



# On the rate of convergence of the $p$ -curve shortening flow

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**Abstract.** In this paper we give rates of convergence for the  $p$ -curve shortening flow for  $p \geq 1$  an integer, which improves on the known estimates and which are probably sharp.

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## 1. Introduction

Let us first introduce the main character of this story, the  $p$ -curve shortening flow, with  $p$  a positive integer. So, we let

$$x : \mathbb{S}^1 \times [0, T) \longrightarrow \mathbb{R}^2$$

be a family of smooth convex embeddings of  $\mathbb{S}^1$ , the unit circle, into  $\mathbb{R}^2$ . We say that  $x$  satisfies the  $p$ -curve shortening flow,  $p \geq 1$ , if  $x$  satisfies

$$\frac{\partial x}{\partial t} = -\frac{1}{p} k^p N, \quad (1)$$

where  $k$  is the curvature of the embedding and  $N$  is the normal vector pointing outwards the region bounded by  $x(\cdot, t)$ .

The  $p$ -curve shortening flow is just a natural generalisation of the well known and well studied curve shortening flow. A solution to (1) starting from an embedded convex simple curve will contract, via embedded convex curves, towards a round point in finite time: this means that if we start with a simple convex curve, via the  $p$ -curve shortening, after a convenient normalisation, which includes a time reparametrisation, the embedded curves converge smoothly to a circle (see [1]). It is also known that this convergence is exponential in the following sense (here  $\tilde{k}$  denotes the curvature of the embedded curves after normalisation)

$$\left\| \tilde{k}^{(n)} \right\|_{\infty} \leq C e^{-\delta t},$$

with where  $\tilde{k}^{(n)}$  represents the  $n$ -derivative of  $\tilde{k}$  with respect to the arclength parameter in  $\mathbb{S}^1$ ,  $n \geq 1$ , and  $\delta > 0$ . For the curve shortening flow, we can use as  $\delta$  any  $2\alpha$  for  $0 < \alpha < 1$ , this was proved by Gage and Hamilton [6] in their by now famous (by mathematical standards) paper. For the  $p$ -curve shortening, Huang in [7] showed that  $\delta$  can be taken as  $2\alpha p$ , with the same restrictions on  $\alpha$ . Interestingly enough, with the exception of the curve shortening flow ( $p = 1$ ), it has not been showed that  $\tilde{k} \rightarrow 1$  exponentially! For the curve shortening flow ( $p = 1$ ), in the book [3] exponential convergence of the curvature towards 1 is shown (see Chapter 3, Section 3 in [3]), and Andrews and Bryan showed in [2] (although they did not stated explicitly) that  $\tilde{k} \rightarrow 1$  as fast as  $e^{-2\tau}$ .

Related to this problem is the mean curvature flow, and Sesum in [9], using Huisken’s work as a departure point, has given sharp rates of convergence for this flow.

The main goal of this paper to give better rates of convergence for the  $p$ -curve shortening flow,  $p \geq 1$  an integer, than the ones previously known. Our main result, from which the said rates of convergence can be deduced, is the following (for the notation employed in the statement of the theorem, in particular the meaning of  $\|\psi\|_2$  and  $\|\cdot\|_{C^l(\mathbb{S}^1)}$ , please see the end of Sect. 2).

**Theorem 1.** *Let  $\psi > 0$  be the curvature of the initial condition to (1). Then there exists a constant  $c_p > 0$  such that if*

$$\hat{\psi}(0) \geq c_p \|\psi\|_2, \tag{2}$$

*then the solution to the normalised  $p$ -curve shortening flow ( $p$  a positive integer), that is for the curvature  $\tilde{k}$  of the curves given by the rescaled embedding  $\left(\frac{p+1}{p}\right)^{\frac{1}{p+1}} (T-t)^{-\frac{1}{p+1}} x$ , with rescaled time parameter  $\tau = -\frac{1}{p+1} \log\left(1 - \frac{t}{T}\right)$ , where  $0 < T < \infty$  is the maximum time of existence for (1), it holds that*

$$\left\| \tilde{k} - 1 \right\|_{C^l(\mathbb{S}^1)} \leq C_{p,l} e^{-(3p-1)\tau},$$

where  $C_{p,l}$  is a constant that only depends on  $p, l$  and  $\psi$ .

Together with Theorem II.1 from [1], this gives the following

**Theorem 2.** *For any simple convex curve as initial data, the normalised version of (1), converges towards a circle smoothly and the curvature of the normalised embeddings satisfy*

$$\left\| \tilde{k} - 1 \right\|_{C^l(\mathbb{S}^1)} \leq C_{p,l} e^{-(3p-1)\tau},$$

where  $C_{p,l}$  is a constant that only depends on  $p, l$  and the curvature of the initial condition.

Indeed, by the theorem of Andrews referred to above, (2) eventually holds if we start (1) with a given convex simple curve as initial data. The rate of convergence given by our main result seems to be the sharpest possible rate of

convergence for the  $p$ -curve shortening flow (see the remark at the end of this paper). Theorem 2 sharpens the convergence results given in [6, 7].

A naive idea for proving Theorem 1 would be to use the Parabolic PDE to which the normalised version of  $p$ -curve shortening flow is equivalent to (see Eq. (14) in Sect. 3.2), and then linearise around the steady solution to obtain exponential convergence. However, if we linearise around the steady solution, the elliptic part of the parabolic operator corresponding to the  $p$ -curveshortening flow has a negative eigenvalue, so no exponential convergence should be expected (see the discussion in [5] right after Theorem 2.2, and notice that when  $\lambda = 1$ , a negative eigenvalue occurs). The good news here is that, being  $k$  a curvature, it satisfies an important identity which is responsible for us being able to obtain this exponential convergence.

Our methods are based on the techniques employed in [5], that is to say on the Fourier method. Hence, we will transform our problem into (finite dimensional) approximations of an infinite dimensional dynamical system, for which appropriate estimates will be proved, and which will finally lead to a proof of Theorem 1, proof which is given in the final section of this paper. The intermediate sections are devoted to show these appropriate estimates, which, in short, amount to controlling the Fourier coefficients of a solution to (1) in terms of the average of the curvature; from this we will be able to show a time decay for the Fourier coefficients different from the average (which in fact blows-up), and which, as we said before, will lead to a proof of Theorem 1.

## 2. Basic definitions and notation

When the initial curve is convex, the  $p$ -curve shortening flow is equivalent to the following boundary value problem:

$$\begin{cases} \frac{\partial k}{\partial t} = k^2 \left( k^{p-1} \frac{\partial^2 k}{\partial \theta^2} + (p-1)k^{p-2} \left( \frac{\partial k}{\partial \theta} \right)^2 + \frac{1}{p}k^p \right) & \text{in } [0, 2\pi] \times (0, T) \\ k(\theta, 0) = \psi(\theta) & \text{on } [0, 2\pi], \end{cases} \tag{3}$$

with  $p \in \mathbb{Z}^+$ , imposing periodic boundary conditions, and with  $\psi$  a *strictly positive function*. Notice that the Maximum Principle implies that  $k$  must remain positive for all times (i.e. a convex curve remains convex). We will need to compute finite dimensional approximations of the previous partial differential equation in Fourier space, so we must establish some definitions and notation. Recall that For  $f \in L^2 [0, 2\pi]$ , its Fourier expansion is given by

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{in\theta},$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta.$$

We shall refer to  $\hat{f}(n)$  as the *Fourier coefficients* of  $f$  (and we call  $n$  the *index of the coefficient*).

We will also adopt the notation

$$\hat{u}^{*(m)}(q_1, q_2, \dots, q_m, t) = \hat{u}(q_1, t)\hat{u}(q_2, t) \cdots \hat{u}(q_m, t),$$

$$H(p, q_1, q_2) = \frac{1}{p} - (p - 1)q_1q_2 - q_1^2,$$

and define the following sets

$$\mathcal{B}_n = \{(q_1, \dots, q_{p+2}) \in \mathbb{Z}^{p+2} : q_{p+2} = n - q_1 - \dots - q_{p+1}\},$$

$$\mathcal{A}_n = \{\mathbf{q} \in \mathcal{B}_n : \text{there are } 1 \leq i < j \leq p + 2 \text{ such that } b_i \neq 0 \text{ and } b_j \neq 0\},$$

and,

$$\mathcal{C}_n = \{\mathbf{q} \in \mathcal{A}_n : q_j \neq 0, \pm 1, \text{ for all } 1 \leq j \leq p + 2\}.$$

From now on  $\mathcal{Z}$  will denote a finite set of integers which contains 0 (i.e.,  $0 \in \mathcal{Z}$ ), and which is symmetric around 0 (i.e., if  $n \in \mathcal{Z}$  then  $-n \in \mathcal{Z}$ ).

Using this notation, in Fourier space, the  $p$ -curve shortening flow can be approximated by the following finite dimensional dynamical system:

$$\begin{cases} \frac{d}{dt} \hat{k}(0, t) = \frac{1}{p} \hat{k}(0, t)^{p+2} + \sum_{\mathbf{q} \in \mathcal{A}_0 \cap \mathcal{Z}^{p+2}} H(p, q_1, q_2) \hat{k}^{*(p+2)}(\mathbf{q}, t), \\ \frac{d}{dt} \hat{k}(n, t) = \left(\frac{p+2}{p} - n^2\right) \hat{k}(0, t)^{p+1} \hat{k}(n, t) \\ \quad + \sum_{\mathbf{q} \in \mathcal{A}_n \cap \mathcal{Z}^{p+2}} H(p, q_1, q_2) \hat{k}^{*(p+2)}(\mathbf{q}, t), \text{ if } n \neq 0, n \in \mathcal{Z}, \end{cases} \tag{4}$$

with initial condition

$$\hat{k}(n, 0) = \hat{\psi}(n), \text{ if } n \in \mathcal{Z},$$

Formally, the  $\hat{k}$  in the system right above should bear, for instance, a subindex which makes its dependence on  $\mathcal{Z}$  explicit, but as this is understood from now on, we will suppress it in what follows (and as our estimates will not depend on  $\mathcal{Z}$ , this should be of no importance).

Notice also that (4) is an autonomous system, so there is a unique and smooth solution for a short time (see [4]). We will also make use of the seminorms  $\|\cdot\|_\beta$ , which are defined as in [5] as follows:

$$\|f\|_\beta = \max \left\{ \sup_{\xi \in \mathbb{Z}} |\xi|^\beta \left| \operatorname{Re} \left( \hat{f}(\xi) \right) \right|, \sup_{\xi \in \mathbb{Z}} |\xi|^\beta \left| \operatorname{Im} \left( \hat{f}(\xi) \right) \right| \right\}.$$

As usual, we define  $C^l([0, 2\pi])$ ,  $l = 0, 1, 2, \dots$ , as the space of functions with continuous derivatives of order  $l$ , equipped with the norm

$$\|f\|_{C^l([0, 2\pi])} = \max_{j=0, \dots, l} \sup_{\theta \in [0, 2\pi]} \left| \frac{d^j f(\theta)}{d\theta^j} \right|.$$

### 3. Technical lemmas and intermediate results

We shall follow closely the arguments presented in [8]. Therefore we must show that for given a solution to (4), we can control the Fourier coefficients different from 0 in terms of the 0-th coefficient. The fact that the first eigenvalue  $\lambda_1$  of the elliptic operator related to (3) satisfies  $\lambda_1 < 0$  is the main difficulty we must face, as this makes difficult to control the  $\pm 1$ -Fourier coefficients in terms of the 0 th Fourier coefficients. Once we have done this, all that is left is to follow the arguments presented in [5, 8]. The key to our proofs is that a curvature function of a locally convex curve satisfies (see [6])

$$Q(k) = \int_0^{2\pi} \frac{e^{i\theta}}{k(\theta, t)} d\theta = 0,$$

since this identity, once we have control over the higher Fourier coefficients (those with  $|n| \geq 2$ ) assuming control over the  $\pm 1$  coefficients, allows us to control the  $\pm 1$  Fourier coefficients.

The careful reader must notice that is some of the proofs given in this paper our estimates are given for system (4), that this estimates are independent of  $\mathcal{Z}$ , and that this allows us to take a limit so the results are valid for the full system (3).

#### 3.1. Controlling the Fourier coefficients

We start with a technical lemma.

**Lemma 1.** *There is a  $\delta > 0$  such that if the initial condition  $\psi$  of (3) satisfies:*

$$2\delta \cdot \psi(0) \geq q^2 |\hat{k}(q, t)|,$$

and

$$\hat{k}(0, t) \geq (1 - \delta) \hat{\psi}(0),$$

for  $t \in (0, \tau)$ , then  $\hat{k}(0, t)$  is non decreasing.

*Proof.* From the hypothesis of the lemma,

$$U := \sum_{\mathbf{q} \in \mathcal{A}_0 \cap \mathcal{Z}^{p+2}} H(p, q_1, q_2) \hat{k}^{*(p+2)}(\mathbf{q}, t) = O\left(\delta \hat{k}(0, t)^{p+2}\right),$$

and the implicit constant in the big  $O$  notation does not depend on  $\mathcal{Z}$ . Hence, for  $\delta > 0$  small enough, the term  $\frac{1}{p} \hat{k}(0, t)$  dominates the the term  $U$  in the differential equation for  $\hat{k}(0, t)$ . This implies that  $\frac{d}{dt} \hat{k}(0, t) > 0$ , and the conclusion of the lemma follows.  $\square$

Now we show some estimates that give us some control over the Fourier coefficients of  $k$ .

**Lemma 2.** *There is a  $\delta > 0$  such that if the initial condition  $\psi$  of (3) satisfies:*

$$2\delta \cdot \hat{\psi}(0) \geq |\hat{\psi}(\pm 1)| \quad \text{and} \quad \delta \cdot \hat{\psi}(0) \geq q^2 |\hat{\psi}(q)| \quad \text{for} \quad |q| \geq 2$$

holds, and for  $t \in (0, \tau)$

$$2\delta \cdot \hat{\psi}(0) \geq |\hat{k}(\pm 1, t)|,$$

and

$$\hat{k}(0, t) \geq (1 - \delta) \hat{\psi}(0).$$

Then

$$\delta \cdot \hat{\psi}(0) \geq q^2 |\hat{k}(q, t)|.$$

*Proof.* Let us consider the quantity  $R_n = \left| \frac{\hat{k}(n, t)}{\hat{\psi}(0)} \right|$ , and we prove that is nonincreasing for  $n$  fixed.

We compute:

$$\frac{d}{dt} \log R_n = \left( \frac{p+2}{p} - n^2 \right) \hat{k}(0, t)^{p+1} + \sum_{i=1}^2 B_i,$$

where the  $B_i$  terms are given by:

$$B_1 = \frac{1}{\hat{k}(n, t)} \sum_{\mathbf{q} \in \mathcal{C}_n \cap \mathcal{Z}^{p+2}} H(p, q_1, q_2) \hat{k}^{*(p+2)}(\mathbf{q}, t),$$

$$B_2 = \frac{1}{\hat{k}(n, t)} \sum_{\mathbf{q} \in \mathcal{A}_n \setminus \mathcal{C}_n \cap \mathcal{Z}^{p+2}} H(p, q_1, q_2) \hat{k}^{*(p+2)}(\mathbf{q}, t).$$

We bound  $B_1$ ,

$$\begin{aligned} \left| \hat{k}(n, t) \right| |B_1| &\leq \frac{1}{p} \sum_{\mathbf{q} \in \mathcal{C}_n \cap \mathcal{Z}^{p+2}} \left| \hat{k}^{*(p+2)}(\mathbf{q}, t) \right| \\ &\quad + (p-1) \sum_{\mathbf{q} \in \mathcal{C}_n \cap \mathcal{Z}^{p+2}} |q_1| |q_2| \left| \hat{k}^{*(p+2)}(\mathbf{q}, t) \right| \\ &\quad + \sum_{\mathbf{q} \in \mathcal{C}_n \cap \mathcal{Z}^{p+2}} q_1^2 \left| \hat{k}^{*(p+2)}(\mathbf{q}, t) \right|. \end{aligned}$$

Now if  $\mathbf{q} \in \mathcal{C}_n \cap \mathcal{Z}^{p+2}$ , then

$$\left| \hat{k}^{*(p+2)}(\mathbf{q}, t) \right| = \frac{2^{p+2} \delta^{p+2} \hat{\psi}(0)^{p+2}}{q_1^2 \cdots q_{p+1}^2 (n - q_1 - \cdots - q_{p+1})^2},$$

and hence

$$\left| \hat{k}(n, t) \right| |B_1| \leq \delta^{p+2} C'_p \hat{\psi}(0)^{p+2},$$

with  $C'_p$  independent of  $\mathcal{Z}$ . Since  $|\hat{k}(n, t)| = |\hat{k}(0, t)|/n^2$ , we get

$$|B_1| \leq \delta^{p+2} C'_p n^2 \hat{\psi}(0)^{p+1}. \tag{5}$$

Splitting the sums and using similar calculations as in (5), we obtain

$$|B_2| \leq \delta C''_p n^2 \hat{\psi}(0)^{p+1}, \tag{6}$$

and again  $C_p''$  is independent of  $\mathcal{Z}$ .

Since the sum of the absolute value of all these terms can be made smaller than

$$\left(n^2 - \frac{p+2}{p}\right) \hat{k}(0, t)^{p+1},$$

for  $n \geq 2$  by taking  $\delta > 0$  small enough, then the  $R_n$  term is non increasing for  $\delta > 0$  small enough. From this the conclusion of the lemma follows.  $\square$

As we have been doing so far, in what follows the indices of the coefficients of our Fourier expansions are restricted to a fixed but arbitrary set  $\mathcal{Z}$ , so keep this in mind. And, as announced at the beginning of this section, since the estimates are independent of  $\mathcal{Z}$ , a limiting procedure will give the result for when we take as  $\mathcal{Z}$  the whole set of integers.

The following lemma shows that once we have control over the Fourier coefficients  $\hat{k}(n, t)$  for  $|n| \geq 2$  on an interval of time, we also have control over the coefficients  $\hat{k}(\pm 1, t)$  on the same interval of time.

**Lemma 3.** *Let  $\psi$  be such that  $Q(\psi) = 0$ . There is a  $\delta' > 0$  such that if  $0 < \delta \leq \delta'$  and  $\delta\hat{\psi}(0) \geq |\hat{\psi}(\pm 1)|$  then whenever  $|\hat{k}(n, t)| n^2 \leq \delta\hat{k}(0, t)$  for all  $|n| \geq 2$ , for all  $t \in [0, \tau]$ , then we also have  $|\hat{k}(q, t)| q^2 \leq \delta\hat{k}(0, t)$  for  $q = \pm 1$  on the same time interval.*

*Proof.* Since we have that  $\delta\hat{\psi}(0) \geq |\hat{\psi}(\pm 1)|$ , we can choose a  $\tau' \in [0, \tau]$  such that  $|\hat{k}(\pm 1, t)| \leq 2\delta\hat{k}(0, t)$  on  $[0, \tau']$  (remember we are working with an arbitrary but final dimensional approximation of the  $p$ -curve shortening flow). We have the following identity

$$\frac{1}{k(\theta, t)} = \frac{1}{\hat{k}(0, t)} \frac{1}{1 + \sum_{q \neq 0} \frac{\hat{k}(q, t)}{\hat{k}(0, t)} e^{iq\theta}} = \frac{1}{\hat{k}(0, t)} \frac{1}{1 + z} = \frac{1}{\hat{k}(0, t)} \sum_{n=0}^{\infty} (-1)^n z^n,$$

where  $z = z(\theta, t) = \sum_{q \neq 0} \frac{\hat{k}(q, t)}{\hat{k}(0, t)} e^{iq\theta}$ . It can be easily seen that for the Fourier coefficients of  $z$  are given by:

$$\hat{z}(p, t) = \begin{cases} 0 & \text{if } p = 0 \\ \frac{\hat{k}(p, t)}{\hat{k}(0, t)} & \text{otherwise} \end{cases}.$$

Taking Fourier transform, this implies

$$\begin{aligned} \widehat{\left(\frac{1}{k}\right)}(-1, t) &= \frac{1}{\hat{k}(0, t)} \\ &\times \left( -\hat{z}(-1, t) + \sum_{m=2}^{\infty} (-1)^m \sum_{q_1 + \dots + q_m = -1} \hat{z}(q_1, t) \cdots \hat{z}(q_m, t) \right). \end{aligned}$$

Since  $k$  is the curvature of a convex curve, we have  $Q(k) = 0$ , so

$$\left(\frac{1}{k}\right)(-1, t) = \int_{\mathbb{S}^1} \frac{e^{-(-1)i\theta}}{k(\theta, t)} d\theta = Q(k) = 0,$$

then

$$|\hat{z}(-1, t)| = \left| \sum_{m=2}^{\infty} (-1)^m \sum_{q_1+\dots+q_m=-1} \hat{z}(q_1, t) \cdots \hat{z}(q_m, t) \right|.$$

In order to estimate the sum in the right side, let us notice that

$$\begin{aligned} \sum_{q_1+\dots+q_m=-1} \hat{z}(q_1, t) \cdots \hat{z}(q_m, t) &= \sum_{j=0}^{m-1} \binom{m}{j} \hat{z}(1, t)^j \sum_{l=0}^{m-j-1} \binom{m-j}{l} \hat{z}(-1, t)^j \\ &\quad \sum_{q_{j+l+1}+\dots+q_m=-1-j+l} \hat{z}(q_{j+l+1}, t) \cdots \hat{z}(q_m, t). \end{aligned}$$

Now we proceed to estimate  $S_m = \sum_{q_1+\dots+q_m=-1} \hat{z}(q_1, t) \cdots \hat{z}(q_m, t)$ ,

$$\begin{aligned} |S_m| &\leq \sum_{j=0}^{m-1} \binom{m}{j} |\hat{z}(1, t)|^j \sum_{l=0}^{m-j-1} \binom{m-j}{l} |\hat{z}(-1, t)|^l \\ &\quad \sum_{\substack{q_{j+l+1}+\dots+q_m \\ =-1-j+l}} \delta^{m-j-l} \frac{1}{q_{j+l+1}^2} \cdots \frac{1}{q_m^2} \\ &\leq \sum_{j=0}^m \binom{m}{j} |\hat{z}(1, t)|^j \sum_{l=0}^{m-j} \binom{m-j}{l} |\hat{z}(-1, t)|^l \delta^{m-j-l} C_1^{m-j-l} \\ &= (|\hat{z}(1, t)| + |\hat{z}(-1, t)| + \delta C_1)^m, \end{aligned}$$

where  $C_1$  is a constant independent of  $\mathcal{Z}$  and  $\delta$ .

As we have that  $|\hat{z}(\pm 1, t)| \leq 2\delta$ , we get

$$|S_m| \leq (|\hat{z}(1, t)| + |\hat{z}(-1, t)| + \delta C_1)^m \leq \delta^m (4 + C_1)^m.$$

Therefore

$$|\hat{z}(-1, t)| \leq \frac{\delta^2(4 + C_1)^2}{1 - \delta(4 + C_1)} \leq \delta$$

as long as  $\delta \leq \frac{1}{2(4 + C_1)}$ . □

**Lemma 4.** *There is a  $\delta > 0$ , independent of  $\mathcal{Z}$ , such that if the initial condition  $\psi$  of (3) satisfies  $Q(\psi) = 0$  and*

$$\delta \cdot \hat{\psi}(0) \geq q^2 |\hat{\psi}(q)|,$$

*then for all times  $t$  (as long as the solution to (4) exists),*

$$\delta \psi(0) \geq q^2 \left| \hat{k}(q, t) \right|.$$



*Proof.* There exists a  $\tau > 0$  such that the interval  $[0, \tau]$  is maximal with respect to the following property: for all  $t \in [0, \tau]$  we have

$$2\delta \cdot \hat{\psi}(0) \geq q^2 |\hat{k}(q, t)|,$$

and

$$\hat{k}(0, t) \geq (1 - \delta) \hat{\psi}(0).$$

Now applying Lemmas 1 and 2, we obtain that

$$\delta\psi(0) \geq q^2 \left| \hat{k}(q, t) \right|,$$

whenever  $|q| \geq 2$ . Applying Lemma 3, we get that

$$\delta\psi(0) \geq \left| \hat{k}(\pm 1, t) \right|,$$

when  $t \in [0, \tau]$ . Hence we have that  $\delta\psi(0) \geq q^2 \left| \hat{k}(q, \tau) \right|$  and if we apply the same arguments as before we can show that there is a  $\tau_1 > \tau$  such that if  $t \in [0, \tau_1]$  then  $\delta \cdot \hat{\psi}(0) \geq q^2 \left| \hat{k}(q, t) \right|$ , contradicting the maximality of  $[0, \tau]$ .  $\square$

For  $\delta > 0$  small enough, assuming that for the initial condition we have  $\delta \hat{\psi}(0) \geq q^2 |\hat{\psi}(q)|$ , we have now control over all the Fourier coefficients of the solution. The arguments in [5] now apply almost verbatim: see the upcoming sections.

### 3.2. Decay of the Fourier coefficients

Again, all the estimates proved in this section are valid for any choice of  $\mathcal{Z}$ , and are also independent of the choice. Our main purpose is to show that the Fourier coefficients  $\hat{k}(n, t)$ ,  $n \neq 0$ , go to 0 as  $t \rightarrow T$ . To begin, we have, as in [5], the Trapping Lemma (Lemma 3.2 in [5]). Keep in mind that we are always under the assumption that  $\psi > 0$  is the curvature function of a simple convex closed curve (or equivalently, the identity  $Q(\psi) = 0$  holds).

**Theorem 3.** (Trapping Lemma) *There exists a constant  $c_p > 0$  independent of the choice of  $\mathcal{Z}$  such that if the initial datum  $\psi$  satisfies the inequality*

$$\hat{\psi}(0) \geq c_p \|\psi\|_2,$$

*then there exists a  $\gamma > 0$  that depends on  $\psi$  such that the solution to (4) satisfies*

$$\left| \hat{k}(n, t) \right| \leq \frac{\hat{\psi}(0) e^{-\gamma|n|t}}{c_p |n|^2}, \quad n \neq 0. \tag{7}$$

Also, in the same way as Lemmas 3.3 and 3.4 are obtained in [5], we have a Blow-up Lemma.

**Lemma 5.** (Blow-up) *There is a  $c_p > 0$  (the same as in the Trapping Lemma) such that if the initial condition  $\psi$  of (3) satisfies*

$$\cdot \hat{\psi}(0) \geq c_p \left\| \hat{\psi} \right\|_2,$$

then there are constants  $c, c' > 0$  such that

$$\frac{c}{T-t} \leq \hat{k}(0, t)^{p+1} \leq \frac{c'}{T-t}. \tag{8}$$

From now on, we assume that  $\psi$  satisfies

$$\hat{\psi}(0) \geq c_p \|\psi\|_2,$$

where  $c_p$  is such that the Trapping Lemma holds.

We have a few important observations. First, integrating the ODE for  $\hat{k}(n, t)$ , we obtain

$$\hat{k}(n, t) = \hat{k}(n, \tau) e^{-(n^2 - \frac{p+2}{p}) \int_\tau^t \hat{k}(0, \sigma)^{p+1} d\sigma} + \int_\tau^t h(s) e^{-(n^2 - \frac{p+2}{p}) \int_s^t \hat{k}(0, \sigma)^{p+1} d\sigma} ds \tag{9}$$

where  $h(t)$  is given by

$$h(t) = \sum_{\mathbf{q} \in \mathcal{A}_n \cap \mathbb{Z}^{p+2}} H(p, q_1, q_2) \Phi(\mathbf{q}, t) \hat{k}^{*(p+2)}(\mathbf{q}, t),$$

Applying the *Trapping Lemma* we get

$$|h(t)| \leq C_p \hat{k}(0, t)^p.$$

Also, from the Trapping Lemma, there exists  $C, \mu > 0$ , such that

$$\left| \hat{k}(n, t) \right| \leq C e^{-|\mu|t}, \quad \text{for } t \geq \frac{T}{2}. \tag{10}$$

We shall use these observations in proving the following decay (in time) estimates for the Fourier coefficients of  $k$ .

**Proposition 6.** *There exists  $\epsilon_0 > 0$  which depends on  $p$ , and a  $\mu > 0$  that depends also on  $p$  and on  $\psi$ , such that if  $t > \frac{T}{2} > 0$  then there is a constant  $b > 0$  such that for any  $0 < \epsilon < \epsilon_0$ , for  $n \neq 0, \pm 1$ , the following estimate holds for the solution of (3),*

$$\left| \hat{k}(n, t) \right| < b e^{-\mu|n|} (T-t)^\epsilon \quad \text{whenever } t > \frac{T}{2}.$$

*Proof.* (See also the proof of Lemma 3.6 in [5]) First we have, for  $\delta > 0$  small enough, by the Blow-up Lemma,

$$\begin{aligned} \left| \hat{k}(n, t) \right| &\leq \left| \hat{k}(n, T-\delta) \right| e^{-(n^2 - \frac{p+2}{p}) \int_{T-\delta}^t \hat{k}(0, s)^{p+1} ds} \\ &\quad + \int_{T-\delta}^t |h(s)| e^{-(n^2 - \frac{p+2}{p}) \int_s^t \hat{k}(0, \sigma)^{p+1} d\sigma} ds \\ &\leq \left| \hat{k}(n, T-\delta) \right| \left( \frac{T-t}{\delta} \right)^{\eta\alpha(n,p)} \\ &\quad + \int_{T-\delta}^t |h(s)| e^{-(n^2 - \frac{p+2}{p}) \int_s^t \hat{k}(0, \sigma)^{p+1} d\sigma} ds, \end{aligned}$$

where  $\eta > 0$  is a constant and

$$\alpha(n, p) = \left( n^2 - \frac{p+2}{p} \right) \frac{p}{p+1}.$$

We are going to estimate the term inside the integral in the last inequality. As before we split  $h(s)$  into sums of the form

$$J_{i_1, i_2, \dots, i_l} = \sum_{\substack{\mathbf{q} \in \mathcal{A}_0 \cap \mathcal{Z}^{p+2}; \\ q_j = 0 \Leftrightarrow j = i_1, i_2, \dots, i_l}} H(p, q_1, q_2) \hat{k}^{*(p+2)}(\mathbf{q}, s).$$

Using this, the Trapping Lemma and the observations after its statement, we get

$$|J_{i_1, i_2, \dots, i_l}| \leq \frac{C e^{-\mu|n|}}{(T-s)^{\frac{l}{p+1}}},$$

and since  $l \leq p$ , we finally obtain

$$|h(s)| \leq \frac{C e^{-\mu|n|}}{(T-s)^{\frac{p}{p+1}}}.$$

Then we have

$$\begin{aligned} |\hat{k}(n, t)| &\leq |\hat{k}(n, T-\delta)| \left( \frac{T-t}{\delta} \right)^{\eta\alpha(n, p)} \\ &\quad + C (T-t)^{\eta\alpha(n, p)} e^{-\mu|n|} \int_{T-\delta}^t \frac{1}{(T-s)^{\eta\alpha(n, p) + \frac{p}{p+1}}} ds. \end{aligned}$$

Using again (10) and the fact that  $\alpha(n, p) \geq \alpha(2, p) > 0$ , we obtain

$$\begin{aligned} |\hat{k}(n, t)| &\leq |\hat{k}(n, T-\delta)| \left( \frac{T-t}{\delta} \right)^{\eta\alpha(n, p)} + C e^{-\mu|n|} (T-t)^{1-\frac{p}{p+1}} \\ &\leq \frac{b}{2} e^{-\mu|n|} \left( (T-t)^{\eta\alpha(n, p)} + (T-t)^{1-\frac{p}{p+1}} \right), \end{aligned}$$

for a constant  $b > 0$ . Then we have

$$|\hat{k}(n, t)| \leq b e^{-\mu|n|} (T-t)^\epsilon,$$

for any  $0 < \epsilon < \min \left\{ \eta\alpha(2, p), 1 - \frac{p}{p+1} \right\} = \epsilon_0$ . □

Next we are going to improve on the decay estimates so far for the Fourier coefficients. To be able to do this, we will need the following lemma.

**Lemma 7.** *There is a  $t_0$  such that if  $t \in (t_0, T)$ , then we have the estimates*

$$\left( \frac{p+1}{p} \right)^{\frac{1}{p+1}} \hat{k}(0, t) \leq \frac{1}{\left[ (T-t) - c_1 (T-t)^{1+\frac{2}{p+1}} \right]^{\frac{1}{p+1}}},$$

and

$$\left(\frac{p+1}{p}\right)^{\frac{1}{p+1}} \hat{k}(0, t) \geq \frac{1}{\left[(T-t) + c_1 (T-t)^{1+\frac{2}{p+1}}\right]^{\frac{1}{p+1}}}.$$

*Proof.* We have that the following differential inequality

$$\frac{d}{dt} \hat{k}(0, t) \leq \frac{1}{p} \hat{k}(0, t)^{p+2} + A \hat{k}(0, t)^p,$$

holds for a constant  $A > 0$  independent of  $t$ . This is equivalent to

$$\frac{1}{\hat{k}(0, t)^{p+2}} \frac{d}{dt} \hat{k}(0, t) \leq \frac{1}{p} + A \hat{k}(0, t)^{-2}.$$

Using Lemma 5, from the previous differential inequality we obtain

$$\frac{1}{\hat{k}(0, t)^{p+2}} \frac{d}{dt} \hat{k}(0, t) \leq \frac{1}{p} + C (T-t)^{\frac{2}{p+1}},$$

The result follows by integration. For the other inequality, notice that there exists a constant  $A'$  so we also have the following differential inequality

$$\frac{d}{dt} \hat{k}(0, t) \geq \frac{1}{p} \hat{k}(0, t)^{p+2} + A' \hat{k}(0, t)^p.$$

□

In order to proceed and to improve on our decay estimates, we need to estimate the integral

$$I = \frac{p+1}{p} \int_{T-\delta}^t \hat{k}(0, \tau)^{p+1} d\tau$$

from below. From the previous lemma, and since  $\delta$  is small, using Taylor's Theorem, we arrive at

$$I \geq \int_{T-\delta}^t \frac{d\tau}{T-\tau + (T-\tau)^{1+\frac{2}{p+1}}} \geq -\ln\left(\frac{T-t}{\delta}\right) - c,$$

where  $c > 0$ , is a constant that only depends on  $p$  and  $\delta$ . Using this and (9), we get

$$\begin{aligned} \left| \hat{k}(n, t) \right| &\leq C \left| \hat{k}(n, T-\delta) \right| \left( \frac{T-t}{\delta} \right)^{\alpha(n,p)} \\ &\quad + C (T-t)^{\alpha(n,p)} \int_{T-\delta}^t \left( \frac{1}{T-s} \right)^{\alpha(n,p)} h(s) ds \end{aligned} \tag{11}$$

here

$$\alpha(n, p) = \left( n^2 - \frac{p+2}{p} \right) \frac{p}{p+1}. \tag{12}$$

Using the estimate of Proposition 6 and the fact that if  $\mathbf{q} \in \mathcal{A}_n$  then  $\mathbf{q}$  has at least two entries different from 0, we get

$$|h(s)| \leq \frac{C e^{-\mu'|n|}}{(T-s)^{\frac{p}{p+1}}} (T-s)^{2\epsilon}.$$

If we introduce the bound from Proposition 6 in (11) we get

$$\begin{aligned} \left| \hat{k}(n, t) \right| &\leq C \left| \hat{k}(n, T - \delta) \right| \left( \frac{T - t}{\delta} \right)^{\alpha(n, p)} \\ &\quad + C(T - t)^{\alpha(n, p)} e^{-\mu' |n|} \int_{T - \delta}^t \frac{(T - t)^{2\epsilon}}{(T - s)^{\alpha(n, p) + \frac{p}{p+1}}} ds, \end{aligned}$$

for a well chosen  $0 < \mu' < \mu$ , and from which we obtain the estimate (we use again the fact  $\alpha(n, p) \geq \alpha(2, p)$ )

$$\left| \hat{k}(n, t) \right| \leq C' e^{-\mu' |n|} (T - t)^{\min\{\alpha(2, p), 1 - \frac{p}{p+1} + 2\epsilon\}}.$$

If it happens that  $\alpha(2, p) > 1 - \frac{p}{p+1} + 2\epsilon$ , using this new bound an pluggin it into (11), we improve again our estimate on  $\hat{k}(n, t)$ :

$$\left| \hat{k}(n, t) \right| \leq C'' e^{-\mu'' |n|} (T - t)^{\min\{\alpha(2, p), 3(1 - \frac{p}{p+1}) + 4\epsilon\}} \quad (0 < \mu'' < \mu').$$

Hence, if we repeat this procedure a finite number of times we arrive at an estimate

$$\left| \hat{k}(n, t) \right| \leq D e^{-\xi |n|} (T - t)^{\alpha(2, p)}, \quad n \neq 0, \pm 1. \tag{13}$$

where  $\xi > 0$  is a constant independent of  $n$ , and  $\alpha(2, p)$  is defined by (12) (so its value is  $(3p - 2) / (p + 1)$ ).

Now, we must show now that the Fourier coefficients  $\hat{k}(\pm 1, t)$  satisfy the same estimate. In this case we write

$$z(n, t) = \left( \frac{p + 1}{p} \right)^{\frac{1}{p+1}} (T - t)^{\frac{1}{p+1}} \hat{k}(n, t).$$

We also have an identity which follows from  $Q(k) = 0$  (here we use that  $\hat{z}(-1, t)$  is the conjugate of  $\hat{z}(1, t)$ , as  $z$  is real valued)

$$\sum_{n=0}^{\infty} \binom{2n}{n} |z(1, t)|^{2n} z(1, t) = \sum_{m=2}^{\infty} \sum'_{q_1 + q_2 + \dots + q_m = 1} z(q_1) \dots z(q_m, t),$$

where the prime ( $'$ ) in the inner sum of the righthand side indicates that at least one of the  $q_j \neq \pm 1$ . Using similar computations as in the proof of Lemma 3, together with (13), we can conclude that

$$\sum_{n=0}^{\infty} \binom{2n}{n} |z(1, t)|^{2n} z(1, t) = O\left( (T - t)^{\frac{3p-1}{p+1}} \right),$$

hence, if  $\delta > 0$  is small enough, we can deduce that

$$z(1, t) = O\left( (T - t)^{\frac{3p-1}{p+1}} \right),$$

which is just that  $\left| \hat{k}(\pm 1, t) \right| \leq C (T - t)^{\frac{3p-2}{p+1}}$ . So we have proved

**Proposition 8.** *Let  $\psi > 0$  is a smooth  $2\pi$ -periodic function which satisfies that  $Q(\psi) = 0$ . There exists a positive constant  $c_p$  such that if*

$$\hat{\psi}(0) \geq c_p \|\psi\|_2,$$

then a solution to (3) satisfies

$$\left\| k(\theta, t) - \hat{k}(0, t) \right\|_{C^k[0, 2\pi]} \leq M_{p,k}(T - t)^{\frac{3p-2}{p+1}}$$

where  $T$  is the blow-up time and  $M_{p,k}$  is a constant that depends only on  $p, k$  and  $\psi$ .

We normalise the solution of (3) by means of the following transformation:

$$\tilde{k}(\theta, t) = \left(\frac{p+1}{p}\right)^{\frac{1}{p+1}} (T-t)^{\frac{1}{p+1}} k(\theta, t), \quad \tau = -\frac{1}{p+1} \log\left(1 - \frac{t}{T}\right).$$

Applying chain rule, we obtain the following normalised version of (3)

$$\begin{cases} \frac{\partial \tilde{k}}{\partial \tau} = p\tilde{k}^{p+1} \frac{\partial^2 \tilde{k}}{\partial \theta^2} + p(p-1)\tilde{k}^p \left(\frac{\partial \tilde{k}}{\partial \theta}\right)^2 + \tilde{k}^{p+2} - \tilde{k} & \text{in } [0, 2\pi] \times (0, \infty) \\ \tilde{k}(\theta, 0) = \left(\frac{(p+1)T}{p}\right)^{\frac{1}{p+1}} \psi(\theta). \end{cases} \tag{14}$$

Using this normalisation, Proposition 8 translates into:

**Corollary 9.** *Let  $\psi > 0$  is a smooth  $2\pi$ -periodic function which satisfies that  $Q(\psi) = 0$ . There exists a positive constant  $c_p$  such that if*

$$\hat{\psi}(0) \geq c_p \|\psi\|_2,$$

then the normalization  $\tilde{k}$  of  $k$  satisfies:

$$\left\| \tilde{k} - \hat{\tilde{k}}(0, t) \right\|_{C^k[0, 2\pi]} \leq M_{p,k} e^{-(3p-1)\tau},$$

where  $M_{p,k}$  is a positive constant that depends only on  $p, k$  and  $\psi$ .

#### 4. Exponential convergence of the normalised curvature towards 1: proof of the main result

We will need the following improvement over Lemma 7.

**Lemma 10.** *There are constants  $c_0, c_1 > 0$  for which the following estimates hold on  $(0, T)$*

$$\left(\frac{p+1}{p}\right)^{\frac{1}{p+1}} \hat{k}(0, t) \leq \frac{1}{\left[(T-t) - c_1 (T-t)^{1+\frac{6p-2}{p+1}}\right]^{\frac{1}{p+1}}},$$

and

$$\left(\frac{p+1}{p}\right)^{\frac{1}{p+1}} \hat{k}(0, t) \geq \frac{1}{\left[(T-t) + c_0 (T-t)^{1+\frac{6p-2}{p+1}}\right]^{\frac{1}{p+1}}}.$$

*Proof.* Notice that using (13) and the equation satisfied by  $\hat{k}(0, t)$ , we have the following differential inequality, which is valid for a constant  $A > 0$ ,

$$\frac{d}{dt} \hat{k}(0, t) \leq \frac{1}{p} \hat{k}(0, t)^{p+2} + A(T-t)^{\frac{6p-4}{p+1}} \hat{k}(0, t)^p.$$

Integrating, from  $t$  to  $T$ , we obtain the second inequality. Analogously, for a constant  $A'$ , we have the differential inequality

$$\frac{d}{dt} \hat{k}(0, t) \geq \frac{1}{p} \hat{k}(0, t)^{p+2} - A'(T-t)^{\frac{6p-4}{p+1}} \hat{k}(0, t)^p,$$

which by integration gives the first inequality. □

Finally we have our main result.

**Theorem 4.** *Let  $\psi > 0$  be the initial condition of (3) (so it is the curvature function of a convex simple curve). Then there exists a constant  $c_p > 0$  such that if*

$$\hat{\psi}(0) \geq c_p \|\psi\|_2,$$

then the solution to (14) satisfies

$$\left\| \tilde{k} - 1 \right\|_{C^l[0, 2\pi]} \leq C_{p,l} e^{-(3p-1)\tau},$$

where  $C_{p,l}$  is a constant that only depends on the initial condition  $\psi$  and  $p$  and  $l$ .

*Proof.* Let

$$\tilde{u}(0, t) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{k}(\theta, t) d\theta = \left( \frac{p+1}{p} \right)^{\frac{1}{p+1}} (T-t)^{\frac{1}{p+1}} \hat{k}(0, t).$$

Using Lemma 10 we can estimate

$$\begin{aligned} \tilde{u}(0, t) - 1 &= \left( \frac{p+1}{p} \right)^{\frac{1}{p+1}} \hat{k}(0, t) (T-t)^{\frac{1}{p+1}} - 1 \\ &\leq \frac{(T-t)^{\frac{1}{p+1}}}{\left[ (T-t) - c_1(T-t)^{1+\frac{6p-2}{p+1}} \right]^{\frac{1}{p+1}}} - 1 \leq C(T-t)^{\frac{6p-2}{p+1}}. \end{aligned}$$

Analogously,

$$1 - \tilde{u}(0, t) \leq C(T-t)^{\frac{6p-2}{p+1}}.$$

Since  $e^{-\tau} = \left( \frac{T-t}{T} \right)^{\frac{1}{p+1}}$ , then

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \tilde{k}(\theta, \tau) d\theta - 1 \right| \leq C e^{-(6p-2)\tau}.$$

Applying the triangular inequality and Corollary 9 we can conclude that

$$\left\| \tilde{k} - 1 \right\|_{C^l[0, 2\pi]} \leq \left\| \tilde{k} - \tilde{u} \right\|_{C^l[0, 2\pi]} + \|\tilde{u} - 1\|_{C^l[0, 2\pi]} \leq C_{p,l} e^{-(3p-1)\tau}$$

for some constant  $C_{p,l}$  that depends on  $p$  and  $l$ . □

#### 4.1. Final remarks

The rate of convergence obtained in Theorem 1, seems to be the best possible in general. We have not been able to produce an example where the rate given in Theorem 1 is met; however, to justify our claim, we refer to the comments after the statement of Theorem 2.2 in [5]: The first positive eigenvalue of the elliptic part of (14), i.e., the left hand side of the equation, when linearised around the steady solution  $\tilde{k} \equiv 1$  is precisely  $3p - 1$ .

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