Nonlinear Differ. Equ. Appl. (2017) 24:41

1021-9722/17/040001-12
published online June 23, 2017
DOI 10.1007/s00030-017-0465-7

Nonlinear Differential Equations and Applications NoDEA

Global existence and blow-up for a fourth order parabolic equation involving the Hessian

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Abstract. This paper deals with a fourth order parabolic equation involving the Hessian, which was studied in Escudero et al. (J Math Pures Appl 103(4):924–957, 2015) recently, where the initial conditions for $W_0^{2,2}$-norm and $W_0^{1,4}$-norm blow-up were got when the initial energy $J(u_0) \leq d$, where $d > 0$ is the mountain-pass level. The purpose of this paper is to study two of the open questions proposed in the paper, that is, L^p-norm blow-up and the behavior of the solutions when $J(u_0) > d$. For the case of $J(u_0) < 0$, we prove the solution blows up in finite time with L^2-norm. Moreover, we estimate the blow-up time and the blow-up rate. For the case of $J(u_0) > d$, we find two sets Ψ_α and Φ_α, and prove that the solution blows up in finite time if the initial value belongs to Ψ_α, while the solution exists globally and tends to zero as time $t \to +\infty$ when the initial value belongs to Φ_α.

Mathematics Subject Classification. 35J30, 35K25, 35K35, 35K55, 35G31, 35Q70.

Keywords. Fourth order parabolic equation, Global existence, Blow-up.

1. Introduction

In this paper, we study the following fourth order parabolic equation

$$\begin{cases} u_t + \Delta^2 u = \det\left(D^2 u\right), & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u = u_\nu = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

This work is partially supported by the Basic and Advanced Research Project of CQC-STC Grant cstc2016jcyjA0018, NSFC Grant 11201380, Fundamental Research Funds for the Central Universities grant XDJK2015A16.

where $\Omega \subset \mathbb{R}^2$ is an open, bounded domain with smooth boundary $\partial\Omega$. The initial value $u_0(x) \in W_0^{2,2}(\Omega)$, ν is the unit out normal vector on $\partial\Omega$.

Problem (1.1) describes the process of epitaxial growth, which is a technique by means of which the deposition of new material on existing layers of the same material takes place under high vacuum conditions. Problem (1.1) and its related problems have been studied in [1–5,10]. Especially, in [5], the authors studied the behavior of the solutions to problem (1.1), and they proposed seven open questions, the purpose of this paper is to study two of them. In order to introduce the main results of [5] and the two open questions, let's firstly introduce some notations, sets and functionals. Throughout this paper, we denote by (x, y) the element of $\mathbb{R}^2$. The norm of $W^{m,p}(\Omega)$ is denoted by $\|\cdot\|_{W^{m,p}(\Omega)}$ except for the $W_0^{2,2}$-norm, which is denoted by $\|\cdot\|$, and

$$\|u\|^2 = \|\Delta u\|_2^2 = \int_\Omega |\Delta u|^2.$$

For $p \in [1, +\infty)$, we denote by $\|\cdot\|_p$ the L^p-norm and

$$\|u\|_p^p = \int_\Omega |u|^p.$$

Next, we define the energy functional related to the stationary equation of (1.1) by

$$J(u) := \frac{1}{2}\|u\|^2 - I(u), \quad \forall u \in W_0^{2,2}(\Omega), \tag{1.2}$$

where

$$I(u) := \int_\Omega u_x u_y u_{xy}.$$

The corresponding mountain-pass level is given by (see [5])

$$d := \inf_{\gamma\in\Gamma} \max_{0\le s\le 1} J(\gamma(s)) \tag{1.3}$$

where

$$\Gamma := \left\{\gamma \in C([0, 1], W_0^{2,2}(\Omega)); \gamma(0) = 0, J(\gamma(1)) < 0\right\}.$$

By [5, Theorem 2.6], we know that d can be lower bounded in terms of the best constant for embedding $W_0^{2,2}(\Omega) \hookrightarrow W_0^{1,4}(\Omega)$, namely

$$d \ge \frac{8}{27} \min_{u\in W_0^{2,2}(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega |\Delta u|^2\right)^2}{\int_\Omega |\nabla u|^4} > 0. \tag{1.4}$$

The Nehari manifold is defined by

$$\mathcal{N} := \left\{u \in W_0^{2,2}(\Omega)\setminus\{0\}, \langle J'(u), u\rangle = \|u\|^2 - 3I(u) = 0\right\}, \tag{1.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{-2,2}(\Omega)$ and $W_0^{2,2}(\Omega)$. Moreover, we define

$$\mathcal{N}_+ := \left\{u \in W_0^{2,2}(\Omega); \|u\|^2 > 3I(u)\right\}, \tag{1.6}$$

$$\mathcal{N}_- := \left\{u \in W_0^{2,2}(\Omega); \|u\|^2 < 3I(u)\right\}. \tag{1.7}$$

In [5], for the subcritical initial energy case ($J(u_0) < d$) and the critical initial energy case ($J(u_0) = d$), the authors obtained the initial conditions for global existence and finite time blow-up of solutions to problem (1.1). In detail, they proved that the solution of problem (1.1) blows up in finite time with $W_0^{2,2}$-norm and $W_0^{1,4}$-norm if $u_0 \in \mathcal{N}_-$ and $J(u_0) \le d$. Moreover, they proved that if $u_0 \in \mathcal{N}_+, J(u_0) \le d$, then the solution $u(t)$ of problem (1.1) exists globally and $u(t) \to 0$ in $W^{4,2}(\Omega)$ as $t \to +\infty$.

Furthermore, the authors proposed seven open questions and two of them are as follows:

(Q1) **Blow-up in L^p-norms:** From [5, Theorems 4.8 and 4.10], we learn that when $W_0^{2,2}$-norm blow-up occurs, then also the $W_0^{1,4}$-norm blows up. What about the L^p-norm blow-up?

(Q2) **High energy initial data:** In order to prove global existence or finite time blow-up for problem (1.1) we assumed that $J(u_0) \le d$. What happens for $J(u_0) > d$?

We will give answers to the above questions in this paper. Let

$$J^\alpha := \left\{u \in W_0^{2,2}(\Omega), J(u) \le \alpha\right\}, \tag{1.8}$$

where α is a positive constant. For all $\alpha > d$, it is easy to see that

$$\mathcal{N}_\alpha := \mathcal{N} \cap J^\alpha = \left\{u \in \mathcal{N} \,\middle|\, \|u\| \le \sqrt{6\alpha}\right\} \ne \emptyset, \tag{1.9}$$

and then we define

$$\lambda_\alpha := \inf\left\{\frac{1}{2}\|u\|_2^2 \,\middle|\, u \in \mathcal{N}_\alpha\right\}, \quad \Lambda_\alpha := \sup\left\{\frac{1}{2}\|u\|_2^2 \,\middle|\, u \in \mathcal{N}_\alpha\right\}. \tag{1.10}$$

Clearly we have the following monotonicity properties

$$\alpha \mapsto \lambda_\alpha \text{ is nonincreasing}, \quad \alpha \mapsto \Lambda_\alpha \text{ is nondecreasing}. \tag{1.11}$$

We denote by $T = T(u_0)$ the maximal existence time of the solutions to problem (1.1). If $T = +\infty$, we denote by

$$\omega(u_0) = \bigcap_{t \ge 0} \overline{\{u(s) : s \ge t\}},$$

the ω-limit set of $u_0 \in W_0^{2,2}(\Omega)$, where the closure is taken in $W_0^{2,2}(\Omega)$.

Now, we are ready to state the main results of this paper. The first result is about **(Q1)**, which can be stated as the following theorem.

Theorem 1.1. *Let $u_0 \in W_0^{2,2}(\Omega)$ be such that $J(u_0) < 0$, then the solution to problem (1.1) blows up at a finite time T with L^2-norm. Moreover, the blow-up time can be estimated by*

$$T \le -\frac{\|u_0\|_2^2}{3J(u_0)},$$

and the blow-up rate can be estimated by

$$\|u(\cdot,t)\|_2 \le -\frac{\|u_0\|_2^3}{3J(u_0)}(T-t)^{-1}.$$

Next, we give the result about **(Q2)**.

Theorem 1.2. *For any $\alpha \in (d, +\infty)$, the following conclusions hold:*

(i) *If $u_0 \in \Phi_\alpha$, then the solution $u(t)$ to problem (1.1) exists globally and $u(t) \to 0$ with $W_0^{2,2}$-norm as $t \to +\infty$;*
(ii) *If $u_0 \in \Psi_\alpha$, then the solution $u(t)$ to problem (1.1) blows up in finite time,*

where

$$\begin{aligned} \Phi_\alpha &:= \mathcal{N}_+ \cap \left\{ \phi \in W_0^{2,2}(\Omega) \,\middle|\, \frac{1}{2}\|\phi\|_2^2 < \lambda_\alpha,\ d < J(\phi) \le \alpha \right\}, \\ \Psi_\alpha &:= \mathcal{N}_- \cap \left\{ \phi \in W_0^{2,2}(\Omega) \,\middle|\, \frac{1}{2}\|\phi\|_2^2 > \Lambda_\alpha,\ d < J(\phi) \le \alpha \right\}, \end{aligned} \tag{1.12}$$

and λ_α, Λ_α are two constants defined in (1.10), J is defined in (1.2).

Remark 1.3. *We make two remarks about* Theorem 1.2*.*

(i) Lemma 2.1 *shows that*

$$\frac{64}{243\alpha\kappa^8} \le \lambda_\alpha \le \Lambda_\alpha \le \frac{3}{\lambda_1}\alpha,$$

where λ_1 and κ are two positive constants given in (2.2) and (2.4) respectively. Then the definitions of Φ_α and Ψ_α make sense.

(ii) *In* [5, Theorem 4.7]*, the authors gave the following result:*

Assume that $u_0 \in W_0^{2,2}(\Omega)$ and

$$\lambda_1 \|u_0\|_2^2 > 6J(u_0), \tag{1.13}$$

where λ_1 is defined in (2.2), then the solution $u = u(t)$ to problem (1.1) blows up in finite time, that is, there exists $T > 0$ such that $\|u\| \to +\infty$ and $\|u\|_{W_0^{1,4}(\Omega)} \to +\infty$ as $t \to T$.

We can prove that if u_0 satisfies (1.13) and $J(u_0) > d$, then $u_0 \in \Psi_{J(u_0)}$. Therefore, by (ii) of Theorem 1.2*, the solution blows up in finite time, which means the blow-up condition of* Theorem 1.2 *is weaker than the blow-up condition of* [5, Theorem 4.7] *under the assumption $J(u_0) > d$.*

In fact, by the definition of λ_1 in (2.2), (1.13) and the definition of $J(u_0)$ we have

$$\|u_0\|^2 \ge \lambda_1 \|u_0\|_2^2 > 6J(u_0),$$

and

$$6J(u_0) = 3\|u_0\|^2 - 6I(u_0).$$

Combining these two relations, we get $3I(u_0) > \|u_0\|^2$, *i.e.,*

$$u_0 \in \mathcal{N}_- \tag{1.14}$$

On the other hand, by (1.13) *we obtain*

$$\frac{1}{2}\|u_0\|_2^2 > \frac{3}{\lambda_1}J(u_0).$$

By Lemma 2.1, *we get* $\frac{3}{\lambda_1}J(u_0) \geq \Lambda_{J(u_0)}$, *then we have*

$$\frac{1}{2}\|u_0\|_2^2 > \Lambda_{J(u_0)}. \tag{1.15}$$

By (1.14) *and* (1.15), *we get* $u_0 \in \Psi_{J(u_0)}$.

The rest of this paper is organized as follows. In Sect. 2, we give some important lemmas, which will be used in the proof of the main results. In Sect. 3, we give the proof of the above theorems.

2. Preliminaries

We begin this section with the following eigenvalue problem:

$$\begin{cases} \Delta^2 u = \lambda u, & x \in \Omega, \\ u = u_\nu = 0, x \in \partial\Omega. \end{cases} \tag{2.1}$$

Let λ_1 be the first eigenvalue of problem (2.1). By [8], we know that λ_1 is positive, simple and it can be characterized in the following variational form

$$\lambda_1 = \inf_{u \in W_0^{2,2}(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}, \tag{2.2}$$

which implies

$$\|u\|_2^2 \leq \frac{1}{\lambda_1}\|u\|^2, \quad \forall u \in W_0^{2,2}(\Omega). \tag{2.3}$$

Let $\kappa > 0$ be the optimal constant of the following Gagliardo–Nirenberg inequality [6], i,e.,

$$\|\nabla u\|_4 \leq \kappa \|u\|^{\frac{3}{4}} \|u\|_2^{\frac{1}{4}}, \quad \forall u \in W_0^{2,2}(\Omega). \tag{2.4}$$

Lemma 2.1. *Let* $\lambda_\alpha, \Lambda_\alpha$ *be defined in* (1.10). *For any constant* $\alpha > d$, *we have*

$$\frac{64}{243\alpha\kappa^8} \leq \lambda_\alpha \leq \Lambda_\alpha \leq \frac{3}{\lambda_1}\alpha,$$

where d,λ_1 *and* κ *are given in* (1.3), (2.2) *and* (2.4) *respectively.*

Remark 2.2. *It holds that*

$$\frac{64}{243\alpha\kappa^8} \leq \frac{3}{\lambda_1}\alpha. \tag{2.5}$$

In fact, by (2.3) *and* (2.4), *we have*

$$\|\nabla u\|_4^4 \sqrt{\lambda_1} \leq \kappa^4 \|u\|^4, \quad \forall u \in W_0^{2,2}(\Omega),$$

then it follows from $\alpha > d$ and (1.4) that

$$\alpha > d \geq \frac{8}{27} \min_{u \in W_0^{2,2}(\Omega)\setminus\{0\}} \frac{\|u\|^4}{\|\nabla u\|_4^4} \geq \frac{8\sqrt{\lambda_1}}{27\kappa^4},$$

i.e., (2.5) *holds.*

Proof of Lemma 2.1. By the definitions of λ_α and Λ_α in (1.10), we obviously have $\lambda_\alpha \leq \Lambda_\alpha$, then it follows from Remark 2.2 that we only need to prove

$$\lambda_\alpha \geq \frac{64}{243\alpha\kappa^8} \text{ and } \Lambda_\alpha \leq \frac{3}{\lambda_1}\alpha.$$

We firstly show that $\Lambda_\alpha \leq \frac{3}{\lambda_1}\alpha$. For any $u \in \mathcal{N}_\alpha \subset W_0^{2,2}(\Omega)$, by (1.9) and (2.3), we know that

$$\frac{1}{2}\|u\|_2^2 \leq \frac{1}{2\lambda_1}\|u\|^2 \leq \frac{3}{\lambda_1}\alpha.$$

So it follows from the definition of Λ_α in (1.10) that $\Lambda_\alpha \leq \frac{3}{\lambda_1}\alpha$.

Next we prove that $\lambda_\alpha \geq \frac{64}{243\alpha\kappa^8}$. By the inequality [5, (17)], we have

$$I(u) \leq \frac{1}{4}\|\nabla u\|_4^2\|u\|, \quad \forall u \in W_0^{2,2}(\Omega),$$

which combines with (2.4) implies

$$I(u) \leq \frac{\kappa^2}{4}\|u\|^{\frac{5}{2}}\|u\|_2^{\frac{1}{2}}, \quad \forall u \in W_0^{2,2}(\Omega). \tag{2.6}$$

By the definition of $\mathcal{N}$ in (1.5) and (2.6), we obtain

$$\|u\|^2 = 3I(u) \leq \frac{3\kappa^2}{4}\|u\|^{\frac{5}{2}}\|u\|_2^{\frac{1}{2}}, \ \forall u \in \mathcal{N} \subset W_0^{2,2}(\Omega),$$

i.e.,

$$\left(\frac{4}{3\kappa^2}\right)^4 \|u\|^{-2} \leq \|u\|_2^2, \quad \forall u \in \mathcal{N}. \tag{2.7}$$

Now, for all $u \in \mathcal{N}_\alpha \subset \mathcal{N}$, by (1.9) we have $\|u\| \leq \sqrt{6\alpha}$, then it follows from (2.7) that

$$\|u\|_2^2 \geq \left(\frac{4}{3\kappa^2}\right)^4 (6\alpha)^{-1} = \frac{128}{243\alpha\kappa^8}.$$

So by the definition of λ_α in (1.10), we have

$$\lambda_\alpha = \inf_{u \in \mathcal{N}_\alpha} \frac{1}{2}\|u\|_2^2 \geq \frac{64}{243\alpha\kappa^8}.$$

□

Lemma 2.3. *For any $\alpha > 0$, if $u \in J^\alpha \cap \mathcal{N}_+$, then*

$$\|u\| < \sqrt{6\alpha}. \tag{2.8}$$

Proof. For any $u \in \mathcal{N}_+$, by the definition of $\mathcal{N}_+$ in (1.6) we have

$$\|u\|^2 > 3I(u).$$

Then it follows from the definition of J^α in (1.8) that

$$\alpha \geq J(u) = \frac{1}{2}\|u\|^2 - I(u) > \frac{1}{6}\|u\|^2, \quad \forall u \in J^\alpha \cap \mathcal{N}_+,$$

which implies (2.8). □

Lemma 2.4. ([5, Theorem 2.5]) *For all $u \in W_0^{2,2}(\Omega)$, the following implications hold:*

(i) *If $0 < \|u\|^2 < 6d$, then $u \in \mathcal{N}_+$;*
(ii) *If $u \in \mathcal{N}_+$ and $J(u) < d$, then $0 < \|u\|^2 < 6d$;*
(iii) *If $u \in \mathcal{N}_-$, then $\|u\|^2 > 6d$.*

By Lemma 2.4(iii) we have

$$\operatorname{dist}(0, \mathcal{N}_-) = \min_{u \in \mathcal{N}_-} \|u\| \geq \sqrt{6d} > 0. \tag{2.9}$$

Lemma 2.5. ([5, Lemma 4.1]) *If $u = u(t)$ solves problem (1.1) then its energy $J(u(t))$ satisfies*

$$\frac{d}{dt}J(u(t)) = -\|u_t(t)\|_2^2 \leq 0. \tag{2.10}$$

Lemma 2.6. ([5, Lemma 4.4]) *Let $u_0 \in W_0^{2,2}(\Omega)$ and let $u = u(t)$ be the corresponding solution to problem (1.1). Then for all $t \in [0, T)$ we have*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|u\|^2 - 3I(u) = 0. \tag{2.11}$$

Lemma 2.7. ([5, Theorem 4.6]) *Let $u_0 \in \mathcal{N}_-$ be such that $J(u_0) \leq d$. Then the solution $u = u(t)$ to problem (1.1) blows up in finite time, that is, there exists $T > 0$ such that $\|u(t)\| \to +\infty$ as $t \to T$. Moreover, the blow up also occurs in the $W_0^{1,4}$-norm, that is, $\|u(t)\|_{W_0^{1,4}(\Omega)} \to +\infty$ as $t \to T$.*

3. Proofs of the main results

In this section we will prove our main results. We firstly prove Theorem 1.1 by utilizing the methods in [9,11,12].

Proof of Theorem 1.1. Let $u(t)$ be the solution of problem (1.1) with initial value $u_0 \in W_0^{2,2}(\Omega)$ satisfying $J(u_0) < 0$. We define

$$f(t) = \frac{1}{2}\|u\|_2^2, \tag{3.1}$$

and

$$g(t) = -3J(u) = 3I(u) - \frac{3}{2}\|u\|^2. \tag{3.2}$$

Then by (2.11) we have

$$f'(t) = 3I(u) - \|u\|^2, \tag{3.3}$$

and by (2.10) we have

$$g'(t) = -3\frac{d}{dt}J(u(t)) = 3\|u_t\|_2^2 \geq 0. \tag{3.4}$$

Since $J(u_0) < 0$, by the definition of $g(t)$ in (3.2) we have $g(0) = -3J(u_0) > 0$. Then it follows from (3.4) that $g(t) > 0$ for all $t \in [0,T)$. Combining (3.2) and (3.3) we get

$$f'(t) \geq g(t) > 0, \quad \forall t \in [0,T), \tag{3.5}$$

then $f(t) > 0$ for all $t \in [0,T)$.

For any $t \in [0,T)$, by (3.1), (3.4), Schwartz's inequality and (3.5) we obtain

$$\begin{aligned} f(t)g'(t) &= \frac{3}{2}\|u\|_2^2\|u_t\|_2^2 \geq \frac{3}{2}\left(\int_\Omega uu_t\right)^2 \\ &= \frac{3}{2}\left[f'(t)\right]^2 \geq \frac{3}{2}f'(t)g(t), \end{aligned}$$

which can be rewritten as

$$\frac{g'(t)}{g(t)} \geq \frac{3}{2}\frac{f'(t)}{f(t)}.$$

Integrating above inequality from 0 to t we get

$$\frac{g(t)}{[f(t)]^{\frac{3}{2}}} \geq \frac{g(0)}{[f(0)]^{\frac{3}{2}}},$$

then by (3.5) we have

$$\frac{f'(t)}{[f(t)]^{\frac{3}{2}}} \geq \frac{g(0)}{[f(0)]^{\frac{3}{2}}}. \tag{3.6}$$

Integrating inequality (3.6) from 0 to t, we see

$$\frac{1}{[f(t)]^{\frac{1}{2}}} \leq \frac{1}{[f(0)]^{\frac{1}{2}}} - \frac{1}{2}\frac{g(0)}{[f(0)]^{\frac{3}{2}}}t. \tag{3.7}$$

Clearly, (3.7) cannot hold for all time, this means $f(t)$ blows up at some finite time T, i.e.,

$$\lim_{t\to T} f(t) = +\infty, \tag{3.8}$$

then by the definition of $f(t)$ in (3.1), we know that $u(t)$ blows up at T with L^2-norm.

Next, we estimate T and the blow-up rate. Let $t \to T$ in (3.7), then by (3.8) and the definition of $f(t), g(t)$ we get

$$T \leq \frac{2f(0)}{g(0)} = -\frac{\|u_0\|_2^2}{3J(u_0)}.$$

Moreover, by integrating the inequality (3.6) from t to T and (3.8) we have

$$f(t) \leq (T-t)^{-2}\left[\frac{g(0)}{2[f(0)]^{\frac{3}{2}}}\right]^{-2},$$

so it follows from the definition of $f(t)$ and $g(t)$ that

$$\|u(\cdot,t)\|_2 \le -\frac{\|u_0\|_2^3}{3J(u_0)}(T-t)^{-1}.$$

□

Next, we prove Theorem 1.2 and the idea of the proof comes from [7].

Proof of Theorem 1.2. Let $u(t)$ be a solution of problem (1.1), it follows from (2.11) that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = 3I(u) - \|u\|^2. \tag{3.9}$$

By Lemma 2.5, we know that $J(u(t))$ is non-increasing with respect to t, so we get

$$J(u(t)) \le J(u_0) \quad \forall t \in [0,T). \tag{3.10}$$

(i). If $u_0 \in \Phi_\alpha$, then by the definition of Φ_α in (1.12) and the properties of λ_α in (1.11), we have $d < J(u_0) \le \alpha$ and

$$u_0 \in \mathcal{N}_+, \quad \frac{1}{2}\|u_0\|_2^2 < \lambda_\alpha \le \lambda_{J(u_0)}. \tag{3.11}$$

We claim that $u(t) \in \mathcal{N}_+$ for all $t \in [0,T)$. Arguing by contradiction, if the claim is not true, then there is a $t_0 \in (0,T)$ such that $u(t) \in \mathcal{N}_+$ for $0 \le t < t_0$ and $u(t_0) \in \mathcal{N}$. Then by the definition of $\mathcal{N}_+$ in (1.6) and (3.9), we know that $\|u(t)\|_2^2$ is strictly decreasing on $[0,t_0]$. So, it follows from (3.10) and (3.11) that

$$\frac{1}{2}\|u(t_0)\|_2^2 < \frac{1}{2}\|u_0\|_2^2 < \lambda_{J(u_0)}, \tag{3.12}$$

$$J(u(t_0)) \le J(u_0). \tag{3.13}$$

Then by $u(t_0) \in \mathcal{N}$ and (3.13), we get $u(t_0) \in \mathcal{N}_{J(u_0)}$. Hence, it follows from the definition of $\lambda_{J(u_0)}$ in (1.10) that

$$\lambda_{J(u_0)} \le \frac{1}{2}\|u(t_0)\|_2^2,$$

which contradicts (3.12), so the claim is true. Then by (3.10) we obtain $u(t) \in J^{J(u_0)} \cap \mathcal{N}_+$. Hence, by Lemma 2.3 we can obtain

$$\|u(t)\| < \sqrt{6J(u_0)}, \quad \forall t \in [0,T). \tag{3.14}$$

Since the right-hand of (3.14) is independent of T, then we get $T = +\infty$, and we further have (3.14) holds for $0 \le t < +\infty$,

$$u(t) \in J^{J(u_0)} \cap \mathcal{N}_+, \quad \forall t \in [0,+\infty), \tag{3.15}$$

and $\|u(t)\|_2^2$ is strictly decreasing on $[0,+\infty)$

Now for any $\omega \in \omega(u_0)$, by the above discussions, we get

$$\frac{1}{2}\|\omega\|_2^2 < \lambda_{J(u_0)} \quad \text{and} \quad J(\omega) \le J(u_0),$$

so we get $\omega \notin \mathcal{N}_{J(u_0)}$ and $\omega \in J^{J(u_0)}$, then it follows from the definition of $\mathcal{N}_{J(u_0)}$ in (1.9) that $\omega(u_0) \cap \mathcal{N} = \emptyset$. Namely, for any $\omega \in \omega(u_0)$ we have $\omega \notin \mathcal{N}$.

Next, we prove that

$$\omega(u_0) = \{0\} \tag{3.16}$$

In fact, it follows from $u(t) \in \mathcal{N}_+$ and the definitions of J and $\mathcal{N}_+$ that

$$J(u(t)) = \frac{1}{2}\|u(t)\|^2 - I(u(t)) > \frac{1}{2}\|u(t)\|^2 - \frac{1}{3}\|u(t)\|^2 = \frac{1}{6}\|u(t)\|^2, \tag{3.17}$$

then we have $J(u(t)) > 0$, i.e., $J(u(t))$ is bounded below. Since the $J(u(t))$ is non-increasing with respect to t we know that there is a constant c such that

$$\lim_{t\to+\infty} J(u(t)) = c.$$

So for any $\omega \in \omega(u_0)$, we have $J(u_\omega(t)) = c$ for all $t \geq 0$, where $u_\omega(t)$ is the solution of (1.1) with initial value ω. Then combining (2.10) we get $u_\omega(t) \equiv \omega$, and then it follows from (3.9) that

$$\|\omega\|^2 - 3I(\omega) = 0, \quad \forall \omega \in \omega(u_0). \tag{3.18}$$

Combining (3.18), $\omega \notin \mathcal{N}$ and the definition of $\mathcal{N}$ in (1.5) we get (3.16). In other words, the solution $u(t) \to 0$ as $t \to +\infty$.

(ii). Similar to the proof of the first part, if $u_0 \in \Psi_\alpha$, then by the definition of Ψ_α in (1.12) and the properties of Λ_α in (1.11), we have $d < J(u_0) \leq \alpha$ and

$$u_0 \in \mathcal{N}_-, \quad \frac{1}{2}\|u_0\|_2^2 > \Lambda_\alpha \geq \Lambda_{J(u_0)}. \tag{3.19}$$

We claim that $u(t) \in \mathcal{N}_-$ for all $t \in [0, T)$. Arguing by contradiction, if the claim is not true, then there exists $t_1 > 0$ such that $u(t) \in \mathcal{N}_-$ for $0 \leq t < t_1$ and $u(t_1) \in \mathcal{N}$. Then by the definition of $\mathcal{N}_-$ in (1.7) and (3.9) we know that $\|u(t)\|_2^2$ is strictly increasing on $[0, t_1]$. So it follows from (3.10) and (3.19) that

$$\frac{1}{2}\|u(t_1)\|_2^2 > \frac{1}{2}\|u_0\|_2^2 > \Lambda_{J(u_0)}, \tag{3.20}$$

$$J(u(t_1)) \leq J(u_0). \tag{3.21}$$

Then by $u(t_1) \in \mathcal{N}$ and (3.21) we get $u(t_1) \in \mathcal{N}_{J(u_0)}$, hence it follows from the definition of $\Lambda_{J(u_0)}$ in (1.10) that

$$\Lambda_{J(u_0)} \geq \frac{1}{2}\|u(t_1)\|_2^2,$$

which contradicts (3.20), then the claim is true.

Now we assume $T = +\infty$, then

$$u(t) \in J^{J(u_0)} \cap \mathcal{N}_-, \quad \forall t \in [0, +\infty), \tag{3.22}$$

and $\|u(t)\|_2^2$ is strictly increasing on $[0, +\infty)$. So for every $\omega \in \omega(u_0)$, have

$$\frac{1}{2}\|\omega\|_2^2 > \frac{1}{2}\|u_0\|_2^2 > \Lambda_{J(u_0)} \quad \text{and} \quad J(\omega) \leq J(u_0),$$

so we get $\omega \in J^{J(u_0)}$ and $\omega \notin \mathcal{N}_{J(u_0)}$, then it follows from the definition of $\mathcal{N}_{J(u_0)}$ in (1.9) that $\omega(u_0) \cap \mathcal{N} = \emptyset$.

Since $J(u(t))$ is non-increasing with respect to t, then we have following two cases:

(a) there is a constant c such that $\lim_{t\to+\infty} J(u(t)) = c$;

(b) $\lim_{t\to+\infty} J(u(t)) = -\infty$.

Next we will prove both the above cases contradict to $T = +\infty$, then we get the solution $u(t)$ blows up in finite time.

We first consider case (a). If $\lim_{t\to+\infty} J(u(t)) = c$, then by the similar discussions as in the proof of (i), we can get (3.18), and then combining $\omega(u_0)\cap \mathcal{N} = \emptyset$ and the definition of $\mathcal{N}$ in (1.5) we get $\omega(u_0) = \{0\}$. However, by (2.9) we know $\text{dist}(0, \mathcal{N}_-) > 0$, which implies $0 \notin \omega(u_0)$, so we get a contradiction.

Finally we consider case (b). If $\lim_{t\to+\infty} J(u(t)) = -\infty$, then must exist a time t_1 such that $J(u(t_1)) \le d$. Since we have proved that $u(t) \in \mathcal{N}_-$ for all $t \in [0, +\infty)$, then $u(t_1) \in \mathcal{N}_-$. Taking $u(t_1)$ as the initial value, by Lemma 2.7 we know that the corresponding solution $U(t) = u(t + t_1)$ blows up in finite time, which contradicts $T = +\infty$, thus Theorem 1.2 is proved. □

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Received: 3 December 2016.
Accepted: 13 June 2017.