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Uniqueness of the 1D compressible to incompressible limit

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Abstract. Consider two compressible immiscible fluids in 1D in the isentropic approximation. The first fluid is surrounded and in contact with the second one. As the Mach number of the first fluid vanishes, the coupled dynamics of the two fluids results as the compressible to incompressible limit and is known to satisfy an ODE–PDE system. Below, a characterization of this limit is provided, ensuring its uniqueness.

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1. Introduction

The literature on the compressible to incompressible limit is vast. We refer for instance to the well known results [15,16,18,19], the more recent [4,21], the review [20] and the references therein.

In this paper, following [8], we consider two compressible immiscible fluids and study the limit as one of the two becomes incompressible. A volume of a compressible inviscid fluid, say the *liquid*, is surrounded by another compressible fluid, say the *gas*. Using the Lagrangian formulation, in the isentropic case, we assume that the gas obeys a fixed pressure law $P_g(\tau)$, while for the liquid we assume a one parameter family of pressure laws $P_\kappa(\tau)$ such that $P'_\kappa(\tau) \to -\infty$ as $\kappa \to 0$. The total mass of the liquid is fixed so that in Lagrangian coordinates the liquid and gas phases fill the fixed sets (see Fig. 1)

$$\mathcal{L} =]0, m[\qquad \text{and} \qquad \mathcal{G} = \mathbb{R} \setminus]0, m[\ .$$

For an Eulerian description, see [8].

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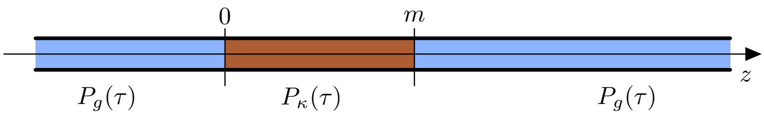

FIGURE 1. In Lagrangian coordinates, the boundaries separating the two fluids are fixed

On $P_g(\tau)$ and $P_\kappa(\tau)$, we require the usual hypotheses:

$$\begin{aligned}
&P_g \in \mathbf{C^4},\ P_g(\tau) > 0,\ P_g'(\tau) < 0,\ P_g''(\tau) > 0;\\
&P_\kappa \in \mathbf{C^4},\ P_\kappa(\tau) > 0,\ P_\kappa'(\tau) < 0,\ P_\kappa''(\tau) > 0;\\
&\text{and the incompressible limit assumption: } P_\kappa'(\tau) \xrightarrow{\kappa\to 0} -\infty\,.
\end{aligned} \tag{1.1}$$

The standard choice $P_g(\tau) = k/\tau^\gamma$ satisfies (1.1) for all $k > 0$ and $\gamma > 0$.

The coupled dynamics of the two fluids is described by the p-system [13, Formula (7.1.11)]

$$\begin{cases}\partial_t\tau - \partial_z v = 0\\ \partial_t v + \partial_z P_\kappa(z,\tau) = 0,\end{cases} \qquad \text{where} \quad P_\kappa(z,\tau) = \begin{cases} P_\kappa(\tau) & \text{for } z \in \mathcal{L}\\ P_g(\tau) & \text{for } z \in \mathcal{G},\end{cases} \tag{1.2}$$

$\tau(t,z)$, $v(t,z)$ being the specific volume and the fluid speed at time t and at the Lagrangian coordinate z.

In Lagrangian coordinates, the conservation of mass and momentum are equivalent to the conservation of τ and v which, in turn, are equivalent along the interfaces $z = 0$ and $z = m$ to the Rankine–Hugoniot conditions for (1.2). Therefore, for a.e. $t \geq 0$,

$$\begin{cases} v(t,0-) = v(t,0+)\\ P_g(\tau(t,0-)) = P_\kappa(\tau(t,0+)),\end{cases} \qquad \begin{cases} v(t,m-) = v(t,m+)\\ P_\kappa(\tau(t,m-)) = P_g(\tau(t,m+)).\end{cases}$$

In other words, pressure and velocity have to be continuous across the interfaces. Hence, the pressure is a natural choice as unknown, rather than the specific volume. Following [8,10,12,14], we introduce the inverse functions of the pressure laws

$$\mathcal{T}_g(p) = P_g^{-1}(p)\,, \quad \mathcal{T}_\kappa(p) = P_\kappa^{-1}(p) \qquad \text{where} \quad \mathcal{T}_\kappa'(p) \xrightarrow{\kappa\to 0} 0\,, \tag{1.3}$$

the last limit being a consequence of (1.1). System (1.2), with (p, v) as unknowns, reads

$$\begin{cases}\partial_t \mathcal{T}_\kappa(z,p) - \partial_z v = 0\\ \partial_t v + \partial_z p = 0\,,\end{cases} \qquad \text{where} \quad \mathcal{T}_\kappa(z,p) = \begin{cases} \mathcal{T}_\kappa(p) & \text{for } z \in \mathcal{L}\\ \mathcal{T}_g(p) & \text{for } z \in \mathcal{G}\,.\end{cases} \tag{1.4}$$

The conditions at the interfaces become continuity requirements on the unknown functions:

$$\begin{cases} v(t,0-) = v(t,0+)\\ p(t,0-) = p(t,0+)\end{cases} \qquad \begin{cases} v(t,m-) = v(t,m+)\\ p(t,m-) = p(t,m+)\end{cases} \qquad \text{for a.e. } t \geq 0\,. \tag{1.5}$$

As in [8], we fix a pressure law P and choose $\mathcal{T} = P^{-1}$, so that

$$\mathcal{T}_\kappa(p) = \mathcal{T}\left(\bar p + \kappa^2 (p - \bar p)\right), \quad \lim_{\kappa\to 0} \mathcal{T}_\kappa(p) = \mathcal{T}(\bar p) = \bar\tau, \tag{1.6}$$

where $\bar\tau$ is the constant specific volume at the incompressible limit and $\bar p = P(\bar\tau)$. For instance, the (modified) Tait equation of state [17, Formula (1)] fits into (1.6) with

$$\mathcal{T}(p) = p^{-1/n} \quad \text{and} \quad \kappa^2 = \frac{n\,\beta_o}{\bar\tau^n}$$

where β_o is the isothermal compressibility, n is a pressure independent parameter and $\beta_o \to 0$ at the incompressible limit.

The main result in [8] states the rigorous convergence (up to a subsequence) at the incompressible limit in the liquid phase of the solutions to (1.4) to solutions to

$$\begin{cases} \begin{cases} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0 \end{cases} & z \in \mathcal{G} \qquad \text{gas;} \\ \dot v_\ell = \dfrac{p(t,0-) - p(t,m+)}{m} & \qquad\qquad \text{liquid;} \\ \begin{cases} v\,(t,0-) = v_\ell(t) \\ v\,(t,m+) = v_\ell(t) \end{cases} & \qquad\qquad \text{interface.} \end{cases} \tag{1.7}$$

The existence of a Lipschitz continuous semigroup generated by (1.7) is proved in [1]. On the other hand, a characterization yielding the uniqueness of solutions to (1.7) is obtained in [9].

In this paper we show that the incompressible limit obtained in [8] satisfies the characterization in [9]. Hence, the solution (p_κ, v_κ) to (1.4) converges as $\kappa \to 0$, the limit being the unique solution to (1.7).

The next Section is devoted to the formal statements, while Sect. 3 contains the technical proofs.

2. Main result

Throughout, we denote by **LC** the set of functions defined on $\mathbb{R} \setminus \,]0, m[$ that are locally constant out of a compact set, i.e., they attain a constant value on $]-\infty, -M]$ and a, possibly different, constant value on $[M, +\infty[$, for a suitable positive M.

Solutions to (1.7) are understood in the following sense, see also also [1, Definition 2.5].

Definition 2.1. ([8, Definition 3.2]) Fix $T > 0$. A *solution* to (1.7) is a pair $((p^*, v^*), v_l)$ of maps $(p^*, v^*) \in \mathbf{C^0}\left([0,T]; (\mathbf{L^1_{loc}} \cap \mathbf{BV})(\mathcal{G}; \mathbb{R}^+ \times \mathbb{R})\right)$ and $v_l \in \mathbf{W^{1,\infty}}([0,T]; \mathbb{R})$ such that:

1. (p^*, v^*) is a weak entropy solution to $\begin{cases} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0 \end{cases}$ in $[0,T] \times \mathcal{G}$;
2. for a.e. $t \in [0,T]$, $\dot v_l(t) = \frac{1}{m}\left(p^*(t,0-) - p^*(t,m+)\right)$;
3. for a.e. $t \in [0,T]$, $v_l(t) = v^*(t,0-) = v^*(t,m+)$.

These solutions are constructed in [8] as limits of solutions to (1.2).

In solutions to (1.2), the propagation speed of waves in the gas region $\mathcal{G}$ is uniformly bounded, independently of κ. Therefore, to prove the uniqueness of solutions to (1.7) obtained as the compressible to incompressible limit, it is sufficient to consider initial data $((\tau_o, v_o), v_{\ell,o})$ such that (τ_o, v_o) is in $\mathbf{LC}$ and $v_{\ell,o} \in \mathbb{R}$.

Given $((\tau, v), v_\ell) \in \mathbf{BV}(\mathcal{G}; \mathbb{R}^2) \times \mathbb{R}$ such that $(\tau, v) \in \mathbf{LC}$, call

$$(\tau_{\pm\infty}, v_{\pm\infty}) = \lim_{x \to \pm\infty} (\tau, v)(x) .$$

Under the transformation

$$U(x) = \begin{bmatrix} \tau(-x) - \tau_{-\infty} \\ v(-x) - v_{-\infty} \\ \tau(x+m) - \tau_{+\infty} \\ v(x+m) - v_{+\infty} \end{bmatrix} \qquad w = \begin{bmatrix} v_\ell - v_{-\infty} \\ v_\ell - v_{+\infty} \end{bmatrix}, \tag{2.1}$$

setting

$$\begin{aligned} f(U) &= \begin{bmatrix} U_2 \\ -P_g\left(U_1 + \tau_{-\infty}\right) \\ -U_4 \\ P_g\left(U_3 + \tau_{+\infty}\right) \end{bmatrix} \\ F(U, w) &= \frac{1}{m} \begin{bmatrix} P_g\left(U_1 + \tau_{-\infty}\right) - P_g\left(U_3 + \tau_{+\infty}\right) \\ P_g\left(U_1 + \tau_{-\infty}\right) - P_g\left(U_3 + \tau_{+\infty}\right) \end{bmatrix} \\ b(U) &= \begin{bmatrix} U_2 \\ U_4 \end{bmatrix} \qquad\qquad g(w) = w \end{aligned} \tag{2.2}$$

the Cauchy Problem

$$\begin{cases} \begin{cases} \partial_t \tau - \partial_z v = 0 \\ \partial_t v + \partial_z P_g(\tau) = 0 \end{cases} & x \in \mathcal{G} \\ \dot{v}_\ell = \dfrac{P_g\left(\tau(t, 0-)\right) - P_g\left(\tau(t, m+)\right)}{m} \\ \begin{cases} v\,(t, 0-) = v_\ell(t) \\ v\,(t, m+) = v_\ell(t) \end{cases} \\ (\tau, v)(0, x) = (\tau_o, v_o)(x) & x \in \mathcal{G} \\ v_\ell(0) = v_{\ell,o} \end{cases} \tag{2.3}$$

is formally equivalent to

$$\begin{cases} \partial_t U(t,x) + \partial_x f\left(U(t,x)\right) = 0 & x \in \mathbb{R}^+ \\ b\left(U(t, 0+)\right) = g\left(w(t)\right) \\ \dot{w}(t) = F\left(U(t, 0+), w(t)\right) \\ U(0, x) = U_o(x) & x \in \mathbb{R}^+ \\ w(0) = w_o \end{cases} \tag{2.4}$$

whose solutions are understood in the following sense:

Definition 2.2. ([9, Definition 1]) A map $(U, w) \in \mathbf{C^0}([0, T]; \mathbf{L^1}(\mathbb{R}^+; \mathbb{R}^4) \times \mathbb{R}^2)$ is a *weak entropy solution* to (2.4) if:

(i) U is a weak entropy solution to $\partial_t U + \partial_x f(U) = 0$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$;
(ii) for almost all $t \in \mathbb{R}^+$, the limit $\lim_{x\to 0+} U(t,x)$ exists and satisfies $b(U(t,0+)) = g(w(t))$;
(iii) w is a Caratheodory solution to $\dot{w} = F(U(t,0+), w)$.

System (2.4) fits in the well posedness theory developed in [9], whose main result is here particularized to the present setting.

In [9] the following hypotheses are considered (here $n = 4$, $l = 2$, $m = 2$):

(**H1**) $f \in \mathbf{C^4}(\mathcal{U};\mathbb{R}^4)$ where $\mathcal{U} \subseteq \mathbb{R}^4$ is open and $0 \in \mathcal{U}$. For all $U \in \mathcal{U}$, the matrix $Df(U)$ admits 4 distinct eigenvalues $\lambda_1(U) < \lambda_2(U) < -c < 0 < c < \lambda_3(U) < \lambda_4(U)$, for a given $c > 0$. Each characteristic field is either genuinely nonlinear or linearly degenerate [3, Definition 5.2].

(**H2**) $b \in \mathbf{C^4}(\mathcal{U};\mathbb{R}^2)$, $b(0) = 0$ and $\det(Db(U)\ R_2(U)) \neq 0$ for all $U \in \mathcal{U}$, the columns of the matrix $R_2(U)$ being the right eigenvectors of $Df(U)$ corresponding to $\lambda_3(U)$ and $\lambda_4(U)$.

(**H3**) $g \in \mathbf{C^{0,1}_{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ and $g(0) = 0$.

(**H4**) $F \in \mathbf{C^{0,1}_{loc}}(\mathcal{U} \times \mathbb{R}^2;\mathbb{R}^2)$.

In the following, we use the usual shock–rarefaction curves $\psi_1, \psi_2, \psi_3, \psi_4$, see [3, Formula (5.38)]. For any U and any i, the $\mathbf{C^2}$ curve $\sigma \to \psi_i(\sigma)(U)$ is the gluing of the ith shock curve and the ith rarefaction curve exiting U and tangent to the ith right eigenvector of the Jacobian matrix $Df(U)$. As in [8] we measure the size of waves through the variation in pressure. We also need to recall the classical Glimm functional and potential adapted to the present case, as in [9, Formula (3)], see also [7, Section 4]. In the case of a piecewise constant function U, they are

$$\begin{aligned} \boldsymbol{V}(U,\bar{w}) &= K \sum_{x \geq 0} \sum_{i=1}^{2} |\sigma_{x,i}| + \sum_{x\geq 0}\sum_{i=3}^{4} |\sigma_{x,i}|, \\ \boldsymbol{Q}(U,\bar{w}) &= \sum_{(\sigma_{x,i},\sigma_{y,j})\in\mathcal{A}} |\sigma_{x,i}\sigma_{y,j}|, \\ \boldsymbol{\Upsilon}(U,\bar{w}) &= \boldsymbol{V}(U,\bar{w}) + H_2\,\boldsymbol{Q}(U,\bar{w}), \end{aligned} \tag{2.5}$$

where $\sigma_{x,i}$ denotes the size of the i-wave in the solution to the Riemann Problem at x with Riemann data $U(x-)$ and $U(x+)$, $\mathcal{A}$ is the usual set of approaching waves, see [3, Chapter 7]. The Riemann problem at the boundary $x = 0$ with Riemann data $U(0+)$ and boundary data $g(\bar{w})$ is solved following [9, Lemma 2] (see also [7, Lemma 4.1]). The constant K, H_2 are defined in [7, Section 4]. As in [5], using [7, Lemma 4.2], the Glimm functional $\boldsymbol{\Upsilon}$ can be uniquely extended in a lower semicontinuous way to all functions with small total variation in $\mathbf{L^1}(\mathbb{R}^+;\mathbb{R}^n)$. Finally define

$$\mathcal{D}_{\bar{\delta}} = \left\{(U,\bar{w}) \in \left(\mathbf{L^1}\cap\mathbf{BV}\right)\left(\mathbb{R}^+;\mathcal{U}\right) \times \mathbb{R}^2 \colon \boldsymbol{\Upsilon}(U,\bar{w}) < \bar{\delta}\right\}. \tag{2.6}$$

The following proposition is a consequence of [9, Theorem 4].

Proposition 2.3. *Let P_g satisfy (1.1). Fix $\tau_{-\infty}, \tau_{+\infty} \in \mathring{\mathbb{R}}^+$. Then, system (2.4) generates a Lipschitz continuous local semigroup (see [9, Definition 3]) S on* *the domain* $\mathcal{D} = \mathcal{D}_{\bar{\delta}}$ *for some positive* $\bar{\delta} > 0$*, uniquely characterized by the following properties:*

(i) *For all* $(U_o, w_o) \in \mathcal{D}$*, the map* $t \to S_t(U_o, w_o)$ *is a solution to* (2.4) *in the sense of* Definition 2.2*.*

(ii) *For all* $(U_o, w_o) \in \mathcal{D}$*, the map* $h \to S_h(U_o, w_o)$ *is first order tangent at* (U_o, w_o) *to* $h \to \mathcal{F}(h)(U_o, w_o)$ *defined by*

$$\begin{aligned} &\mathcal{F}(h)(U_o, w_o) = \left(\overline{S}_h \bar{U}_o, w_o + h\, F(U^\sigma, w_o)\right), \text{ where} \\ &\bar{U}_o = \begin{cases} U^\sigma & x \le 0 \\ U_o(x) & x > 0 \end{cases} \quad \text{and} \quad \begin{cases} b(U^\sigma) = g(w_o) \\ U_o(0+) = \psi_4(\sigma_4) \circ \psi_3(\sigma_3)\,(U^\sigma) \end{cases} \end{aligned} \tag{2.7}$$

$\overline{S}$ *being the Standard Riemann Semigroup generated by* $\partial_t U + \partial_x f(U) = 0$ *and the relations to the right in* (2.7) *being the definition of the boundary state* U^σ *corresponding to the boundary data* $g\,(w_o)$*. First order tangent means that* $\lim_{h\to 0+} \frac{1}{h} d\left(S_h(U_o, w_o), \mathcal{F}_h(U_o, w_o)\right) = 0$*, where* $d\left((U, w), (U', w')\right) = \|U - U'\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R}^2)} + \|w - w'\|$.

(iii) *If an a-priori estimate* $\boldsymbol{\Upsilon}\left(S_t(U_o, w_o)\right) \le \tilde{\delta} < \bar{\delta}$ *hold, then the trajectory* $t \to S_t(U_o, w_o)$ *is defined and solves* (2.4) *for all* $t \ge 0$*.*

Moreover, for a suitable constant $\delta > 0$*:*

$$\begin{aligned} \mathcal{D}_{\bar{\delta}} \supseteq \big\{ (U, w) \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R}^+; \mathbb{R}^4) \times \mathbb{R}^2 \colon \\ \mathrm{TV}(U) + \|b\left(U(0+)\right) - g(w)\| < \delta \big\}. \end{aligned} \tag{2.8}$$

For more details on first order tangency in metric space we defer to [2,6]. We are now ready to state the main result of this paper.

Theorem 2.4. *Let* $t \to \left((\tau, v), v_\ell\right)(t)$ *be a solution to* (1.7) *obtained as limit for* $\kappa \to 0$ *of solutions to* (1.2)*, with an initial datum in* **LC** *and satisfying for all* $t \in \mathbb{R}_+$

$$\mathrm{TV}\left((\tau, v)(t); \mathcal{G}\right) + \left\| \begin{bmatrix} v(t, 0-) - v_\ell(t) \\ v(t, m+) - v_\ell(t) \end{bmatrix} \right\| < \delta \tag{2.9}$$

with δ *as in* (2.8)*. Correspondingly, define* $t \to (U, w)(t)$ *as in* (2.1)*. Then,*

1. *For all* $t \in \mathbb{R}_+$*, the map* $t \to (U, w)(t)$ *coincides with an orbit of the semigroup* S *defined in Proposition* 2.3*.*
2. *The semigroup* S *is defined globally in time for all initial data with sufficiently small total variation.*

Remark 2.5. If the initial datum satisfies (2.9) with a possibly smaller $\delta^* > 0$, then the solution satisfies (2.9) for all times $t \ge 0$ (see [8]). Hence, requirement (2.9) is indeed a condition on the initial datum $\left((\tau, v), v_\ell\right)(0)$.

In the above statement, as well as below, we use the obvious notation

$$\mathrm{TV}\left((\tau, v); \mathcal{G}\right) = \mathrm{TV}\left((\tau, v); \left]-\infty, 0\right]\right) + \mathrm{TV}\left((\tau, v); \left[m, +\infty\right[\right).$$

3. Technical proofs

Proof of Proposition 2.3. On the basis of (2.2) and with the help of (1.1), we verify that (2.4) satisfies the assumptions (**H1**)–(**H4**) which are the hypotheses of [9, Theorem 4]. Clearly, f is of class $\mathbf{C^4}$ by (1.1). The strict hyperbolicity of (2.4) can easily be recovered through a rescaling of the space variable, since the different p-systems in (1.7) ($x < 0$ and $x > m$) interact only through the boundary, see [11, Lemma 4.1]. Again (1.1) ensures that two characteristic speeds are negative, while the other two are positive and that all the four fields are genuinely nonlinear. This concludes the proof of (**H1**).

Concerning (**H2**), b is clearly of class $\mathbf{C^4}$ and $b(0) = 0$. With standard notation, we have:

$$\lambda_1(U) = -\sqrt{-P_g'\left(U_1 + \tau_{-\infty}\right)}\ \lambda_2(U) = -\sqrt{-P_g'\left(U_3 + \tau_{+\infty}\right)}$$
$$\lambda_3(U) = \sqrt{-P_g'\left(U_1 + \tau_{-\infty}\right)}\quad \lambda_4(U) = \sqrt{-P_g'\left(U_3 + \tau_{+\infty}\right)}$$
$$r_1 = \begin{bmatrix} 1 \\ \lambda_1(U) \\ 0 \\ 0 \end{bmatrix}\ r_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\lambda_2(U) \end{bmatrix}\ r_3 = \begin{bmatrix} 1 \\ \lambda_3(U) \\ 0 \\ 0 \end{bmatrix}\ r_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\lambda_4(U) \end{bmatrix}.$$

Therefore,

$$\det\left[Db(U)\left[r_3(U) \quad r_4(U)\right]\right] = \det\begin{bmatrix} \lambda_3(U) & 0 \\ 0 & -\lambda_4(U) \end{bmatrix} = -\lambda_3(U)\,\lambda_4(U)$$

and the latter expression above is non zero by (1.1).

Assumptions (**H3**) and (**H4**) are immediate by (2.2) and (1.1).

We have shown that the hypotheses of [9, Theorem 4] are satisfied, therefore there exists a unique Lipschitz continuous local semigroup S defined on a domain $\mathcal{D}_{\bar\delta}$ for some $\bar\delta > 0$ enjoying properties (i)–(iii). The inclusion (2.8) follows from (2.5) and (2.6). □

Proof of Theorem 2.4. Given $t \to ((\tau, v), v_\ell)(t)$, define $t \to (U, w)(t)$ by means of (2.1). Since

$$\begin{aligned} &\mathrm{TV}\left(U(t)\right) + \left\|b\left(U(t, 0+)\right) - g\left(w(t)\right)\right\| \\ &\quad = \mathrm{TV}\left((\tau, v)(t); \mathcal{G}\right) + \left\|\begin{bmatrix} v(t, 0-) - v_\ell(t) \\ v(t, m+) - v_\ell(t) \end{bmatrix}\right\| \end{aligned}$$

thanks to (2.8) we obtain that for all $t \in \mathbb{R}_+$, $(U, w)(t) \in \mathcal{D}_{\tilde\delta} \subset \mathcal{D}$, $\mathcal{D}$ being the domain defined in Proposition 2.3.

For $\kappa > 0$ and $\varepsilon > 0$, call $(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})$ the corresponding piecewise constant wave front tracking approximate solutions to (1.4), as defined in [8, Section 4].

We recall here some properties enjoyed by the approximations $(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})$ that will be used in the proof.

(a) By [8, Proof of Theorem 3.3]: $\lim_{\varepsilon\to 0}(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(t) = (p^\kappa, v^\kappa)(t)$ for all $t \geq 0$ in $\mathbf{L^1_{loc}}(\mathbb{R}; \mathbb{R}^2)$, where (p^κ, v^κ) is a weak solution to (1.4), satisfies (1.5) and its restrictions to $\mathcal{L}$ and $\mathcal{G}$ is a weak *entropy* solutions to (1.4), see [8, Definition 3.1] for more details.

(b) Up to subsequences (to simplify notations we will omit the introduction of new indexes for subesequences), in the limit $\kappa \to 0$, we have that

$$\begin{aligned} \lim_{\kappa\to 0}(p^\kappa, v^\kappa)(t,\cdot) &= (p,v)(t,\cdot) && \text{for all } t \geq 0 \text{ in } \mathbf{L^1_{loc}}(\mathcal{G};\mathbb{R}^2) \\ \lim_{\kappa\to 0} v^\kappa(t,\cdot) &= v_\ell(t) && \text{for all } t \geq 0 \text{ in } \mathbf{L^1}(\mathcal{L};\mathbb{R}). \end{aligned}$$

(c) The discontinuities of $(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})$ are localized along straight lines, the two states at the sides of the discontinuities are connected by a Lax curve for system (1.4) (the pressure law for the construction of Lax curves is $\mathcal{T}_g$ in the gas region and $\mathcal{T}_\kappa$ in the liquid region).

(d) To avoid the accumulation of interactions points, in the strips $-\varepsilon^2 < z < \varepsilon^2$, $m - \varepsilon^2 < z < m + \varepsilon^2$, all waves travel with speed equal either to 1 or to -1, while outside these strips, they travel with the usual speed given to the discontinuities of wave front tracking approximate solutions ([3, Section 7.2], [8, Section 4]).

(e) By (1.1), all the discontinuities travel with speeds different from zero, in particular, $(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})$ is continuous along the vertical lines $z = 0$ and $z = m$ with the exception of a finite number of points in which discontinuities may cross these lines.

Introduce now

$$\begin{aligned} \overline{v}^{\kappa,\varepsilon}(t) &= \frac{1}{m}\int_0^m v^{\kappa,\varepsilon}(t,z)\,\mathrm{d}z \\ u^{\kappa,\varepsilon}(t) &= \left(\left(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon}\right)(s)_{|\mathcal{G}},\ \overline{v}^{\kappa,\varepsilon}(t) \right). \end{aligned} \tag{3.1}$$

Because of (a) and (b), we have, for all $t \geq 0$ (the second limit holds up to subsequences):

$$\begin{aligned} u^{\kappa,\varepsilon}(t) \to u^\kappa(t) &= \left((p^\kappa, v^\kappa)(t)_{|\mathcal{G}},\ \frac{1}{m}\int_0^m v^\kappa(t,z)\,\mathrm{d}z \right) && \text{as } \varepsilon \to 0, \\ u^\kappa(t) \to\ \ u(t) &= \left((p,v), v_\ell\right)(t) && \text{as } \kappa \to 0. \end{aligned} \tag{3.2}$$

Following (2.1) and (3.1), introduce the variables

$$U^{\kappa,\varepsilon}(t,x) = \begin{bmatrix} \mathcal{T}_g\left(p^{\kappa,\varepsilon}(t,-x)\right) - \tau_{-\infty} \\ v^{\kappa,\varepsilon}(t,-x) - v_{-\infty} \\ \mathcal{T}_g\left(p^{\kappa,\varepsilon}(t,x+m)\right) - \tau_{+\infty} \\ v^{\kappa,\varepsilon}(t,x+m) - v_{+\infty} \end{bmatrix} \quad w^{\kappa,\varepsilon}(t) = \begin{bmatrix} \overline{v}^{\kappa,\varepsilon}(t) - v_{-\infty} \\ \overline{v}^{\kappa,\varepsilon}(t) - v_{+\infty} \end{bmatrix} \tag{3.3}$$

and the distance

$$d\left((U,w), \left(\tilde{U},\tilde{w}\right)\right) = \left\|\tilde{U} - U\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R}^4)} + \left\|\tilde{w} - w\right\|.$$

By the convergences (3.2), the definition (3.3) and the continuity of S_t

$$\begin{aligned} & d\left((U,w)(t), S_t\left((U,w)(0)\right)\right) \\ & \quad \leq \lim_{\kappa\to 0}\lim_{\varepsilon\to 0} d\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(t), S_t\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(0)\right)\right). \end{aligned} \tag{3.4}$$

By [3, Theorem 2.9], denoting by L a Lipschitz constant of S_t,

$$\begin{aligned} & d\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(t), S_t\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(0)\right)\right) \\ & \quad \le L \int_0^t \liminf_{h\to 0} \frac{1}{h}\, d\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s+h), S_h\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s)\right)\right) \mathrm{d}s \\ & \quad = L \int_0^t \liminf_{h\to 0} \frac{1}{h}\, d\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s+h), \mathcal{F}(h)\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s)\right)\right) \mathrm{d}s \end{aligned} \tag{3.5}$$

where $\mathcal{F}$ is the local flow defined in Proposition 2.3 using definitions (2.2).

Choose a time s at which, in the wave front tracking approximate solution $(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})$, no interaction takes place and choose h sufficiently small so that in the time interval $[s, s+h]$ no interaction takes place and no wave hits any of the lines $z = \pm\varepsilon^2$, $z = 0$, $z = m \pm \varepsilon^2$ and $z = m$. In particular $(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s', z)$ is constant on a neighborhood of $z = 0,\ s \le s' \le s+h$ and of $z = m,\ s \le s' \le s+h$.

By construction, the last term in the integrand above is

$$\begin{aligned} & d\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s+h), \mathcal{F}(h)\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s)\right)\right) \\ & \quad = \left\| U^{\kappa,\varepsilon}(s+h) - \bar{S}_h\left(\bar{U}^{\kappa,\varepsilon}(s)\right)\right\|_{\mathbf{L}^1(\mathbb{R}_+;\mathbb{R}^4)} \\ & \qquad + \left\| w^{\kappa,\varepsilon}(s+h) - \left[w^{\kappa,\varepsilon}(s) + hF\left(U^\sigma, w^{\kappa,\varepsilon}(s)\right)\right]\right\| \end{aligned} \tag{3.6}$$

where

$$\bar{U}^{\kappa,\varepsilon}(s,x) = \begin{cases} U^\sigma & x \le 0 \\ U^{\kappa,\varepsilon}(s,x) & x > 0 \end{cases} \quad \text{and} \quad U^\sigma = \begin{bmatrix} \mathcal{T}_g\left(p_0^\sigma\right) - \tau_{-\infty} \\ v_0^\sigma - v_{-\infty} \\ \mathcal{T}_g\left(p_m^\sigma\right) - \tau_{+\infty} \\ v_m^\sigma - v_{+\infty} \end{bmatrix}$$

is the unique state satisfying $b(U^\sigma) = g\left(w^{\kappa,\varepsilon}(s)\right)$ that can be connected to $U^{\kappa,\varepsilon}(s, 0+)$ by means of Lax waves with positive speed (see (2.7)). This, translated in the (p, v) variables, means that $v_0^\sigma = v_{-\infty} + w_1^{\kappa,\varepsilon}(s) = \overline{v}^{\kappa,\varepsilon}(s)$, $v_m^\sigma = v_{+\infty} + w_2^{\kappa,\varepsilon}(s) = \overline{v}^{\kappa,\varepsilon}(s)$ and the pressure p_0^σ, respectively p_m^σ, is such that the Riemann Problem

$$\begin{cases} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0 \\ (p,v)(0,x) = \begin{cases} (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s, 0-) & x < 0 \\ (p_0^\sigma, \overline{v}^{\kappa,\varepsilon})(s) & x > 0, \end{cases} \end{cases}$$

$$\text{resp.} \quad \begin{cases} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0 \\ (p,v)(0,x) = \begin{cases} (p_m^\sigma, \overline{v}^{\kappa,\varepsilon})(s) & x < 0 \\ (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s, m+) & x > 0, \end{cases} \end{cases}$$

is solved by waves with negative, respectively positive, speed ($\overline{v}^{k,\varepsilon}$ is defined in (3.1)). Since $p_0^\sigma = p^{\kappa,\varepsilon}(s, 0-)$ whenever $\overline{v}^{\kappa\varepsilon}(s, 0-) = v^{\kappa,\varepsilon}(s)$, the Lipschitz continuity of the solution to the Riemann problem at the boundary, [7, Lemma 4.1], implies

$$\begin{aligned} \left|p^{\kappa,\varepsilon}(s,0) - p_0^\sigma\right| &= \mathcal{O}(1)\ \left|v^{\kappa,\varepsilon}(s,0) - \overline{v}^{\kappa,\varepsilon}(s)\right|, \\ \left|p^{\kappa,\varepsilon}(s,m) - p_m^\sigma\right| &= \mathcal{O}(1)\ \left|v^{\kappa,\varepsilon}(s,m) - \overline{v}^{\kappa,\varepsilon}(s)\right|. \end{aligned} \tag{3.7}$$

Introduce

$$(\bar p,\bar v)(z) = \begin{cases} (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s,z) & z<0 \\ (p_0^\sigma, \overline{v}^{k,\varepsilon})(s) & z\in [0,m/2[\\ (p_m^\sigma, \overline{v}^{k,\varepsilon})(s) & z\in [m/2,m] \\ (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s,z) & z>m \end{cases}$$

Call Σ the Standard Riemann Semigroup [3, Chapter 9] generated by the p-system

$$\begin{cases} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0 \end{cases}$$

for z varying on all the real line. Observe that, if L_g is a Lipschitz constant for $\mathcal{T}_g$, the first addend in (3.6) reads

$$\begin{aligned} &\left\| U^{\kappa,\varepsilon}(s+h) - \bar S_h\left(\bar U^{\kappa,\varepsilon}(s)\right)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R}^4)} \\ &= \int_{\mathcal{G}} \left\| \left(\mathcal{T}_g\left(p^{\kappa,\varepsilon}\right), v^{\kappa,\varepsilon}\right)(s+h,z) - \left(\mathcal{T}_g\left(\Sigma_{h,1}(\bar p,\bar v)\right), \Sigma_{h,2}(\bar p,\bar v)\right)(z)\right\| \mathrm{d}z \\ &\le L_g \int_{\mathcal{G}} \left\| \left(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon}\right)(s+h,z) - \left(\Sigma_h(\bar p,\bar v)\right)(z)\right\| \mathrm{d}z\,. \end{aligned} \tag{3.8}$$

We now continue to estimate the right hand side in (3.8) limited to $]-\infty,0]$. Let $z_1, z_2, \ldots$ be the points of jump of the map $z \to (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s,z)$. Denote by $\widehat\lambda$ an upper bound for the characteristic speeds in the gas phase. Then, we have

$$\int_{-\infty}^0 \left\| \left(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon}\right)(s+h,z) - \left(\Sigma_h(\bar p,\bar v)\right)(z)\right\| \mathrm{d}z$$

$$= \sum_{z_i<-\varepsilon^2} \int_{z_i-\widehat\lambda h}^{z_i+\widehat\lambda h} \left\| (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s+h,z) - \left(\Sigma_h\left(\left(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon}\right)(s)\right)\right)(z)\right\| \mathrm{d}z \tag{3.9}$$

$$+ \sum_{-\varepsilon^2<z_i<0} \int_{z_i-\widehat\lambda h}^{z_i+\widehat\lambda h} \left\| \left(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon}\right)(s+h,z) - \left(\Sigma_h\left(\left(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon}\right)(s)\right)\right)(z)\right\| \mathrm{d}z \tag{3.10}$$

$$+ \int_{-\widehat\lambda h}^0 \left\| \left(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon}\right)(s+h,z) - \Sigma_h\left(\left(\bar p,\bar v\right)(s)\right)\right\| \mathrm{d}z\,. \tag{3.11}$$

A standard procedure yields the estimate of (3.9) by means of [3, (ii) in Lemma 9.1], so that

$$[(3.9)] = \mathcal{O}(1)\, \varepsilon\, h\ \mathrm{TV}(p^{\kappa,\varepsilon}(s);\,]-\infty, -\varepsilon^2[\,)\,.$$

By property (d), all waves in the strip $\left]-\varepsilon^2, 0\right[$ have speed ± 1, hence by [3, (i) in Lemma 9.1] we have

$$[(3.10)] = \mathcal{O}(1)\ h\ \mathrm{TV}(p^{\kappa,\varepsilon}(s);\,]-\varepsilon^2, 0[\,).$$

Consider now (3.11). We use [7, Point 2) in Theorem 2.2] to estimate the difference between $(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})$ and $\Sigma_h(\bar p,\bar v)$ that are solutions, respectively, to the two initial-boundary value problems

$$\begin{cases} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0 \\ (p,v)(0,z) = (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s,0) \\ v(t,0) = v^{\kappa,\varepsilon}(s,0) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0 \\ (p,v)(0,z) = (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})(s,0) \\ v(t,0) = \overline{v}^{\kappa,\varepsilon}(s) \end{cases}$$

with the mean value $\overline{v}^{\kappa,\varepsilon}$ as defined in (3.1). Then, we apply [8, Proposition 4.9] to obtain

$$\begin{aligned} [(3.11)] &\le \mathcal{O}(1)\ \widehat{\lambda}\, h\, |v^{\kappa,\varepsilon}(s,0) - \overline{v}^{\kappa,\varepsilon}(s)| && \text{by continuity with respect to the boundary data, see [7, Point 2) in Theorem 2.2]} \\ &\le \mathcal{O}(1)\ \widehat{\lambda}\, h\ \mathrm{TV}(v^{\kappa,\varepsilon}(s);\mathcal{L}) && \text{by (3.1)} \\ &\le \mathcal{O}(1)\ h\,\kappa && \text{by [8, Proposition 4.9].} \end{aligned}$$

Entirely analogous estimates can be applied to bound the similar terms on $[m,+\infty[$. We thus continue (3.8) as follows:

$$\begin{aligned} &\left\| U^{\kappa,\varepsilon}(s+h) - \bar{S}_h\left(\bar{U}^{\kappa,\varepsilon}(s)\right)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R}^4)} \\ &\quad\le \mathcal{O}(1)\ h\,\varepsilon\ \mathrm{TV}(p^{\kappa,\varepsilon}(s);\,]-\infty,-\varepsilon^2[\cup]m+\varepsilon^2,+\infty[\,) \\ &\qquad + \mathcal{O}(1)\ h\ \mathrm{TV}(p^{\kappa,\varepsilon}(s);\,]-\varepsilon^2,0[\cup]m,m+\varepsilon^2[\,) \\ &\qquad + \mathcal{O}(1)\ h\,\kappa\,. \end{aligned}$$

We pass now to the second addend in (3.6). Inequalities (3.7) and [8, Proposition 4.9] imply

$$|p^{\kappa,\varepsilon}(s,0) - p_0^\sigma|, \quad |p^{\kappa,\varepsilon}(s,m) - p_m^\sigma| = \mathcal{O}(1)\,\kappa.$$

Using the fact that $p^{\kappa,\varepsilon}(\sigma,z)$ is constant on $z=0$, $\sigma\in[s,s+h]$ and on $z=m$, $\sigma\in[s,s+h]$ and using a telescopic expansion, we compute

$$\begin{aligned} &\|w^{\kappa,\varepsilon}(s+h) - [w^{\kappa,\varepsilon}(s) + hF\left(U^\sigma, w^{\kappa,\varepsilon}(s)\right)]\| \qquad (3.12)\\ &= \left\| \begin{bmatrix} \overline{v}^{\kappa,\varepsilon}(s+h) - \overline{v}^{\kappa,\varepsilon}(s) - hF_1\left(U^\sigma, w^{k,\varepsilon}(s)\right) \\ \overline{v}^{\kappa,\varepsilon}(s+h) - \overline{v}^{\kappa,\varepsilon}(s) - hF_2\left(U^\sigma, w^{k,\varepsilon}(s)\right) \end{bmatrix} \right\| \\ &= \sqrt{2}\left| \overline{v}^{\kappa,\varepsilon}(s+h) - \overline{v}^{\kappa,\varepsilon}(s) - \frac{1}{m}\,h\,(p_0^\sigma - p_m^\sigma)\right| \\ &= \frac{\sqrt{2}}{m}\left| \int_{\mathcal{L}} v^{\kappa,\varepsilon}(s+h,z)\,\mathrm{d}z - \int_{\mathcal{L}} v^{\kappa,\varepsilon}(s,z)\,\mathrm{d}z - \int_s^{s+h}\Big(p^{\kappa,\varepsilon}(\sigma,0) \right. \\ &\quad \left. - p^{\kappa,\varepsilon}(\sigma,m)\Big)\,\mathrm{d}\sigma \right| + \frac{\sqrt{2}\,h}{m}\left|(p_0^\sigma - p_m^\sigma) - (p^{\kappa,\varepsilon}(s,0) - p^{\kappa,\varepsilon}(s,m))\right| \\ &\le \frac{\sqrt{2}}{m}\left| \int_s^{s+h} \frac{\mathrm{d}}{\mathrm{d}\sigma}\int_{\mathcal{L}} v^{\kappa,\varepsilon}(\sigma,z)\,\mathrm{d}z\,\mathrm{d}\sigma - \int_s^{s+h}(p^{\kappa,\varepsilon}(\sigma,0) - p^{\kappa,\varepsilon}(\sigma,m))\,\mathrm{d}\sigma\right| \\ &\quad + \mathcal{O}(1)\ h\,\kappa \\ &\le \frac{\sqrt{2}}{m}\left| \int_s^{s+h}\left[\sum_{z_i\in\,]0,m[} (v^{\kappa,\varepsilon}(\sigma,z_i-) - v^{\kappa,\varepsilon}(\sigma,z_i+))\,\dot{z}_i \right.\right. \end{aligned}$$

$$
\begin{aligned}
& + \sum_{z_i \in \,]0,m[} \left(p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-)\right)\Bigg] \mathrm{d}\sigma \Bigg| + \mathcal{O}(1)\ h\,\kappa \\
\leq\ & \frac{\sqrt{2}}{m} \int_s^{s+h} \sum_{z_i \in \,]0,m[} \Big| \left(v^{\kappa,\varepsilon}(\sigma, z_i-) - v^{\kappa,\varepsilon}(\sigma, z_i+)\right) \dot z_i \\
& + \left(p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-)\right) \Big| \mathrm{d}\sigma + \mathcal{O}(1)\ h\,\kappa
\end{aligned}
$$

where z_i are the discontinuity points of $(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})$. We estimate the integral in the latter term above in different ways, depending on the location of z_i:

$$
\begin{aligned}
& \int_s^{s+h} \sum_{z_i \in \,]0,\varepsilon^2[\,\cup\,]m-\varepsilon^2,m[} \left| \left(v^{\kappa,\varepsilon}(\sigma, z_i-) - v^{\kappa,\varepsilon}(\sigma, z_i+)\right) \dot z_i \right. \\
& \qquad \left. + \left(p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-)\right) \right| \mathrm{d}\sigma \\
\leq\ & \int_s^{s+h} \sum_{z_i \in \,]0,\varepsilon^2[\,\cup\,]m-\varepsilon^2,m[} \left| v^{\kappa,\varepsilon}(\sigma, z_i-) - v^{\kappa,\varepsilon}(\sigma, z_i+) \right| \\
& \qquad + \left| p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-) \right| \mathrm{d}\sigma \\
=\ & \mathcal{O}(1)\ h\ \mathrm{TV}\left(p^{\kappa,\varepsilon}(s);\,]0,\varepsilon^2[\,\cup\,]m-\varepsilon^2,m[\right)
\end{aligned}
$$

since, by property (d) in $]0,\varepsilon^2[\,\cup\,]m-\varepsilon^2,m[$ we have $|\dot z_i| = 1$ and because in the liquid, the variation of $v^{\kappa,\varepsilon}$ is controlled by the variation of $\kappa p^{\kappa,\varepsilon}$ (see [8, (4.6)]). To bound the remaining terms in (3.12), observe that, in $]\varepsilon^2, m-\varepsilon^2[$, $(p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})$ is a standard wave front tracking approximate solution to (1.4) whose Lax curves are described in [8, Lemma 4.1]. Hence let $z_i \in \left]\varepsilon^2, m-\varepsilon^2\right[$. First assume that the jump at z_i is solved by a 2-rarefaction. Since $\left|\dot z_i - \lambda_2^\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i-), v^{\kappa,\varepsilon}(\sigma, z_i-)\right)\right| \leq \varepsilon$ (as usual to avoid multiple interactions, the speed of the waves can be slightly different from the right shock/approximate rarefaction speed), using the explicit expression of the Lax curves in [8, Section 4, Lemma 4.1 and Formula (4.3)], we compute

$$
\begin{aligned}
& \left| -\left(v^{\kappa,\varepsilon}(\sigma, z_i+) - v^{\kappa,\varepsilon}(\sigma, z_i-)\right) \dot z_i + \left(p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-)\right) \right| \\
& \leq \varepsilon \left| v^{\kappa,\varepsilon}(\sigma, z_i+) - v^{\kappa,\varepsilon}(\sigma, z_i-) \right| \\
& \quad + \Bigg| -\frac{1}{\kappa} \sqrt{-\frac{1}{\mathcal{T}'\left(\Pi_\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i-)\right)\right)}}\ \kappa\ \left(p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-)\right) \\
& \qquad \times F\left(\Pi_\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i-)\right), \Pi_\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i+)\right)\right) \\
& \quad + \left(p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-)\right) \Bigg| \\
& \leq \varepsilon \left| v^{\kappa,\varepsilon}(\sigma, z_i+) - v^{\kappa,\varepsilon}(\sigma, z_i-) \right| \\
& \quad + \frac{\left| p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-) \right|}{F\left(\Pi_\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i-)\right), \Pi_\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i-)\right)\right)} \\
& \quad \times \left| -F\left(\Pi_\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i-)\right), \Pi_\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i+)\right)\right) \right. \\
& \qquad \left. + F\left(\Pi_\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i-)\right), \Pi_\kappa\left(p^{\kappa,\varepsilon}(\sigma, z_i-)\right)\right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon \left| v^{\kappa,\varepsilon}(\sigma, z_i+) - v^{\kappa,\varepsilon}(\sigma, z_i-) \right| \\
&\quad + \mathcal{O}(1) \left| p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-) \right| \\
&\quad \times \left| \Pi_\kappa \left(p^{\kappa,\varepsilon}(\sigma, z_i-) \right) - \Pi_\kappa \left(p^{\kappa,\varepsilon}(\sigma, z_i-) \right) \right| \\
&= \varepsilon \left| v^{\kappa,\varepsilon}(\sigma, z_i+) - v^{\kappa,\varepsilon}(\sigma, z_i-) \right| + \mathcal{O}(1)\ \kappa^2 \left| p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-) \right|^2 \\
&= \varepsilon \left| v^{\kappa,\varepsilon}(\sigma, z_i+) - v^{\kappa,\varepsilon}(\sigma, z_i-) \right| + \mathcal{O}(1)\ \kappa^2\, \varepsilon \left| p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-) \right|.
\end{aligned}
$$

When dealing with a 2-shock we obtain the simpler estimate

$$
\begin{aligned}
&\left| - \left(v^{\kappa,\varepsilon}(\sigma, z_i+) - v^{\kappa,\varepsilon}(\sigma, z_i-) \right) \dot{z}_i + \left(p^{\kappa,\varepsilon}(\sigma, z_i+) - p^{\kappa,\varepsilon}(\sigma, z_i-) \right) \right| \\
&\quad \leq \varepsilon \left| v^{\kappa,\varepsilon}(\sigma, z_i+) - v^{\kappa,\varepsilon}(\sigma, z_i-) \right|
\end{aligned}
$$

while the cases of waves of the first family are entirely analogous.

Summarizing:

$$
\begin{aligned}
[(3.12)] \leq{}& \mathcal{O}(1)\ h\ \mathrm{TV}\left(p^{\kappa,\varepsilon}(s);\ ]0, \varepsilon^2[\cup]m - \varepsilon^2, m[\right) \\
&+ \mathcal{O}(1)\ h\,\varepsilon\ \left(\mathrm{TV}\left(v^{\kappa,\varepsilon}(s);\ ]\varepsilon^2, m - \varepsilon^2[\right) \right. \\
&\left. + \mathrm{TV}\left(p^{\kappa,\varepsilon}(s);\ ]\varepsilon^2, m - \varepsilon^2[\right) \right) + \mathcal{O}(1)\ h\,\kappa
\end{aligned}
$$

By [8, Formula (4.32) in Proposition 4.9] we finally obtain,

$$
\begin{aligned}
&d\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s+h), \mathcal{F}(h)\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(s) \right) \right) \\
&\quad \leq \mathcal{O}(1)\ h\,\varepsilon\ \mathrm{TV}(p^{\kappa,\varepsilon}(s);\ ]-\infty, -\varepsilon^2[\cup]m + \varepsilon^2, +\infty[\,) \\
&\quad\quad + \mathcal{O}(1)\ h\ \mathrm{TV}(p^{\kappa,\varepsilon}(s);\ ]-\varepsilon^2, 0[\cup]m, m + \varepsilon^2[\,) \\
&\quad\quad + \mathcal{O}(1)\ h\,\kappa \\
&\quad\quad + \mathcal{O}(1)\ h\ \mathrm{TV}\left(p^{\kappa,\varepsilon}(s);\ ]0, \varepsilon^2[\cup]m - \varepsilon^2, m[\right) \\
&\quad\quad + \mathcal{O}(1)\ h\,\varepsilon\ \left(\mathrm{TV}\left(v^{\kappa,\varepsilon}(s);\ ]\varepsilon^2, m - \varepsilon^2[\right) + \mathrm{TV}\left(p^{\kappa,\varepsilon}(s);\ ]\varepsilon^2, m - \varepsilon^2[\right) \right) \\
&\quad = \mathcal{O}(1)\ h\left(\varepsilon + \kappa + \mathrm{TV}\left(p^{\kappa,\varepsilon}(s);\ [-\varepsilon^2, \varepsilon^2[\cup]m - \varepsilon^2, m + \varepsilon^2[\right) \right)
\end{aligned}
$$

whence, by (3.5)

$$
\begin{aligned}
&d\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(t), S_t\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(0) \right) \right) \\
&\quad \leq \mathcal{O}(1) \int_0^t \left(\varepsilon + \kappa + \mathrm{TV}\left(p^{\kappa,\varepsilon}(s);\ [-\varepsilon^2, \varepsilon^2[\cup]m - \varepsilon^2, m + \varepsilon^2[\right) \right) \mathrm{d}s
\end{aligned}
$$

Changing the order of integration and using [8, Formula (4.33) in Proposition 4.9], we get

$$
\begin{aligned}
&\int_0^t \mathrm{TV}\left(p^{\kappa,\varepsilon}(s);\ [-\varepsilon^2, \varepsilon^2[\cup]m - \varepsilon^2, m + \varepsilon^2[\right) \mathrm{d}s \\
&\quad = \int_{[-\varepsilon^2, \varepsilon^2[\cup]m - \varepsilon^2, m + \varepsilon^2[} \mathrm{TV}\left(p^{\kappa,\varepsilon}(\cdot, z); [0, t] \right) \mathrm{d}z \\
&\quad = \mathcal{O}(1)\ \frac{\varepsilon^2}{\kappa}
\end{aligned}
$$

so that

$$
d\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(t), S_t\left((U^{\kappa,\varepsilon}, w^{\kappa,\varepsilon})(0) \right) \right) = \mathcal{O}(1) \left((\varepsilon + \kappa) t + \frac{\varepsilon^2}{\kappa} \right)
$$

Using (3.4), the proof of 1. is completed.

Properties (2.9) and (2.8) imply $(U, w)(t) \in \mathcal{D}_{\tilde{\delta}}$ for all $t \geq 0$, but we have just proved that $S_t((U, w)(0)) = (U, w)(t)$, therefore *iii)* of Proposition 2.3 concludes the proof. □

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