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# Nonlocal heat equations in the Heisenberg group

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**Abstract.** We study the following nonlocal diffusion equation in the Heisenberg group $\mathbb{H}_n$,

$$u_t(z, s, t) = J * u(z, s, t) - u(z, s, t),$$

where $*$ denote convolution product and $J$ satisfies appropriated hypothesis. For the Cauchy problem we obtain that the asymptotic behavior of the solutions is the same form that the one for the parabolic equation for the fractional laplace operator. To obtain this result we use the spherical transform related to the pair $(U(n), \mathbb{H}_n)$. Finally we prove that solutions of properly rescaled nonlocal Dirichlet problem converge uniformly to the solution of the corresponding Dirichlet problem for the classical heat equation in the Heisenberg group.

**Mathematics Subject Classification.** 47G10, 47J35, 45G10.

**Keywords.** Nonlocal diffusion, Spherical transform, Heisenberg group.

## 1. Introduction and preliminaries

During the last years, many authors have studied the asymptotic behavior for several nonlocal diffusion models in the whole $\mathbb{R}^n$. In some cases, this behavior is related with the asymptotic behavior of the local diffusion model.

In [13] the authors consider some $u(x, t)$ that models the probabilist density function of a single population at the point $x$ at time $t$. Let $J$ be a function with $\int_{\mathbb{R}^n} J(x)\,dx = 1$; $J(x - y)$ as a probability distribution of jumping from location $y$ to location $x$; $J * u(x, t) = \int_{\mathbb{R}^n} J(y - x)u(y, t)\,dy$ is the rate at which individuals are arriving to position $x$ from all other places, and $u(x, t) = \int_{\mathbb{R}^n} J(x - y)u(x, t)\,dy$ is the rate at which they are leaving location $x$ to travel to all other sites. Then $u$ satisfies a nonlocal evolution equation of the form

$$u_t(x, t) = J * u(x, t) - u(x, t), \tag{1.1}$$

In [9] the authors study the Cauchy problem associated to (1.1). They prove that the long time behavior of the solutions is determined by the behavior of the Fourier transform $\widehat{J}$ of $J$ near the origin. If $\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \quad (0 < \alpha \le 2)$, the asymptotic behavior is the same as the one for solutions of the evolution given by the $\alpha/2$ fractional power of the Laplacian. Concerning the Dirichlet problem for the nonlocal model they prove that the asymptotic behavior is given by an exponential decay to zero at a rate given by the first eigenvalue of an associated eigenvalue problem with profile an eigenfunction of the first eigenvalue. Finally, they analyse the Neumann problem and find an exponential convergence to the mean value of the initial condition.

In the work [10] the authors prove that solutions of properly rescaled nonlocal Dirichlet problems of the Eq. (1.1) approximate uniformly the solution of the corresponding Dirichlet problem for the classical heat equation in $\mathbb{R}^n$.

These type of problems have been studied for the case of different nonlocal problems and equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^n} J(x,y)G(u(y,t) - u(x,t))\,dy, \tag{1.2}$$

have been considered. Depending on how is $G$, the operators behave in a similar way to elliptic operators: see [6,22] and [24]. If we choose $G(s) = |s|^{p-2}s$, the solution of the problems involving the Eq. (1.2) have similar behavior to the $p$-Laplacian operator, see [2–4,17,19].

Although these problems were very study in the literature in recent years, only in the case of the (1.1) it was determined asymptotic profile and the asymptotic behavior in $L^\infty$ norm for of the solution of the Cauchy problem. As mentioned above, in this case, the nonlocal operator behaves equally to the fractional Laplace operator.

At the present work we consider a nonlocal operator defined by

$$u_t(x,t) = J * u(x,t) - u(x,t), \tag{1.3}$$

where the convolution product is in the Heisenberg group and $J$ satisfies appropriate hypothesis. We show that a properly rescaled of the kernel $J$ converges to the Heisenberg Laplacian, see Lemma 3.6. Following the ideas of [10] we prove that solutions of the rescaled nonlocal Dirichlet problem converge uniformly to the solution of the corresponding Dirichlet problem for the classical heat equation in the Heisenberg group. We also study similar problems to the ones in [9]: for the Cauchy problem we determine the asymptotic profile and we prove that the asymptotic behavior in $L^\infty$ norm for the solution is the same that the solution of the evolution problem involving the fractional Laplace in the Heisenberg group. In order to do this we have to consider the results obtained in [23], the fact that $\mathbb{H}_n$ is a homogeneous group and the harmonic analysis related to the action of the unitary group $U(n)$ by automorphism on $\mathbb{H}_n$.

In the Heisenberg group has not yet been studied this type of nonlocal problems. The Cauchy problem involving the Eq. (1.3) may be useful to model populations whose density functions have an asymptotic behavior equally to the solutions of the parabolic problem concerning the fractional Laplacian in the Heisenberg group.

Concerning the local problems, in [23] the author considers the classic heat equation for Carnot groups and settles the asymptotic behavior of the solution. In [26] the authors study the existence and large-time behavior of global positive solutions to a semilinear parabolic equation on Carnot groups. In [7] the authors determined the asymptotic profile of the solution of the Dirichlet problem for the $p$-Laplacian in Carnot groups and they investigate the limit as $p$ goes to infinity. The Heisenberg group is the main example of the Carnot groups.

Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ the $2n+1$ dimensional Heisenberg group, with law group $(z, s) \cdot (\tilde{z}, \tilde{s}) = (z + \tilde{z}, s + \tilde{s} + \frac{1}{2}\mathrm{Im}\langle z, \tilde{z}\rangle)$, where $\langle z, \tilde{z}\rangle$ denote the Hermitian inner product of $\mathbb{C}^n$. The Haar measure of the group is de Lebesgue measure. If we write $z = x + iy$, with $x, y$ in $\mathbb{R}^n$ we have a global coordinate system $(x, y, s)$ and the vector fields $X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial s}$, $Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial s}$, and $S = \frac{\partial}{\partial s}$ form a basis for the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$.

The Heisenberg Laplacian is $L := \sum_{j=1}^{n} X_j^2 + Y_j^2$. In coordinates is given by

$$L = \sum_{j=1}^{n}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) + \frac{1}{4}\frac{\partial^2}{\partial s^2}\sum_{j=1}^{n}\left(x_j^2 + y_j^2\right) + \frac{\partial}{\partial s}\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right), \tag{1.4}$$

The Laplacian $L$ is a second order degenerate elliptic operator of Hörmander type and hence it is hypoelliptic see [18].

We recall that a Lie group is called a *homogeneous group* if it is a connected, simply connected, nilpotent Lie group $G$, whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilatation $\{\delta_r\}_{r\in\mathbb{N}}$. Let exp: $\mathfrak{g} \to G$ be the exponential map, which in this case is a diffeomorphism. The maps $\exp\delta_r\exp^{-1}$ are group automorphisms of $G$ also denoted by $\delta_r$ and called dilations of $G$. A standard example is given by $\delta_r(z, s) = (r^{\frac{1}{2}}z, rs)$, $r > 0$ and $(z, s) \in \mathbb{H}_n$. If $f$ is a function defined in the Heisenberg group we denote by $\delta_r f$ the function $\delta_r f(z, s) := f(r^{\frac{1}{2}}z, rs)$.

Let $U(n)$ the unitary group, which acts by automorphism on $\mathbb{H}_n$ by $g \cdot (z, s) = (gz, s)$, $g \in U(n)$ and $(z, s) \in \mathbb{H}_n$. We will denote by $\mathcal{S}(\mathbb{H}_n)^{U(n)}$ the space of functions in the Schwartz space that are invariant by the action of $U(n)$ and we will denote by $L^1(\mathbb{H}_n)^{U(n)}$ the space of $L^1(\mathbb{H}_n)$ the functions that are invariant by the action of $U(n)$. Since $(L^1(\mathbb{H}_n)^{U(n)}, *)$ is a commutative algebra, its spectrum $\Sigma$ is given by the family of the spherical functions $\{\varphi_{\lambda,k}\}_{\lambda\in\mathbb{R}\setminus\{0\},k\in\mathbb{N}} \cup \{\eta_r\}_{r\in\mathbb{R}^{\geq 0}}$ associated with the Gelfand pair $(\mathbb{H}_n, U(n))$, see [16,20,25].

As usual, $\mathcal{U}(\mathfrak{h}_n)$ will denote its universal enveloping algebra, which can be identified with the algebra of left invariant differential operators on $\mathbb{H}_n$. It is well known that the commutative subalgebra $\mathcal{U}(\mathfrak{h}_n)^{U(n)}$ of the elements which commute with the action of $U(n)$ is generated by $S$ and the Heisenberg

Laplacian $L$. The spherical functions are eigenfunction of the operators $L$ and $S$, they satisfying

$$\begin{cases} L\varphi_{\lambda,k} = -|\lambda|(2k+n)\varphi_{\lambda,k}, \quad \lambda \in \mathbb{R}\backslash\{0\}, \quad k \in \mathbb{N} \\ iS\varphi_{\lambda,k} = \lambda\varphi_{\lambda,k}. \end{cases} \tag{1.5}$$

and

$$\begin{cases} L\eta_r = -r^2\eta_r, \quad r \in \mathbb{R}, r \geq 0 \\ iS\eta_r = 0. \end{cases}$$

Explicitly

$$\varphi_{\lambda,k}(z,s) = e^{i\lambda s} L_k^{n-1}\left(\frac{|\lambda|}{2}|z|^2\right) e^{-\frac{|\lambda|}{4}|z|^2},$$
$$\eta_r(z,s) = \frac{2^{n-1}(n-1)!}{(r|z|)^{n-1}} J_{n-1}(r|z|),$$

where $L_k^{n-1}$ denotes, as usual, a Laguerre polynomial of order $n-1$ and degree $k$ normalized by $L_k^{n-1}(0) = 1$ and $J_{n-1}$ is a Bessel function of order $n-1$ of the first kind. The $\varphi_{\lambda,k}$ functions satisfy the following properties:

$$\text{If } c \in \mathbb{R} \text{ then } \varphi_{c\lambda,k}(z,s) = \delta_c\varphi_{\lambda,k}(z,s) = \varphi_{\lambda,k}(\sqrt{c}\,z, cs), \tag{1.6}$$
$$\|\varphi_{\lambda,k}\|_{L^\infty(\mathbb{H}_n)} = 1.$$

The spectrum $\Sigma$ is identify with the set of eigenvalues, $\Sigma = \{(\lambda, |\lambda|(2k+n)) : \lambda \in \mathbb{R}\backslash\{0\}, k \in \mathbb{N}\} \cup \{r \in \mathbb{R}, r \geq 0\}$, with the following measure, if $g \in L^1(\Sigma)$ we have

$$\|g\|_{L^1(\Sigma)} = \sum_{k\in\mathbb{N}} \int_{\mathbb{R}} |g(\lambda,k)|\,|\lambda|^n d\lambda < \infty.$$

For $f \in \mathcal{S}(\mathbb{H}_n)$ we define the spherical transform, $\widehat{f} : \Sigma \longrightarrow \mathbb{R}$, by

$$\widehat{f}(\lambda,k) = \int_{\mathbb{H}_n} f(z,s)\varphi_{\lambda,k}(-z,-s)\,dz\,ds \tag{1.7}$$
$$\widehat{f}(0,r) = \int_{\mathbb{H}_n} f(z,s)\eta_r(-z,-s)\,dz\,ds.$$

If $f \in L^1(\mathbb{H}_n)^{U(n)}$ and $\widehat{f} \in L^1(\Sigma)$, (for example $f \in \mathcal{S}(\mathbb{H}_n)^{U(n)}$), we use the next Plancherel inversion formula to decompose $f$, see [25],

$$f(z,s) = \sum_{k\geq 0} \int_{-\infty}^{\infty} (f * \varphi_{\lambda,k})(z,s)|\lambda|^n d\lambda \tag{1.8}$$
$$= \sum_{k\geq 0} \int_{-\infty}^{\infty} \widehat{f}(\lambda,k)\varphi_{\lambda,k}(z,s)|\lambda|^n d\lambda.$$

Also, spherical functions determine the spectral decomposition of the operator $L$, see p. 353 of [25], and for $0 < \alpha \leq 1$ the following formula is valid for the fractional Laplacian, see p. 181 of [14], also see [15]. If $f \in L^1(\mathbb{H}_n)^{U(n)}$ and $\widehat{f} \in L^1(\Sigma)$ we have

$$(-L)^\alpha f(z,s) = \sum_{k\geq 0} \int_{-\infty}^{\infty} |\lambda|^\alpha (2k+n)^\alpha (f * \varphi_{\lambda,k})(z,s)|\lambda|^n d\lambda \tag{1.9}$$

$$= \sum_{k\geq 0} \int_{-\infty}^{\infty} |\lambda|^\alpha (2k+n)^\alpha \widehat{f}(\lambda,k)\varphi_{\lambda,k}(z,s)|\lambda|^n d\lambda.$$

Now let us consider the classical heat equation for the Heisenberg group, defined by

$$\begin{cases} v_t(z,s,t) = Lv(z,s,t), \\ v(z,s,0) = u_0(z,s). \end{cases} \tag{1.10}$$

In [14] the author proved there is a unique heat kernel $P : \mathbb{H}_n \times (0,\infty) \longrightarrow \mathbb{R}$, $P(z,s,t) = P_t(z,s)$ with $P_0 = \delta_0$, $P_t \geq 0$ and $\int_{\mathbb{H}_n} P_t = 1$. The solution of the Eq. (1.10) is given by $v(z,s,t) = P_t * u_0(z,s)$, where the convolution product is in the Heisenberg group. He also proves that $P_t$ is $C^\infty$, (see also [1,8,11,12,21]).

In [23] the author proves that if $u_0 \in L^1(\mathbb{H}_n)$ then

$$\|v(\cdot,\cdot,t)\|_\infty \leq C t^{\frac{-(2n+2)}{2}}, \tag{1.11}$$

where the constant $C$ depends on the norm $\|u_0\|_{L^1(\mathbb{H}_n)}$.

Finally, for an initial data $u_0 \in L^1(\mathbb{H}_n)^{U(n)}$ and $\widehat{u} \in L^1(\Sigma)$, if we consider the equation

$$\begin{cases} v_t(z,s,t) = -(-L)^\alpha v(z,s,t), \\ v(z,s,0) = u_0(z,s), \end{cases} \tag{1.12}$$

we can apply the spherical transform

$$\begin{cases} \widehat{v}_t(\lambda,k,t) = -|\lambda|^\alpha (2k+n)^\alpha \widehat{v}(\lambda,k,t), \\ \widehat{v}(\lambda,k,0) = \widehat{u_0}(\lambda,k). \end{cases} \tag{1.13}$$

Then

$$\widehat{v}(\lambda,k,t) = e^{-|\lambda|^\alpha (2k+n)^\alpha t} \widehat{u_0}(\lambda,k), \tag{1.14}$$

and as $\widehat{u} \in L^1(\Sigma)$ we can apply the inverse of the spherical transform to obtain and solution of (1.12).

Now, we are going to establish the behavior of the solution $v$ of the problem (1.12).

**Lemma 1.1.** *Let $u_0 \in L^1(\mathbb{H}_n)^{U(n)}$ with $\widehat{u} \in L^1(\Sigma)$ and let $v$ be the solution of the problem (1.12). Then*

$$\|v\|_\infty \leq C\|u_0\|_\infty t^{\frac{-(n+1)}{\alpha}}. \tag{1.15}$$

*Proof.* Let $v(z,s,t)$ be a solution of (1.12), with initial datum $v(z,s,0) = u_0(z,s)$. Then by Eqs. (1.8) and (1.14) we have

$$\begin{aligned}|v(z,s,t)| &= \left|\sum_{k\geq 0}\int_{-\infty}^{\infty}\widehat{v}(\lambda,k,t)\varphi_{\lambda,k}(z,s)|\lambda|^n d\lambda\right| \\ &\leq \sum_{k\geq 0}\int_{-\infty}^{\infty}\left|e^{-(|\lambda|(2k+n))^\alpha t}\widehat{u}_0(\lambda,k)\right||\lambda|^n d\lambda.\end{aligned}$$

We make the change of variables $\lambda(2k+n)t^{1/\alpha} = \eta$, and then $(|\lambda|(2k+n))^\alpha t = |\eta|^\alpha$ and $d\lambda(2k+n)t^{1/\alpha} = d\eta$. Thus,

$$\begin{aligned}|v(z,s,t)| &\leq \sum_{k\geq 0}\int_{-\infty}^{\infty}\left|e^{-(|\lambda|(2k+n))^\alpha t}\widehat{u}_0(\lambda,k)\right||\lambda|^n d\lambda \\ &\leq t^{\frac{-(n+1)}{\alpha}}||\widehat{u}_0||_\infty\left[\int_{-\infty}^{\infty}e^{-|\eta|^\alpha}|\eta|^n d\eta\right]\sum_{k\geq 0}\frac{1}{(2k+n)^{n+1}} \\ &\leq C\|u_0\|_\infty t^{\frac{-(n+1)}{\alpha}}\end{aligned}$$

□

In this work we consider the nonlocal equation given by (1.3)

$$u_t(z,s,t) = J * u(z,s,t) - u(z,s,t),$$

where $J$ satisfies the following hypothesis:

$(H)$ $J : \mathbb{H}_n \to \mathbb{R}$ is a positive function invariant by the action of $U(n)$ with $\int_{\mathbb{H}_n} J(z,s)\,dzds = 1$.

We will assume $(H)$ throughout the paper.

Let us now state our results concerning the asymptotic behavior.

The first problem to be addressed is the Cauchy diffusion problem in $\mathbb{H}_n$. We consider the equation

$$\begin{cases} u_t(z,s,t) = J * u(z,s,t) - u(z,s,t), \\ u(z,s,0) = u_0(z,s). \end{cases} \tag{1.16}$$

For this problem we study the asymptotic behavior in the infinity and use the spherical transform to prove the following result

**Theorem 1.2.** *Let $u$ be the solution of the problem (1.16) with $u_0$ in $L^1(\mathbb{H}_n)^{U(n)}$ and $\widehat{u_0}$ in $L^1(\Sigma)$. Assume that $J$ satisfies (H) and we assume*

$$\widehat{J}(\lambda,k) = 1 - (|\lambda|(2k+n))^\alpha + o((|\lambda|(2k+n))^\alpha),$$

$$\text{with} \quad \lim_{|\lambda|(2k+n)\to 0}\frac{o((|\lambda|(2k+n))^\alpha)}{(|\lambda|(2k+n))^\alpha} = 0.$$

*Then the asymptotic behavior of $u(z,s,t)$ is given by*

$$\lim_{t\longrightarrow\infty} t^{\frac{n+1}{\alpha}}\max_{(z,s)}|u(z,s,t) - v(z,s,t)| = 0, \tag{1.17}$$

*where $v$ is the solution of parabolic equation for the Heisenberg group (1.12).*

*The asymptotic profile is given by:*

$$\lim_{t\longrightarrow\infty}\max_{(z,s)}|t^{\frac{n+1}{\alpha}}\delta_{t^\alpha}u(z,s,t)-G_{u_0}(z,s)|=0,$$

*where* $G_{u_0}(z,s)$ *satisfies* $\widehat{G_{u_0}}(\lambda,k)=e^{-|\lambda|^\alpha(2k+n)^\alpha}\widehat{u_0}(0,k)$.
*We also have,*

$$\|u(\cdot,\cdot,t)\|_{L^\infty(\mathbb{H}_n)}\le Ct^{-\frac{(n+1)}{\alpha}},$$

*and by interpolation for* $2<p<\infty$,

$$\|u(\cdot,\cdot,t)\|_{L^p(\mathbb{H}_n)}\le Ct^{-\frac{(n+1)}{\alpha}\frac{p-2}{p}}.$$

**Remark 1.3.** In the literature, estimates of the decay in infinite norm have been obtained only for nonlocal equation that approximate the fractional laplacian operator and not for a more general fractional elliptic operator. The Heisenberg laplacian operator for a function $u$ invariant by the action of $U(n)$, is given in polar coordinates by

$$Lu=\frac{\partial^2u}{\partial r^2}+\frac{2n-2}{r}\frac{\partial u}{\partial r}+\frac{r^2}{4}\frac{\partial^2u}{\partial s^2}.$$

where $r^2=\sum_{i=1}^n x^2+y^2$. For this reason, Theorem 1.2 gives an example of another fractional elliptic operator that can be approximated by a nonlocal equation in infinite norm.

Let us see the existence of a function $J$ that satisfies the hypotheses of the Theorem 1.17. We are only going to consider the case $\alpha=1$.

**Lemma 1.4.** *There exists a positive function $J$ invariant by the action of $U(n)$ with $\int_{\mathbb{H}_n}J(z,s)\,dzds=1$, and the spherical transform of $J$ is of the form*

$$\widehat{J}(\lambda,k)=1-|\lambda|(2k+n)+o(|\lambda|(2k+n)),$$
$$\textit{with}\quad\lim_{|\lambda|(2k+n)\to0}\frac{o(|\lambda|(2k+n))}{|\lambda|(2k+n)}=0.$$

*Proof.* Let

$$g(\lambda,k)=e^{-|\lambda|(2k+n)}=\sum_{j\ge0}\frac{(-|\lambda|(2k+n))^j}{j!}=1-|\lambda|(2k+n)+o(|\lambda|(2k+n)).$$

Then by (1.14) $g$ is the spherical transform of the heat kernel $P_1$, and so $J(z,s)=P_1(z,s)$ is a positive function invariant by the action of $U(n)$ and $\int_{\mathbb{H}_n}J(z,s)\,dzds=1$ with $\widehat{J}(\lambda,k)=g(\lambda,k)$. □

Next we consider a bounded smooth domain $\Omega\subset\mathbb{H}_n$ and impose boundary conditions to our model. From now on we assume that $J$ is continuous. We consider the next Dirichlet problem

$$\begin{cases}u_t(z,s,t)=J*u(z,s,t)-u(z,s,t), & \text{for}\quad (z,s)\in\Omega\quad\text{and}\quad t>0,\\ u(z,s,t)=g(z,s,t), & \text{for}\quad (z,s)\notin\Omega\quad\text{and}\quad t>0,\\ u(z,s,0)=u_0(z,s), & \text{for}\quad (z,s)\in\Omega.\end{cases}\tag{1.18}$$

If $J$ satisfies the following hypothesis

$(\tilde{H})$ $J$ is continuous, no negative with $J(0,0)>0$; $J$ have compact support and is symmetric in the variable $s$. We assume there exists a constant $C_1$ with $\int_{\mathbb{H}_n} J(z,s)x_j^2\,dzds = C_1$, $\int_{\mathbb{H}_n} J(z,s)y_j^2\,dzds = C_1$, $\int_{\mathbb{H}_n} J(z,s)s^2\,dzds = C_1$.

We will consider the rescaled kernel

$$J^\epsilon(z,s) = \frac{2C_1^{-1}}{\epsilon^{2n+2}}\delta_{\epsilon^{-2}}J(z,s) = \frac{2C_1^{-1}}{\epsilon^{2n+2}}J\left(\frac{z}{\epsilon},\frac{s}{\epsilon^2}\right)$$

and the problem

$$\begin{cases} u_t^\epsilon(z,s,t) = \dfrac{1}{\epsilon^2}J^\epsilon * u(z,s,t) - u(z,s,t), & \text{for} \quad (z,s)\in\Omega \quad \text{and} \quad t>0,\\ u^\epsilon(z,s,t) = g(z,s,t), & \text{for} \quad (z,s)\notin\Omega \quad \text{and} \quad t>0,\\ u^\epsilon(z,s,0) = u_0(z,s), & \text{for} \quad (z,s)\in\Omega. \end{cases} \tag{1.19}$$

We prove that the solution of (1.19) approximate uniformly to the solution of the corresponding Dirichlet problem for the classical heat equation, given by

$$\begin{cases} v_t(z,s,t) = Lv(z,s,t), & \text{for} \quad (z,s)\in\Omega \quad \text{and} \quad t>0,\\ v_t(z,s,t) = g(z,s,t), & \text{for} \quad (z,s)\in\partial\Omega \quad \text{and} \quad t>0,\\ v(z,s,0) = u_0(z,s), & \text{for} \quad (z,s)\in\Omega. \end{cases} \tag{1.20}$$

Our result are as follows.

**Theorem 1.5.** *Let $\Omega$ be a bounded $C^{2+\alpha}$ domain for some $0<\alpha<1$. Let $v\in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ be the solution to (1.20) and let $u^\epsilon$ be the solution to (1.19) with $J^\epsilon$ as above and $J$ satisfying (H) and $(\tilde{H})$. Then, there exists $C=C(T)$ such that*

$$\sup_{t\in[0,T]}\|u^\epsilon(\cdot,\cdot,t)-v(\cdot,\cdot,t)\|_{L^\infty(\Omega)} \le C\,\epsilon^\alpha, \quad \textit{as} \quad \epsilon\to 0.$$

**Remark 1.6.** Observe that since the initial data $u_0(z,s)$ is not necessarily invariant by the action of $U(n)$, $L$ is given by the formula (1.4) and the solution of problem (1.19) approaches to the solution of a more irregular equation, given in (1.20).

Finally we observe, that if $J$ is symmetric in the variable $s$ and as $J$ is invariant by the action of $U(n)$, we have

$$\begin{aligned} J\left(z-\tilde{z},s-\tilde{s}-\frac{1}{2}\mathrm{Im}\langle z,\tilde{z}\rangle\right) &= J\left(\tilde{z}-z,\tilde{s}-s-\left(-\frac{1}{2}\mathrm{Im}\langle z,\tilde{z}\rangle\right)\right)\\ &= J\left(\tilde{z}-z,\tilde{s}-s-\frac{1}{2}\mathrm{Im}\langle\tilde{z},z\rangle\right). \end{aligned} \tag{1.21}$$

Then, if we write $K((z,s),(\tilde{z},\tilde{s})) = J\left(z-\tilde{z},s-\tilde{s}-\frac{1}{2}\mathrm{Im}\langle z,\tilde{z}\rangle\right)$, $K$ is a non-negative and symmetric Kernel. Therefore Theorem 2 of [9] is true for the nonlocal equation defined by the kernel $K$. That is to say that $g(z,s,t)\equiv 0$ in (1.7) and $J$ is also symmetric in the variable $s$, we find an exponential decay given by the first eigenvalue of an associated problem and the asymptotic behavior of solutions is described by the unique (up to a constant) associated eigenfunction. Let $\lambda_1 = \lambda_1(\Omega)$ be given by

$$\lambda_1 = \inf_{u \in L^2(\mathbb{H}^n)} \frac{\frac{1}{2}\int_{\mathbb{H}_n}\int_{\mathbb{H}_n} J\left((z-\tilde{z}, s-\tilde{s}-\frac{1}{2}\mathrm{Im}\langle z,\tilde{z}\rangle)\right)(u(z,s)-u(\tilde{z},\tilde{z}))^2\, dzds\, d\tilde{z}d\tilde{s}}{\int_\Omega (u(z,s)^2) dzds}. \tag{1.22}$$

**Theorem 1.7.** *Let $u_0 \in L^1(\Omega) \cap L^2(\Omega)$. Assume that $J$ is continuous, satisfies (H) and is symmetric in the variable $s$. Then the solutions of (1.18), with $g(z,s,t) \equiv 0$, decay to zero as $t \to \infty$ with an exponential rate*

$$\|u(\cdot,\cdot,t)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)} e^{-\lambda_1 t}.$$

*If $u_0$ is continuous, positive and bounded then there exist positive constants $C$ and $\tilde{C}$ such that*

$$\|u(\cdot,\cdot,t)\|_{L^\infty(\Omega)} \le C e^{-\lambda_1 t},$$

*and*

$$\lim_{t\to 0} \max_{(z,s)} |e^{\lambda_1 t} u(z,s,t) - \tilde{C}\phi_1(z,t)| = 0,$$

*where $\phi_1$ is the eigenfunction associated to $\lambda_1$.*

We consider next the Neumann boundary conditions:

$$\begin{cases} u_t(z,s,t) = \displaystyle\int_\Omega J\left(z-\tilde{z}, s-\tilde{s}-\frac{1}{2}\mathrm{Im}\langle z,\tilde{z}\rangle\right)[u(\tilde{z},\tilde{s},t)-u(z,s,t)]d\tilde{z}d\tilde{s}, \\ u(z,s,0) = u_0(z,s). \end{cases} \tag{1.23}$$

If we impose that $J$ is symmetric in the variable $s$ by Eq. (1.21) the Theorem 3 of [9] is true. And, in this case, we find that the asymptotic behavior is given by an exponential decay determined by an eigenvalue problem. Let $\beta_1$ be given by:

$$\beta_1 = \inf_{u \in L^2(\Omega),\, \int u = 0} \frac{\frac{1}{2}\int_\Omega\int_\Omega J\left(z-\tilde{z}, s-\tilde{s}-\frac{1}{2}\mathrm{Im}\langle z,\tilde{z}\rangle\right)[u(\tilde{z},\tilde{s})-u(z,s)]^2 d\tilde{z}d\tilde{s}dzds}{\int_\Omega (u(z,s))^2 dzds}. \tag{1.24}$$

**Theorem 1.8.** *Let $J$ be a continuous kernel symmetric in the variable $s$ that satisfies (H). For every $u_0 \in L^1(\Omega)$ there exists a unique solution $u$ of (1.23) such that $u \in C([0,\infty); L^1(\Omega))$. This solution preserves the total mass in $\Omega$:*

$$\int_\Omega u(z,s,t)\, dzds = \int_\Omega u_0(z,s)\, dzds.$$

*Moreover, let $\mathcal{M} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0(z,s)\, dzds$. Then the asymptotic behavior of solutions of (1.23) is described as follows: if $u_0 \in L^2(\Omega)$,*

$$\|u(\cdot,\cdot,t) - \mathcal{M}\|_{L^2(\Omega)} \le e^{-\beta_1 t}\|u_0 - \mathcal{M}\|_{L^2(\Omega)},$$

*and if* $u_0$ *is continuous and bounded there exists a positive constant* $C$ *such that*

$$\|u(\cdot,\cdot,t)-\mathcal{M}\|_{L^\infty(\Omega)} \le Ce^{-\beta_1 t}.$$

The rest of the paper is organized as follows: in Sect. 2, we prove existence and uniqueness of the Cauchy problem given by (1.16) and we also prove Theorem 1.2. In Sect. 3 we prove existence, uniqueness and a comparison principle of the Dirichlet problem given by (1.18), and we also prove the convergence result for Dirichlet problem, Theorem 1.5.

## 2. The Cauchy problem

In this section, we will use that the function $u_0$ is invariant by the action of $U(n)$, this allows us to use the spherical transform of $\mathbb{H}_n$ in order to obtain explicit solutions of Cauchy problem (1.16).

**Theorem 2.1.** *Let* $u_0$ *in* $L^1(\mathbb{H}_n)^{U(n)}$ *and* $\widehat{u_0}$ *in* $L^1(\Sigma)$. *Let* $J$ *satisfy (H). Then there exists a unique solution* $u \in C^0([0,\infty), L^1(\mathbb{H}_n))$ *of problem* (1.16) *and it is given by:*

$$\widehat{u}(\lambda,k,t) = e^{(\widehat{J}(\lambda,k)-1)t}\widehat{u_0}(\lambda,k).$$

*Proof.* First observe that since $\int_{\mathbb{H}_n} J(z,s)\,dzds = 1$, then $\widehat{J} \in C_0(\Sigma)$ and $\widehat{J}(0,0)=1$.

We have

$$u_t(z,s,t) = J * u(z,s,t) - u(z,s,t).$$

Applying the spherical transform to this equation, we obtain:

$$\widehat{u}_t(\lambda,k,t) = (\widehat{J}(\lambda,k)-1)\widehat{u}(\lambda,k,t).$$

Hence,

$$\widehat{u}(\lambda,k,t) = e^{(\widehat{J}(\lambda,k)-1)t}\widehat{u}_0(\lambda,k).$$

Since $\widehat{u_0} \in L^1(\Sigma)$ and $e^{(\widehat{J}-1)t}$ is continuous and bounded, the result follows by taking the inverse of the spherical transform. □

**Lemma 2.2.** *Let* $J \in \mathcal{S}(\mathbb{H}_n)^{U(n)}$ *satisfy (H) and* $u_0 = \delta_0$ *(the Dirac delta in* $\mathbb{H}_n$*). Then the fundamental solution of* (1.16) *can be decomposed as*

$$w(z,s,t) = e^{-t}\delta_0 + \nu(z,s,t),$$

*with* $\nu(z,s,t)$ *smooth. Moreover, if* $u$ *is a solution of* (1.16) *with initial condition a function* $u_0$ *invariant by the action of* $U(n)$, *it can be written as*

$$u(z,s,t) = w * u_0(z,s,t).$$

*Proof.* By the previous result, we have

$$\widehat{w}_t(\lambda,k,t) = (\widehat{J}(\lambda,k)-1)\widehat{w}(\lambda,k,t).$$

Hence $\widehat{\delta_0} = 1$, in the sense of distributions, we have

$$\widehat{w}(\lambda,k,t) = e^{(\widehat{J}(\lambda,k)-1)t} = e^{-t}(e^{\widehat{J}(\lambda,k)t}-1) + e^{-t}.$$

Now let us prove that, for each fixed $t$, $e^{\widehat{J}(\lambda,k)t}-1 \in L^1(\Sigma)$. By the mean value theorem

$$\begin{aligned}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}} |e^{\widehat{J}(\lambda,k)t}-1|\,|\lambda|^n d\lambda &= \sum_{k\in\mathbb{N}}\int_{\mathbb{R}} |\widehat{J}(\lambda,k)te^{\widehat{J}(\lambda',k')t}|\,|\lambda|^n d\lambda \\ &\le C\sum_{k\in\mathbb{N}}\int_{\mathbb{R}} |\widehat{J}(\lambda,k)t|\,|\lambda|^n d\lambda\end{aligned}$$

By [5] exist a function $g \in S(\mathbb{R}^2)$ such that $\widehat{J}(\lambda,k) = g(\lambda, |\lambda|(2k+n))$ then

$$\begin{aligned}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}} |e^{\widehat{J}(\lambda,k)t}-1|\,|\lambda|^n d\lambda &\le C\sum_{k\in\mathbb{N}}\int_{\mathbb{R}} |g(\lambda,|\lambda|(2k+n)t)|\,|\lambda|^n d\lambda \\ &= C\frac{1}{t^{n+1}}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}} \left|g\left(\frac{\eta}{(2k+n)t},\eta\right)\right| \frac{|\eta|^n}{(2k+n)^{n+1}}d\eta\end{aligned}$$

As $g$ belongs in $S(\mathbb{R}^2)$ exist constants $C_1$ and $C_2$ such that $|g(x,y)| \le C_1$ and $|g(x,y)| \le \frac{C_2}{|y|^{n+2}}$, then

$$\begin{aligned}&\sum_{k\in\mathbb{N}}\int_{\mathbb{R}} |e^{\widehat{J}(\lambda,k)t}-1|\,|\lambda|^n d\lambda \\ &\quad\le C\frac{1}{t^{n+1}}\sum_{k\in\mathbb{N}}\frac{1}{(2k+n)^{n+1}}\left[\int_{|\eta|\le 1} C_1\,|\eta|^n d\eta + \int_{|\eta|>1}\frac{C_2}{|\eta|^2}\,d\eta\right] < \infty\end{aligned}$$

Therefore the first part of the lemma follows applying the inverse spherical transform.

Note that since $J$ and $u_0$ are invariant by the action of $U(n)$ it is enough to show that there exist $L^{(r)}(\nu)$ and $S^r(\nu)$, for all $r \in \mathbb{N}$, to prove that $\nu \in C^\infty(\mathbb{H}_n)^{U(n)}$. this is shown similarly to the previous account using (1.5).

To finish the proof, we observe, that

$$\widehat{w * u_0}(\lambda,k,t) = \widehat{w}(\lambda,k,t)\widehat{u_0}(\lambda,k) = e^{(\widehat{J}(\lambda,k)-1)t}\widehat{u_0}(\lambda,k).$$

By Theorem (2.1) the solution of problem (1.16) satisfies

$$\widehat{u}(\lambda,k,t) = e^{(\widehat{J}(\lambda,k)-1)t}\widehat{u}_0(\lambda,k).$$

Then the result is followed since the spherical transform is injective. □

Next we will prove the asymptotic behavior for the nonlocal diffusion equation (1.16).

*Proof of Theorem 1.2.* We remark that from our hypotheses on $J$,

$$\widehat{J}(\lambda, k) = 1 - |\lambda|^{\alpha}(2k+n)^{\alpha} + o(|\lambda|^{\alpha}(2k+n)^{\alpha}),$$
$$\text{with} \quad \lim_{|\lambda|(2k+n)\longrightarrow 0} \frac{o((|\lambda|(2k+n))^{\alpha})}{(|\lambda|(2k+n))^{\alpha}} = 0.$$

We have that,

$$\hat{J}(\lambda, k) \leq 1 - |\lambda|^{\alpha}(2k+n)^{\alpha} + |\lambda|^{\alpha}(2k+n)^{\alpha} h(|\lambda|(2k+n)), \tag{2.1}$$

where $h$ is a bounded positive function and $\lim_{|\lambda|(2k+n)\longrightarrow 0} h(|\lambda|(2k+n)) = 0$. As $\int_{\mathbb{H}_n} J(z,s)\,dzds = 1$ we have that $|\hat{J}(\lambda,k)| \leq 1$, then there exist a number $\xi > 0$ and constants $D > 0$ and $E > 0$ such that

$$\begin{aligned} \hat{J}(\lambda, k) &\leq 1 - D|\lambda|^{\alpha}(2k+n)^{\alpha}, \quad &\text{if} \quad |\lambda|(2k+n) \leq \xi, \\ \hat{J}(\lambda, k) &\leq 1 - E, &\text{if} \quad |\lambda|(2k+n) > \xi. \end{aligned} \tag{2.2}$$

As in the proof of the Theorem (2.1), we have

$$\widehat{u}(\lambda, k, t) = e^{(\widehat{J}(\lambda,k)-1)t}\widehat{u_0}(\lambda, k).$$

On the other hand, let $v(z,s,t)$ be a solution of the problem given by the Eq. (1.12), with the same initial datum $v(z,s,0) = u_0(z,s)$. By Eq. (1.14) we get

$$\widehat{v}(\lambda, k, t) = e^{-|\lambda|^{\alpha}(2k+n)^{\alpha}t}\widehat{u_0}(\lambda, k).$$

Then, by (1.8) and (1.6), we have

$$\begin{aligned} |u(z,s,t) - v(z,s,t)| &= \left| \sum_{k\geq 0} \int_{-\infty}^{\infty} (\widehat{u} - \widehat{v})(\lambda, k, t)\varphi_{\lambda,k}(z,s)|\lambda|^n d\lambda \right| \\ &\leq \sum_{k\geq 0} \int_{-\infty}^{\infty} \left| \left( e^{(\widehat{J}(\lambda,k)-1)t} - e^{-|\lambda|^{\alpha}(2k+n)^{\alpha}t} \right) \widehat{u_0}(\lambda, k) \right| \\ &\quad \times |\lambda|^n d\lambda. \end{aligned}$$

We decompose the equation in two parts, when $|\lambda|^{\alpha}(2k+n)^{\alpha}\sqrt{t} \geq 1$ and $|\lambda^{\alpha}|(2k+n)^{\alpha}\sqrt{t} < 1$.

$$\begin{aligned} &|u(z,s,t) - v(z,s,t)| \\ &\quad \leq \sum_{k\geq 0} \int_{|\lambda| \geq \frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left| \left( e^{(\widehat{J}(\lambda,k)-1)t} - e^{-|\lambda|^{\alpha}(2k+n)^{\alpha}t} \right) \widehat{u_0}(\lambda, k) \right| |\lambda|^n d\lambda \\ &\qquad + \sum_{k\geq 0} \int_{|\lambda| < \frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left| \left( e^{(\widehat{J}(\lambda,k)-1)t} - e^{-|\lambda|^{\alpha}(2k+n)^{\alpha}t} \right) \widehat{u_0}(\lambda, k) \right| |\lambda|^n d\lambda \\ &\quad := I + II. \end{aligned}$$

First we work with $I$,

$$
\begin{aligned}
I &\leq \sum_{k\geq 0}\int_{|\lambda|>\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left|\left(e^{(\widehat{J}(\lambda,k)-1)t}-e^{-|\lambda|^{\alpha}(2k+n)^{\alpha}t}\right)\widehat{u}_0(\lambda,k)\right| |\lambda|^n d\lambda\\
&\leq \sum_{k\geq 0}\int_{|\lambda|>\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left|e^{(\widehat{J}(\lambda,k)-1)t}\widehat{u}_0(k,\lambda)\right| |\lambda|^n d\lambda\\
&\quad+\sum_{k\geq 0}\int_{|\lambda|>\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left|e^{-|\lambda|^{\alpha}(2k+n)^{\alpha}t}\widehat{u}_0(\lambda,k)\right| |\lambda|^n d\lambda\\
&:= I_1+I_2.
\end{aligned}
$$

For $I_2$, we make the change of variables $\lambda(2k+n)t^{\frac{1}{\alpha}}=\eta$, then $|\lambda|(2k+n)t^{\frac{1}{\alpha}}=|\eta|$ and $d\lambda(2k+n)t^{\frac{1}{\alpha}}=d\eta$, and

$$
\begin{aligned}
t^{\frac{n+1}{\alpha}}I_2 &= t^{\frac{n+1}{\alpha}}\sum_{k\geq 0}\int_{|\lambda|>\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left|e^{-|\lambda|^{\alpha}(2k+n)^{\alpha}t}\widehat{u}_0(\lambda,k)\right| |\lambda|^n d\lambda\\
&\leq \|\widehat{u}_0\|_{\infty}\left[\int_{|\eta|>t^{\frac{1}{2\alpha}}} e^{-|\eta|^{\alpha}}|\eta|^n d\eta\right]\sum_{k\geq 0}\frac{1}{(2k+n)^{n+1}}.
\end{aligned}
$$

Note that the sum is finite and by the dominated convergence theorem,

$$
\lim_{t\to\infty} t^{\frac{n+1}{\alpha}}I_2=0.
$$

Now, we work with $I_1$. By (2.2) $I_1$ is bounded by

$$
\begin{aligned}
I_1 &= \sum_{k\geq 0}\int_{|\lambda|>\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left|e^{(\hat{J}(\lambda,k)-1)t}\hat{u}_0(\lambda,k)\right| |\lambda|^n d\lambda\\
&= \sum_{k\geq 0}\int_{\frac{\xi}{2k+n}>|\lambda|>\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left|e^{(\hat{J}(\lambda,k)-1)t}\hat{u}_0(\lambda,k)\right| |\lambda|^n d\lambda\\
&\quad+\sum_{k\geq 0}\int_{|\lambda|\geq\frac{\xi}{2k+n}} \left|e^{(\hat{J}(\lambda,k)-1)t}\hat{u}_0(\lambda,k)\right| |\lambda|^n d\lambda\\
&\leq \sum_{k\geq 0}\int_{\frac{\xi}{2k+n}>|\lambda|>\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left|e^{-D|\lambda|^{\alpha}(2k+n)^{\alpha}t}\hat{u}_0(\lambda,k)\right| |\lambda|^n d\lambda+\|\hat{u}_0\|_{L^1(\Sigma)}e^{-Et}.
\end{aligned}
$$

We now make the change of variables $\lambda(2k+n)t^{\frac{1}{\alpha}}=\eta$, and then

$$
\begin{aligned}
t^{\frac{n+1}{\alpha}}I_1 &\leq \|\hat{u}_0\|_{\infty}\sum_{k\geq 0}\frac{1}{(2k+n)^{n+1}}\int_{\xi t^{\frac{1}{\alpha}}>|\eta|>t^{\frac{1}{2\alpha}}} e^{-D|\eta|^{\alpha}}|\eta|^n d\eta\\
&\quad+\|\hat{u}_0\|_{L^1(\Sigma)}t^{\frac{n+1}{\alpha}}e^{-Et}\\
&\leq \|\hat{u}_0\|_{\infty}\left[\int_{|\eta|>t^{\frac{1}{2\alpha}}} e^{-D|\eta|^{\alpha}}|\eta|^n d\eta\right]\sum_{k\geq 0}\frac{1}{(2k+n)^{n+1}}\\
&\quad+\|\hat{u}_0\|_{L^1(\Sigma)}t^{\frac{n+1}{\alpha}}e^{-Et}.
\end{aligned}
$$

Therefore $t^{\frac{n+1}{\alpha}} I_1 \to 0$ when $t \to \infty$.

Finally we will estimate $II$. Again we make the change of variables $\lambda(2k+n)t^{\frac{1}{\alpha}} = \eta$, if $t^{\frac{1}{\alpha}}$ is a sufficiently large number, by (2.1), we have

$$\begin{aligned}
&t^{\frac{n+1}{\alpha}} II \\
&= t^{\frac{n+1}{\alpha}} \sum_{k\geq 0} \int_{|\lambda|<\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left| \left( e^{[\hat{J}-1+|\lambda|^\alpha (2k+n)^\alpha]t} - 1 \right) e^{-|\lambda|^\alpha (2k+n)^\alpha t} \hat{u}_0(k,\lambda) \right| \\
&\quad \times |\lambda|^n d\lambda \\
&\leq t^{\frac{n+1}{\alpha}} \|\hat{u}_0\|_\infty \sum_{k\geq 0} \int_{|\lambda|<\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} \left| e^{|\lambda|^\alpha (2k+n)^\alpha h(|\lambda|(2k+n))t} - 1 \right| e^{-|\lambda|^\alpha (2k+n)^\alpha t} \\
&\quad \times |\lambda|^n d\lambda \\
&\leq C t^{\frac{n+1}{\alpha}} \|\hat{u}_0\|_\infty \sum_{k\geq 0} \int_{|\lambda|<\frac{1}{(2k+n)t^{\frac{1}{2\alpha}}}} |\lambda|^\alpha (2k+n)^\alpha h(|\lambda|(2k+n)) t e^{-|\lambda|^\alpha (2k+n)^\alpha t} \\
&\quad \times |\lambda|^n d\lambda \\
&\leq C \|\hat{u}_0\|_\infty \sum_{k\geq 0} \int_{|\eta|<t^{\frac{1}{\alpha}}} \frac{1}{(2k+n)^{n+1}} h\left( \frac{|\eta|}{t^{\frac{1}{\alpha}}} \right) e^{-|\eta|^\alpha} |\eta|^{n+1} d\eta \\
&\leq C \|\hat{u}_0\|_\infty \left( \sum_{k\geq 0} \frac{1}{(2k+n)^{n+1}} \right) \int_{\mathbb{R}} h\left( \frac{|\eta|}{t^{\frac{1}{\alpha}}} \right) e^{-|\eta|^\alpha} |\eta|^{n+1} d\eta.
\end{aligned}$$

Observe that $h\left( \frac{|\eta|}{t^{\frac{1}{\alpha}}} \right) \to 0$ when $t \to \infty$. Also

$$h\left( \frac{|\eta|}{t^{\frac{1}{\alpha}}} \right) e^{-|\eta|^\alpha} |\eta|^{n+1} \leq \|h\|_\infty |\eta|^{n+1} e^{-|\eta|^\alpha},$$

and then by convergence dominated theorem $t^{\frac{n+1}{\alpha}} II \to 0$ when $t \to \infty$.

Thus we have showed that

$$\lim_{t\longrightarrow\infty} t^{\frac{n+1}{\alpha}} \max_{(z,s)} |u(z,s,t) - v(z,s,t)| = 0$$

since

$$\begin{aligned}
&\lim_{t\longrightarrow\infty} t^{\frac{n+1}{\alpha}} \max_{(z,s)} |u(z,s,t) - v(z,s,t)| \\
&\quad \leq \lim_{t\longrightarrow\infty} t^{\frac{n+1}{\alpha}} \sum_{k\geq 0} \int_{-\infty}^{\infty} |\hat{u} - \hat{v}|(\lambda,k,t) |\lambda|^n d\lambda = 0.
\end{aligned}$$

Now we will prove that the asymptotic profile is given by

$$\lim_{t\to\infty} \max_{(z,s)} |t^{\frac{n+1}{\alpha}} u(t^{\frac{\alpha}{2}} z, t^\alpha s, t) - G_{u_0}(z,s)| = 0,$$

where $G_{u_0}(z,s)$ is the function such that $\widehat{G_{u_0}}(\lambda,k) = e^{-|\lambda|^\alpha (2k+n)^\alpha} \widehat{u_0}(0,k)$.

Indeed, we have

$$\widehat{v}(t^{-\alpha}\lambda, k, t) = e^{-|\lambda|^\alpha (2k+n)^\alpha} \widehat{u_0}(t^{-\alpha}\lambda, k) \to e^{-|\lambda|^\alpha (2k+n)^\alpha} \widehat{u_0}(0,k). \tag{2.3}$$

Now, taking the spherical transform and by (1.6) and (1.7), we get

$$t^{\frac{n+1}{\alpha}}\delta_{t^\alpha}\widehat{v(\cdot,\cdot,t)}(\lambda,k,t) = t^{\frac{n+1}{\alpha}}\int_{\mathbb{H}_n} v(t^{\frac{\alpha}{2}}z, t^\alpha s, t)\varphi_{t^\alpha t^{-\alpha}\lambda,k}(-z,-s)\,dz\,ds \quad (2.4)$$
$$= t^{\frac{n+1}{\alpha}}\int_{\mathbb{H}_n} v(t^{\frac{\alpha}{2}}z, t^\alpha s, t)\varphi_{t^{-\alpha}\lambda,k}(-t^{\frac{\alpha}{2}}z,-t^\alpha s)\,dz\,ds$$
$$= \int_{\mathbb{H}_n} v(z,s,t)\varphi_{t^{-\alpha}\lambda,k}(-z,-s)\,dz\,ds$$
$$= \widehat{v}(t^{-\alpha}\lambda,k,t).$$

By (1.17), (2.3) and (2.4) we have

$$\lim_{t\to\infty}\max_{(z,s)}|t^{\frac{n+1}{\alpha}}\delta_{t^\alpha}u(z,s,t) - G_{u_0}(z,s)|$$
$$\le \lim_{t\to\infty}\max_{(z,s)}|t^{\frac{n+1}{\alpha}}\delta_{t^\alpha}u(z,s,t) - t^{\frac{n+1}{\alpha}}\delta_{t^\alpha}v(z,s,t)|$$
$$+ \lim_{t\to\infty}\max_{(z,s)}|t^{\frac{n+1}{\alpha}}\delta_{t^\alpha}v(z,s,t) - G_{u_0}(z,s)| = 0.$$

Finally, since $\|v(\cdot,\cdot,t)\|_{L^\infty(\mathbb{H}_n)} \le Ct^{-\frac{n+1}{\alpha}}$, (see (1.15)), we have

$$\|u(\cdot,\cdot,t)\|_{L^\infty(\mathbb{H}_n)} \le Ct^{-\frac{n+1}{\alpha}}.$$

and by interpolation for $2 < p < \infty$,

$$\|u(\cdot,\cdot,t)\|_{L^p(\mathbb{H}_n)} \le \|u(\cdot,\cdot,t)\|^{\frac{2}{p}}_{L^2(\mathbb{H}_n)}\|u(\cdot,\cdot,t)\|^{\frac{p-2}{p}}_{L^\infty(\mathbb{H}_n)}.$$

As (1.16) has bounded the $L^2(\mathbb{H}_n)$ norm, because it is the solution given through the spherical transform, we have

$$\|u(\cdot,\cdot,t)\|_{L^p(\mathbb{H}_n)} \le Ct^{-\frac{n+1}{\alpha}\frac{p-2}{p}}.$$

□

## 3. The Dirichlet problem

### 3.1. Existence and properties of solutions

We shall first derive the existence and uniqueness of solutions of (1.18), which is a consequence of Banach's fixed point theorem. The main arguments are basically the same of [9] or [10], but we write them here to make the paper self-contained.

**Theorem 3.1.** *Let $u_0 \in L^1(\Omega)$ and be $J$ a kernel that verifies (H) and ($\tilde{H}$). Then there exists a unique solution $u$ of* (1.18) *such that $u \in C([0,\infty), L^1(\Omega))$.*

Recall that a solution of the Dirichlet problem is defined as a $u \in C([0,\infty), L^1(\Omega))$ satisfying (1.18).

*Proof.* We use the Banach's fixed point theorem. Fix $t_0 > 0$ and consider the Banach space

$$X_{t_0} := \left\{w \in C([0,t_0]; L^1(\Omega)), \text{and } w(z,s,t) = g(z,s,t) \text{ if } (z,s) \notin \Omega\right\},$$

with the norm

$$|||w||| := \max_{0\le t\le t_0} \|w(\cdot,\cdot,t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $\mathfrak{T} : X_{t_0} \to X_{t_0}$ defined by

$$\mathfrak{T}(w)(z,s,t) := \begin{cases} w_0(z,s) + \displaystyle\int_0^t J * w(z,s,r) - w(z,s,r)\,dr & \text{if} \quad (z,s)\in\Omega, \\ g(z,s,t) & \text{if} \quad (z,s)\notin\Omega, \end{cases}$$

where $w_0(z,s) = w(z,s,0)$.

Let $w$, $v \in X_{t_0}$. Then there exists a constant $C$ depending on $J$ and $\Omega$ such that

$$|||\mathfrak{T}(w) - \mathfrak{T}(v)||| \le Ct_0|||w-v||| + \|w_0 - v_0\|_{L^1(\Omega)}. \tag{3.1}$$

We will prove (3.1). Indeed,

$$\begin{aligned} \int_\Omega \left|\mathfrak{T}(w) - \mathfrak{T}(v)\right|(z,s,t)dzds &\le \int_\Omega |w_0 - v_0|(z,s)dzds \\ &\quad + \int_\Omega \left|\int_0^t J*(w-v)(z,s,r) - (w-v)(z,s,r)\,dr\right| dzds \\ &\le \|w_0 - v_0\|_{L^1(\Omega)} + t(\|J\|_{L^\infty(\Omega)} + 1)|||(w-v)|||. \end{aligned}$$

Taking the maximum in $t$ (3.1) follows.

Now, taking $v_0 \equiv v \equiv 0$ in (3.1) we get that $\mathfrak{T}(w) \in C([0,t_0];L^1(\Omega))$ and this says that $\mathfrak{T}$ maps $X_{t_0}$ into $X_{t_0}$.

Finally, we will consider $X_{t_0,u_0} = \{u \in X_{t_0} : u(z,s,0) = u_0(z,s)\}$. $\mathfrak{T}$ maps $X_{t_0,u_0}$ into $X_{t_0,u_0}$ and taking $t_0$ such that $(C+1)t_0 < 1$, where $C$ is the constant given in (3.1) we can apply the Banach's fixed point theorem in the interval $[0,t_0]$ because $\mathfrak{T}$ is a strict contraction in $X_{t_0,u_0}$. From this we get the existence and uniqueness of the solution in $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial data $u(x,t_0)\in L^1(\Omega)$ and obtain a solution up to $[0,2t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$. □

In order to prove a comparison principle of problem given by (1.18) we need to introduce the definition of sub and super solutions.

**Definition 3.2.** A function $u \in C([0,T];L^1(\Omega))$ is a supersolution of (1.18) if

$$\begin{cases} u_t(z,s,t) \ge J*u(z,s,t) - u(z,s,t), & \text{for} \quad (z,s)\in\Omega \quad \text{and} \quad t>0, \\ u_t(z,s,t) \ge g(z,t), & \text{for} \quad (z,s)\notin\Omega \quad \text{and} \quad t>0, \\ u(z,s,0) \ge u_0(z,s), & \text{for} \quad (z,s)\in\Omega. \end{cases} \tag{3.2}$$

As usual, subsolutions are defined analogously by reversing the inequalities.

**Lemma 3.3.** *Let $u_0 \in C(\overline{\Omega})$, $u_0 \ge 0$, and $u \in C(\overline{\Omega}\times[0,T])$ a supersolution of (1.18) with $g \ge 0$. Then, $u \ge 0$.*

*Proof.* Assume to the contrary that $u(z,s,t)$ is negative in some point. Let $v(z,s,t) = u(z,s,t) + \epsilon t$ with $\epsilon > 0$ small such that $v$ is still negative somewhere.

Then, if $(z_0, s_0, t_0)$ is a point where $v$ attains its negative minimum, there it holds that $t_0 > 0$ and

$$\begin{aligned} v_t(z_0, s_0, t_0) &= u_t(z_0, s_0, t_0) + \epsilon > J * u(z_0, s_0, t_0) - u(z_0, s_0, t_0) \\ &= \int_{\mathbb{H}_n} J(\tilde{z} - z, \tilde{s} - s - \frac{1}{2}\mathrm{Im}\langle \tilde{z}, z\rangle)(v(\tilde{z}, \tilde{s}, t_0) - v(z_0, s_0, t_0))d\tilde{z}d\tilde{s} \\ &\geq 0. \end{aligned}$$

This contradicts that $(z_0, s_0, t_0)$ is a minimum of $v$. Thus, $u \geq 0$. □

**Corollary 3.4.** *Let $J \in L^\infty(\mathbb{H}_n)$. Let $u_0$ and $v_0$ in $L^1(\Omega)$ with $u_0 \geq v_0$ and $g, h \in L^\infty((0, T); L^1(\mathbb{H}_n \backslash \Omega))$ with $g \geq h$. Let $u$ be a solution of (1.18) with $u(z, s, 0) = u_0(z, s)$ and Dirichlet datum $g$, and let $v$ be a solution of (1.18) with $v(z, s, 0) = v_0(z, s)$ and datum $h$. Then, $u \geq v$ a.e. $\Omega$.*

*Proof.* Let $w = u - v$. Then, $w$ is a supersolution with initial datum $u_0 - v_0 \geq 0$ and datum $g - h \geq 0$. Using the continuity of the solutions with respect to the data and the fact that $J \in L^\infty(\mathbb{H}_n)$, we may assume that $u, v \in C(\Omega \times [0, T])$. By Lemma (3.3) we obtain that $w = u - v \geq 0$. So the corollary is proved. □

**Corollary 3.5.** *Let $u \in C(\Omega \times [0, T])$ (resp., $v$) be a supersolution (resp., subsolution) of (1.18). Then, $u \geq v$.*

*Proof.* It follows from the proof of the previous corollary. □

### 3.2. Convergence to the heat equation

In order to prove a to prove Theorem 1.5, let $\tilde{v}$ be a $C^{2+\alpha,1+\alpha/2}$ extension of $v$ to $\mathbb{H}_n \times [0, T]$, where $v$ is the solution of (1.20). Let us define the operator

$$\tilde{L}_\epsilon(w)(z, s, t) := \frac{1}{\epsilon^2} J_\epsilon * w(z, s, t) - w(z, s, t).$$

Then $\tilde{v}$ verifies

$$\begin{cases} \tilde{v}_t(z, s, t) = \tilde{L}_\epsilon(v)(z, s, t) + F_\epsilon(z, s, t), & \text{for} \quad (z, s) \in \Omega \quad \text{and} \quad t \in [(0, T], \\ \tilde{v}(z, s, t) = g(z, s, t) + G(z, s, t), & \text{for} \quad (z, s) \notin \Omega \quad \text{and} \quad t \in, (0, T] \\ \tilde{v}(z, s, 0) = u_0(z, s), & \text{for} \quad (z, s) \in \Omega. \end{cases} \tag{3.3}$$

Since $Lv(z, s, t) = L\tilde{v}(z, s, t)$ for $(z, s) \in \Omega$, we have

$$F_\epsilon(z, s, t) = L\tilde{v}(z, s, t) - \tilde{L}_\epsilon(v)(z, s, t).$$

Moreover, as $G$ is smooth and $G(z, s, t) = 0$ if $(z, s) \in \partial\Omega$ we have

$$G(z, s, t) = o(\epsilon) \quad \text{for } (z, s) \text{ such that } \mathrm{dist}((z, s), \partial\Omega) \leq \epsilon d.$$

We set $w^\epsilon = \tilde{v} - u^\epsilon$ and we note that

$$\begin{cases} w^\epsilon_t(z, s, t) = \tilde{L}_\epsilon(w^\epsilon)(z, s, t) + F_\epsilon(z, s, t), & \text{for} \quad (z, s) \in \Omega \quad \text{and} \quad t \in (0, T], \\ w^\epsilon(z, s, t) = G(z, s, t), & \text{for } (z, s) \notin \Omega \quad \text{and} \quad t \in, (0, T], \\ w^\epsilon(z, s, 0) = 0, & \text{for} \quad (z, s) \in \Omega. \end{cases} \tag{3.4}$$

**Lemma 3.6.** *Let $\tilde{v}$, $\tilde{L}_\epsilon$ and $F_\epsilon$ be as previously defined. Then we have that*

$$\sup_{t\in[0,T]} \|F_\epsilon\|_{L^\infty(\Omega)} = o(\epsilon^\alpha). \tag{3.5}$$

*Proof.* By $\tilde{v} \in C^{2+\alpha,1+\alpha/2}(\mathbb{H}_n \times [0,T])$, we have that

$$F_\epsilon(z,s,t) = L\tilde{v}(z,s,t) - \tilde{L}_\epsilon(v)(z,s,t)$$

In the global coordinate system $(x,y,s)$, we obtain

$$\begin{aligned}\tilde{L}_\epsilon(v)(x,y,s,t) = \frac{2C_1^{-1}}{\epsilon^{2n+4}} \int_{\mathbb{R}^{2n+1}} & J\left(\frac{\tilde{x}-x}{\epsilon}, \frac{\tilde{y}-y}{\epsilon}, \frac{\tilde{s}-s}{\epsilon^2} - \frac{1}{2}\frac{\tilde{x}y-\tilde{y}x}{\epsilon^2}\right) \\ & \cdot (v(\tilde{x},\tilde{y},\tilde{s},t) - v(x,y,s,t))\, d\tilde{x}d\tilde{y}d\tilde{s}.\end{aligned}$$

We now make the change of variables $\frac{\tilde{x}-x}{\epsilon} = \hat{x}$, $\frac{\tilde{y}-y}{\epsilon} = \hat{y}$ and $\frac{\tilde{s}-s}{\epsilon^2} = \hat{x}$, and so,

$$\begin{aligned}\tilde{L}_\epsilon(v)(x,y,s,t) &= \frac{2C_1^{-1}}{\epsilon^2} \int_{\mathbb{R}^{2n+1}} J\left(\hat{x},\hat{y},\hat{s} - \frac{1}{2}\frac{(\epsilon\hat{x}+x)y - (\epsilon\hat{y}+y)x}{\epsilon^2}\right) \\ &\quad \cdot (v(\epsilon\hat{x}+x,\epsilon\hat{y}+y,\epsilon\hat{s}+s,t) - v(x,y,s,t))\, d\hat{x}d\hat{y}d\hat{s} \\ &= \frac{2C_1^{-1}}{\epsilon^2} \int_{\mathbb{R}^{2n+1}} J\left(\hat{x},\hat{y},\hat{s} - \frac{\epsilon\hat{x}y - \epsilon\hat{y}x}{2\epsilon^2}\right) \\ &\quad \cdot (v(\epsilon\hat{x}+x,\epsilon\hat{y}+y,\epsilon\hat{s}+s,t) - v(x,y,s,t))\, d\hat{x}d\hat{y}d\hat{s}.\end{aligned}$$

By a simple Taylor expansion we have

$$\begin{aligned}& v(\epsilon\hat{x}+x,\epsilon\hat{y}+y,\epsilon\hat{s}+s,t) - v(x,y,s,t) \\ &\quad = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} v(x,y,s,t)\epsilon\hat{x}_j + \sum_{j=1}^{n} \frac{\partial}{\partial y_j} v(x,y,s,t)\epsilon\hat{y}_j \\ &\qquad + \frac{\partial}{\partial s} v(x,y,s,t)\epsilon^2\hat{s} + \frac{1}{2}\sum_{j,i} \frac{\partial^2}{\partial x_j \partial x_i} v(x,y,s,t)\epsilon^2\hat{x}_j\hat{x}_i \\ &\qquad + \frac{1}{2}\sum_{j,i} \frac{\partial^2}{\partial y_j \partial y_i} v(x,y,s,t)\epsilon^2\hat{y}_j\hat{y}_i + \sum_{j,i} \frac{\partial^2}{\partial x_j \partial y_i} v(x,y,s,t)\epsilon^2\hat{x}_j\hat{y}_i \\ &\qquad + \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial s} v(x,y,s,t)\epsilon^3\hat{x}_j\hat{s} + \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j \partial s} v(x,y,s,t)\epsilon^3\hat{y}_j\hat{s} \\ &\qquad + \frac{1}{2}\frac{\partial^2}{\partial s^2} v(x,y,s,t)\epsilon^4\hat{s}^2 + o(\epsilon^{2+\alpha}).\end{aligned}$$

By the fact that $J$ verifies the hypothesis $(\tilde{H})$,

$$\begin{aligned}
&\tilde{L}_\epsilon(v)(x,y,s,t)\\
&\quad = \left[\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + \epsilon^2 \frac{\partial^2}{\partial s^2}\right] v(x,y,s,t)\\
&\qquad + \left[\frac{1}{4}\frac{\partial^2}{\partial s^2}\sum_{j=1}^n \left(x_j^2+y_j^2\right) + \frac{\partial}{\partial s}\sum_{j=1}^n \left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)\right] v(x,y,s,t) + o(\epsilon^\alpha)\\
&\quad = Lv(x,y,s,t) + o(\epsilon^\alpha) = L\tilde{v}(x,y,s,t) + o(\epsilon^\alpha).
\end{aligned}$$

□

*Proof of Theorem 1.5.* In order to prove the theorem by a comparison argument we first look for a supersolution. Let $\overline{w}$ be given by

$$\overline{w}(z,s,t) := K_1\epsilon^\alpha t + K_2\epsilon. \tag{3.6}$$

For $(z,s,t) \in \Omega \times [0,T]$ we have $\tilde{L}_\epsilon(\overline{w})(z,s,t) = 0$, and if $K_1$ is large by Lemma 3.6 and Eq. (3.4):

$$\overline{w}_t(z,s,t) - \tilde{L}_\epsilon(\overline{w})(z,s,t) = K_1\epsilon^\alpha \geq F_\epsilon(z,s,t) = w_t^\epsilon(z,s,t) - \tilde{L}_\epsilon(w^\epsilon)(z,s,t). \tag{3.7}$$

Since

$$G(z,s,t) = o(\epsilon) \quad \text{for } (z,s) \text{ such that } \operatorname{dist}((z,s),\partial\Omega) \leq \epsilon d,$$

choosing $K_2$ large, we obtain

$$\overline{w}(z,s,t) \geq w_t^\epsilon(z,s,t), \tag{3.8}$$

for $(z,s) \notin \Omega$ such that $\operatorname{dist}((z,s),\partial\Omega) \leq \epsilon$ and $t \in [0,T]$. Moreover, it is clear that

$$\overline{w}(z,s,0) = K_2\epsilon > 0 = w_t^\epsilon(z,s,0). \tag{3.9}$$

By (3.7), (3.8) and (3.9) we can apply the comparison result, Corollary 3.4, and conclude that

$$w^\epsilon(z,s,t) \leq \overline{w}(z,s,t) = K_1\epsilon^\alpha t + K_2\epsilon. \tag{3.10}$$

In a similar way we prove that $\underline{w}(x,t) = -K_1\epsilon^\alpha t - K_2\epsilon$ is a subsolution and hence,

$$w^\epsilon(z,s,t) \geq \underline{w}(z,s,t) = -K_1\epsilon^\alpha t - K_2\epsilon. \tag{3.11}$$

Therefore by (3.10), (3.11) and since $0 < \alpha < 1$, we get

$$\sup_{t\in[0,T]} \|v - u^\epsilon\|_{L^\infty(\Omega)} = \sup_{t\in[0,T]} \|w^\epsilon\|_{L^\infty(\Omega)} \leq C(T)\epsilon^\alpha.$$

This proves the theorem. □

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