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A mathematical model for piracy control through police response

Dedicated to Professor Alberto Bressan on the occasion of his 60th anniversary.

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Abstract. We introduce a model describing the dynamics and interactions of three populations of ships (pirates ships, commercial cargos, and police watercrafts) in a marine region. We establish well-posedness of the coupled ODE-PDEs system describing the ships dynamics and we discuss a related optimal control problem.

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1. Introduction and description of the model

In the last few years several mathematical models describing criminality and police response have been introduced. For an overview, we refer to the review paper [12], which distinguishes between models motivated by economic theories (see, for instance [4]) and social models (see, for instance, [5–7, 11, 16, 17, 20, 21]).

In this note we take inspiration from a model introduced in [21] to describe the criminal behavior in an urban contest. Here we want to describe

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the behavior and interactions of three populations of ships: pirates ships, commercial cargos, and police watercrafts. The ships move in a given bounded marine region, which is modeled by a set $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, that is open, bounded, connected, and with smooth boundary. We use a macroscopic approach to describe the dynamics of the pirates and commercial ships and a microscopic approach for the police watercrafts. In the following, we assume that there are exactly M police watercrafts. Also, we denote by

$$\rho : [0, \infty) \times \Omega \longrightarrow \mathbb{R}, \quad A : [0, \infty) \times \Omega \longrightarrow \mathbb{R}, \quad \vec{d} : [0, \infty) \longrightarrow \Omega^M,$$

the density of the pirates ships, the density of the cargo ships, and the position of the M police watercrafts, respectively. In the following paragraphs we separately describe the evolution equations for ρ, A and $\vec{d}$. Moreover, we explicitly point out that the main issue this paper addresses are the open see dynamics. For this reason we augment our equations with the simplest possible boundary conditions, which allow us to avoid the non trivial challenges arising when balance laws satisfy non trivial boundary conditions, see for instance [3,9] for a related discussion. However, from the modeling viewpoint, it could be interesting to consider other boundary conditions, describing the evolution of the dynamics near to the coasts, see for instance [1].

Dynamics of pirate ships. In [21], the authors model the evolution of the criminal density by the diffusion equation

$$\partial_t \rho = \Delta \rho - \operatorname{div}\left(\rho \nabla \ln(A(t,\cdot) * \mathcal{K} + \varepsilon)\right) - f(x, \vec{d})\rho. \tag{1.1}$$

In the previous expression and in the following, $\mathcal{K}$ and ε are a given smooth positive kernel and a given positive number, respectively. Here we assume that the evolution of the density of pirate ships is governed by the equation

$$\partial_t \rho = \Delta \rho - \operatorname{div}\left(\rho \kappa\left(|\nabla A(t,\cdot) * \mathcal{K}|\right) \frac{\nabla A(t,\cdot) * \mathcal{K}}{|\nabla A(t,\cdot) * \mathcal{K}|}\right) - f(x, \vec{d})\rho. \tag{1.2}$$

In the previous expression, the symbol $*$ denotes the convolution with respect to the space variable only. The function $\kappa \in C^\infty(\mathbb{R})$ is non decreasing and satisfies

$$\kappa(x) = \begin{cases} 0, & \text{if } x < \varepsilon, \\ 1, & \text{if } x > 1. \end{cases} \tag{1.3}$$

Finally, the function f has the following structure:

$$f(x, \vec{d}(t)) = \sum_{i=1}^{M} \mathcal{C}(|x - d_i(t)|), \tag{1.4}$$

where the function $\mathcal{C} : \mathbb{R}^+ \to \mathbb{R}$ is compactly supported, smooth, positive, constant in a neighborhood of 0 and monotone non increasing on $\mathbb{R}^+$. Note that we could use a different cut-off $\mathcal{C}_i$ for every $i = 1, \ldots, M$, but to simplify the notation we have used the same for every i. The rationale underpinning equation (1.2) is the following:

- $\Delta \rho$ takes in account the stochastic behavior of the dynamics of pirates.

- As in (1.1), Eq. (1.2) contains both a divergence term and a convolution kernel $\mathcal{K}$. The kernel $\mathcal{K}$ takes into account non-local effects. The divergence term in (1.2) only depends on ∇A, while in (1.1) it also depends on A. The rationale underpinning (1.1) is that pirates ships, if located in areas crowded with commercial cargos, are less motivated to move towards more attractive surroundings. Moreover, the speed towards commercial cargos is not a-priori bounded in (1.1), while it is in (1.2). Note furthermore that we introduce the cut-off function κ and the positive threshold ε to avoid the singularities that might be generated by the absence of commercial ships (i.e., $A = 0$). From the modeling viewpoint, the presence of κ can be justified by postulating that, if the commercial cargos density is extremely low, then it does not affect the dynamics of pirate ships, which in this case move randomly.
- The term $-f(x, \vec{d})\rho$ models the effect of the anti-piracy police watercrafts.

We augment (1.2) with homogeneous Neumann boundary conditions, prescribing that no pirate ships are leaving or entering Ω:

$$\nabla\rho \cdot \vec{n} = 0 \quad \text{on } (0, \infty) \times \partial\Omega. \tag{1.5}$$

In the previous expression, $\vec{n}$ denotes the outward pointing, unit normal vector to $\partial\Omega$.

Dynamics of cargo ships. We model the evolution of the density of commercial cargos by using the conservation law

$$\partial_t A + \operatorname{div}\Big(U(A)\Big(\vec{r}(x) + \mathcal{J}(x, \rho(t,\cdot), \vec{d}(t))\Big)\Big) = 0. \tag{1.6}$$

In the previous expression, the flux function U has the same properties as in the LWR traffic model [10,15,18], namely $U(A) = Av(A)$ for some function v representing the speed and satisfying

$$v \in C^2(\mathbb{R}), \qquad v' < 0, \qquad v(A_{max}) = 0, \tag{1.7}$$

where A_{max} represents the maximal density of commercial cargos. Without loss of generality, we assume that $A_{max} = 1$. The vector field $\vec{r}$ represents the commercial routes and satisfies the following assumptions:

$$\vec{r} \in C^1(\mathbb{R}^n; \mathbb{R}^n), \qquad \vec{r}(x) \cdot \vec{n}(x) \le 0, \text{ for every } x \in \partial\Omega. \tag{1.8}$$

In the previous expression, $\vec{n}$ is the outward pointing, unit normal vector to $\partial\Omega$. Finally, the functional $\mathcal{J}$ is defined as follows:

$$\begin{aligned}&\mathcal{J}(x, \rho(t,\cdot), \vec{d}(t))\\ &\quad := \chi(x)\left(-\int_\Omega \rho(t,y)(y-x)\mathcal{C}(|x-y|)dy + \sum_{i=1}^M \mathcal{C}(|x-d_i(t)|)(d_i(t)-x)\right).\end{aligned} \tag{1.9}$$

In the previous expression, χ is a cut-off function that is compactly supported in Ω and the function $\mathcal{C} : \mathbb{R}^+ \to \mathbb{R}$ is as before compactly supported, smooth, positive, constant in a neighborhood of 0 and monotone non increasing on $\mathbb{R}^+$. Note that, although the function ρ is only defined on the set Ω, by appropriately choosing the cut-off functions $\mathcal{C}$ and χ we can define the functional $\mathcal{J}$ for every $x \in \mathbb{R}^n$.

In the following, we will set the equation (1.6) on the whole space-time domain $(0, \infty) \times \mathbb{R}^n$, but we will take an initial datum A_0 which is 0 almost everywhere outside Ω. Owing to condition (1.8), one can show (see [8, Proposition 3.1]) that, for every $t \geq 0$, the entropy admissible solution satisfies $A(t, x) = 0$ for a.e. $x \in \mathbb{R}^n \backslash \Omega$.

The rationale underpinning equation (1.6) is the following:

- The total amount of commercial cargos is conserved, and hence we model the evolution of A with a conservation law.
- The commercial cargos tend to follow the commercial routes $\vec{r}$, but can deviate either to be closer to the police watercrafts or to go further away from the pirate ships: we model this possibility by introducing the nonlocal term $\mathcal{J}$.
- The presence of the cut-off function χ takes in account the fact that we want to focus on the open sea dynamics and not on the dynamics close to the boundary of Ω.

Dynamics of the police watercrafts. We model the dynamics of the police watercrafts by using the equation

$$\vec{d}'(t) = \vec{F}(\vec{d}(t), \rho(t, \cdot), A(t, \cdot), u(t)). \tag{1.10}$$

The function $\vec{F}$ attains values in $(\mathbb{R}^n)^M$ and the i-th component F_i has the following expression:

$$\begin{aligned} &F_i(\vec{d}(t), \rho(t, \cdot), A(t, \cdot), u(t)) \\ &\quad = \chi(d_i(t)) \Bigg(\int_\Omega \mathcal{C}(|d_i(t) - y|) \rho(t, y) A(t, y)(y - d_i(t)) dy \\ &\qquad - \sum_{j \neq i} \mathcal{C}(|d_i(t) - d_j(t)|)(d_i(t) - d_j(t)) + u_i(t) \Bigg) - (1 - \chi(d_i(t)) \vec{s}(d_i(t)). \end{aligned} \tag{1.11}$$

In the previous expression, χ and $\mathcal{C}$ are as before cut-off functions, the functions $u_i \in L^\infty((0, \infty); \mathbb{R}^n)$ are bounded controls and the vector field $\vec{s}$ satisfies

$$\vec{s}(x) \cdot \vec{n}(x) \leq 0, \text{ for every } x \in \partial\Omega.$$

The rationale underpinning (1.11) is the following:

- The first term in (1.11) is attractive and models the fact that police watercrafts are attracted by the regions where pirate ships and commercial cargos are simultaneously present.
- The second term in (1.11) is repulsive and models the fact that police watercrafts tend to stay away from each other.

- The presence of the cut-off functions χ models the fact that we want to focus on the open sea dynamics and not on the dynamics close to the boundary of Ω, which might be different. The vector field $\vec{s}$ represents the dynamics of police watercrafts close to the boundaries of the marine region Ω.

Complete model and control problem. By combining (1.1), (1.6) and (1.10) we arrive at the following system:

$$\begin{cases} \partial_t \rho = \Delta \rho - \operatorname{div}\left(\rho \kappa \left(|\nabla A(t,\cdot) * \mathcal{K}|\right) \dfrac{\nabla A(t,\cdot) * \mathcal{K}}{|\nabla A(t,\cdot) * \mathcal{K}|}\right) - f(x,d)\rho, \\ \partial_t A + \operatorname{div}\Big(U(A)\Big(\vec{r} + \mathcal{J}\Big)\Big) = 0, \\ \vec{d}' = \vec{F}(\vec{d}, \rho, A, u), \end{cases} \tag{1.12}$$

which is augmented with the boundary and initial conditions

$$\begin{cases} \nabla \rho \cdot \vec{n} = 0, & \text{on } (0,\infty) \times \partial\Omega, \\ \rho(0,\cdot) = \rho_0, & \text{in } \Omega, \\ A(0,\cdot) = A_0, & \text{in } \mathbb{R}^n, \\ \vec{d}(0) = \vec{d}_0. \end{cases} \tag{1.13}$$

We recall that by assumption the set Ω is open, bounded, connected and with a smooth boundary. We prescribe the following conditions on the initial data:

$$\begin{gathered} \rho_0 \geq 0, \qquad \rho_0 \in L^2(\Omega), \\ 0 \leq A_0 \leq 1, \qquad A_0 \in BV(\mathbb{R}^n), \qquad A_0(x) = 0 \text{ for a.e. } x \in \mathbb{R}^n \setminus \Omega, \\ \vec{d}_0 \in \Omega^M, \end{gathered} \tag{1.14}$$

and on the control function

$$u \in L^\infty((0,\infty)\,; (\mathbb{R}^n)^M).$$

In the first part of the paper we rely on a fixed point argument and we establish well-posedness of the mixed PDE-ODE system (1.12), (1.13) for every given control $u \in L^\infty((0,\infty)\,; (\mathbb{R}^n)^M)$.

In the second part of the paper we discuss a related control problem. More precisely, we introduce the cost functional

$$J(u) = \sum_{i=1}^{M} \int_0^\infty \omega(t) |d_i'(t)| dt + \int_0^\infty \omega(t) \int_\Omega \rho(t,x) A(t,x) dt dx \tag{1.15}$$

and we focus on the problem of minimizing J. In the above expression, the first term measures the length of the routes described by the police watercrafts, while the second term takes in account the interactions between pirates' and cargo ships. Finally, ω is a measurable weight function satisfying

$$\begin{gathered} \omega(t) \geq 0 \text{ for every } t, \ \omega \text{ is non increasing}, \\ \lim_{t \to \infty} \exp(t^\alpha) \omega(t) = 0 \text{ for some } \alpha > 1. \end{gathered} \tag{1.16}$$

Note that the only reason why we introduce the third condition in (1.16) is to ensure that the cost functional J is finite. Also, from the modeling point of view the monotonicity of ω expresses the fact that the police response is more concerned with the close future than with the far one. Note that the case of a finite time horizon can be taken into account by choosing a compactly supported weight function ω. Finally, we point out that one could choose different weights ω_1 and ω_2 for the first and the second term in (1.15), but since the analysis is completely analogous to simplify the notation we assume that the weight is the same in the two terms.

Remark 1.1. Note that in the functional (1.15) the number M of police watercrafts is fixed. It could be interesting to consider other functionals, depending on both u and M and taking into account the operating costs of the police watercrafts. In this way the optimal strategy would provide information on the optimal number of police watercrafts. However, the analysis of these functionals would likely require rather different techniques than those used in the present paper and hence we do not pursue it here.

1.1. Paper outline

The paper is organized as follows. In Sect. 2 we prove the well-posedness of (1.12) for every given bounded control u. In Sect. 3 we discuss the control problem for the functional J. In Sect. 4 we present some numerical examples. For the reader's convenience we conclude the introduction by collecting the main notation used in the present paper.

1.2. Notation

We denote by c any constant which only depends on the coefficient of the problem, i.e. for instance on the C^0 norm on the convolution kernel $\mathcal{K}$, on the diameter of the set Ω, or on the number of police watercrafts, but does not depend on the control u and on the initial data $(\rho_0, A_0, \vec{d}_0)$. The precise value of c can vary from occurrence to occurrence.

1.2.1. Quantities introduced in the present paper.

- Ω: the marine region where the dynamics occur.
- ρ: the density of pirate ships.
- A: the density of commercial cargos.
- $\vec{d}$: the position of the police watercrafts.
- M: the number of police watercrafts.
- $\mathcal{K}$: the convolution kernel in the equation for the pirate ships density (1.2).
- κ: the cut-off function in (1.3).
- ε: the threshold in (1.3).
- f: the repulsive term defined in (1.4).
- $\mathcal{C}$: a cut-off function that is compactly supported, smooth, positive, constant in a neighborhood of 0 and monotone non increasing on $\mathbb{R}^+$, see (1.4) and (1.9).
- $\vec{n}$: the outward pointing, unit normal vector to the boundary $\partial\Omega$.
- $U(A) = Av(A)$: the flux function in (1.6). The speed function v satisfies (1.7).
- $\vec{r}$: the commercial routes, see (1.6) and (1.8).

- $\mathcal{J}$: the functional taking into account possible deviations from the commercial routes in (1.9).
- χ: a cut-off function compactly supported in Ω, see (1.9) and (1.11).
- $\vec{s}$: the dynamics of police watercrafts close to the boundary $\partial\Omega$.
- J: the cost functional in (1.15).
- ω: the weight function in the cost functional J, satisfying (1.16).

2. Well-posedness of the model

In this section we fix a control function $u \in L^\infty((0,\infty);(\mathbb{R}^n)^M)$ and we establish existence, uniqueness and stability results for the model (1.12)–(1.13) under the assumptions (1.14).

First, we provide the precise definition of solution of (1.12)–(1.13).

Definition 2.1. Given $T > 0$, we term the triple (ρ, A, d) a solution of (1.12)–(1.13) on $(0,T)\times\Omega$ if

(i) $\rho \in C^0([0,T];L^2(\Omega)) \cap L^2((0,T);H^1(\Omega))$, $\rho \geq 0$ almost everywhere and ρ is a solution in the sense of distributions of the initial-boundary value problem obtained by coupling (1.2) with (1.13).

(ii) $A \in C^0([0,T];L^1(\Omega)) \cap BV((0,T)\times\Omega)$, $0 \leq A \leq 1$ almost everywhere and A is a Kružkov entropy solution of the system obtained by coupling (1.6) with (1.13).

(iii) $\vec{d} \in C^0([0,T];\Omega^M)$, and

$$d_i(t) = d_{0,i} + \int_0^t F_i(d(s), \rho(s,\cdot), A(s,\cdot), u(s))ds,$$

for every $t \geq 0$ and $i \in \{1,\dots,M\}$.

In the previous expression, BV is the space of function with bounded total variation, see [2, Chapter 3] for the precise definition. Also, we refer to [13] for the definition of Kružkov entropy solution of a scalar conservation law.

The main result of this section is the following.

Theorem 2.2. *Fix $T > 0$, $u \in L^\infty((0,\infty);(\mathbb{R}^n)^M)$ and let $(\rho_0, A_0, \vec{d_0})$ satisfy (1.14). Then the initial-boundary value problem (1.12) has a unique solution (ρ, A, d) in the sense of Definition 2.1. Also, assume that (ρ, A, d) and $(\widetilde{\rho}, \widetilde{A}, \widetilde{\vec{d}})$ are the two solutions of (1.12)–(1.13) corresponding to the initial data ρ_0, A_0, $\vec{d_0}$ and $\widetilde{\rho}_0$, $\widetilde{A}_0$, $\widetilde{\vec{d}}_0$, respectively. Then*

$$\begin{aligned}\|\rho(t,\cdot) - \widetilde{\rho}(t,\cdot)\|_{L^2(\Omega)} &+ \left\|A(t,\cdot) - \widetilde{A}(t,\cdot)\right\|_{L^1(\Omega)} + \left|\vec{d}(t) - \widetilde{\vec{d}}\,(t)\right| \\ &\leq C\left(\|\rho_0 - \widetilde{\rho}_0\|_{L^2(\Omega)} + \left\|A_0 - \widetilde{A}_0\right\|_{L^1(\Omega)} + \left|d_0 - \widetilde{\vec{d}}_0\right|\right), \end{aligned} \tag{2.1}$$

for every $t \in [0,T]$. In the previous expression, the constant C only depends on T, $\|u\|_{L^\infty((0,T);(\mathbb{R}^n)^M)}$, $\|A_0\|_{L^1(\Omega)}$, $\|\widetilde{A}_0\|_{L^1(\Omega)}$, $\mathrm{TotVar}(A_0)$, $\mathrm{TotVar}(\widetilde{A}_0)$, $\|\rho_0\|_{L^2(\Omega)}$, and $\|\widetilde{\rho}_0\|_{L^2(\Omega)}$.

Note that here and in the following we denote by $\mathrm{TotVar}(A_0)$ the total variation of the function A_0, see [2, Definition 3.4] for the precise definition. The proof of Theorem 2.2 is organized as follows:

- in Sect. 2.1 we introduce some notation and provide the proof outline;
- in Sect. 2.1.1 we establish a priori estimates on the solutions of (1.2), (1.6) and (1.10).
- in Sect. 2.2 we conclude the existence proof by relying on the Schauder Fixed Point Theorem. We also establish uniqueness and stability by combining suitable a-priori estimates with a Gronwall Lemma argument.

Note that, to simplify the notation, in the following we always assume that $M = 1$, namely that there is only one police watercraft. In this case $\vec{d}$ attains values in $\Omega \subseteq \mathbb{R}^n$ and we denote it by d. The analysis of the case $M > 1$ does not pose additional challenges.

2.1. Proof outline

The proof relies on a fixed point argument. More precisely, given $B > 0$, we consider the domains

$$
\begin{aligned}
\mathcal{D} =&\mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3,\\
\mathcal{D}_1 =& \left\{ \rho \in L^2((0,T)\times\Omega) : \begin{array}{ll} \|\rho(t,\cdot)\|_{L^2(\Omega)} \le B, & \text{for a.e. } t \\ \rho(t,x) \ge 0 & \text{for a.e. } (t,x) \end{array} \right\},\\
\mathcal{D}_2 =& \left\{ A \in C^0([0,T];L^1(\mathbb{R}^n)) : \begin{array}{ll} 0 \le A(t,x) \le 1 & t \in [0,T], \text{a.e. } x \in \mathbb{R}^n \\ A(t,x) = 0 & t \in [0,T], \text{a.e. } x \in \mathbb{R}^n \setminus \Omega \end{array} \right\},\\
\mathcal{D}_3 =& \left\{ d \in C^0([0,T]) : d(t) \in \bar{\Omega} \text{ for every } t \right\}.
\end{aligned}
\tag{2.2}
$$

The exact value of the constant B will be determined in the following and will depend on $\|\rho_0\|_{L^2(\Omega)}$ and on T. On $\mathcal{D}$ we define the norm

$$
\|(\rho, A, d)\|_{\mathcal{D}} = \|\rho\|_{L^2((0,T)\times\Omega)} + \|A\|_{C^0([0,T];L^1(\Omega))} + \|d\|_{C^0([0,T];\mathbb{R}^n)}
$$

and the operator

$$
\mathcal{T} : (\rho, A, d) \longmapsto (\mathcal{T}_1(A, d), \mathcal{T}_2(\rho, d), \mathcal{T}_3(\rho, A)) \tag{2.3}
$$

by proceeding as follows. We term $\mathcal{T}_1(A, d)$ the unique solution of

$$
\begin{cases}
\partial_t \rho = \Delta \rho - \operatorname{div}\left(\rho \kappa \left(|\nabla A(t,\cdot) * \mathcal{K}| \right) \dfrac{\nabla A(t,\cdot) * \mathcal{K}}{|\nabla A(t,\cdot) * \mathcal{K}|} \right) - f(x,d)\rho, & \text{in } (0,T)\times\Omega,\\
\partial_\nu \rho(t,x) = 0, & \text{on } (0,T)\times\partial\Omega,\\
\rho(0,\cdot) = \rho_0, & \text{at } t = 0.
\end{cases}
$$

Also, we term $\mathcal{T}_2(\rho, d)$ the unique Kružkov entropy solution of

$$
\begin{cases}
\partial_t A + \operatorname{div}\left(U(A)(\vec{r}(x) + \mathcal{J}(x, \rho(t,\cdot), d(t))) \right) = 0, & \text{in } (0,T)\times\mathbb{R}^n,\\
A(0,\cdot) = A_0, & \text{at } t = 0.
\end{cases}
$$

Finally, $\mathcal{T}_3(\rho, A)$ is the unique solution of

$$
\begin{cases}
d'(t) = F(d(t), \rho(t,\cdot), A(t,\cdot), u(t)), & \text{in } (0,T),\\
d(0) = d_0.
\end{cases}
$$

In Sect. 2.1.1 we establish a-priori estimates on $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ and in Sect. 2.2 we use them to apply a fixed point argument.

2.1.1. A-priori estimates. First, we establish a-priori estimates on $\mathcal{T}_1$. We denote with $H^*(\Omega)$ the dual space of $H^1(\Omega)$.

Lemma 2.3. *Under the same assumptions as in the statement of Theorem 2.2 we have that*

$$\mathcal{T}_1(A,d) \in \mathcal{D}_1, \qquad \text{for every } (A,\, d) \in \mathcal{D}_2 \times \mathcal{D}_3. \tag{2.4}$$

In particular,

$$\|\mathcal{T}_1(A,d)(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla\mathcal{T}_1(A,d)(s,\cdot)\|^2_{L^2(\Omega)}\, ds \le \|\rho_0\|^2_{L^2(\Omega)}\, e^{ct}, \tag{2.5}$$

for a.e. $t \in (0,T)$ *and every* $(A,\, d) \in \mathcal{D}_2 \times \mathcal{D}_3$. *Also, we have that* $\partial_t \mathcal{T}_1(A,d) \in L^2((0,T); H^*(\Omega))$ *and*

$$\|\partial_t \mathcal{T}_1(A,d)\|_{L^2((0,t);H^*(\Omega))} \le c\, \|\rho_0\|_{L^2(\Omega)}\, e^{ct}. \tag{2.6}$$

Finally,

$$\begin{aligned} &\left\|\mathcal{T}_1(A,d)(t,\cdot) - \mathcal{T}_1(\bar{A},\bar{d})(t,\cdot)\right\|^2_{L^2(\Omega)} \\ &\quad + \int_0^t \left\|(\nabla\mathcal{T}_1(A,d) - \nabla\mathcal{T}_1(\bar{A},\bar{d}))(s,\cdot)\right\|^2_{L^2(\Omega)}\, ds \\ &\le c e^{ct}\, \|\rho_0\|^2_{L^2(\Omega)} \left(\left\|A-\bar{A}\right\|^2_{C^0([0,T];L^1(\mathbb{R}^n))} + \left\|d-\bar{d}\right\|^2_{C^0([0,T];\mathbb{R}^n)} \right), \end{aligned} \tag{2.7}$$

for a.e. $t \in (0,T)$ *and for every* $(A,\, d)$, $(\bar{A},\, \bar{d}) \in \mathcal{D}_2 \times \mathcal{D}_3$.

Proof. In the proof of this lemma we use the following notation:

$$\rho := \mathcal{T}_1(A,d), \qquad V := \kappa\left(|\nabla A(t,\cdot) * \mathcal{K}|\right) \frac{\nabla A(t,\cdot) * \mathcal{K}}{|\nabla A(t,\cdot) * \mathcal{K}|}.$$

Since $0 \le A \le 1$, then

$$\|V\|_{C^0([0,T]\times\Omega)},\ \|\mathrm{div}\,(V)\,\|_{C^0([0,T]\times\Omega)} \le c. \tag{2.8}$$

Also, by recalling (1.4) we get

$$\begin{aligned} 0 \le f(x,d(t)) = \mathcal{C}(|x-d(t)|) &\le \left|\mathcal{C}(|x-d(t)|) - \mathcal{C}(0)\right| + c \\ &\le c|x-d(t)| + c \le c\ \mathrm{diameter}(\Omega) + c \le c. \end{aligned} \tag{2.9}$$

Note that (1.2) can be written as

$$\partial_t \rho = \Delta\rho - \mathrm{div}\,(\rho V) - f\rho. \tag{2.10}$$

Owing to (2.8) and (2.9), we can apply classical results on parabolic equations (we refer to the book by Salsa [19, Chapter 9] for a detailed exposition). In particular, we obtain that (2.10) has a unique solution

$$\rho \in L^2((0,T); H^1(\Omega)),\ \ \partial_t\rho \in L^2((0,T); H^*(\Omega)) \tag{2.11}$$

provided that V and f are fixed and the initial datum ρ_0 satisfies (1.14). Also, since the constant 0 is a subsolution of (2.10) and $\rho_0 \ge 0$ by (1.14), the comparison principle for parabolic equations gives that $\rho \ge 0$ (see the book of Ladyženskaja et al. [14, Chapter 1] for a comprehensive discussion on the maximum principle). Also, note that, if ρ satisfies (2.11), then owing to [19, Theorem 7.22] in the equivalence class of ρ there is a representative satisfying $\rho \in C^0([0,T];L^2(\Omega))$. Thus, in the following, we will always identify ρ with its L^2-continuous representative.

We now establish (2.5) by using some formal computations, which can be made rigorous by relying on suitable approximation arguments. We have that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{\rho^2}{2}dx &= \int_\Omega \rho\partial_t\rho dx \\
&\overset{(2.10)}{=} \int_\Omega \rho\Delta\rho dx - \int_\Omega \rho \operatorname{div}(\rho V)\,dx - \int_\Omega f(x,d)\rho^2 dx \\
&\overset{(1.5)}{=} -\int_\Omega |\nabla\rho|^2 dx - \int_\Omega \rho\nabla\rho\cdot V dx - \int_\Omega \rho^2 \operatorname{div}(V)\,dx - \int_\Omega f(x,d)\rho^2 dx \\
&\le -\int_\Omega |\nabla\rho|^2 dx + \frac{1}{2}\int_\Omega \rho^2|V|^2 dx + \frac{1}{2}\int_\Omega |\nabla\rho|^2 dx - \int_\Omega f(x,d)\rho^2 dx \\
&\overset{(2.8),(2.9)}{\le} -\frac{1}{2}\int_\Omega |\nabla\rho|^2 dx + c\int_\Omega \rho^2 dx.
\end{aligned}$$

Owing to the Gronwall Lemma, this implies (2.5) and hence that $\rho \in \mathcal{D}_1$ provided that we appropriately choose the constant B.

To establish (2.6), we use the equation for ρ, which implies that for a.e. $t \in (0,T)$

$$\|\partial_t\rho(t,\cdot)\|_{H^*(\Omega)} \le \|\nabla\rho(t,\cdot)\|_{L^2(\Omega)} + \|\rho(t,\cdot)V\|_{L^2(\Omega)} + \|f\rho(t,\cdot)\|_{L^2(\Omega)}$$

and by combining (2.8) and (2.9) with (2.5) we eventually arrive at (2.6).

We are left to establish (2.7). We introduce the notation

$$\bar{V} := \kappa\left(\left|\nabla\bar{A}(t,\cdot)*\mathcal{K}\right|\right)\frac{\nabla\bar{A}(t,\cdot)*\mathcal{K}}{\left|\nabla\bar{A}(t,\cdot)*\mathcal{K}\right|}$$

and we point out that

$$\begin{aligned}
\|V-\bar{V}\|_{C^0([0,T]\times\mathbb{R}^n)} &\le c\|A-\bar{A}\|_{C^0([0,T];L^1(\mathbb{R}^n))}, \\
\|\operatorname{div}\left(V-\bar{V}\right)\|_{C^0([0,T]\times\mathbb{R}^n)} &\le c\|A-\bar{A}\|_{C^0([0,T];L^1(\mathbb{R}^n))}
\end{aligned} \tag{2.12}$$

and

$$|f(x,d)-f(x,\bar{d})| \le c\|d-\bar{d}\|_{C^0([0,T];\mathbb{R}^n)}. \tag{2.13}$$

We now use a formal computation (which can be made rigorous by relying on a suitable approximation argument) and we point out that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{(\rho-\bar{\rho})^2}{2}dx &= \int_\Omega (\rho-\bar{\rho})\,\partial_t(\rho-\bar{\rho})dx \\
&= \int_\Omega (\rho-\bar{\rho})\Delta(\rho-\bar{\rho})dx - \int_\Omega (\rho-\bar{\rho})\operatorname{div}((\rho-\bar{\rho})V)\,dx \\
&\quad - \int_\Omega (\rho-\bar{\rho})\operatorname{div}\left(\bar{\rho}(V-\bar{V})\right)dx
\end{aligned}$$

$$
\begin{aligned}
&- \int_\Omega f(x,d)(\rho-\bar\rho)^2 dx - \int_\Omega (f(x,d)-f(x,\bar d))\bar\rho(\rho-\bar\rho)dx \\
&= - \int_\Omega |\nabla(\rho-\bar\rho)|^2 dx - \int_\Omega (\rho-\bar\rho)\nabla(\rho-\bar\rho)\cdot V dx - \int_\Omega (\rho-\bar\rho)^2 \mathrm{div}\,(V)\,dx \\
&- \int_\Omega (\rho-\bar\rho)\nabla\bar\rho\cdot(V-\bar V)dx - \int_\Omega (\rho-\bar\rho)\bar\rho\,\mathrm{div}\left(V-\bar V\right)dx \\
&- \int_\Omega f(x,d)(\rho-\bar\rho)^2 dx - \int_\Omega (f(x,d)-f(x,\bar d))\bar\rho(\rho-\bar\rho)dx.
\end{aligned}
$$

By recalling (2.8), (2.9) we then get

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{(\rho-\bar\rho)^2}{2}dx \le& -\frac{1}{2}\int_\Omega |\nabla(\rho-\bar\rho)|^2 dx + c\int_\Omega (\rho-\bar\rho)^2 dx + c\int_\Omega |\nabla\bar\rho|^2|V-\bar V|^2 dx \\
&+ c\int_\Omega \bar\rho^2(\mathrm{div}\left(V-\bar V\right))^2 dx + c\|f(\cdot,d)-f(\cdot,\bar d)\|_{C^0}\int_\Omega \bar\rho^2 dx.
\end{aligned}
$$

Next, we use the Fundamental Theorem of Calculus and we recall that ρ and $\bar\rho$ have the same initial datum ρ_0. By recalling (2.12) and (2.13) we obtain

$$
\begin{aligned}
&\|\rho(t,\cdot)-\bar\rho(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega |\nabla\rho-\nabla\bar\rho|^2 dxds \\
&\le c\int_0^t \|\rho(s,\cdot)-\bar\rho(s,\cdot)\|^2_{L^2(\Omega)}\,ds + c\|d-\bar d\|_{C^0([0,T])}\int_0^t\int_\Omega \bar\rho^2 dxds \\
&\quad + c\|A-\bar A\|^2_{C^0([0,T];L^1(\mathbb{R}^n))}\int_0^t\int_\Omega \Big[|\nabla\bar\rho|^2+\bar\rho^2\Big]dxds.
\end{aligned}
$$

By using the Gronwall Lemma we eventually arrive at (2.7). □

Next, we establish a-priori estimates on $\mathcal{T}_2$.

Lemma 2.4. *Under the same assumptions as in the statement of Theorem 2.2 we have that*

$$\mathcal{T}_2(\rho,d)\in\mathcal{D}_2,\ \textit{for every}\ (\rho,\,d)\in\mathcal{D}_1\times\mathcal{D}_3.$$

In particular,

$$\|\mathcal{T}_2(\rho,d)(t,\cdot)\|_{L^1(\mathbb{R}^n)} \le \|A_0\|_{L^1(\mathbb{R}^n)}\,, \tag{2.14}$$

$$\mathrm{TotVar}(\mathcal{T}_2(\rho,d)(t,\cdot)) \le \mathrm{TotVar}(A_0), \tag{2.15}$$

$$0 \le \mathcal{T}_2(\rho,d)(t,\cdot) \le 1, \tag{2.16}$$

$$\mathcal{T}_2(\rho,d)(t,x) = 0 \ \textit{for every}\ t\ge 0\ \textit{and a.e.}\ x\in\mathbb{R}^n\backslash\Omega, \tag{2.17}$$

$$\|\mathcal{T}_2(\rho,d)(s,\cdot)-\mathcal{T}_2(\rho,d)(t,\cdot)\|_{L^1} \le c\,\mathrm{TotVar}(A_0)|t-s|, \tag{2.18}$$

for every $(\rho,\,d)\in\mathcal{D}_1\times\mathcal{D}_3$ and $t,s\in[0,T]$. Also,

$$
\begin{aligned}
&\|\mathcal{T}_2(\rho,d)(t,\cdot)-\mathcal{T}_2(\bar\rho,\bar d)(t,\cdot)\|_{L^1(\mathbb{R}^n)} \\
&\le c\sqrt{t}\Big(\|\rho-\bar\rho\|_{L^2((0,T)\times\Omega)} + \sqrt{t}\|d-\bar d\|_{C^0}\Big)\Big(1+\mathrm{TotVar}(A_0)\Big).
\end{aligned}
\tag{2.19}
$$

for every $t\in[0,T]$ and every (ρ,d), $(\bar\rho,\bar d)\in\mathcal{D}_1\times\mathcal{D}_3$.

Proof. We fix $(\rho,\, d) \in \mathcal{D}_1 \times \mathcal{D}_3$ and we introduce the vector field W by setting

$$W(t,x) := \vec{r}(x) + \mathcal{J}(x, \rho(t,\cdot), d(t)) \tag{2.20}$$

and we point out that the vector field W is smooth. Also, note that, although ρ is only defined on Ω, owing to the presence of the cut-off function χ in the definition of $\mathcal{J}$ [see (1.9)] the vector field W is well defined for every $x \in \mathbb{R}^n$. By definition, $\mathcal{T}_2(\rho, d) = A$ is the entropy admissible solution of the Cauchy problem

$$\begin{cases} \partial_t A + \operatorname{div}\left(U(A)W(t,x)\right) = 0, \\ A(0,\cdot) = A_0, \end{cases} \tag{2.21}$$

where the initial datum A_0 satisfies (1.14), see [13]. The by-now classical theory by Kružkov provides existence and uniqueness results for the above Cauchy problem, and implies that estimates (2.14), (2.15), (2.16), and (2.18) hold true. The rigorous proof of (2.17) is given in [8, §5.3], here we provide an heuristic justification based on a formal computation. We set $\Omega^c := \mathbb{R}^n \backslash \Omega$, we recall that, owing to the finite propagation speed, $A(t,\cdot)$ is compactly supported for every $t > 0$. We infer that

$$\begin{aligned} \frac{d}{dt}\int_{\Omega^c} A(t,x)dx &= \int_{\Omega^c} \partial_t A(t,x)dx = -\int_{\Omega^c} \operatorname{div}\left(U(A)W(x)\right)dx \\ &= \int_{\partial\Omega} U(A)W(x)\cdot\vec{n}(x)\; d\sigma \le 0. \end{aligned}$$

To establish the third equality we have used the fact that $\vec{n}$ is the outward pointing, normal vector to $\partial\Omega$. To establish the final inequality, we have used the fact that $U(A) \ge 0$ since $A \le 1$ and on the boundary $\partial\Omega$ we have $W(x) = \vec{r}(x)$, which satisfies (1.8). Since

$$\int_{\Omega^c} A_0(x)dx = 0,$$

from the above inequality we infer that, since $A \ge 0$, then $A = 0$ almost everywhere on $\mathbb{R}^n \backslash \Omega$.

We are left to establish (2.19). We set

$$\bar{W}(t,x) := \vec{r}(x) + \mathcal{J}(x, \bar{\rho}(t,\cdot), \bar{d}(t)) \tag{2.22}$$

and we term $\bar{A}$ the corresponding entropy admissible solution of the equation (2.21). We recall that Kružkov entropy solutions are limit of vanishing viscosity approximations (see again Kružkov [13]) and we term A_m and $\bar{A}_m$ a sequence of vanishing viscosity solutions approximating A and $\bar{A}$, respectively. This implies, in particular, that the equation

$$\partial_t(A_m - \bar{A}_m) + \operatorname{div}\left(WU(A_m) - \bar{W}U(\bar{A}_m)\right) = \frac{1}{m}\Delta(A_m - \bar{A}_m) \tag{2.23}$$

holds in the sense of distributions. To establish (2.19), we employ some *formal* computations, which can be made rigorous by introducing a smooth regularization of the initial data. We recall that $U(A) = Av(A)$ and that v satisfies (1.7). This implies that

$$\begin{aligned}\operatorname{div}\left(WU(A_m)-\bar{W}U(\bar{A}_m)\right)&=\operatorname{div}\left((W-\bar{W})U(A_m)\right)\\&\quad+\operatorname{div}\left(\bar{W}v(A_m)(A_m-\bar{A}_m)\right)+\operatorname{div}\left(\bar{W}\bar{A}_m(v(A_m)-v(\bar{A}_m))\right).\end{aligned}\tag{2.24}$$

Since $v'<0$, then

$$\operatorname{sign}(A_m-\bar{A}_m)=-\operatorname{sign}(v(A_m)-v(\bar{A}_m))$$

and hence by using (2.24) we arrive at

$$\begin{aligned}&\operatorname{div}\left(WU(A_m)-\bar{W}U(\bar{A}_m)\right)\operatorname{sign}(A_m-\bar{A}_m)\\&=\Big[\operatorname{div}\left((W-\bar{W})\right)U(A_m)+(W-\bar{W})U'(A_m)\nabla A_m\Big]\operatorname{sign}(A_m-\bar{A}_m)\\&\quad+\operatorname{div}\left(\bar{W}v(A_m)|A_m-\bar{A}_m|\right)-\operatorname{div}\left(\bar{W}\bar{A}_m|v(A_m)-v(\bar{A}_m)|\right).\end{aligned}$$

By recalling the expressions of W and $\bar{W}$ [see (2.20) and (2.22)] and by combining the explicit expression of $\mathcal{J}$ with the classical properties of convolution we eventually get that

$$\|W(s,\cdot)-\bar{W}(s,\cdot)\|_{C^0(\Omega)}\le c\|\rho(t,\cdot)-\bar{\rho}(t,\cdot)\|_{L^2(\Omega)}+c\|d-\bar{d}\|_{C^0}\tag{2.25}$$

and that

$$\|\operatorname{div}\left((W-\bar{W})(s,\cdot)\right)\|_{L^1(\mathbb{R}^n)}\le c\|\rho(t,\cdot)-\bar{\rho}(t,\cdot)\|_{L^2(\Omega)}+c\|d-\bar{d}\|_{C^0}.\tag{2.26}$$

By using the above formulas we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}|A_m-\bar{A}_m|dx=\int_{\mathbb{R}^n}\operatorname{sign}(A_m-\bar{A}_m)\partial_t(A_m-\bar{A}_m)dx\\&\overset{(2.23)}{=}-\int_{\mathbb{R}^n}\operatorname{sign}(A_m-\bar{A}_m)\operatorname{div}\left(WU(A_m)-\bar{W}U(\bar{A}_m)\right)dx\\&\quad+\int_{\mathbb{R}^n}\operatorname{sign}(A_m-\bar{A}_m)\frac{1}{m}\Delta(A_m-\bar{A}_m)dx\\&\le\int_{\mathbb{R}^n}\Big|\operatorname{div}\left((W-\bar{W})\right)U(A_m)+(W-\bar{W})U'(A_m)\nabla A_m\Big|dx\\&\le c\|\operatorname{div}\left((W-\bar{W})\right)\|_{L^1(\mathbb{R}^n)}\|A_m\|_{L^\infty(\mathbb{R}^n)}+c\|W-\bar{W}\|_{L^\infty(\mathbb{R}^n)}\|\nabla A_m\|_{L^1}\\&\overset{(2.25),(2.26)}{\le}c(1+\operatorname{TotVar}(A_0))\Big[\|\rho(t,\cdot)-\bar{\rho}(t,\cdot)\|_{L^2(\Omega)}+\|d-\bar{d}\|_{C^0}\Big].\end{aligned}$$

By applying the Fundamental Theorem of Calculus and letting $m\to\infty$ we eventually arrive at (2.19). □

Lemma 2.5. *Under the same assumptions as in the statement of Theorem 2.2 we have that*

$$\mathcal{T}_3(\rho,A)\in\mathcal{D}_3,\ \textit{for every}\ (\rho,\,A)\in\mathcal{D}_1\times\mathcal{D}_2.\tag{2.27}$$

In particular,

$$|\mathcal{T}_3(\rho,A)(t)-\mathcal{T}_3(\rho,A)(s)|\le c\Big(B+\|u\|_{L^\infty((0,\infty);\mathbb{R}^n)}+1\Big)|t-s|,\tag{2.28}$$

for every t, $s \in [0, T]$ *and* $(\rho, A) \in \mathcal{D}_1 \times \mathcal{D}_2$. *Also,*

$$\begin{aligned} &\left\| \mathcal{T}_3(\rho, A) - \mathcal{T}_3(\bar\rho, \bar A) \right\|_{C^0([0,T];\mathbb{R}^n)} \\ &\leq c \exp\left(cT (B + \|u\|_{L^\infty((0,\infty);\mathbb{R}^n)} + 1) \right) \\ &\quad \times \left(\sqrt{T} \, \|\rho - \bar\rho\|_{L^2((0,T)\times\Omega)} + T \left\| A - \bar A \right\|_{C^0([0,T];L^1(\mathbb{R}^n))} \right), \end{aligned} \tag{2.29}$$

for every (ρ, A), $(\bar\rho, \bar A) \in \mathcal{D}_1 \times \mathcal{D}_2$.

Proof. We recall (1.10) and (1.11) and we straightforwardly obtain (2.28). To establish (2.29) we introduce the notation $d := \mathcal{T}_3(\rho, A)$ and $\bar d := \mathcal{T}_3(\bar\rho, \bar A)$ and we point out that

$$\begin{aligned} \frac{d}{dt} |d(t) - \bar d(t)| &\overset{(1.10)}{\leq} |F(d, \rho, A, u) - F(\bar d, \bar\rho, \bar A, u)| \\ &\overset{(1.11)}{\leq} c |d(t) - \bar d(\bar t)| \Big(B + \|u\|_{L^\infty((0,\infty);\mathbb{R}^n)} + 1 \Big) \\ &\quad + c \left(\|\rho(t,\cdot) - \bar\rho(t,\cdot)\|_{L^2(\Omega)} + \left\| A(t,\cdot) - \bar A(t,\cdot) \right\|_{L^1(\Omega)} \right). \end{aligned}$$

By using the Gronwall Lemma, we arrive at (2.29). □

2.2. Conclusion of the proof

We can now provide the proof of Theorem 2.2: we first establish existence of a solution, next we establish stability (and henceforth uniqueness).

EXISTENCE: we apply the Schauder Fixed Point Theorem to the map $\mathcal{T}$. First, we recall the definition of $\mathcal{T}$ and of the domain $\mathcal{D} := \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3$, see (2.3) and (2.2), respectively. Note that $\mathcal{D}$ is convex and that $\mathcal{T}$ is continuous owing to (2.7), (2.19) and (2.29). Hence, it suffices to prove that that map $\mathcal{T}$ is compact, namely that the components $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are all compact:

- To show that $\mathcal{T}_1$ is compact it suffices to recall (2.5) and (2.6) and to apply the Aubin-Lions Lemma.
- To show that $\mathcal{T}_2$ is compact we rely on the following version of the Ascoli-Arzelà Theorem. If X is a Banach Space, a set $K \subseteq C^0([0, T]; X)$ is compact provided that (i) K is equicontinuous (ii) there is a compact set C such that for every $t \in [0, T]$ and every $h \in K$, $h(t) \in C$. We term C the set of functions $A \in L^1(\Omega)$ such that

  $$0 \leq A(x) \leq 1 \text{ for a.e. } x, \quad A(x) = 0 \text{ for a.e. } x \notin \Omega$$

  and

  $$\text{TotVar} A \leq \text{TotVar} A_0.$$

  Note that C is compact in $L^1(\mathbb{R}^N)$ owing to the Fréchet–Kolmogorov Theorem. We then conclude that $\mathcal{T}_2$ is compact by recalling (2.14), (2.16), (2.17), and (2.18).
- To show that $\mathcal{T}_3$ is compact we rely on the Ascoli-Arzelà Theorem: it suffices to combine the fact that $\bar\Omega$ is bounded with (2.27).

This concludes the proof of the existence part.
STABILITY: let (ρ, A, d) and $(\widetilde{\rho}, \widetilde{A}, \widetilde{d})$ be as in the statement of Theorem 2.2. We introduce the quantity Λ by setting

$$\begin{aligned}\Lambda(t) : &= \|\rho(t,\cdot) - \widetilde{\rho}(t,\cdot)\|^2_{C^0([0,t];L^2(\Omega))} + \|A(t,\cdot) - \widetilde{A}(t,\cdot)\|_{C^0([0,t];L^1(\mathbb{R}^n))} \\ &\quad + \|d - \widetilde{d}\|_{C^0([0,t];\mathbb{R}^n)}.\end{aligned}$$

By arguing as in the proof of (2.7), (2.19), (2.29) one can show that

$$\Lambda(t) \le (1 + B + \|u\|_{L^\infty} + \mathrm{TotVar}(A_0)) \int_0^t \Big(|\nabla \bar{\rho}|^2 + \bar{\rho}^2 + 1\Big)\Lambda(s)ds. \tag{2.30}$$

By combining (2.30) with the Gronwall Lemma we obtain (2.1), which in particular implies uniqueness.

3. Optimization problem

In this section we discuss the optimal control problem obtained by minimizing the functional J defined in (1.15). The main result is the following.

Theorem 3.1. *Assume that ρ_0, A_0, d_0 satisfy (1.14). Fix a constant $\theta \ge 0$, then there is $u_\theta \in \mathcal{B}_\theta$ such that*

$$J(u_\theta) = \min_{u \in \mathcal{B}_\theta} J(u),$$

where

$$\mathcal{B}_\theta := \Big\{u \in L^\infty((0,\infty);\mathbb{R}^n) \colon \|u\|_{L^\infty((0,\infty);\mathbb{R}^n)} \le \theta\Big\}.$$

Proof. Since J is nonnegative, we can fix a minimizing sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathcal{B}_\theta$ such that

$$\lim_{k\to\infty} J(u_k) = \inf_{u \in \mathcal{B}_\theta} J(u). \tag{3.1}$$

There is $u_\theta \in \mathcal{B}_\theta$ such that, up to subsequences (which we do not relabel), we have

$$u_k \overset{*}{\rightharpoonup} u_\theta, \quad \text{weakly-}^* \text{ in } L^\infty((0,+\infty);\mathbb{R}^n)).$$

Owing to Theorem 2.2, for every u_k there is a unique solution (ρ_k, A_k, d_k) of (1.12). Also, by arguing as in the proof of Lemma 2.3 we get that (2.4) and (2.6) hold with $\mathcal{T}_1$ replaced by ρ_k. Owing to the Aubin-Lions we conclude that there exists $\rho_\theta \in \mathcal{D}_1$ such that, up to subsequences (that we do not relabel), we have

$$\begin{aligned}&\rho_k \to \rho_\theta \text{ strongly in } L^2((0,T)\times\Omega), \text{ for every } T > 0,\\ &\nabla\rho_k \rightharpoonup \nabla\rho_\theta \text{ weakly in } L^2((0,T)\times\Omega;\mathbb{R}^n), \text{ for every } T > 0.\end{aligned}$$

By arguing as in the proof of Lemma 2.4, we get that A_k satisfies (2.14)–(2.18) with $\mathcal{T}_2$ replaced by A_k and by combining the Ascoli–Arzelà Theorem with the Fréchet–Kolmogorov Theorem as in the proof of Theorem 2.2 we conclude that there is $A_\theta \in \mathcal{D}_2$ such that

$$\begin{aligned}&A_k \to A_\theta \text{ strongly in } L^1((0,T)\times\Omega) \text{ for every } T > 0,\\ &TV(A_\theta(t,\cdot)) \le TV(A_0), \text{ for every } t \ge 0.\end{aligned}$$

Finally, by arguing as in the proof of Lemma 2.5 we infer that d_k satisfies (2.28) with $\mathcal{T}_3$ replaced by d_k. Owing to the Ascoli–Arzelà Theorem we conclude that there is $d_\theta \in \mathcal{D}_3$ such that

$$d_k \to d_\theta, \text{ uniformly in } C^0([0,T];\mathbb{R}^n), \text{ for every } T > 0.$$

We recall that ω satisfies (1.16) and that d_k attains values in $\bar{\Omega}$, which is bounded, and we conclude that, owing to the Lebesgue Dominated Convergence Theorem,

$$\lim_{k\to\infty} \int_0^{+\infty} |d_k(t)|\omega(t)dt = \int_0^{+\infty} |d_\theta(t)|\omega(t)dt. \tag{3.2}$$

Also, we have

$$\left| \int_\Omega A_k \rho_k(t,\cdot)dx \right| \overset{A_k \le 1}{\le} \left| \int_\Omega \rho_k(t,\cdot)dx \right| \le c\|\rho_k(t,\cdot)\|_{L^2(\Omega)}$$

and by combining (1.16) with the Lebesgue Dominated Convergence Theorem we get

$$\lim_{k\to\infty} \int_0^{+\infty} \int_\Omega \omega(t) A_k \rho_k(t,\cdot)dx = \lim_{k\to\infty} \int_0^{+\infty} \int_\Omega \omega(t) A_\theta \rho_\theta(t,\cdot)dx. \tag{3.3}$$

By using analogous arguments, one can show that $(\rho_\theta, A_\theta, d_\theta)$ is a solution of (1.12) with $u = u_\theta$. By combining (3.1), (3.2) and (3.3) we eventually conclude that

$$J(u_\theta) = \inf_{u\in\mathcal{B}_\theta} J(u)$$

and this concludes the proof of the theorem. □

4. Numerical experiments

This section is devoted to the discussion of numerical integrations of system (1.12). We consider a two-dimensional marine region $\Omega = (a,b) \times (\alpha,\beta)$, discretized by a numerical grid with N_x and N_y equi-spaced points in the x and y direction respectively, and a discrete time sequence $t^n = n\Delta t$, for $n \in \mathbb{N}$, with $\Delta t > 0$. Define the mesh points (x_j, y_k), for $j \in \{0,\ldots,N_x\}$ and $k \in \{0,\ldots,N_y\}$, as

$$x_j = a + j\Delta x \qquad \text{and} \qquad y_k = \alpha + k\Delta y$$

where $\Delta x = \frac{b-a}{N_x}$ and $\Delta y = \frac{\beta-\alpha}{N_y}$. For every $j \in \{0,\ldots,N_x-1\}$ and every $k \in \{0,\ldots,N_y-1\}$, the cell $C_{j,k}$ is defined by $C_{j,k} = [x_j, x_{j+1}] \times [y_k, y_{k+1}]$.

With $\rho^n_{j,k}$ and $A^n_{j,k}$ we denote an approximation of the densities ρ and A, respectively, at the discrete time t_n and in the cell $C_{j,k}$. Moreover with $\vec{d}_n \in \Omega^M$, we denote an approximation of the position of the police vessels at time t^n. The solution $\rho^n_{j,k}$ of the diffusion equation (1.2) for the evolution of pirate ships is calculated by an explicit finite difference method using a five-point numerical Laplacian. Here we treat the Neumann boundary conditions by using ghost cells. The solution $A^n_{j,k}$ of the conservation law (1.6) for the evolution

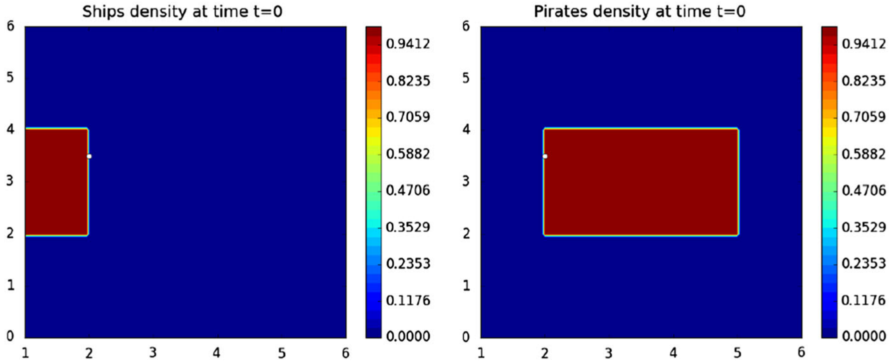

FIGURE 1. The initial densities A_0 (*left*) and ρ_0 (*right*) for the simulation of Sect. 4.1

of commercial cargos is obtained by using a dimensional splitting Godunov-type finite volume method. Finally, the solution $\vec{d}_n$ for the ordinary differential equation (1.10) is obtained through a first-order explicit Euler method.

4.1. Different control policies

We consider here the marine region $\Omega = (1, 6) \times (0, 6)$, i.e. $a = 1$, $b = 6$, $\alpha = 0$, and $\beta = 6$, with only one police watercraft, i.e. $M = 1$, with the following initial conditions

$$\begin{aligned} \rho(x, y) &= \chi_{[2,5]\times[2,4]}(x, y), \\ A(x, y) &= \chi_{[1,2]\times[2,4]}(x, y) \\ \vec{d}_0 &= (2, 3.5)\,, \end{aligned}$$

where χ is the characteristic function (see Fig. 1), and with the following functions:

$$\vec{r} = (1, 0)\,, \qquad v(A) = 1 - A.$$

Finally, $\mathcal{K}$ is a standard mollifier with support contained in $B\,(0, 0.5)$, while $\mathcal{C}$ in (1.9) and in (1.11) are standard mollifiers with support contained in $B(0, 2)$ and in $B(0, 1)$, respectively. We compare several simulations according to the following control strategies:

1. $u(t) \equiv (-0.3, 0)$ corresponding to the police watercrafts pointing to the left;
2. $u(t) \equiv (0.3, 0)$ corresponding to the police watercrafts pointing to the right;
3. $u(t) \equiv (0, 0.3)$ corresponding to the police watercrafts pointing to the north;
4. $u(t) \equiv (0, -0.3)$ corresponding to the police watercrafts pointing to the south;
5. $u(t) = (0.3\cos(t), -0.3\sin(t))$ corresponding to the police watercrafts moving along a circle.

Table 1. The cost obtained by simulations of Sect. 4.1 according to the different control policies

Control	Cost
$(-0.3, 0)$	424.79
$(0.3, 0)$	469.11
$(0, 0.3)$	514.25
$(0, -0.3)$	**422.86**
$(0.3\cos(t), -0.3\sin(t))$	429.97

Among the selected strategies, the best one is the one where the police watercraft moves to the south

Bold value indicates the lowest cost

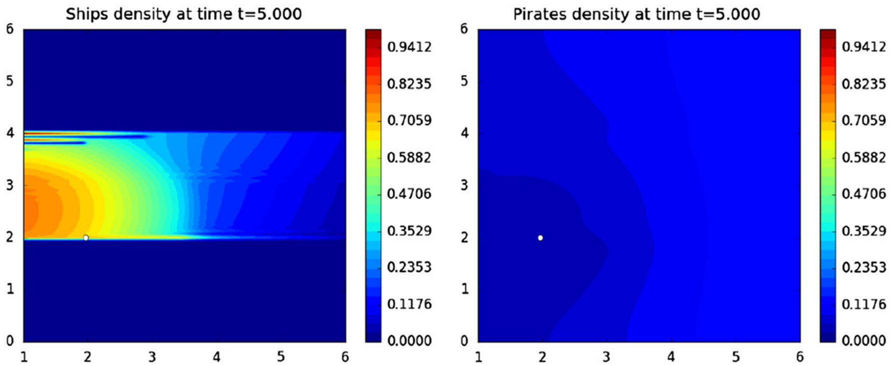

Figure 2. The densities at the final time $T = 5$ for the commercial cargos (*left*) and for the pirates (*right*). Note that around the position of the police watercraft (*the white point*) the pirate density is low. Moreover the cargo ships adjust their positions according to the one of the police

All the simulations are done on the time interval $(0, T)$ with $T = 5$ and with $N_x = N_y = 100$. The comparison is done according to the cost functional

$$J(u) = \int_0^T \left|\vec{d}(t)\right| dt + \int_0^T \int_\Omega \rho(t,x) A(t,x)\, dx dt.$$

In Table 1 one can find the costs obtained by the different strategies. The *optimal* strategy, among those considered in this part, is the one corresponding to the south movement of the police watercraft. Note that the north strategy is far from optimal; indeed, if the police watercraft moves towards the north, then it moves in a region of few ships and pirates. In Fig. 2, the densities at time $T = 5$ of commercial and pirates ships are plotted.

4.2. Different number of police watercrafts

We consider here, as in Sect. 4.1, the marine region $\Omega = (1, 6)\times(0, 6)$, i.e. $a = 1$, $b = 6$, $\alpha = 0$, and $\beta = 6$, with the following initial conditions for ρ and A

Table 2. The cost obtained by simulations of Sect. 4.2

Case	Cost
$M = 0$	1130.03
$M = 1$	422.86
$M = 2$, $u_1 \equiv (0, -0.3)$, $u_2 \equiv (0, 0.3)$	**212.78**
$M = 2$, $u_1 \equiv (0, -0.3)$, $u_2 \equiv (0.3, 0)$	228.99

The difference of the cost between the cases $M = 0$, $M = 1$, and $M = 2$ is substantial. The two cases with $M = 2$ produces similar results
Bold value indicates the lowest cost

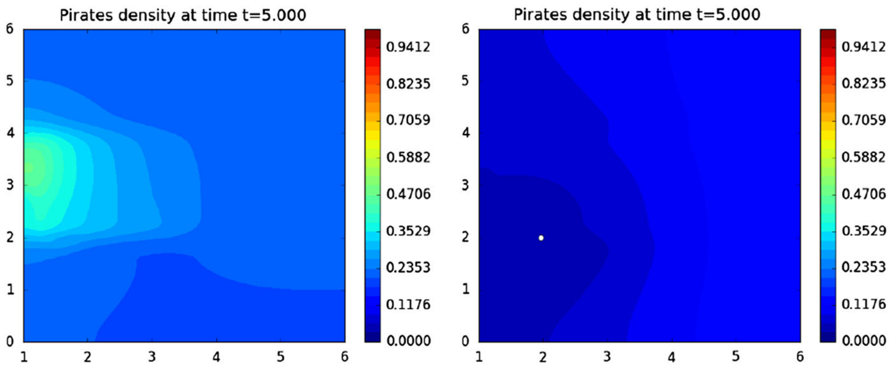

Figure 3. The density at the final time $T = 5$ for the pirates. On the *left* the case of $M = 0$ police watercrafts. On the *right* the case of $M = 1$ police watercraft

$$\begin{aligned} \rho(x, y) &= \chi_{[2,5]\times[2,4]}(x, y), \\ A(x, y) &= \chi_{[1,2]\times[2,4]}(x, y), \end{aligned}$$

where χ is the characteristic function, and with the following functions:

$$\vec{r} = (1, 0), \qquad v(A) = 1 - A.$$

Finally, $\mathcal{K}$ is a standard mollifier with support contained in $B(0, 0.5)$, while $\mathcal{C}$ in (1.9) and in (1.11) are standard mollifiers with supports contained in $B(0, 2)$ and in $B(0, 1)$, respectively. We consider the following situations.

1. $M = 0$, i.e. no police watercraft is present.
2. $M = 1$ and the control is $u_1(t) \equiv (0, -0.3)$, i.e. there is one police watercraft moving towards the south. The initial condition is $\vec{d_1}(0) = (2, 3.5)$.
3. $M = 2$ with initial positions $\vec{d_1}(0) = (2, 3.5)$. $\vec{d_2}(0) = (2, 2.5)$. The controls are $u_1(t) \equiv (0, -0.3)$ and $u_2(t) \equiv (0, 0.3)$, i.e. one moving towards the south and one towards the north.
4. $M = 2$ with initial positions $\vec{d_1}(0) = (2, 3.5)$. $\vec{d_2}(0) = (2, 2.5)$. The controls are $u_1(t) \equiv (0, -0.3)$ and $u_2(t) \equiv (0.3, 0)$, i.e. one moving towards the south and one towards the east.

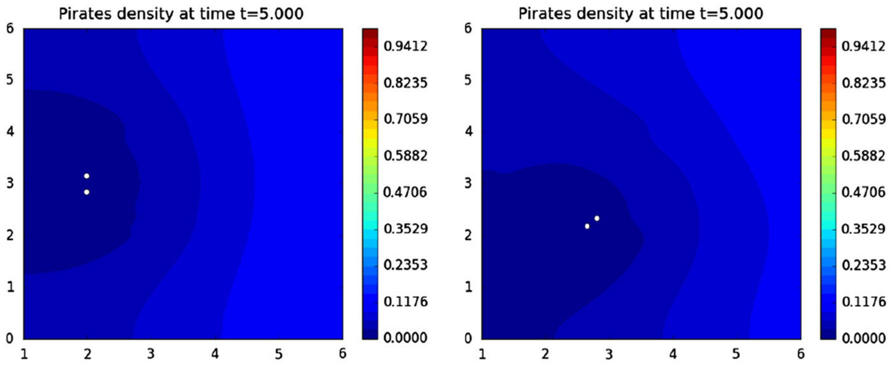

FIGURE 4. The density at the final time $T = 5$ for the pirates with $M = 2$ police watercrafts. On the *left*, the case of one control pointing to the north and one to the south. On the *right*, the case of one control pointing to the south and one to the east

All the simulations are done on the time interval $(0, T)$ with $T = 5$ and with $N_x = N_y = 100$. The comparison is done using to the cost functional

$$J(u) = \sum_{i=1}^{M} \int_0^T \left| \vec{d}_i(t) \right| dt + \int_0^T \int_\Omega \rho(t, x) A(t, x)\, dxdt.$$

In Table 2 one can find the costs obtained by the different strategies. Moreover in Figs. 3 and 4 the final densities of the pirates ships in the different cases are plotted.

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