocalness of the fractional Laplacian, they introduced the function space $(X_0, \|.\|_{X_0})$. The space X is defined as

$$X=\left\{u|\ u:\mathbb{R}^n\to\mathbb{R} \text{ is measurable, } u|_\Omega\in L^2(\Omega) \text{ and } \frac{(u(x)-u(y))}{|x-y|^{\frac{n}{2}+s}}\in L^2(Q)\right\},$$

where $Q = \mathbb{R}^{2n} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \backslash \Omega$. The space X is endowed with the norm

$$\|u\|_X = \|u\|_{L^2(\Omega)} + [u]_X ,$$

where

$$[u]_X = \left(\int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}} .$$

Then we define $X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \backslash \Omega\}$. Also we have the Poincare type inequality: there exists a constant $C > 0$ such that $\|u\|_{L^2(\Omega)} \leq C[u]_X$, for all $u \in X_0$. Hence, $\|u\| = [u]_X$ is a norm on X_0. Moreover, X_0 is a Hilbert space and $C_c^\infty(\Omega)$ is dense in X_0. Note that the norm $\|.\|$ involves the interaction between Ω and $\mathbb{R}^n \backslash \Omega$. We denote $\|.\| = [.]_X$ for the norm in X_0 and $|.|_p$ for the L_p norm. From the embedding results, we know that X_0 is continuously and compactly embedded in $L^r(\Omega)$ when $1 \leq r < 2_s^*$, where $2_s^* = 2n/(n-2s)$ and the embedding is continuous but not compact if $r = 2_s^*$. We define

$$S_s = \inf_{u \in X_0 \backslash \{0\}} \frac{\|u\|^2}{|u|_{2^*}^2}$$

The key point to apply variational approach for the problem (P_λ) is the following well-known Hardy–Littlewood–Sobolev inequality.

Proposition 2.1. [20] *Let $t, r > 1$ and $0 < \mu < n$ with $1/t + \mu/n + 1/r = 2$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. There exists a sharp constant $C(t, n, \mu, r)$, independent of f, h such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leq C(t, n, \mu, r)|f|_t |h|_r. \tag{2.1}$$

If $t = r = \frac{2n}{2n-\mu}$ then $C(t, n, \mu, r) = C(n, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{n}{2} - \frac{\mu}{2}\right)}{\Gamma\left(n - \frac{\mu}{2}\right)} \left\{ \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)} \right\}^{-1+\frac{\mu}{n}}$. In this case there is equality in (2.1) if and only if $f \equiv (constant)h$ and

$$h(x) = A(\gamma^2 + |x-a|^2)^{\frac{-(2n-\mu)}{2}}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^n$.

In general, let $f = h = |u|^q$ then by Hardy–Littlewood–Sobolev inequality we get,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y$$

is well defined if $|u|^q \in L^t(\mathbb{R}^n)$ for some $t > 1$ satisfying

$$\frac{2}{t} + \frac{\mu}{n} = 2.$$

Thus, for $u \in H^s(\mathbb{R}^n)$, by Sobolev Embedding theorems, we must have

$$\frac{2n-\mu}{n} \leq q \leq \frac{2n-\mu}{n-2s}.$$

From this, for $u \in X_0$ we have

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_{\mu,s}}} \leq C(n,\mu)^{\frac{1}{2^*_{\mu,s}}}|u|^2_{2^*_s},$$

where $C(n,\mu)$ is a suitable constant, $2^*_{\mu,s}=(2n-\mu)/(n-2)$ and $2^*_s=2n/(n-2s)$. We define

$$S_s^H := \inf_{H^s(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y}{\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_{\mu,s}}}}$$

as the best constant which is achieved if and only if u is of the form

$$C\left(\frac{t}{t^2+|x-x_0|^2}\right)^{\frac{n-2s}{2}}, \quad \text{for all } x \in \mathbb{R}^n,$$

for some $x_0 \in \mathbb{R}^n$, $C > 0$ and $t > 0$ (following Lemma 1.2 of [14]). It is well-known that this characterization of u provides the minimizer for S_s which is the best Sobolev constant for the embedding $X_0 \hookrightarrow L^{2^*_s}(\Omega)$. Also, when $\mu = 4s$ (refer Theorem 2.15 of [13]) it satisfies

$$(-\Delta)^s u = \left(\int_{\mathbb{R}^n} \frac{|u|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}y\right)|u|^{2^*_{\mu,s}-2}u \ \text{ in } \mathbb{R}^n. \tag{2.2}$$

Moreover,

$$S_s^H = \frac{S_s}{C(n,\mu)^{\frac{1}{2^*_{\mu,s}}}}. \tag{2.3}$$

Consider the family of functions $\{U_\epsilon\}$ defined as

$$U_\epsilon(x) = \epsilon^{-\frac{(n-2s)}{2}}\left(\frac{u^*\left(\frac{x}{\epsilon}\right)}{|u^*|_{2^*_s}}\right), \ x \in \mathbb{R}^n,$$

where $u^*(x) = \alpha\left(\beta^2 + \left|\frac{x}{S_s^{\frac{1}{2s}}}\right|^2\right)^{-\frac{n-2s}{2}}$ with $\alpha \in \mathbb{R}\setminus\{0\}$ and $\beta > 0$ are fixed constants. Then for each $\epsilon > 0$, U_ϵ satisfies

$$(-\Delta)^s u = |u|^{2^*_s-2}u \ \text{ in } \mathbb{R}^n$$

and verifies the equality

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U_\epsilon(x)-U_\epsilon(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^n}|U_\epsilon|^{2^*_s}\,\mathrm{d}x = S_s^{\frac{n}{2s}}.$$

(For a proof, we refer to [33].) Next lemma gives a property about S_s^H which is known to be true for S_s.

Lemma 2.2. *Let*

$$S_s^H(\Omega) := \inf_{X_0\setminus\{0\}} \frac{\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y}{\left(\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_{\mu,s}}}}.$$

Then $S_s^H(\Omega) = S_s^H$ *and* $S_s^H(\Omega)$ *is never achieved except* $\Omega = \mathbb{R}^n$*.*

Proof. Clearly $S_s^H \leq S_s^H(\Omega)$. Let $\{u_k\} \subset C_c^\infty(\mathbb{R}^n)$ be a minimizing sequence for S_s^H. We choose $\tau_k \in \mathbb{R}^n$ and $\theta_k > 0$ such that

$$v_k(x) := \theta_k^{\frac{n-2s}{2}} u_k(\theta_k x + \tau_k) \in C_c^\infty(\Omega)$$

which satisfies

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y$$

and

$$\int_{\Omega}\int_{\Omega} \frac{|v_k(x)|^{2^*_{\mu,s}}|v_k(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)|^{2^*_{\mu,s}}|u_k(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y.$$

We see that v_k's are translations and dilations of u_k's. By definition,

$$S_s^H(\Omega) \leq \frac{\int_Q \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y}{\int_\Omega\int_\Omega \frac{|v_k(x)|^{2^*_{\mu,s}}|v_k(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y}$$

which implies $S_s^H(\Omega) \leq S_s^H$. Thus, $S_s^H(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^n$ because $\{U_\epsilon\}$ are the only form of minimizers for which the equality holds in Hardy–Littlewood–Sobolev inequality and the best constant is achieved. □

Definition 2.3. *We say that* $u \in X_0$ *is a weak solution of* (P_λ) *if*

$$\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}-2}u(y)\varphi(y)}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y + \lambda\int_\Omega u\varphi\; dx,$$

for every $\varphi \in X_0$*.*

The corresponding energy functional associated to the problem (P_λ) is given by

$$I_\lambda(u) = I(u) := \frac{\|u\|^2}{2} - \frac{1}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y - \frac{\lambda}{2}\int_\Omega |u|^2\mathrm{d}x.$$

Using Hardy–Littlewood–Sobolev inequality, we can show that $I \in C^1(X_0, \mathbb{R})$ and

$$\langle I'(u), \varphi\rangle = \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y$$
$$- \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}-2}u(y)\varphi(y)}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y - \lambda\int_\Omega u\varphi\; dx,$$

for every $\varphi \in X_0$. Thus, u is a weak solution of (P_λ) if and only if u is a critical point of the functional I. We now state the main results of this paper.

Theorem 2.4. *Let* $n \geq 4s$ *for* $s \in (0,1)$*, then* (P_λ) *has a nontrivial weak solution for every* $\lambda > 0$ *such that* λ *is not an eigenvalue of* $(-\Delta)^s$ *with homogenous Dirichlet boundary condition in* $\mathbb{R}^n\backslash\Omega$*.*

Theorem 2.5. *Let $s \in (0,1)$ and $2s < n < 4s$, then there exist $\bar{\lambda} > 0$ such that for any $\lambda > \bar{\lambda}$ different from the eigenvalues of $(-\Delta)^s$ with homogenous Dirichlet boundary condition in $\mathbb{R}^n \backslash \Omega$, (P_λ) has a nontrivial weak solution.*

Theorem 2.6. *Let $\lambda < 0$ and $\Omega \not\equiv \mathbb{R}^n$ be a strictly star shaped bounded domain (with respect to origin) with $C^{1,1}$ boundary, then (P_λ) cannot have a nonnegative nontrivial solution.*

3. Preliminary results

We consider Ω to be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with Lipschitz boundary and λ to be a real parameter throughout this paper.

Definition 3.1. *Let I be a C^1 functional defined on Banach space X, we say that $\{v_k\}$ is a Palais-Smale sequence of I at c (denoted by $(PS)_c$) if*

$$I(v_k) \to c \ \ \text{and} \ \ I'(v_k) \to 0, \ \text{as } k \to +\infty.$$

And we say that I satisfies the Palais-Smale condition at the level c, if every Palais-Smale sequence at c has a convergent subsequence.

The following lemmas can be proved using the standard methods but we give some of their proof here for the sake of completeness. To begin, we recall that pointwise convergence of a bounded sequence implies weak convergence.

Lemma 3.2. *(Proposition 5.4.7, see [37]) Let $q \in (1,\infty)$ and $\{u_k\}$ be a bounded sequence in $L^q(\mathbb{R}^n)$. If $u_k \to v$ almost everywhere in $\mathbb{R}^n$ as $k \to \infty$, then $u_k \rightharpoonup u$ weakly in $L^q(\mathbb{R}^n)$.*

Lemma 3.3. *Let $n > 2s$, $0 < \mu < n$ and $\{u_k\}$ be a bounded sequence in $L^{2^*_s}(\mathbb{R}^n)$ such that $u_k \to u$ almost everywhere in $\mathbb{R}^n$ as $n \to \infty$, then the following hold,*

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)|^{2^*_{\mu,s}}|u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|(u_k-u)(x)|^{2^*_{\mu,s}}|(u_k-u)(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y \\
&\quad \to \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y \ \ \text{as } k \to \infty.
\end{aligned}$$

Proof. Proof follows similarly as proof of Lemma 2.2 [14]. □

Lemma 3.4. *Let*

$$\|u\|_0 := \left(\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{22^*_{\mu,s}}}$$

then $\|.\|_0$ defines a norm on $Y_0 := \{u : \Omega \to \mathbb{R} : \ u \text{ is measurable}, \ \|u\|_0 < +\infty\}$. Also $(Y_0, \|\cdot\|_0)$ is complete.

Proof. Let $u, v \in Y_0$. We have

$$\int_\Omega \int_\Omega \frac{|u(x)|^{2^*_{\mu,s}} |u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\, \mathrm{d}y\mathrm{d}x = \int_\Omega \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\, \mathrm{d}y \right) |u(x)|^{2^*_{\mu,s}}\, \mathrm{d}x.$$

Using semigroup property of Reisz potential and Plancheral theorem, we get

$$\begin{aligned} \int_\Omega \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^2 \mathrm{d}x &= \int_\Omega \left(\mathcal{F} \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right) \right)^2 \mathrm{d}x \\ &= \int_\Omega \mathcal{F} \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\, \mathrm{d}y \right) \mathcal{F} \left(|u(x)|^{2^*_{\mu,s}} \right)\, \mathrm{d}x, \end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform map. Therefore, we get

$$\int_\Omega \int_\Omega \frac{|u(x)|^{2^*_{\mu,s}} |u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\, \mathrm{d}y\mathrm{d}x = \int_\Omega \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^2 \mathrm{d}x$$

So using Minskowski's inequality, we get

$$\begin{aligned} \int_\Omega \left(\int_\Omega \frac{|(u+v)(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^2 &\leq \int_\Omega \left(\int_\Omega \left| \frac{|u(y|)}{|x-y|^{\frac{n+\mu}{22^*_{\mu,s}}}} + \frac{|v(y)|}{|x-y|^{\frac{n+\mu}{22^*_{\mu,s}}}} \right|^{2^*_{\mu,s}} \mathrm{d}y \right)^{22^*_{\mu,s}\frac{1}{2^*_{\mu,s}}} \\ \leq \left(\left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^{2\frac{1}{22^*_{\mu,s}}} \right. &+ \left. \left(\int_\Omega \frac{|v(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^{2\frac{1}{22^*_{\mu,s}}} \right)^{22^*_{\mu,s}}. \end{aligned}$$

As the integrals above are all non-negative, again using the Minskowski's inequality we get

$$\begin{aligned} &\left(\int_\Omega \left(\int_\Omega \frac{|(u+v)(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^2 \mathrm{d}x \right)^{\frac{1}{22^*_{\mu,s}}} \\ &\leq \left(\int_\Omega \left(\left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^{\frac{1}{2^*_{\mu,s}}} + \left(\int_\Omega \frac{|v(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^{\frac{1}{2^*_{\mu,s}}} \right)^{22^*_{\mu,s}} \mathrm{d}x \right)^{\frac{1}{22^*_{\mu,s}}} \\ &\leq \left(\int_\Omega \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^2 \mathrm{d}x \right)^{\frac{1}{22^*_{\mu,s}}} + \left(\int_\Omega \left(\int_\Omega \frac{|v(y)|^{2^*_{\mu,s}}}{|x-y|^{\frac{n+\mu}{2}}}\, \mathrm{d}y \right)^2 \mathrm{d}x \right)^{\frac{1}{22^*_{\mu,s}}}. \end{aligned}$$

Therefore, we get $\|u+v\|_0 \leq \|u\|_0 + \|v\|_0$ and other properties of norm are also satisfied by $\|\cdot\|_0$. So, $\|\cdot\|_0$ is a norm on Y_0 and Y_0 is a Banach space under this norm(proof can be sketched using the techniques to prove $L^p(\Omega)$ is a Banach space with the usual L^p-norm). □

Lemma 3.5. *Let $n > 2s$, $\lambda > 0$ and $0 < \mu < n$. Then every Palais-Smale sequence of I is bounded and its weak limit is a weak solution of (P_λ).*

Proof. Let $\{u_k\}$ be a Palais-Smale sequence of I at $c \in \mathbb{R}^n$. There exist positive constant C_1 such that

$$|I(u_k)| \leq C_1, \quad \text{and} \quad \left| \left\langle I'(u_k), \frac{u_k}{\|u_k\|} \right\rangle \right| \leq C_1.$$

We have

$$\frac{1}{2}\langle I'(u_k), u_k\rangle = I(u_k) - \frac{n+2s-\mu}{2(2n-\mu)} \int_\Omega \int_\Omega \frac{|u_k(x)|^{2^*_{\mu,s}} |u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \,\mathrm{d}x\mathrm{d}y$$

which implies

$$\|u_k\|_0^{22^*_{\mu,s}} = \int_\Omega \int_\Omega \frac{|u_k(x)|^{2^*_{\mu,s}} |u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \,\mathrm{d}x\mathrm{d}y \leq C_2(1 + \|u_k\|), \tag{3.1}$$

for some positive constant C_2. If $\lambda \in (0, \lambda_1)$ for n large enough we get

$$\begin{aligned} C_3(1 + \|u_k\|) &\geq I(u_k) - \frac{1}{22^*_{\mu,s}}\langle I'(u_k), u_k\rangle \\ &= \left(\frac{1}{2} - \frac{1}{22^*_{\mu,s}}\right)(\|u_k\|^2 - \lambda|u_k|_2^2) \geq C_4\|u_k\|^2. \end{aligned}$$

for positive constants C_3, C_4. Let $e_j's$ denote the eigenfunctions of $(-\Delta)^s$ with homogenous Dirichlet boundary condition in $\mathbb{R}^n \backslash \Omega$ corresponding to λ_j. Then when $\lambda \in [\lambda_r, \lambda_{r+1})$ for $r \in \mathbb{N}$ we choose a $M < \frac{1}{22^*_{\mu,s}} < \frac{1}{2}$ and taking $u_k = w_k + v_k$ where $v_k \in span\{e_1, \ldots, e_r\}$ and $w_k \in X_0$ is such that $\langle w_k, e_j\rangle_{X_0} = 0,\ j = 1, \ldots, r$ we obtain

$$\begin{aligned} C_1(1 + \|u_k\|) \geq I(u_k) - M\langle I'(u_k), u_k\rangle &= \left(\frac{1}{2} - M\right)(\|u_k\|^2 - \lambda|u_k|_2^2) \\ &\quad + \left(M - \frac{1}{22^*_{\mu,s}}\right)\|u_k\|_0^{22^*_{\mu,s}} \\ &= \left(\frac{1}{2} - M\right)(\|w_k\|^2 - \lambda\|v_k\|^2 + |w_k|_2^2 - \lambda|u_k|_2^2) \\ &\quad + \left(M - \frac{1}{22^*_{\mu,s}}\right)\|u_k\|_0^{22^*_{\mu,s}} \\ &\geq \left(\frac{1}{2} - M\right)\left(\|w_k\|^2 - \frac{\lambda}{\lambda_{r+1}}\|w_k\|^2 + (\lambda_1 - \lambda)|v_k|_2^2\right) \\ &\quad + \left(M - \frac{1}{22^*_{\mu,s}}\right).\|u_k\|_0^{22^*_{\mu,s}} \end{aligned}$$

Since span$\{e_1, \ldots, e_r\}$ is finite dimensional, $|y_n|_2$, $\|y_n\|$ and $\|y_n\|_0$ are all equivalent. So using (3.1) we get

$$\left(C_1 + \left(\frac{1}{22^*_\mu} - M\right)\right)(1 + \|u_k\|) + \lambda C_5(1 + \|u_k\|)^{1/2^*_{\mu,s}} \geq C_6\|u_k\|^2$$

for appropriate positive constants C_5, C_6. Thus, using all above inequalities we get $\{u_k\}$ to be a bounded sequence in X_0 which implies that there exist a subsequence, still denoted by u_k and $u \in X_0$ such that $u_k \rightharpoonup u$ in X_0 and also $u_k \rightharpoonup u$ in $L^{2^*_s}(\Omega)$ as $k \to +\infty$. Then

$$|u_k|^{2^*_{\mu,s}} \rightharpoonup |u|^{2^*_{\mu,s}} \text{ in } L^{\frac{2n}{2n-\mu}}(\Omega) \quad \text{and} \quad |u_k|^{2^*_{\mu,s}-2}u_k \rightharpoonup |u|^{2^*_{\mu,s}-2}u \text{ in } L^{\frac{2n}{n+2s-\mu}}(\Omega)$$

as $k \to +\infty$. The Reisz potential defines a linear and continuous map from $L^{\frac{2n}{2n-\mu}}(\Omega)$ to $L^{\frac{2n}{\mu}}(\Omega)$, which can be shown using Hardy–Littlewood–Sobolev inequality. This gives

$$\int_\Omega \frac{|u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}y \rightharpoonup \int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}y \text{ weakly in } L^{\frac{2n}{\mu}}(\Omega) \text{ as } k \to +\infty.$$

Combining all these, we get

$$\begin{aligned}&\int_\Omega \frac{|u_k(y)|^{2^*_{\mu,s}}|u_k(x)|^{2^*_{\mu,s}-2}u_k(x)}{|x-y|^\mu}\,\mathrm{d}y\\ &\quad \rightharpoonup \int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}|u(x)|^{2^*_{\mu,s}-2}u(x)}{|x-y|^\mu}\,\mathrm{d}y \text{ weakly in } L^{\frac{2n}{n+2s}}(\Omega)\end{aligned}$$

as $k \to +\infty$. Since $I'(u_k) \to 0$ as $k \to +\infty$, for any $\varphi \in X_0$, we get

$$\begin{aligned}&\lim_{k\to+\infty}\Bigg(\int_Q \frac{(u_k(x)-u_k(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\\ &-\int_\Omega\int_\Omega \frac{|u_k(x)|^{2^*_{\mu,s}}|u_k(y)|^{2^*_{\mu,s}-2}u_k(y)\varphi(y)}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y - \lambda\int_\Omega u_k\varphi\,dx\Bigg) = 0.\end{aligned}$$

This gives

$$\begin{aligned}0 = &\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\\ &-\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}-2}u(y)\varphi(y)}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y - \lambda\int_\Omega u\varphi\,dx\end{aligned}$$

for any $\varphi \in X_0$. Thus, u is a weak solution of (P_λ). □

Let u be the solution obtained in above lemma and we take $\varphi = u \in X_0$ as the test function in (P_λ), then we get

$$\|u\|^2 = \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + \lambda\int_\Omega u^2\,\mathrm{d}x.$$

So,

$$I(u) = \frac{n+2s-\mu}{2(2n-\mu)}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y \geq 0. \tag{3.2}$$

Lemma 3.6. *Let $\lambda > 0$, $0 < \mu < n$ and $\{u_k\}$ be a $(PS)_c$ sequence of I with*

$$c < \frac{n+2s-\mu}{2(2n-\mu)}(S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}.$$

Then $\{u_k\}$ has a convergent subsequence.

Proof. Let u be the weak limit of $\{u_k\}$ obtained using Lemma 3.5. We set $w_k := u_k - u$, then $w_k \rightharpoonup 0$ weakly in X_0 and $w_k \to 0$ a.e. in Ω as $k \to +\infty$. By Brezis–Lieb Lemma, we have

$$\|u_k\|^2 = \|w_k\|^2 + \|u\|^2 + o_k(1), \text{ and } |u_k|_2^2 = |w_k|_2^2 + |u|_2^2 + o_k(1).$$

Also, using Lemma 3.3, we have

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|u_k(x)|^{2^*_{\mu,s}}|u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\ &\quad= \int_\Omega\int_\Omega \frac{|w_k(x)|^{2^*_{\mu,s}}|w_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + o_k(1)\end{aligned}$$

Since $I(u_k) \to c$ as $k \to +\infty$, we get

$$\begin{aligned}c &= \lim_{k\to+\infty} I(u_k)\\ &= \lim_{k\to+\infty}\left(\frac{\|u_k\|^2}{2} - \frac{1}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|u_k(x)|^{2^*_{\mu,s}}|u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y - \frac{\lambda}{2}\int_\Omega |u_k|^2\mathrm{d}x\right)\\ &= \frac{\|w_k\|^2}{2} - \frac{\lambda}{2}\int_\Omega w_k^2\,\mathrm{d}x + \frac{\|u\|^2}{2} - \frac{\lambda}{2}\int_\Omega u^2\mathrm{d}x\\ &\quad - \frac{1}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|w_k(x)|^{2^*_{\mu,s}}|w_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\ &\quad - \frac{1}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + o_k(1)\\ &= I(u) + \frac{\|w_k\|^2}{2} - \frac{\lambda}{2}\int_\Omega w_k^2\,\mathrm{d}x\\ &\quad - \frac{1}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|w_k(x)|^{2^*_{\mu,s}}|w_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + o_k(1)\\ &\geq \frac{\|w_k\|^2}{2} - \frac{1}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|w_k(x)|^{2^*_{\mu,s}}|w_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + o_k(1),\end{aligned} \tag{3.3}$$

using (3.2) and the fact that $\int_\Omega w_k^2\,\mathrm{d}x \to 0$ as $k \to +\infty$ (because $X_0 \hookrightarrow L^2(\Omega)$ compactly). In a similar manner, since u is a weak solution of (P_λ), u must be a critical point of I which gives $\langle I'(u), u\rangle = 0$ that is

$$\begin{aligned}o_k(1) &= \|u_k\|^2 - \int_\Omega\int_\Omega \frac{|u_k(x)|^{2^*_{\mu,s}}|u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y - \lambda\int_\Omega |u_k|^2\mathrm{d}x\\ &= \|w_k\|^2 - \lambda\int_\Omega |w_k|^2\mathrm{d}x + \|u\|^2 - \lambda\int_\Omega |u|^2\mathrm{d}x\\ &\quad - \int_\Omega\int_\Omega \frac{|w_k(x)|^{2^*_{\mu,s}}|w_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\ &\quad - \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + o_k(1)\end{aligned}$$

$$
\begin{aligned}
&= \langle I'(u), u\rangle + \|w_k\|^2 - \lambda \int_\Omega |w_k|^2 \mathrm{d}x \\
&\quad - \int_\Omega\int_\Omega \frac{|w_k(x)|^{2^*_{\mu,s}}|w_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + o_k(1) \\
&= \|w_k\|^2 - \int_\Omega\int_\Omega \frac{|w_k(x)|^{2^*_{\mu,s}}|w_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + o_k(1).
\end{aligned} \tag{3.4}
$$

This implies

$$
\lim_{k\to+\infty} \|w_k\|^2 = \lim_{k\to+\infty} \int_\Omega\int_\Omega \frac{|w_k(x)|^{2^*_{\mu,s}}|w_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y = a,
$$

where a is nonnegative constant. From (3.3) and (3.4), we deduce

$$
c \geq \frac{n+2s-\mu}{2(2n-\mu)} a.
$$

Using definition of S_s^H, we get

$$
S_s^H \left(\int_\Omega\int_\Omega \frac{|w_k(x)|^{2^*_{\mu,s}}|w_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y \right)^{\frac{n-2s}{2n-\mu}} \leq \|w_k\|^2,
$$

which gives $a \geq S_s^H a^{\frac{n-2s}{2n-\mu}}$. Thus, either $a = 0$ or $a \geq (S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}$. If $a = 0$ we are done, else $a \geq (S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}$ gives

$$
\frac{n+2s-\mu}{2(2n-\mu)} (S_s^H)^{\frac{2n-\mu}{n+2s-\mu}} \leq c.
$$

This contradicts the hypothesis that

$$
c < \frac{n+2s-\mu}{2(2n-\mu)} (S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}.
$$

Thus, $a = 0$ which implies $\|u_k - u\| \to 0$ as $k \to +\infty$. □

4. Proof of Theorem 2.4

We fix $n \geq 4s$ and Ω to be a smooth bounded domain in $\mathbb{R}^n$. We divide the proof of Theorem 2.4 considering two cases.

4.1. Case (1): $\lambda \in (0, \lambda_1)$

Without loss of generality, we assume $0 \in \Omega$ and fix $\delta > 0$ such that $B_\delta \subset \Omega \subset B_{\hat{k}\delta}$, for some $\hat{k} > 1$. Let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $0 \leq \eta \leq 1$ in $\mathbb{R}^n$, $\eta \equiv 1$ in B_δ and $\eta \equiv 0$ in $\mathbb{R}^n \backslash \Omega$. For $\epsilon > 0$, we define the function u_ϵ as follows

$$
u_\epsilon(x) := \eta(x) U_\epsilon(x),
$$

for $x \in \mathbb{R}^n$, where U_ϵ has been defined in Sect. 2. We have the following results for u_ϵ using Proposition 21 and 22 of [33].

Proposition 4.1. *Let $s \in (0, 1)$ and $n > 2s$. Then, the following estimates holds true as $\epsilon \to 0$*

(i) $\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{n+2s}}\ \mathrm{d}x\mathrm{d}y \leq S_s^{n/(2s)} + o(\epsilon^{n-2s})$,

(ii) $\displaystyle\int_\Omega |u_\epsilon|^{2_s^*}\ \mathrm{d}x = S_s^{n/(2s)} + o(\epsilon^n)$,

(iii)

$$\int_\Omega |u_\epsilon(x)|^2\ \mathrm{d}x \geq \begin{cases} C_s\epsilon^{2s} + o(\epsilon^{n-2s}) & \text{if } n > 4s \\ C_s\epsilon^{2s}|\log\epsilon| + o(\epsilon^{2s}) & \text{if } n = 4s \\ C_s\epsilon^{n-2s} + o(\epsilon^{2s}) & \text{if } n < 4s \end{cases},$$

for some positive constant C_s, depending on s.

Using (2.3), Proposition 4.1(i) can be written as

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{n+2s}}\ \mathrm{d}x\mathrm{d}y &\leq S_s^{n/(2s)} + o(\epsilon^{n-2s}) \\ &= (C(n,\mu))^{\frac{n(n-2s)}{2s(2n-\mu)}}(S_s^H)^{\frac{n}{2s}} + o(\epsilon^{n-2s}).\end{aligned} \tag{4.1}$$

We now prove the following proposition in the spirit of Sect. 3 of [14].

Proposition 4.2. *The following estimates holds true:*

$$\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\ \mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}} \leq (C(n,\mu))^{\frac{n(n-2s)}{2s(2n-\mu)}}(S_s^H)^{\frac{n-2s}{2}} + o(\epsilon^{n-2s}),$$

and

$$\begin{aligned}&\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\ \mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}} \\ &\quad\geq \left((C(n,\mu))^{\frac{n}{2s}}(S_s^H)^{\frac{2n-\mu}{2s}} - o\left(\epsilon^{\frac{n(2n-\mu-\eta)}{2n-\eta}}\right)\right)^{\frac{n-2s}{2n-\mu}}\end{aligned}$$

for each $0<\eta<2n-\mu$.

Proof. By Hardy–Littlewood–Sobolev inequality, Proposition 4.1(ii) and 2.3, we get

$$\begin{aligned}&\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\ \mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}} \\ &\quad\leq (C(n,\mu))^{\frac{n-2s}{2n-\mu}}|u_\epsilon|^2_{2_s^*} = (C(n,\mu))^{\frac{n-2s}{2n-\mu}}\left(S_s^{n/(2s)} + o(\epsilon^n)\right)^{\frac{n-2s}{n}} \\ &\quad= (C(n,\mu))^{\frac{n-2s}{2n-\mu}}\left((C(n,\mu))^{\frac{n(n-2s)}{2s(2n-\mu)}}(S_s^H)^{\frac{n}{2s}} + o(\epsilon^n)\right)^{\frac{n-2s}{n}} \\ &\quad\leq (C(n,\mu))^{\frac{n(n-2s)}{2s(2n-\mu)}}(S_s^H)^{\frac{n-2s}{2s}} + o(\epsilon^{n-2s}).\end{aligned}$$

Next, we consider

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\ &\quad\geq \int_{B_\delta}\int_{B_\delta} \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y = \int_{B_\delta}\int_{B_\delta} \frac{|U_\epsilon(x)|^{2^*_{\mu,s}}|U_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\ &\quad= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U_\epsilon(x)|^{2^*_{\mu,s}}|U_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y - 2\int_{\mathbb{R}^n\setminus B_\delta}\int_{B_\delta} \frac{|U_\epsilon(x)|^{2^*_{\mu,s}}|U_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\ &\qquad - \int_{\mathbb{R}^n\setminus B_\delta}\int_{\mathbb{R}^n\setminus B_\delta} \frac{|U_\epsilon(x)|^{2^*_{\mu,s}}|U_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{4.2}$$

We estimate the integrals in R.H.S. of (4.2) separately. Firstly, consider

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U_\epsilon(x)|^{2^*_{\mu,s}}|U_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y = \left(\frac{\|U_\epsilon\|^2}{S_s^H}\right)^{\frac{2n-\mu}{n-2s}} = (C(n,\mu))^{\frac{n}{(2s)}}\left(S_s^H\right)^{\frac{2n-\mu}{2s}}. \tag{4.3}$$

Secondly, fix $0<\eta<2n-\mu$ and consider

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B_\delta}\int_{B_\delta} \frac{|U_\epsilon(x)|^{2^*_{\mu,s}}|U_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\ &\quad\leq C_{1,s}\int_{\mathbb{R}^n\setminus B_\delta}\int_{B_\delta} \frac{\epsilon^{\mu-2n}}{|x-y|^\mu\left(1+|\frac{x}{\epsilon}|^2\right)^{\frac{2n-\mu}{2}}\left(1+|\frac{y}{\epsilon}|^2\right)^{\frac{2n-\mu}{2}}}\,\mathrm{d}x\mathrm{d}y\\ &\quad= o(\epsilon^{2n-\mu})\int_{\mathbb{R}^n\setminus B_\delta}\int_{B_\delta} \frac{1}{|x-y|^\mu\left(\epsilon^2+|x|^2\right)^{\frac{2n-\mu}{2}}\left(\epsilon^2+|y|^2\right)^{\frac{2n-\mu}{2}}}\,\mathrm{d}x\mathrm{d}y\\ &\quad\leq o(\epsilon^{2n-\mu})\left(\int_{\mathbb{R}^n\setminus B_\delta}\int_{B_\delta}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{2n-\eta}}\right)^{\frac{\mu}{2n-\eta}}\left(\int_{\mathbb{R}^n\setminus B_\delta}\frac{1}{(\epsilon^2+|x|^2)^{\frac{(2n-\mu)(2n-\eta)}{2(2n-\mu-\eta)}}}\mathrm{d}x\right)^{\frac{2n-\mu-\eta}{2n-\eta}}\\ &\qquad\left(\int_{B_\delta}\frac{1}{(\epsilon^2+|y|^2)^{\frac{(2n-\mu)(2n-\eta)}{2(2n-\mu-\eta)}}}\mathrm{d}y\right)^{\frac{2n-\mu-\eta}{2n-\eta}}\\ &\quad\leq o(\epsilon^{2n-\mu})\left(\int_{\mathbb{R}^n\setminus B_\delta}\frac{1}{|x|^{\frac{(2n-\mu)(2n-\eta)}{2(2n-\mu-\eta)}}}\mathrm{d}x\right)^{\frac{2n-\mu-\eta}{2n-\eta}}\left(\int_0^\delta\frac{t^{n-1}}{(\epsilon^2+t^2)^{\frac{(2n-\mu)(2n-\eta)}{2(2n-\mu-\eta)}}}\mathrm{d}t\right)^{\frac{2n-\mu-\eta}{2n-\eta}}\\ &\quad\leq o\left(\epsilon^{2n-\mu}\right)\left(\frac{\epsilon^n}{\epsilon^{\frac{(2n-\mu)(2n-\eta)}{2n-\mu-\eta}}}\int_0^\infty\frac{1}{(1+t^2)^{\frac{(2n-\mu)(2n-\eta)}{2(2n-\mu-\eta)}}}\,\mathrm{d}t\right)^{\frac{2n-\mu-\eta}{2n-\eta}}\leq o\left(\epsilon^{\frac{n(2n-\mu-\eta)}{2n-\eta}}\right)\end{aligned} \tag{4.4}$$

where $C_{1,s}$ is an appropriate positive constant. Lastly, in a similar manner we have

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B_\delta}\int_{\mathbb{R}^n\setminus B_\delta} \frac{|U_\epsilon(x)|^{2^*_{\mu,s}}|U_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\ &\quad\leq C_{1,s}\int_{\mathbb{R}^n\setminus B_\delta}\int_{\mathbb{R}^n\setminus B_\delta} \frac{\epsilon^{\mu-2n}}{|x-y|^\mu\left(1+|\frac{x}{\epsilon}|^2\right)^{\frac{2n-\mu}{2}}\left(1+|\frac{y}{\epsilon}|^2\right)^{\frac{2n-\mu}{2}}}\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$= \epsilon^{2n-\mu} C_{2,s} \int_{\mathbb{R}^n \setminus B_\delta} \int_{\mathbb{R}^n \setminus B_\delta} \frac{1}{|x-y|^\mu \left(\epsilon^2 + |x|^2\right)^{\frac{2n-\mu}{2}} \left(\epsilon^2 + |y|^2\right)^{\frac{2n-\mu}{2}}} \, \mathrm{d}x\mathrm{d}y$$

$$\leq \epsilon^{2n-\mu} C_{2,s} \int_{\mathbb{R}^n \setminus B_\delta} \int_{\mathbb{R}^n \setminus B_\delta} \frac{1}{|x-y|^\mu |x|^{2n-\mu} |y|^{2n-\mu}} \mathrm{d}x\mathrm{d}y = o(\epsilon^{2n-\mu}). \quad (4.5)$$

Using (4.3), (4.4) and (4.5) in (4.2), we get

$$\left(\int_\Omega \int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}} |u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{n-2s}{2n-\mu}}$$

$$\geq \left((C(n,\mu))^{\frac{n}{2s}} (S_s^H)^{\frac{2n-\mu}{2s}} - o\left(\epsilon^{\frac{n(2n-\mu-\eta)}{2n-\eta}}\right) \right)^{\frac{n-2s}{2n-\mu}}. \quad (4.6)$$

This completes the proof. □

Remark 4.3. (4.6) and (4.1) still holds when $2s < n < 4s$.

We prove the existence of solution to (P_λ) using an invariant of mountain pass lemma.

Lemma 4.4. *If $\lambda \in (0, \lambda_1)$, then the energy functional I satisfies the following properties:*

(i) *there exist $\beta, \rho > 0$ such that $I(u) \geq \beta$ when $\|u\| = \rho$,*
(ii) *there exists $\tilde{u} \in X_0$ such that $\|\tilde{u}\| > \rho$ and $I(\tilde{u}) < 0$.*

Proof. (i) Since $\lambda \in (0, \lambda_1)$, using Sobolev embedding and Hardy–Littlewood –Sobolev inequality, we get

$$I(u) \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^2 - \frac{1}{2 2^*_{\mu,s}} C_1 |u|_{2^*_s}^{\frac{2(2n-\mu)}{n-2s}}$$

$$\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^2 - \frac{1}{2 2^*_{\mu,s}} C_1 C_2 \|u\|^{\frac{2(2n-\mu)}{n-2s}},$$

for all $u \in X_0 \setminus \{0\}$, where C_1, C_2 are positive constants. Since $0 < \mu < n$, so $2 < 2\left(\frac{2n-\mu}{n-2s}\right)$. Thus, some $\beta, \rho > 0$ can be chosen such that $I(u) \geq \beta$ when $\|u\| = \rho$.

(ii) We fix $u_0 \in X_0 \setminus \{0\}$ and since $I(tu_0) \to -\infty$ as $t \to \infty$, we get

$$I(tu_0) = \frac{t^2 \|u_0\|^2}{2} - \frac{t^{2 2^*_{\mu,s}}}{2 2^*_{\mu,s}} \int_\Omega \int_\Omega \frac{|u_0(x)|^{2^*_{\mu,s}} |u_0(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \, \mathrm{d}x\mathrm{d}y - \frac{\lambda t^2}{2} \int_\Omega |u_0|^2 \mathrm{d}x < 0$$

for sufficiently large $t > 0$. This implies, we can obtain $\tilde{u} = t_0 u_0 \in X_0$ for some $t_0 > 0$ such that $\|\tilde{u}\| > \rho$ and $I(\tilde{u}) < 0$. □

Proposition 4.5. (Proposition 3.3, [14]) *Using lemma 4.4 and the mountain pass lemma without (PS) condition [37], there exists a (PS) sequence $\{u_k\}$ such that $I(u_k) \to c$ and $I'(u_k) \to 0$ in X_0^* (dual of X_0) at the minimax level*

$$c^* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0,$$

where

$$\Gamma := \{\gamma \in C([0,1], X_0) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Proof of Theorem 2.4. ($n \geq 4s$, $\lambda \in (0,\lambda_1)$) Before proving this theorem, we claim that there exists $w \in X_0\backslash\{0\}$ such that

$$\frac{\|w\|^2 - \lambda\int_\Omega |w|^2\mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|w(x)|^{2^*_{\mu,s}}|w(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}} < S_s^H \tag{4.7}$$

If $n = 4s$, using Proposition 4.1(iii), (4.1), (4.6) and $0 < \mu < n$, for sufficiently small $\eta > 0$ we get

$$\begin{aligned}
&\frac{\|u_\epsilon\|^2 - \lambda\int_\Omega |u_\epsilon|^2\mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{2s}{8s-\mu}}}\\
&\quad\leq \frac{(C(4s,\mu))^{\frac{4s}{8s-\mu}}(S_s^H)^2 - \lambda C_s\epsilon^{2s}|\log\epsilon| + o(\epsilon^{2s})}{\left((C(4s,\mu))^2(S_s^H)^{\frac{8s-\mu}{2s}} - o(\epsilon^{\frac{4s(8s-\mu-\eta)}{8s-\eta}}\right)^{\frac{2s}{8s-\mu}}}\\
&\quad\leq S_s^H - \lambda C_s\epsilon^{2s}|\log\epsilon| + o(\epsilon^{2s}) < S_s^H
\end{aligned} \tag{4.8}$$

for $\epsilon > 0$ small enough. If $n > 4s$, then again using Proposition 4.1(iii), (4.1), (4.6) and $n > 4s$, for sufficiently small $\eta > 0$ we get

$$\begin{aligned}
&\frac{\|u_\epsilon\|^2 - \lambda\int_\Omega |u_\epsilon|^2\mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}}\\
&\quad\leq \frac{(C(n,\mu))^{\frac{n(n-2s)}{2s(2n-\mu)}}(S_s^H)^{\frac{n}{2s}} - \lambda C_s\epsilon^{2s} + o(\epsilon^{n-2s})}{\left((C(n,\mu))^{\frac{n}{2s}}(S_s^H)^{\frac{2n-\mu}{2s}} - o(\epsilon^{\frac{n(2n-\mu-\eta)}{2n-\eta}})\right)^{\frac{n-2s}{2n-\mu}}}\\
&\quad\leq S_s^H - \lambda C_s\epsilon^{2s} + o(\epsilon^{n-2s}) < S_s^H
\end{aligned} \tag{4.9}$$

for $\epsilon > 0$ small enough. So (4.7) holds true if we take $w = u_\epsilon$ for appropriately chosen $\epsilon > 0$. Consequently, we have

$$\begin{aligned}
&\max_{t\geq 0}\left(\frac{t^2\|w\|^2}{2} - \frac{t^{22^*_{\mu,s}}}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|w(x)|^{2^*_{\mu,s}}|w(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y - \frac{\lambda t^2}{2}\int_\Omega w^2\mathrm{d}x\right)\\
&= \frac{n+2s-\mu}{2(2n-\mu)}\left(\frac{\|w\|^2 - \lambda\int_\Omega w^2\mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|w(x)|^{2^*_{\mu,s}}|w(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}}\right)^{\frac{2n-\mu}{n+2s-\mu}}\\
&< \frac{n+2s-\mu}{2(2n-\mu)}(S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}.
\end{aligned}$$

This implies

$$0 < \max_{t\geq 0} I(tw) < \frac{n+2s-\mu}{2(2n-\mu)}(S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}.$$

From the definition c^*, we can say that $c^* < \frac{n+2s-\mu}{2(2n-\mu)}(S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}$. Then using Proposition 4.5, there exists a (PS) sequence, say $\{u_k\}$ at c^*. We know $\{u_k\}$ has a convergent subsequence, using Lemma 3.6 and thus, I has a critical value $c^* \in \left(0, \frac{n+2s-\mu}{2(2n-\mu)}(S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}\right)$ which gives a nontrivial weak solution for (P_λ). □

4.2. Case (2): $\lambda \geq \lambda_1$

Let us consider the sequence of eigenvalues of the operator $(-\Delta)^s$ with homogenous Dirichlet boundary condition in $\mathbb{R}^n \backslash \{\Omega\}$, denoted by

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots$$

and $\{e_j\}_{j\in\mathbb{N}} \subset L^\infty(\Omega)$ be the corresponding sequence of eigenfunctions. We also consider this sequence of e_j's to form an orthonormal basis of $L^2(\Omega)$ and orthogonal basis of X_0.

In this case, without loss of generality, we can assume $\lambda \in [\lambda_r, \lambda_{r+1})$ for some $r \in \mathbb{N}$ and e_r denote the eigenfunction corresponding to λ_r. We define

$$\mathbb{M}_{r+1} := \left\{ u \in X_0 : \langle u, e_i \rangle := \int_Q \frac{(u(x)-u(y))(e_i(x)-e_i(y))}{|x-y|^{n+2s}} \mathrm{d}x\mathrm{d}y \right.$$
$$\left. = 0,\ i = 1, 2, \ldots, r \right\},$$

and

$$\mathbb{D}_r := \text{span}\{e_1, e_2, \ldots, e_r\}.$$

Clearly, $\mathbb{D}_r$ is finite dimensional and $\mathbb{D}_r \oplus \mathbb{M}_{r+1} = X_0$.

Lemma 4.6. *Let $\lambda \in [\lambda_r, \lambda_{r+1})$ for some $r \in \mathbb{N}$. Then the energy functional I satisfies the following properties :*

(i) *There exists $\beta, \rho > 0$ such that $I(u) \geq \beta$, for any $u \in \mathbb{M}_{r+1}$ with $\|u\| = \rho$.*
(ii) *If $u \in \mathbb{D}_r$, then $I(u) < 0$.*
(iii) *If $\mathbb{E}$ is any finite dimensional subspace of X_0, then there exists $R > \rho$ such that for any $u \in \mathbb{E}$ with $\|u\| \geq R$, we have $I(u) \leq 0$.*

Proof. (i) Since $\lambda \in [\lambda_r, \lambda_{r+1})$, using Sobolev embedding and Hardy–Littlewood–Sobolev inequality, we get

$$I(u) \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{r+1}}\right)\|u\|^2 - \frac{1}{22^*_{\mu,s}} C_1 |u|_{2^*_s}^{\frac{2(2n-\mu)}{n-2s}}$$
$$\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{r+1}}\right)\|u\|^2 - \frac{1}{22^*_{\mu,s}} C_1 C_2 \|u\|^{\frac{2(2n-\mu)}{n-2s}},$$

for all $u \in \mathbb{M}_{r+1}\backslash\{0\}$, where C_1, C_2 are positive constants. Since $0 < \mu < n$, so $2 < 2\left(\frac{2n-\mu}{n-2s}\right)$ and thus, some $\beta, \rho > 0$ can be chosen such that $I(u) \geq \beta$ for $u \in M_{r+1}$ and $\|u\| = \rho$.

(ii) Let $u \in \mathbb{D}_r$, then there exists $a_i \in \mathbb{R}$ such that $u = \sum_{i=1}^r a_i e_i$. Since e_j's form an orthonormal basis of X_0 and $L^2(\Omega)$, we get

$$\int_\Omega u^2\ \mathrm{d}x = \sum_{i=1}^r a_i^2 \text{ and } \|u\|^2 = \sum_{i=1}^r a_i^2 \|e_i\|^2.$$

This implies

$$I(u) = \frac{1}{2}\sum_{i=1}^{r} a_i^2(\|e_i\|^2 - \lambda) - \frac{1}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y$$
$$< \frac{1}{2}\sum_{i=1}^{r} a_i^2(\lambda_i - \lambda) \leq 0,$$

because $\lambda \in [\lambda_r, \lambda_{r+1})$.

(iii) Let $u \in \mathbb{E}\backslash\{0\}$. Recalling that $\lambda > 0$, all norms on a finite dimensional space are equivalent and Lemma 3.4, we get

$$I(u) = \frac{\|u\|^2}{2} - \frac{\lambda|u|_2^2}{2} - \frac{1}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y$$
$$\leq \frac{\|u\|^2}{2} - \frac{C_1}{22^*_{\mu,s}}\|u\|^{22^*_{\mu,s}},$$

for some positive constant C_1. This implies $I(u) \to -\infty$ as $\|u\| \to +\infty$. Thus there exists $R > \rho$ such that for any $u \in \mathbb{E}$ with $\|u\| \geq R$, we have $I(u) \leq 0$ and (iii) follows. □

Before proceeding further, we define the linear space

$$\mathbb{G}_{r,\epsilon} := \mathrm{span}\{e_1, e_2, \ldots, e_r, u_\epsilon\}$$

and set

$$g_{r,\epsilon} := \max_{u\in M}\left(\|u\|^2 - \lambda\int_\Omega |u|^2\,\mathrm{d}x\right),$$

where $M = \{u \in \mathbb{G}_{r,\epsilon} : \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y = 1\}$ and u_ϵ (using (4.7)) is such that

$$\frac{\|u_\epsilon\|^2 - \lambda\int_\Omega |u_\epsilon|^2\mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}} < S_s^H.$$

Lemma 4.7. *Let $n \geq 4s$ for $s \in (0,1)$ and $\lambda \in [\lambda_r, \lambda_{r+1})$ for some $r \in \mathbb{N}$, then the following holds true:*

(i) *There exist $u_g \in \mathbb{G}_{r,\epsilon}$ such that $g_{r,\epsilon}$ is achieved at u_g and*

$$u_g = w + tu_\epsilon$$

with $w \in \mathbb{D}_r$ and $t \geq 0$.

(ii) *As $\epsilon \to 0$, we have*

$$g_{r,\epsilon} \leq \begin{cases} (\lambda_r - \lambda)|w|_2^2 & \text{if } t = 0 \\ (\lambda_r - \lambda)|w|_2^2 + F_\epsilon(1 + |w|_2 o(\epsilon^{\frac{n-2s}{2}})) + o(\epsilon^{\frac{n-2s}{2}})|w|_2 & \text{if } t > 0, \end{cases}$$

where w is given in (i) and F_ϵ is given by

$$F_\epsilon = \frac{\|u_\epsilon\|^2 - \lambda\int_\Omega |u_\epsilon|^2\mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}}.$$

Proof. (i) Clearly $\mathbb{G}_{r,\epsilon}$ is finite dimensional, so $g_{r,\epsilon}$ is achieved at some $u_g \in G_{r,\epsilon}$ such that $u_g \in M$. By definition of $\mathbb{G}_{r,\epsilon}$, there exist $w \in \mathbb{D}_r$ and $t \in \mathbb{R}$ such that $u_g = w + tu_\epsilon$. We can assume $t \geq 0$ because if $t < 0$, then we can replace u_g by $-u_g$.

(ii) To prove this, first let $t = 0$, then $u_g = w \in \mathbb{D}_r$ and

$$g_{r,\epsilon} = \|w\|^2 - \lambda \int_\Omega |w|^2 \,\mathrm{d}x \leq (\lambda_r - \lambda)|w|_2^2.$$

Now, suppose $t > 0$. Since $w \in \mathrm{Span}\{e_1, e_2, \ldots, e_r\}$ and $e_j \in L^\infty(\Omega)$ for $j = 1, 2, \ldots, r$, we get $w \in L^\infty(\Omega)$. Consider

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}} |u_\epsilon(y)|^{2^*_{\mu,s}-1}}{|x-y|^\mu} \,\mathrm{d}x\mathrm{d}y \\
&\quad \leq \int_{B_{\hat{k}\delta}} \int_{B_{\hat{k}\delta}} \frac{|u_\epsilon(x)|^{2^*_{\mu,s}} |u_\epsilon(y)|^{2^*_{\mu,s}-1}}{|x-y|^\mu} \,\mathrm{d}x\mathrm{d}y \\
&\quad = \int_{B_{\hat{k}\delta}} \int_{B_{\hat{k}\delta}} \frac{|U_\epsilon(x)|^{2^*_{\mu,s}} |U_\epsilon(y)|^{2^*_{\mu,s}-1}}{|x-y|^\mu} \,\mathrm{d}x\mathrm{d}y \\
&\quad = C_{1,s} \epsilon^{\frac{-(3n-2\mu+2s)}{2}} \int_{B_{\hat{k}\delta}} \int_{B_{\hat{k}\delta}} \frac{1}{\left(1+\left|\frac{x}{\epsilon}\right|^2\right)^{\frac{2n-\mu}{2}} |x-y|^\mu \left(1+\left|\frac{y}{\epsilon}\right|^2\right)^{\frac{n-\mu+2s}{2}}} \,\mathrm{d}x\mathrm{d}y \\
&\quad = C_{1,s} \epsilon^{\frac{n-2s}{2}} \int_{B_{\frac{\hat{k}\delta}{\epsilon}}} \int_{B_{\frac{\hat{k}\delta}{\epsilon}}} \frac{1}{\left(1+|x|^2\right)^{\frac{2n-\mu}{2}} |x-y|^\mu \left(1+|y|^2\right)^{\frac{n-\mu+2s}{2}}} \,\mathrm{d}x\mathrm{d}y \\
&\quad \leq C_{1,s} \epsilon^{\frac{n-2s}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{\left(1+|x|^2\right)^{\frac{2n-\mu}{2}} |x-y|^\mu \left(1+|y|^2\right)^{\frac{n-\mu+2s}{2}}} \,\mathrm{d}x\mathrm{d}y,
\end{aligned} \tag{4.10}$$

where $C_{1,s}$ is positive constant. If $\mu > 1$, then using Hardy–Littlewood–Sobolev inequality we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{\frac{2n-\mu}{2}} |x-y|^\mu (1+|y|^2)^{\frac{n-\mu+2s}{2}}} \,\mathrm{d}x\mathrm{d}y \\
&\quad \leq \left(\int_{\mathbb{R}^n} \left(\frac{1}{(1+|x|^2)^{\frac{2n-\mu}{2}}} \right)^{\frac{n}{n-s}} \mathrm{d}x \right)^{\frac{n-s}{n}} \left(\int_{\mathbb{R}^n} \left(\frac{1}{(1+|y|^2)^{\frac{n-\mu+2s}{2}}} \right)^{\frac{n}{n-\mu+s}} \mathrm{d}y \right)^{\frac{n-\mu+s}{n}}
\end{aligned} \tag{4.11}$$

where each of the above integrals are finite. Else if $\mu \leq 1$, then using Hardy–Littlewood–Sobolev inequality we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{\left(1+|x|^2\right)^{\frac{2n-\mu}{2}}|x-y|^{\mu}\left(1+|y|^2\right)^{\frac{n-\mu+2s}{2}}}\,\mathrm{d}x\mathrm{d}y$$
$$\leq\left(\int_{\mathbb{R}^n}\left(\frac{1}{(1+|x|^2)^{\frac{2n-\mu}{2}}}\right)^{\frac{2n}{2n-\mu}}\mathrm{d}x\right)^{\frac{2n-\mu}{2n}}\left(\int_{\mathbb{R}^n}\left(\frac{1}{(1+|y|^2)^{\frac{n-\mu+2s}{2}}}\right)^{\frac{2n}{2n-\mu}}\mathrm{d}y\right)^{\frac{2n-\mu}{2n}} \tag{4.12}$$

where each of the above integrals are finite. Using (4.10), (4.11) and (4.12) we get

$$\int_{\Omega}\int_{\Omega}\frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}-1}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y\leq o(\epsilon^{\frac{n-2s}{2}}). \tag{4.13}$$

Moreover, using simple computations we see that for $\epsilon>0$ sufficiently small

$$\int_{\Omega}\int_{\Omega}\frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}-1}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y$$
$$\geq\int_{B_\delta}\int_{B_\delta}\frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}-1}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y$$
$$=C_{1,s}\epsilon^{\frac{n-2s}{2}}\int_{B_{\frac{\delta}{\epsilon}}}\int_{B_{\frac{\delta}{\epsilon}}}\frac{1}{\left(1+|x|^2\right)^{\frac{2n-\mu}{2}}|x-y|^{\mu}\left(1+|y|^2\right)^{\frac{n-\mu+2s}{2}}}\,\mathrm{d}x\mathrm{d}y$$
$$\geq C_{1,s}\epsilon^{\frac{n-2s}{2}}\int_{B_\delta}\int_{B_\delta}\frac{1}{\left(1+|x|^2\right)^{\frac{2n-\mu}{2}}|x-y|^{\mu}\left(1+|y|^2\right)^{\frac{n-\mu+2s}{2}}}\,\mathrm{d}x\mathrm{d}y=o(\epsilon^{\frac{n-2s}{2}}), \tag{4.14}$$

where $C_{1,s}$ is a positive constant. Therefore using (4.13) and (4.14), we get

$$\int_{\Omega}\int_{\Omega}\frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}-1}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y=o(\epsilon^{\frac{n-2s}{2}}).$$

Using the fact that the map $t\mapsto t^{22^*_{\mu,s}}$ in convex for $t\geq 0$, (4.14) and all norms on $\mathbb{D}_r$ are equivalent, we get

$$1=\int_{\Omega}\int_{\Omega}\frac{|u_g(x)|^{2^*_{\mu,s}}|u_g(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y$$
$$=\int_{\Omega}\int_{\Omega}\frac{|(w+tu_\epsilon)(x)|^{2^*_{\mu,s}}|(w+tu_\epsilon)(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y$$
$$\geq\int_{\Omega}\int_{\Omega}\frac{|tu_\epsilon(x)|^{2^*_{\mu,s}}|tu_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y$$
$$+22^*_{\mu,s}\int_{\Omega}\int_{\Omega}\frac{|tu_\epsilon(x)|^{2^*_{\mu,s}-1}w(x)|tu_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y$$
$$=t^{22^*_{\mu,s}}\int_{\Omega}\int_{\Omega}\frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,\mathrm{d}x\mathrm{d}y$$

$$
\begin{aligned}
&+ 22^*_{\mu,s} t^{22^*_{\mu,s}-1} \int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}-1} w(x) |u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\
&\geq t^{22^*_{\mu,s}} \int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}} |u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\
&\quad - 22^*_{\mu,s} t^{22^*_{\mu,s}-1} |w|_\infty \int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}-1} |u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\
&\geq t^{22^*_{\mu,s}} \int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}} |u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y - K_1 t^{22^*_{\mu,s}-1}|w|_2 o(\epsilon^{\frac{n-2s}{2}}),
\end{aligned}
$$

for some positive constant K_1. This implies $t < K_2$ for some constant $K_2 > 0$ and

$$
\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}} |u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y \leq 1 + K_3|w|_2 o(\epsilon^{\frac{n-2s}{2}}).
$$

Hence, using the definition of F_ϵ and w being linear combination of finitely many eigenfunctions, we get

$$
\begin{aligned}
g_{r,\epsilon} &\leq (\lambda_r - \lambda)|w|_2^2 + F_\epsilon \left(\int_\Omega\int_\Omega \frac{|tu_\epsilon(x)|^{2^*_{\mu,s}} |tu_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}} + K_4\int_\Omega u_\epsilon w\mathrm{d}x\\
&\leq (\lambda_r - \lambda)|w|_2^2 + F_\epsilon\left(1 + K_3|w|_2 o(\epsilon^{\frac{n-2s}{2}})\right) + K_5|u_\epsilon|_1|w|_2\\
&\leq (\lambda_r - \lambda)|w|_2^2 + F_\epsilon\left(1 + K_3|w|_2 o(\epsilon^{\frac{n-2s}{2}})\right) + o(\epsilon^{\frac{n-2s}{2}})|w|_2,
\end{aligned}
$$

where K_4, K_5 are positive constants and we used $|u_\epsilon|_1 = o(\epsilon^{\frac{n-2s}{2}})$ (which can be derived as other estimates done before). This completes the proof. □

Lemma 4.8. *If $n \geq 4s$ and $\lambda \in [\lambda_r, \lambda_{r+1})$, for some $r \in \mathbb{N}$, then for every $u \in G_{r,\epsilon}$ we have*

$$
\frac{\|u\|^2 - \lambda\int_\Omega |u|^2\mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}} < S_s^H.
$$

Proof. It is enough to show that $g_{r,\epsilon} < S_s^H$. From lemma 4.7, if $t = 0$ we have

$$
g_{r,\epsilon} \leq (\lambda_r - \lambda)|w|_2^2 < 0 < S_s^H.
$$

Else if $t > 0$, then we consider the cases $n = 4s$ and $n > 4s$ separately.
Case 1: ($n = 4s$) By lemma 4.7(ii) and estimates in (4.8), we have

$$
\begin{aligned}
g_{r,\epsilon} &\leq (\lambda_r - \lambda)|w|_2^2 + \frac{\|u_\epsilon\|^2 - \lambda\int_\Omega |u_\epsilon|^2\mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{2s}{8s-\mu}}}\\
&\qquad (1 + |w|_2 o(\epsilon^s)) + |w|_2 o(\epsilon^s)\\
&\leq S_s^H - \lambda C_s \epsilon^{2s}|\log\epsilon| + o(\epsilon^{2s}) + (\lambda_r - \lambda)|w|_2^2 + |w|_2 o(\epsilon^s)\\
&\leq S_s^H - \lambda C_s \epsilon^{2s}|\log\epsilon| + o(\epsilon^{2s}) < S_s^H,
\end{aligned}
$$

for sufficiently small $\epsilon > 0$.

Case 2: ($n > 4s$) By lemma 4.7(ii) and estimates in (4.9), we have

$$
\begin{aligned}
g_{r,\epsilon} &\leq (\lambda_r - \lambda)|w|_2^2 + \frac{\|u_\epsilon\|^2 - \lambda \int_\Omega |u_\epsilon|^2 \mathrm{d}x}{\left(\int_\Omega \int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}}(1+|w|_2)o(\epsilon^{\frac{n-2s}{2}}) \\
&\quad + o(\epsilon^{\frac{n-2s}{2}})|w|_2 \\
&\leq S_s^H - \lambda C_s \epsilon^{2s} + o(\epsilon^{n-2s}) + (\lambda_r - \lambda)|w|_2^2 + |w|_2 o(\epsilon^{\frac{n-2s}{2}})
\end{aligned}
$$

for sufficiently small $\epsilon > 0$. Also for $n \geq 4s$, we have that

$$
(\lambda_r - \lambda)|w|_2^2 + |w|_2 o(\epsilon^{\frac{n-2s}{2}}) \leq \frac{1}{4(\lambda_r - \lambda)} o(\epsilon^{n-2s}) = o(\epsilon^{n-2s}),
$$

which implies $g_{r,\epsilon} < S_s^H$ for both the cases. This completes the proof. □

Proof of Theorem 2.4. ($n \geq 4s$, $\lambda > \lambda_1$)

Since we know $u_g = w + tu_\epsilon$, we can also write $u_g = \widehat{w} + t\widehat{u_\epsilon}$ where

$$
\widehat{w} = w + t\sum_{i=1}^r \left(\int_\Omega u_\epsilon e_i \mathrm{d}x\right) e_i \in \mathbb{D}_r \text{ and } \widehat{u}_\epsilon = u_\epsilon - \sum_{i=1}^r \left(\int_\Omega u_\epsilon e_i \mathrm{d}x\right) e_i
$$

and find that $\widehat{w}$ and $\widehat{u}_\epsilon$ are orthogonal in $L^2(\Omega)$. Then, $|u_g|_2^2 = |\widehat{w}|_2^2 + t^2|\widehat{u}_\epsilon|_2^2$. From the definition of $\mathbb{G}_{r,\epsilon}$, we can write that

$$
\mathbb{G}_{r,\epsilon} = \mathbb{D}_r \oplus u_\epsilon \mathbb{R} = \mathbb{D}_r \oplus \hat{u}_\epsilon \mathbb{R},
$$

where $u_\epsilon \mathbb{R} = \{ru_\epsilon : r \in \mathbb{R}\}$ and similarly, $z_\epsilon \mathbb{R}$. By lemma 4.6, we have

(i) $\inf_{u \in \mathbb{M}_{r+1}, \|u\|=\rho} I(u) \geq \beta > 0$,
(ii) $\sup_{u \in \mathbb{D}_r} I(u) < 0$, and
(iii) $\sup_{u \in \mathbb{G}_{r,\epsilon}, \|u\| \geq R} I(u) \leq 0$,

where β, ρ are defined in Lemma 4.6. Therefore, I satisfies the geometric structure of the linking theorem (Theorem 5.3,[32]). We define

$$
\bar{c} = \inf_{\gamma \in \Gamma} \max_{u \in A} I(\gamma(u)) > 0,
$$

where $\gamma := \{\gamma \in C(\bar{A}, X_0) : \gamma = id \text{ on } \partial A\}$ and $A := (\bar{B}_R \cap \mathbb{D}_r) \oplus \{r\hat{u}_\epsilon : r \in (0, R)\}$. By definition, for any $\gamma \in \Gamma$, we have $\bar{c} \leq \max_{u \in A} I(\gamma(u))$ and particularly, if we take $\gamma = id$ on $\bar{A}$, then

$$
\bar{c} \leq \max_{u \in A} I(u) \leq \max_{\mathbb{G}_{r,\epsilon}} I(u).
$$

As we saw earlier, for any $u \in X_0 \backslash \{0\}$,

$$
\max_{t \geq 0} I(tu) = \frac{n+2s-\mu}{2(2n-\mu)} \left(\frac{\|w\|^2 - \lambda \int_\Omega |w|^2 \mathrm{d}x}{\left(\int_\Omega \int_\Omega \frac{|w(x)|^{2^*_{\mu,s}}|w(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}} \right)^{\frac{2n-\mu}{n+2s-\mu}}. \tag{4.15}
$$

Since $\mathbb{G}_{r,\epsilon}$ is a linear space, we have

$$
\max_{u \in \mathbb{G}_{r,\epsilon}} I(u) = \max_{u \in \mathbb{G}_{r,\epsilon}, t \neq 0} I\left(\frac{|t|u}{|t|}\right) \leq \max_{u \in \mathbb{G}_{r,\epsilon}, t \geq 0} I(tu).
$$

Hence, using lemma 4.8 and (4.15), we get

$$\bar{c} \leq \max_{u\in\mathbb{G}_{r,\epsilon},t\geq 0} I(tu) < \frac{n+2s-\mu}{2(2n-\mu)}(S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}.$$

Finally, using Linking theorem and Lemma 3.6, we conclude that (P_λ) has a nontrivial weak solution in X_0 with critical value $\bar{c} \geq \beta$. □

5. Proof of Theorem 2.5

We will prove this theorem using the Mountain Pass theorem and Linking theorem in a combined way.

Lemma 5.1. *Let $2s < n < 4s$ and u_ϵ be as defined in Sect. 4, case 1. Then there exists $\bar{\lambda} > 0$ such that for $\lambda > \bar{\lambda}$,*

$$\frac{\|u_\epsilon\|^2 - \lambda\int_\Omega |u_\epsilon|^2 \mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}} < S_s^H,$$

for sufficiently small $\epsilon > 0$.

Proof. Using Proposition 4.1 and 4.2, we get

$$\begin{aligned}
&\frac{\|u_\epsilon\|^2 - \lambda\int_\Omega |u_\epsilon|^2 \mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}}|u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}}\\
&\quad\leq \frac{(C(n,\mu))^{\frac{n(n-2s)}{2s(2n-\mu)}}(S_s^H)^{\frac{n}{2s}} - \lambda C_s\epsilon^{n-2s} + o(\epsilon^{2s})}{\left(((C(n,\mu))^{\frac{n}{2s}}(S_s^H)^{\frac{2n-\mu}{2s}} - o(\epsilon^{\frac{n(2n-\mu-\eta)}{2n-\eta}})\right)^{\frac{n-2s}{2n-\mu}}}\\
&\quad\leq S_s^H + \frac{\epsilon^{n-2s}(o(1)-\lambda C_s)}{((C(n,\mu))^{\frac{n}{2s}}(S_s^H)^{\frac{2n-\mu}{2s}} - o(\epsilon^{\frac{n(2n-\mu-\eta)}{2n-\eta}}))^{\frac{n-2s}{2n-\mu}}} + o(\epsilon^{2s})\\
&\quad< S_s^H,
\end{aligned}$$

when we choose $\lambda > 0$ large enough, say $\lambda > \bar{\lambda}$ and provided $\epsilon > 0$ be sufficiently small. This completes the proof. □

We have already seen in previous sections that the functional I satisfies geometry of Mountain Pass theorem when $\lambda < \lambda_1$ (using Lemma 4.4). When $\lambda > \lambda_1$, without loss of generality, we assume $\lambda \in [\lambda_r, \lambda_{r+1})$, for some $r \in \mathbb{N}$. Then using Lemma 4.6, we get that I satisfies geometry of Linking theorem. Also, by Lemma 3.6, we get that I satisfies the $(PS)_c$ condition when

$$c < \frac{n+2s-\mu}{2(2n-\mu)}) = (S_s^H)^{\frac{2n-\mu}{n+2s-\mu}}.$$

So, in order to apply the classical critical point theorems, we need the Mountain Pass critical level and Linking critical level of I to stay below this threshold. Consider $\mathbb{M}_{r+1}, \mathbb{D}_r$ and $G_{r,\epsilon}$ to be as defined in earlier section. Note that Lemma 4.7 holds true in this case also and we have the following lemma.

Lemma 5.2. *If $2s < n < 4s$ and $\lambda \in (\lambda_r, \lambda_{r+1})$, for some $r \in \mathbb{N}$, then for every $u \in G_{r,\epsilon}$ we have*

$$\frac{\|u\|^2 - \lambda \int_\Omega |u|^2 \mathrm{d}x}{\left(\int_\Omega \int_\Omega \frac{|u(x)|^{2^*_{\mu,s}} |u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \,\mathrm{d}x\mathrm{d}y\right)^{\frac{n-2s}{2n-\mu}}} < S_s^H.$$

Proof. If $t = 0$ then since $\lambda \in (\lambda_r, \lambda_{r+1})$, we get

$$g_{r,\epsilon} \leq (\lambda_r - \lambda)|w|_2^2 \leq 0 < S_s^H.$$

When $t > 0$, then

$$\begin{aligned}
g_{r,\epsilon} &\leq (\lambda_r - \lambda)|w|_2^2 + \frac{\|u_\epsilon\|^2 - \lambda \int_\Omega |u_\epsilon|^2 \mathrm{d}x}{\left(\int_\Omega \int_\Omega \frac{|u_\epsilon(x)|^{2^*_{\mu,s}} |u_\epsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \,\mathrm{d}x\mathrm{d}y\right)^{\frac{2s}{8s-\mu}}} \\
&\quad (1 + |w|_2) o(\epsilon^{\frac{n-2s}{2}}) + o(\epsilon^{\frac{n-2s}{2}})|w|_2 \\
&\leq S_s^H + \left(\frac{(o(1) - \lambda C_s)\epsilon^{n-2s}}{\left(((C(n,\mu))^{\frac{n}{2s}} (S_s^H)^{\frac{2n-\mu}{2s}} - o(\epsilon^{\frac{n(2n-\mu-\eta)}{2n-\eta}})\right)^{\frac{n-2s}{2n-\mu}}} + o(\epsilon^{2s}) \right) \\
&\quad (1 + |w|_2) o(\epsilon^{\frac{n-2s}{2}}) + (\lambda_r - \lambda)|w|_2^2 + |w|_2 o(\epsilon^{\frac{n-2s}{2}}) \\
&\leq S_s^H + \frac{(o(1) - \lambda C_s)\epsilon^{n-2s}}{\left(((C(n,\mu))^{\frac{n}{2s}} (S_s^H)^{\frac{2n-\mu}{2s}} - o(\epsilon^{\frac{n(2n-\mu-\eta)}{2n-\eta}})\right)^{\frac{n-2s}{2n-\mu}}} + (\lambda_r - \lambda)|w|_2^2 \\
&\quad + |w|_2 o(\epsilon^{\frac{n-2s}{2}}) < S_s^H,
\end{aligned}$$

for sufficiently small $\epsilon > 0$ because we already considered $\lambda > \bar{\lambda}$ and $\lambda \in (\lambda_r, \lambda_{r+1})$. Hence, the result follows. □

Proof of Theorem 2.5. We consider two cases:

Case 1. $(\lambda_1 > \bar{\lambda})$ For this case, we use Mountain Pass theorem if $\lambda \in (\bar{\lambda}, \lambda_1)$ and Linking theorem if $\lambda \in (\lambda_r, \lambda_{r+1})$, for some $r \in \mathbb{N}$.

If $\lambda \in (\bar{\lambda}, \lambda_1)$, using Remark 4.3, Lemma 4.4 and Proposition 4.5, following the same arguments as Case 1 in proof of Theorem 2.4, we get that (P_λ) admits a nontrivial weak solution.

Otherwise if $(\lambda_1 > \bar{\lambda})$, we assume $\lambda \in (\lambda_r, \lambda_{r+1})$ for some $r \in \mathbb{N}$ (since λ can no longer be an eigenvalue of $(-\Delta)^s$). Here, following the arguments as in Case2 in proof of Theorem 2.4, we get that (P_λ) admits a nontrivial weak solution.

Case 2. $(\lambda_1 < \bar{\lambda})$ In this case, we can assume $\lambda \in (\lambda_r, \lambda_{r+1})$ for some $r \in \mathbb{N}$ and $\lambda > \bar{\lambda}$. Here again, following the arguments as in Case2 in proof of Theorem 2.4, we get that (P_λ) has a nontrivial weak solution. □

6. Regularity of weak solutions

In this section, we prove that any non-negative weak solution of (P_λ) is bounded and moreover, a Holder continuous function. First, we prove Theorem 6.1.

Theorem 6.1. *Suppose u is a nonnegative weak solution of (P_λ), then for each $\varphi \in X_0$, it satisfies*

$$\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, \mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}-2}u(y)\varphi(y)}{|x-y|^{\mu}}\, \mathrm{d}x\mathrm{d}y + \lambda\int_\Omega u\varphi\, dx.$$

Proof. Let $\varphi \in X_0, \varphi \geq 0$. By Lemma 2.2 of [28], we obtain a sequence $\{w_k\} \in X_0$ such that $\{w_k\} \to \varphi$ strongly in X_0, each w_k has compact support in Ω and $0 \leq w_1 \leq w_2 \leq \ldots$. For each fixed k, we can find a sequence $\{\psi_m^k\} \subset C_c^\infty(\Omega)$ such that $\psi_m^k \geq 0$, $\bigcup_m \mathrm{Supp}\ \psi_m^k$ is contained in a compact subset of Ω, $\{\|\psi_m^k\|_\infty\}$ is bounded and $\|\psi_m^k - w_k\| \to 0$ strongly as $m \to \infty$. Since u is a weak solution of P_λ, we get

$$\int_Q \frac{(u(x)-u(y))(\psi_m^k(x)-\psi_m^k(y))}{|x-y|^{n+2s}}\, \mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}|u(x)|^{2^*_{\mu,s}-2}u(x)\psi_m^k(x)}{|x-y|^{\mu}}\, \mathrm{d}y\mathrm{d}x + \lambda\int_\Omega u\psi_m^k\, \mathrm{d}x.$$

By Lebesgue dominated convergence Theorem, as $m \to \infty$ we get

$$\int_Q \frac{(u(x)-u(y))(w_k(x)-w_k(y))}{|x-y|^{n+2s}}\, \mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}-2}u(y)w_k(y)}{|x-y|^{\mu}}\, \mathrm{d}x\mathrm{d}y + \lambda\int_\Omega uw_k\, \mathrm{d}x.$$

Using monotone convergence Theorem and the nonnegativity of u, we obtain

$$\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, \mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}-2}u(y)\varphi(y)}{|x-y|^{\mu}}\, \mathrm{d}x\mathrm{d}y + \lambda\int_\Omega u\varphi\, dx,$$

If $\varphi \in X_0$, then $\varphi = \varphi^+ - \varphi^-$ and $\varphi^+, \varphi^- \geq 0$. Since we proved the lemma for each $\varphi \in X_0$, $\varphi \geq 0$, we obtain the conclusion. □

Theorem 6.2. *Let $0 \leq u \in X_0$ be a weak solution of (P_λ). Then $u \in L^\infty(\Omega)$.*

Proof. We may assume that u does not vanish identically (otherwise the proof is trivial) and let u be nonnegative. Let $\delta > 0$, to be chosen appropriately small whose choice will be done in (6.15) later in proof. Now, let $c > 0$ be a constant chosen in such a way that for any $x \in \mathbb{R}^n$, $v(x) := \frac{u(x)}{c} \in X_0$ satisfies

$$\begin{aligned} &\int_Q \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ &\quad \le \int_\Omega\int_\Omega \frac{|v(y)|^{2^*_{\mu,s}}|v(x)|^{2^*_{\mu,s}-2}v(x)\varphi(x)}{|x-y|^{\mu}}\,\mathrm{d}y\mathrm{d}x + \lambda\int_\Omega v\varphi\,dx, \end{aligned} \tag{6.1}$$

for every $0 \le \varphi \in C_c^\infty(\Omega)$ and $|v|_{2^*_s} = \delta$. It is a simple observation that if $v \in X_0$, then $v^+ := \max\{v, 0\}$ satisfies

$$(v(x)-v(y))(v^+(x)-v^+(y)) \ge |v^+(x)-v^+(y)|^2, \tag{6.2}$$

for any $x, y \in \mathbb{R}^n$. Let us set $C_k := 1-2^{-k}$, $v_k := v - C_k$, $w_k := v_k^+ \in X_0$ and $U_k := |w_k|_{2^*_s}$. We get that

$$0 \le |v| + C_k \le |v| + 1 \in L^{2^*_s}(\Omega),$$

being Ω bounded, and

$$\lim_{k\to+\infty} w_k = (v-1)^+.$$

Therefore, by the Dominated Convergence Theorem,

$$\lim_{k\to+\infty} U_k = \left(\int_\Omega [(v-1)^+]^{2^*_s}\,\mathrm{d}x\right)^{\frac{1}{2^*_s}}. \tag{6.3}$$

For any $k \in \mathbb{N}$, $C_{k+1} > C_k$ and so $w_{k+1} \le w_k$ a.e. in $\mathbb{R}^n$. Also let $A_k := C_{k+1}/(C_{k+1}-C_k) = 2^{k+1}-1$, for any $k \in \mathbb{R}^n$. We claim that for any $k \in \mathbb{N}$

$$v < A_k w_k \text{ on } \{w_{k+1} > 0\}. \tag{6.4}$$

To check this, let $x \in \{w_{k+1} > 0\}$. Then $v(x) > C_{k+1} > C_k$, so $w_k(x) = v_k(x) = v(x) - C_k$ and

$$A_k w_k(x) = v(x) + \frac{C_k}{C_{k+1}-C_k}(v(x)-C_{k+1}) > v(x).$$

Notice also that $v_{k+1}(x) - v_{k+1}(y) = v(x) - v(y)$, for any $x, y \in \mathbb{R}^n$. Using this, (6.1), (6.4), (6.2), Hölder's inequality, Lemma 6.1 and the fact that $w_{k+1} = v_{k+1}^+ \in X_0$, we get

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \frac{|w_{k+1}(x)-w_{k+1}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y = \int_Q \frac{|v_{k+1}^+(x)-v_{k+1}^+(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ &\le \int_{\mathbb{R}^{2n}} \frac{(v_{k+1}(x)-v_{k+1}(y))(v_{k+1}^+(x)-v_{k+1}^+(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(v_{k+1}^+(x)-v_{k+1}^+(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ &\le \int_\Omega\int_\Omega \frac{|v(y)|^{2^*_{\mu,s}}|v(x)|^{2^*_{\mu,s}-2}v(x)w_{k+1}(x)}{|x-y|^{\mu}}\,\mathrm{d}y\mathrm{d}x + \lambda\int_\Omega v(x)w_{k+1}(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\{w_{k+1}(x)>0\}} \int_{\Omega} \frac{|v(y)|^{2^*_{\mu,s}}|v(x)|^{2^*_{\mu,s}-2}v(x)w_{k+1}(x)}{|x-y|^{\mu}}\ \mathrm{d}y\mathrm{d}x \\
&\quad + \lambda \int_{\{w_{k+1}>0\}} v(x)w_{k+1}(x)\ dx \\
&\leq A_k^{2^*_{\mu,s}-1} \int_{\{w_{k+1}(x)>0\}} \int_{\Omega} \frac{|v(y)|^{2^*_{\mu,s}}|w_k(x)|^{2^*_{\mu,s}-1}w_{k+1}(x)}{|x-y|^{\mu}}\ \mathrm{d}y\mathrm{d}x \\
&\quad + \lambda A_k \int_{\{w_{k+1}>0\}} w_k^2(x)\ dx \\
&\leq A_k^{2^*_{\mu,s}-1} \int_{\{w_{k+1}(x)>0\}} \int_{\Omega} \frac{|v(y)|^{2^*_{\mu,s}}|w_k(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\ \mathrm{d}y\mathrm{d}x \\
&\quad + \lambda 2^{k+1}|w_k|^2_{2^*_s}|\{w_{k+1}>0\}|^{\frac{2s}{n}}. \qquad (6.5)
\end{aligned}
$$

Let us consider the first integral of R.H.S. of above inequality separately and we get that

$$
\begin{aligned}
&\int_{\{w_{k+1}(x)>0\}} \int_{\Omega} \frac{|v(y)|^{2^*_{\mu,s}}|w_k(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\ \mathrm{d}y\mathrm{d}x \\
&\quad \leq \left(\int_{\{w_{k+1}(x)>0\}} \int_{\{v(y)\geq C_{k+1}\}} + \int_{\{w_{k+1}(x)>0\}} \int_{\{v(y)<C_{k+1}\}} \right) \qquad (6.6) \\
&\quad\quad \frac{|v(y)|^{2^*_{\mu,s}}|w_k(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\ \mathrm{d}y\mathrm{d}x = I_1 + I_2,\ (\text{say}).
\end{aligned}
$$

Now using (6.4) and Hardy–Littlewood–Sobolev inequality, we have

$$
\begin{aligned}
I_1 &= \int_{\{w_{k+1}(x)>0\}} \int_{\{w_{k+1}(y)\geq 0\}} \frac{|v(y)|^{2^*_{\mu,s}}|w_k(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\ \mathrm{d}y\mathrm{d}x \\
&\leq A_k^{2^*_{\mu,s}} \int_{\{w_{k+1}(x)>0\}} \int_{\{w_{k+1}(y)\geq 0\}} \frac{|w_k(y)|^{2^*_{\mu,s}}|w_k(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\ \mathrm{d}y\mathrm{d}x \qquad (6.7) \\
&\leq A_k^{2^*_{\mu,s}} C(n,\mu)|w_k|_{2^*_s}^{22^*_{\mu,s}}.
\end{aligned}
$$

Next, again using (6.4) and Hölder's inequality we have

$$
\begin{aligned}
I_2 &= \int_{\{w_{k+1}(x)>0\}} \int_{\{w_{k+1}(y)<0\}} \frac{|v(y)|^{2^*_{\mu,s}}|w_k(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\ \mathrm{d}y\mathrm{d}x \\
&\leq C_{k+1}^{2^*_{\mu,s}} \int_{\{w_{k+1}(x)>0\}} |w_k(x)|^{2^*_{\mu,s}} \int_{\Omega} \frac{\mathrm{d}y}{|x-y|^{\mu}}\ \mathrm{d}x \\
&\leq MC_{k+1}^{2^*_{\mu,s}} \int_{\{w_{k+1}(x)>0\}} |w_k(x)|^{2^*_{\mu,s}}\mathrm{d}x \qquad (6.8) \\
&\leq MC_{k+1}^{2^*_{\mu,s}} |\{w_{k+1}>0\}|^{\frac{\mu}{2n}} |w_k|_{2^*_s}^{2^*_{\mu,s}}.
\end{aligned}
$$

Using (6.6), (6.7), (6.8) and Sobolev inequality in (6.5), we get

$$\begin{aligned} S_s|w_{k+1}|_{2_s^*}^2 &\leq \int_{\mathbb{R}^{2n}} \frac{|w_{k+1}(x)-w_{k+1}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ &\leq A_k^{2^*_{\mu,s}-1}\left(A_k^{2^*_{\mu,s}}C(n,\mu)|w_k|_{2_s^*}^{22^*_{\mu,s}} + MC_{k+1}^{2^*_{\mu,s}}|\{w_{k+1}>0\}|^{\frac{\mu}{2n}}|w_k|_{2_s^*}^{2^*_{\mu,s}}\right. \\ &\quad \left. + \lambda 2^{k+1}|w_k|_{2_s^*}^2|\{w_{k+1}>0\}|^{\frac{2s}{n}}\right). \end{aligned} \tag{6.9}$$

Now we claim that

$$\{w_{k+1}>0\} \subset \{w_k > 2^{-(k+1)}\}. \tag{6.10}$$

To establish this, we observe that if $x \in \{w_{k+1}>0\}$ then

$$v(x) - C_{k+1} > 0.$$

Accordingly, $v_k(x) = v(x) - C_k > C_{k+1} - C_k = 2^{-(k+1)}$, so that,

$$w_k(x) = v_k(x) > 2^{-(k+1)}.$$

Thus, (6.10) gives

$$U_k^{2_s^*} = |w_k|_{2_s^*}^{2_s^*} \geq \int_{\{w_k>2^{-(k+1)}\}} w_k^{2_s^*}\,\mathrm{d}x \geq 2^{-2_s^*(k+1)}|\{w_{k+1}>0\}|. \tag{6.11}$$

If $\lambda > 0$, as a consequence of (6.11), from (6.9) we get

$$\begin{aligned} S_s|w_{k+1}|_{2_s^*}^2 &\leq A_k^{2^*_{\mu,s}-1}\left(A_k^{2^*_{\mu,s}}C(n,\mu)|w_k|_{2_s^*}^{22^*_{\mu,s}} + MC_{k+1}^{2^*_{\mu,s}}2^{\frac{\mu 2_s^*(k+1)}{2n}}|w_k|_{2_s^*}^{2_s^*}\right. \\ &\quad \left. + \lambda 2^{k+1}|w_k|_{2_s^*}^{2_s^*}2^{\frac{2s2_s^*(k+1)}{n}}\right) \\ &\leq 2^{(2^*_{\mu,s}-1)(k+1)}\left(2^{2^*_{\mu,s}(k+1)}C(n,\mu)|w_k|_{2_s^*}^{22^*_{\mu,s}} + M2^{\frac{\mu 2_s^*(k+1)}{2n}}|w_k|_{2_s^*}^{2_s^*}\right. \\ &\quad \left. + \lambda 2^{k+1}|w_k|_{2_s^*}^{2_s^*}2^{\frac{2s2_s^*(k+1)}{n}}\right) \\ &\leq 2^{(2^*_{\mu,s}-1)(k+1)}\max\left\{2^{2^*_{\mu,s}(k+1)}C(n,\mu), M2^{\frac{\mu 2_s^*(k+1)}{2n}}\right. \\ &\quad \left. + \lambda 2^{(k+1)\left(1+\frac{2s2_s^*}{n}\right)}\right\} \times \left(|w_k|_{2_s^*}^{22^*_{\mu,s}} + |w_k|_{2_s^*}^{2_s^*}\right). \end{aligned} \tag{6.12}$$

Otherwise if $\lambda \leq 0$, then using (6.9) and (6.11) we have

$$\begin{aligned} S_s|w_{k+1}|_{2_s^*}^2 &\leq A_k^{2^*_{\mu,s}-1}\left(A_k^{2^*_{\mu,s}}C(n,\mu)|w_k|_{2_s^*}^{22^*_{\mu,s}} + MC_{k+1}^{2^*_{\mu,s}}2^{\frac{\mu 2_s^*(k+1)}{2n}}|w_k|_{2_s^*}^{2_s^*}\right) \\ &\leq 2^{(2^*_{\mu,s}-1)(k+1)}\left(2^{2^*_{\mu,s}(k+1)}C(n,\mu)|w_k|_{2_s^*}^{22^*_{\mu,s}} + M2^{\frac{\mu 2_s^*(k+1)}{2n}}|w_k|_{2_s^*}^{2_s^*}\right) \end{aligned}$$

$$\leq 2^{(2^*_{\mu,s}-1)(k+1)} \max\left\{2^{2^*_{\mu,s}(k+1)}C(n,\mu), M2^{\frac{\mu 2^*_s(k+1)}{2n}}\right\}$$
$$\left(|w_k|_{2^*_s}^{22^*_{\mu,s}} + |w_k|_{2^*_s}^{2^*_s}\right). \tag{6.13}$$

Therefore using definition of U_k in (6.12), we get

$$U_{k+1} \leq D^{(k+1)}\left(U_k^{2^*_{\mu,s}} + U_k^{\frac{2^*_s}{2}}\right), \tag{6.14}$$

where,

$$D = \begin{cases} \left(1 + \frac{1}{S_s}\left(2^{(2^*_{\mu,s}-1)} \max\left\{2^{2^*_{\mu,s}}C(n,\mu), M2^{\frac{\mu}{2n}} + 2^{\left(1+\frac{2s}{n}\right)}\right\}\right)^{1/2}\right) & \text{if } \lambda \geq 0 \\ \left(1 + \frac{1}{S_s}\left(2^{(2^*_{\mu,s}-1)} \max\left\{2^{2^*_{\mu,s}}C(n,\mu), M2^{\frac{\mu}{2n}}\right\}\right)^{1/2}\right) & \text{if } \lambda < 0 \end{cases}.$$

We have $D > 1$ and $2^*_{\mu,s} > 2^*_s/2 > 1$.

Now we are ready to perform our choice of δ: namely we assume that $\delta > 0$ is so small that

$$\delta^{\frac{2^*_s}{2}-1} < \frac{1}{(2^{2^*_{\mu,s}}D)^{\frac{1}{(2^*_s/2)-1}}}. \tag{6.15}$$

We also fix $\eta \in \left(\delta^{\frac{2^*_s}{2}-1}, \frac{1}{(2^{2^*_{\mu,s}}D)^{\frac{1}{(2^*_s/2)-1}}}\right)$. Since $D > 1$ and $2^*_s/2 > 1$, we get $\eta \in (0,1)$. Moreover,

$$\delta^{\frac{2^*_s}{2}-1} \leq \eta \text{ and } \left(2^{2^*_{\mu,s}}D\eta\right)^{\frac{2^*_s}{2}-1} \leq 1. \tag{6.16}$$

We claim that

$$U_k \leq 2\delta\eta^{k+1}. \tag{6.17}$$

The proof is by induction. First of all

$$U_0 = |v^+|_{2^*_s} \leq |v|_{2^*_s} = \delta \leq 2\delta$$

which is (6.17) when $k = 0$. Let us now suppose that (6.17) holds true for k and let us prove it for $k+1$. Using (6.14) and (6.16), we get

$$U_{k+1} \leq D^{k+1}\left(U_k^{2^*_{\mu,s}} + U_k^{\frac{2^*_s}{2}}\right) \leq 2^{2^*_{\mu,s}+1}D^{k+1}(\delta\eta^{k+1})^{\frac{2^*_s}{2}}$$
$$\leq 2\delta\left(2^{2^*_{\mu,s}}D\eta^{\frac{2^*_s}{2}-1}\right)^{k+1}\delta^{\frac{2^*_s}{2}-1}\eta^{k+1} \leq 2\delta\eta^{k+1}.$$

This proves our claim (6.17). Then using $\eta \in (0,1)$ and (6.17), we conclude that

$$\lim_{k\to+\infty} U_k = 0.$$

Hence, by (6.3), $(v-1)^+ = 0$ a.e. in Ω, that is $v \leq 1$ a.e. in Ω. Therefore, $u \leq c$ a.e. in Ω which implies $|u|_\infty \leq c$. This completes the proof. □

Theorem 6.3. *Let Ω be a bounded $C^{1,1}$ domain and u be a non-negative weak solution of (P_λ). Then $u/\delta^s \in C^\alpha(\bar{\Omega})$ for some $\alpha > 0$(depending on Ω and s) satisfying $\alpha < \min\{s, 1-s\}$, where $\delta(x) = dist(x, \partial\Omega)$ for $x \in \Omega$.*

Proof. For any non-negative weak solution u of (P_λ), from previous theorem we have

$$\left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}y\right)|u|^{2^*_{\mu,s}-2}u + \lambda u \in L^\infty(\Omega).$$

So using Proposition 1.2 of [31], we obtain the conclusion. □

7. Nonexistence result

In this section, we prove a non-existence result for $\lambda \leq 0$ when Ω is a star shaped domain. At first, we prove the Pohozaev type identity:

Proposition 7.1. *If $\lambda < 0$, Ω be bounded $C^{1,1}$ domain and $u \in L^\infty(\Omega)$ solves (P_λ), then*

$$\frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,\mathrm{d}x - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x.\nu)\mathrm{d}\sigma$$
$$= -\left(\frac{2n-\mu}{22^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y + \frac{\lambda n}{2}\int_\Omega |u|^2\mathrm{d}x\right),$$

where ν denotes the unit outward normal to $\partial\Omega$ at x and Γ is the Gamma function.

Proof. Since u solves (P_λ), u satisfies the equation

$$(-\Delta)^s u = \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{2^*_{\mu,s}-2}u + \lambda u \ \text{ in } \Omega.$$

Multiplying both sides of the above equation by $(x.\nabla u)$ and integrating, we get

$$\int_\Omega (x.\nabla u)(-\Delta)^s u\,\mathrm{d}x = \int_\Omega (x.\nabla u)\left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{2^*_{\mu,s}-1}\mathrm{d}x$$
$$+\lambda\int_\Omega (x.\nabla u)u\,\mathrm{d}x. \tag{7.1}$$

Using Theorem 1.4 and 1.6 of [30], we get

$$\int_\Omega (x.\nabla u)(-\Delta)^s u\,\mathrm{d}x = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,\mathrm{d}x - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x.\nu)\mathrm{d}\sigma.$$

Now, consider the term

$$\int_\Omega (x.\nabla u)\left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{2^*_{\mu,s}-1}\mathrm{d}x$$

$$
\begin{aligned}
&= -\int_\Omega u(x)\left(n\left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{2^*_{\mu,s}-1}\right.\\
&\quad +(2^*_{\mu,s}-1)|u(x)|^{2^*_{\mu,s}-2}x.\nabla u(x)\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\\
&\quad \left. +\ |u(x)|^{2^*_{\mu,s}-1}\int_\Omega(-\mu)x.(x-y)\frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu+2}}\mathrm{d}y\right)\mathrm{d}x\\
&= -n\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\ \mathrm{d}x\mathrm{d}y\\
&\quad -(2^*_{\mu,s}-1)\int_\Omega x.\nabla u(x)\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y|u(x)|^{2^*_{\mu,s}-1}\mathrm{d}x\\
&\quad +\mu\int_\Omega\int_\Omega x.(x-y)\frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu+2}}|u(x)|^{2^*_{\mu,s}}\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&2^*_{\mu,s}\int_\Omega(x.\nabla u(x))\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y|u(x)|^{2^*_{\mu,s}-1}\mathrm{d}x\\
&\quad = -n\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\ \mathrm{d}x\mathrm{d}y\\
&\qquad +\ \mu\int_\Omega\int_\Omega x.(x-y)\frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu+2}}|u(x)|^{2^*_{\mu,s}}\mathrm{d}y\mathrm{d}x,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&2^*_{\mu,s}\int_\Omega(y.\nabla u(y))\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}x|u(y)|^{2^*_{\mu,s}-1}\mathrm{d}y\\
&\quad = -n\int_\Omega\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}|u(x)|^{2^*_{\mu,s}}}{|x-y|^\mu}\ \mathrm{d}y\mathrm{d}x\\
&\qquad +\ \mu\int_\Omega\int_\Omega y.(y-x)\frac{|u(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu+2}}|u(y)|^{2^*_{\mu,s}}\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_\Omega(x.\nabla u(x))\left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\right)|u(x)|^{2^*_{\mu,s}-1}\mathrm{d}x\\
&\quad = \frac{\mu-2n}{22^*_{\mu,s}}\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(x)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Since

$$
\int_\Omega(x.\nabla u)u\mathrm{d}x = -\frac{n}{2}\int_\Omega u^2\mathrm{d}x,
$$

using (7.1), the result follows. □

Proof of Theorem 2.6. Let $u \geq 0$ be a nontrivial weak solution of (P_λ), then by Theorem 6.2, $u \in L^\infty(\Omega)$. Therefore, we have

$$\|u\|^2 = \int_\Omega \int_\Omega \frac{|u(x)|^{2^*_{\mu,s}} |u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \, \mathrm{d}x\mathrm{d}y + \lambda \int_\Omega u^2 \mathrm{d}x.$$

Using Proposition 7.1, we get

$$\frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x.\nu) \mathrm{d}\sigma = \lambda s \int_\Omega u^2 \mathrm{d}x.$$

But, since Ω is star shaped with respect to origin in $\mathbb{R}^n$, so $x.\nu > 0$. From above equation and $\lambda < 0$, we have $u \equiv 0$, which is a contradiction. This completes the proof. □

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