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Metastability for nonlinear convection–diffusion equations

Special issue in honor of Alberto Bressan.

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Abstract. We study the metastable dynamics of solutions to nonlinear evolutive equations of parabolic type, with a particular attention to the case of the viscous scalar Burgers equation with small viscosity ε. In order to describe rigorously such slow motion, we adapt the strategy firstly proposed in Mascia and Strani (SIAM J Math Anal 45:3084–3113, 2013) by linearizing the original equation around a *metastable state* and by studying the system obtained for the couple (ξ, v), where ξ is the position of the internal shock layer and v is a perturbative term. The main result of this paper provides estimates for the speed of the shock layer and for the error v; in particular, in the case of the viscous Burgers equation, we prove they are exponentially small in ε. As a consequence, the time taken for the solution to reach the unique stable steady state is exponentially large, and we have *exponentially slow motion*.

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1. Introduction

In the analysis of PDEs, **metastability** is a broad term describing the persistence of unsteady structures for a very long time. We refer to *metastable dynamics* when, in a first stage, the evolution of a (non-stationary) solution is

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so slow that the solution *appears* to be stable, and it is only after a very long time that it converges to its asymptotic limit. To be more precise, the qualitative features of metastable dynamics are the following: through a transient process a pattern of internal layers is formed from initial data over a $\mathcal{O}(1)$ time interval; at this point of the dynamics the solution is far from any stable configuration of the system. Once this pattern is formed, the subsequent motion of the interfaced solution is exceedingly slow, converging to its asymptotic limit. As a consequence, two different time scales emerge: a short time phase where one observes the formation of this pattern of internal interfaces (the so called metastable state), and a subsequent long time phase where these interfaces interact until the solution stabilizes to the stable steady state of the system.

A large class of evolutive PDEs exhibits such phenomenon; far from being exhaustive, we here recall the study of metastability for the Allen–Cahn equation performed in [5,6,12], for the Cahn–Hilliard equation [1–3], for the Jin–Xin systems [19], for convection-reaction-diffusion and reaction-diffusion equations [20–22] and for hyperbolic variations of the Allen–Cahn equation [9–11].

Given $\ell > 0$, $I = [-\ell, \ell]$ and the space $X := L^2(I)$, in this paper we mean to investigate the metastable dynamics of solutions $u : [0, +\infty) \to X$ to the following Cauchy problem

$$u_t = \mathcal{F}^\varepsilon[u], \qquad u(\cdot, 0) = u_0. \tag{1.1}$$

Here, $\mathcal{F}^\varepsilon$ denotes a nonlinear differential operator, complemented with appropriate boundary conditions, depending in a singular way with respect to ε, meaning that the operator $\mathcal{F}^0$ is of lower order with respect to $\mathcal{F}^\varepsilon$.

In particular, the main example we have in mind is the initial-boundary value problem for the viscous Burgers equation, i.e.

$$\begin{cases} \partial_t u = \varepsilon\, \partial_x^2 u - \partial_x f(u), & x \in I, t > 0, \\ u(x,0) = u_0(x), & x \in I, \\ u(-\ell, t) = u_-,\ u(\ell, t) = u_+, & t > 0, \end{cases} \tag{1.2}$$

where the unknown $u \in C^0([0, +\infty); L^2(I))$ and the the flux function is $f(u) = u^2/2$. Formally, in the limit $\varepsilon \to 0$, problem (1.2) reduces to the initial-boundary value problem for the hyperbolic Burgers equation

$$\partial_t u + \partial_x \left(\tfrac{1}{2} u^2\right) = 0, \tag{1.3}$$

for which it is well known that solutions with discontinuities may appear, provided they satisfy the Rankine–Hugoniot and the entropy conditions. In particular, in this setting these conditions state that only jumps from a value u_- greater than u_+ are admitted and their speed of propagation is given by

$$s = \frac{[\![f(u)]\!]}{[\![u]\!]} = \frac{u_- + u_+}{2}.$$

When looking for stationary solutions, because of the above discussion, for any given $\xi \in (-\ell, \ell)$ the one parameter family

$$U_{hyp}(x;\xi) := u_- \chi_{(-\ell,\xi)} + u_+ \chi_{(\xi,\ell)}$$

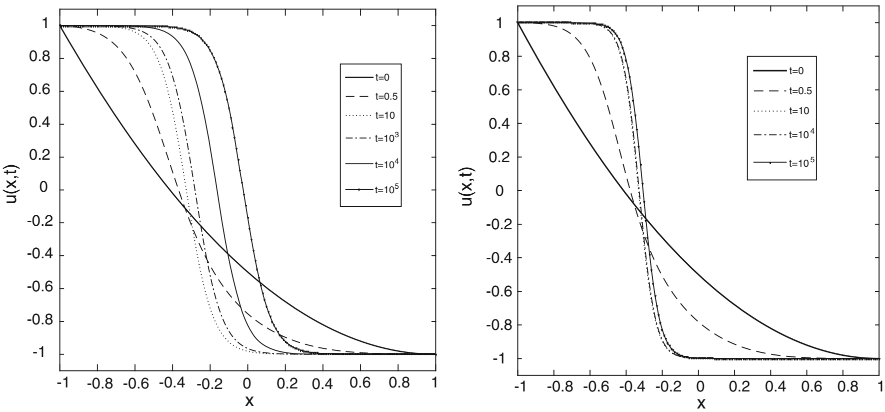

FIGURE 1. The solutions to (1.2) with $\varepsilon = 0.07$ and $\varepsilon = 0.045$, respectively. The initial datum is $u_0(x) = \frac{1}{2}x^2 - x - \frac{1}{2}$

is composed by stationary solutions to (1.3) if and only if $u_- = -u_+$, i.e. $s = 0$. From now on, we will thus consider boundary conditions of the form $u_\pm = \mp u_*$ for some $u_* > 0$.

When $\varepsilon > 0$, the situation is very different: indeed, in this case it is possible to prove that (1.2) possesses a unique stationary solution, that can be explicitly computed and reads

$$\bar{U}^\varepsilon(x) = -\kappa \tanh\left(\frac{\kappa x}{2\varepsilon}\right),$$

where the constant $\kappa = \kappa(u_*)$ is univocally determined by the boundary conditions. In [14] the authors proved the asymptotic stability of $\bar{U}^\varepsilon$ by analyzing spectral properties of the linearized operator around $\bar{U}^\varepsilon$; precisely, in [14] it has been shown that the eigenvalues of

$$\mathcal{L}^\varepsilon u := \varepsilon \partial_x^2 u - \partial_x(\bar{U}^\varepsilon(x)u),$$

are real, negative and have the following distribution as with respect to ε

$$\lambda_1 = \mathcal{O}(e^{-1/\varepsilon}), \qquad \lambda_k < -\frac{1}{\varepsilon} \quad \forall\, k \geq 2.$$

Negativity of the spectrum implies the asymptotic stability of the steady state $\bar{U}^\varepsilon$; the precise distribution of the eigenvalues and, in particular, the order of the first eigenvalue, suggest that the speed rate of convergence of the time-dependent solution towards the steady state $\bar{U}^\varepsilon$ is exponentially small in ε, precisely of order $\mathcal{O}(e^{-1/\varepsilon})$.

Numerical simulations validate such hypothesis, as they show that, even if starting from an initial datum localized far from the stable steady state $\bar{U}^\varepsilon$, the corresponding time-dependent solution develops into a layered function on a short time scale, while on a longer time scale proportional to $e^{1/\varepsilon}$, it moves towards the location corresponding to the equilibrium solution.

To have an idea of how the size of the parameter ε influences the speed rate of convergence of the solution towards its steady state, Fig. 1 shows the solution to (1.2) with $u_\pm = \mp 1$ and initial datum $u_0(x) = \frac{1}{2}x^2 - x - \frac{1}{2}$ for two different values of ε: when $\varepsilon = 0.07$, for times of order 10^5 the solution is already very close to its asymptotic configuration or in other words, the location of the shock is already very close to zero. On the other hand, when $\varepsilon = 0.045$, and for times of the same order, the shock layer location is still quite far from its equilibrium.

Many papers have been devoted to the study of metastability for the viscous Burgers equation. Without claiming to be complete, we list some of the contributions. Slow motion for the viscous Burgers equation in the case of the whole real line have been analized in [4,13]. If $u^\varepsilon = u^\varepsilon(x,t)$ is the solution of the viscous Burgers equation and $u^0 = u^0(x,t)$ is the solution of the corresponding inviscous case (1.3), then it is known that $u^\varepsilon \to u^0$ in an appropriate sense for any fixed $t > 0$ as $\varepsilon \to 0$. However, for fixed ε, the long-time behavior of u^ε and u^0 is different and they converge to solutions known as diffusion waves and N-waves, respectively. In [13], the authors studied generation of N-waves and their evolution to the final stage of a diffusion wave by using a transformed version of the Burgers equation (in self-similar variables). The same problem has been studied in [4], where the large-time behavior of solutions is described using invariant manifolds; in particular, the authors provided a geometric explanation for the metastability.

The case of bounded intervals has been examined in [7,8,15,17,23]. In [15,17,23] the analysis is conducted at a formal level and validated numerically by means of comparison with significant computations. In particular, specific examples are solved explicitly by using the Cole–Hopf transformation and the "supersensitivity" of the metastable motion is analyzed. On the other hand, a rigorous analysis has been performed in [7,8], where the approach is based on the use of traveling wave solutions satisfying the boundary conditions and with small velocity. The authors used such traveling waves and maximum principle methods to obtain rigorous asymptotic formulae for the velocity of convergence of solutions.

In order to describe rigorously the metastable behavior of solutions to (1.1) without relying on such techniques, which are typical of scalar equations, in this paper we use more flexible strategies and techniques, firstly performed in [16]. Precisely, the key idea relies on the construction of what we refer to as a *one-parameter family of approximated steady states* $\{U^\varepsilon(x;\xi)\}_{\xi\in I}$ (that is, a family whose elements satisfies the stationary equation up to an error that is small in ε in a sense that will be specified later on), and on a subsequent linearization of the original equation around an element of this family. The idea of a linearization around U^ε is developed in order to separate the two distinct phases of the dynamics of the solution, so that to understand firstly what happens far from the stable equilibrium solution, when the shock layer is formed, and, subsequently, to study the evolution of such a layer towards the asymptotic limit. In particular, in the case of the viscous Burgers equation, an element of the family $\{U^\varepsilon(x;\xi)\}_{\xi\in I}$ is a layered function and the parameter ξ represents the position of the unique internal interface, so that the function $\xi(t)$ describes the evolution of the position of the internal shock layer and, consequently, the time dependent evolution of the layered solution. As it was in [16], after the linearization, that is after representing the solution as the sum of an element $U^\varepsilon(x;\xi)$ moving along the family $\{U^\varepsilon\}$, plus a perturbation term v, we end up with a coupled system for the shock layer position ξ and the perturbation v; in [16], only a reduced version of this system, obtained by disregarding quadratic terms in v, has been analyzed. On the other hand, in the present paper we state and prove a general result, analogous to the one proved in [16, Theorem 2.1], concerning the complete nonlinear system for the couple (ξ, v), that better suites the behavior of the solutions to the original equation. Of course, dealing with the complete system brings into the analysis the specific form of the quadratic terms arising from the linearization; when these terms do not depend on the first order derivative of the solution, similar results can be proven (see, for example, [21]). In the case analyzed in this paper a nonlinear first-order space derivative term is present, as in the case of the viscous Burgers equation (1.2). Our main result provides an estimate for the L^2-norm of the perturbation v and for the speed of the internal shock layer ξ. In particular, in the case of the viscous Burgers equation, if we start from initial data that are exponentially close to an element of the family $U^\varepsilon(\cdot;\xi)$, then the perturbation v remains exponentially small for all times and the speed of the internal shock layer is exponentially small. As a consequence, the time taken for the solution to reach the equilibrium is exponentially large, leading to an *exponentially slow motion.* In [18] an algebraic slow motion result for Burgers equation is proven, with slightly stronger assumption and by analyzing the H^1-norm of the perturbation.

The rest of the paper is organized as follows. Section 2 contains the general strategy and the main result of the paper, Theorem 2.1, providing the above sketched estimates on the couple (ξ, v). In Sect. 3 we illustrate how this procedure can be applied to the case of the viscous Burgers equation. We prove exponentially slow motion for the latter equation and present some numerical explorations, showing the metastable dynamics of the solutions to (1.2).

2. General framework

We here mean to describe the general strategy proposed in [16] to study the metastable dynamics of solutions to evolutive equations of the form

$$u_t = \mathcal{F}^\varepsilon[u], \qquad u(\cdot,0) = u_0. \tag{2.1}$$

Here and in what follows, $\ell > 0$, $I = [-\ell, \ell]$, the space $X := L^2(I)$ and the solution $u : [0, +\infty) \to X$. As previously mentioned, our aim is to adapt this strategy in order to describe metastability for the solutions to (1.2).

Given an open interval $J \subset I$, consider a one-parameter family $\{U^\varepsilon(\cdot;\xi) : \xi \in J\}$ in X, satisfying

$$|\langle \psi, \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]\rangle| \le \Omega^\varepsilon(\xi)|\psi|_{L^\infty}, \quad \forall\, \psi \in C(I), \forall\, \xi \in J, \tag{2.2}$$

where $\Omega^\varepsilon(\xi)$ is a family of smooth positive functions converging to zero as $\varepsilon \to 0$, uniformly with respect to $\xi \in J$. Formula (2.2) states that each element of the family satisfies the stationary equation up to an error that is small in ε, and that is measured by Ω^ε. Then, the family $U^\varepsilon(\cdot;\xi)$ is composed by *approximate steady states* of (2.1).

Once the one-parameter family $U^\varepsilon(\cdot;\xi)$ is chosen, the second step of our strategy is the linearization of the original equation (2.1) around an element of the family; then, we look for a solution of the form

$$u(x,t) = U^\varepsilon(x;\xi(t)) + v(x,t), \tag{2.3}$$

where the perturbation $v \in X$ and the parameter ξ has to be determined. In particular, the key point here is that the parameter ξ, describing the position of the internal shock layer, depends on time, so that by following its dynamics we mean to describe the time dependent evolution of the layered solution. Hence, the idea is to reduce the dynamics of the PDE (2.1) into a one-dimensional dynamics for the parameter ξ.

By substituting (2.3) into (2.1) we obtain

$$\partial_t v = \mathcal{L}^\varepsilon_\xi v + \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] - \partial_\xi U^\varepsilon(\cdot;\xi)\,\frac{d\xi}{dt} + \mathcal{Q}^\varepsilon[v,\xi], \tag{2.4}$$

where $\mathcal{L}^\varepsilon_\xi v$ is the linearized operator arising from the linearization around U^ε, and

$$\mathcal{Q}^\varepsilon[v,\xi] := \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi) + v] - \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] - \mathcal{L}^\varepsilon_\xi v.$$

Having in mind the case of the Burgers equation, where $\mathcal{Q}^\varepsilon[v,\xi] = -\partial_x\left(\frac{1}{2}v^2\right)$ is a divergence term, and thus integrable by parts, in the following we shall assume

$$|\langle \psi, \mathcal{Q}^\varepsilon[v,\xi]\rangle| \le C|v|^2_{L^2}|\psi'|_{L^\infty}, \quad \forall\, \psi \in C^1(I),\, \forall\, \xi \in J. \tag{2.5}$$

Next, following the line of [16], we require that $\mathcal{L}^\varepsilon_\xi$ has a discrete spectrum composed by semi-simple eigenvalues $\lambda^\varepsilon_k := \lambda^\varepsilon_k(\xi)$ with corresponding eigenfunctions $\varphi^\varepsilon_k := \varphi^\varepsilon_k(\cdot;\xi)$. Moreover, we assume that λ^ε_k are real, negative and diverge to $-\infty$ as k^α, $\alpha > 1$, that is

$$|\lambda^\varepsilon_k| = \mathcal{O}(k^\alpha), \quad \text{for some } \alpha > 1 \text{ and for } k \to +\infty. \tag{2.6}$$

Finally, we require that the first eigenvalue is small in ε and tends to 0 as $\varepsilon \to 0$. Let now $\psi^\varepsilon_k(\cdot;\xi)$ be the eigenfunctions of the adjoint operator $\mathcal{L}^{\varepsilon,*}_\xi$ of $\mathcal{L}^\varepsilon_\xi$. In what follows, we also assume that we can expand v in terms of the eigenfunctions φ^ε_k, namely

$$v(x,t) = \sum_k v_k(t)\,\varphi^\varepsilon_k(x,\xi(t)) \quad \text{with} \quad v_k(t) = \langle \psi^\varepsilon_k(\cdot,\xi(t)), v(\cdot,t)\rangle. \tag{2.7}$$

The latter can be seen as an assumption on the linearized operator $\mathcal{L}^\varepsilon_\xi$ and/or on the perturbation term v we are dealing with.

In order to remove the singular part of the operator $\mathcal{L}^\varepsilon_\xi$ in the limit $\varepsilon \to 0$ (i.e. the one corresponding to the first singular eigenvalue λ^ε_1) we set an algebraic condition ensuring orthogonality between the solution v and ψ^ε_1; in particular, the equation for the parameter $\xi(t)$ is chosen in such a way that the slower decaying terms in the perturbation v are canceled out. In formula, we have

$$\frac{d}{dt}\langle\psi_1^\varepsilon(\cdot;\xi(t)),v(\cdot,t)\rangle=0 \qquad \text{and} \qquad \langle\psi_1^\varepsilon(\cdot;\xi(0)),v(\cdot,0)\rangle=0,$$

so that, using (2.4), we obtain

$$\langle\psi_1^\varepsilon(\cdot;\xi),\mathcal{L}_\xi^\varepsilon v+\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]-\partial_\xi U^\varepsilon(\cdot;\xi)\frac{d\xi}{dt}+\mathcal{Q}^\varepsilon[v,\xi]\rangle+\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi)\frac{d\xi}{dt},v\rangle=0.$$

Since $\langle\psi_1^\varepsilon,\mathcal{L}_\xi^\varepsilon v\rangle=\lambda_1^\varepsilon\langle\psi_1^\varepsilon,v\rangle=0$, we end up with a scalar nonlinear differential equation for the variable ξ, that reads

$$\left(\langle\psi_1^\varepsilon(\cdot;\xi),\partial_\xi U^\varepsilon(\cdot;\xi)\rangle-\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi),v\rangle\right)\frac{d\xi}{dt}=\langle\psi_1^\varepsilon(\cdot;\xi),\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]+\mathcal{Q}^\varepsilon[v,\xi]\rangle. \tag{2.8}$$

To rewrite (2.8) in a normal form, we assume that there exists a constant $c_0>0$ (independent on ξ) such that

$$|\langle\psi_1^\varepsilon(\cdot;\xi),\partial_\xi U^\varepsilon(\cdot;\xi)\rangle|\geq c_0.$$

Therefore, we can renormalize the first adjoint eigenfunction in such a way

$$\langle\psi_1^\varepsilon(\cdot;\xi),\partial_\xi U^\varepsilon(\cdot;\xi)\rangle=1, \quad \forall\ \xi\in J.$$

It follows that

$$\frac{d\xi}{dt}=\frac{\langle\psi_1^\varepsilon(\cdot;\xi),\mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)]+\mathcal{Q}^\varepsilon[v,\xi]\rangle}{1-\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi),v\rangle}. \tag{2.9}$$

Using that

$$\frac{1}{1-\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi),v\rangle}=1+\langle\partial_\xi\psi_1^\varepsilon,v\rangle+\frac{\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi),v\rangle^2}{1-\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi),v\rangle},$$

and inserting in (2.9), we obtain an ODE for the variable $\xi(t)$ that reads

$$\frac{d\xi}{dt}=\theta^\varepsilon(\xi)\big(1+\langle\partial_\xi\psi_1^\varepsilon,v\rangle\big)+\rho^\varepsilon[\xi,v], \tag{2.10}$$

where

$$\begin{aligned}\theta^\varepsilon(\xi)&:=\langle\psi_1^\varepsilon,\mathcal{F}^\varepsilon[U^\varepsilon]\rangle,\\ \rho^\varepsilon[\xi,v]&:=\frac{\langle\psi_1^\varepsilon,\mathcal{Q}^\varepsilon[v,\xi]\rangle}{1-\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi),v\rangle}+\theta^\varepsilon(\xi)\frac{\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi),v\rangle^2}{1-\langle\partial_\xi\psi_1^\varepsilon(\cdot;\xi),v\rangle}.\end{aligned} \tag{2.11}$$

We notice that, heuristically and for small perturbation v, the speed of the parameter ξ is given by $\theta^\varepsilon(\xi)$, the leading order term in the equation. In particular, by the assumption (2.2) it follows that

$$|\theta^\varepsilon(\xi)|\leq C\,\Omega^\varepsilon(\xi), \qquad \xi\in J,$$

suggesting that the speed of the shock layer location ξ is indeed small in ε.

Equations (2.4) and (2.10) form a system for the couple (ξ,v). This system is obtained by linearizing with respect to v, and by keeping the nonlinear dependence on ξ, in order to describe the evolution of the interface when it is localized far from the equilibrium location. It reads

$$\begin{cases} \dfrac{d\xi}{dt} = \theta^\varepsilon(\xi)\big(1 + \langle \partial_\xi \psi_1^\varepsilon, v\rangle\big) + \rho^\varepsilon[\xi, v], \\ \partial_t v = \mathcal{L}_\xi^\varepsilon v + \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] - \partial_\xi U^\varepsilon \dfrac{d\xi}{dt} + \mathcal{Q}^\varepsilon[v, \xi], \end{cases} \tag{2.12}$$

together with initial data

$$\langle \psi_1^\varepsilon(\cdot;\ \xi_0), v_0\rangle = 0, \quad v_0 = u_0 - U^\varepsilon(\cdot;\xi_0). \tag{2.13}$$

The aim of this section is to analyze the behavior of the solutions to (2.12) in the limit $\varepsilon \to 0$. The novelty with respect to [16] is that the complete system is considered; indeed in [16] the terms ρ^ε and $\mathcal{Q}^\varepsilon$ are neglected.

From now on, we will make use of the following properties on the eigenfunctions ψ_k^ε and the metastable state U^ε:

$$|\psi_k^\varepsilon|_{L^2} \le C, \quad |\partial_x \psi_k^\varepsilon|_{L^\infty} + |\partial_\xi \psi_k^\varepsilon|_{L^\infty} \le C\sqrt{|\lambda_k^\varepsilon|}, \quad |\partial_\xi U^\varepsilon|_{L^2} \le \frac{C}{\sqrt{\varepsilon}}. \tag{2.14}$$

We now state our main result, giving an estimate for the L^2-norm of the perturbation v and for the speed of the internal shock layer ξ.

Theorem 2.1. *Let (ξ, v) be the solution to the initial-value problem (2.12)–(2.13), with v_0 and Ω^ε satisfying*

$$|v_0|_{L^2} = o(\sqrt{\varepsilon}) \quad \textit{and} \quad |\Omega^\varepsilon|_{L^\infty} = o(\varepsilon). \tag{2.15}$$

Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ it holds

$$|v|_{L^2}(t) \le C\left(|v_0|_{L^2} + \frac{1}{\sqrt{\varepsilon}}|\Omega^\varepsilon|_{L^\infty}\right) \quad \textit{and} \quad \left|\frac{d\xi}{dt}\right| \le C\left(|v_0|^2_{L^2} + |\Omega^\varepsilon|_{L^\infty}\right), \tag{2.16}$$

for all $t \ge 0$ and for some constant $C > 0$.

Proof of Theorem 2.1. By using (2.7) and differentiating with respect to t,

we obtain an infinite-dimensional differential system for the coefficients v_k

$$\begin{aligned} \frac{dv_k}{dt} &= \langle \psi_k^\varepsilon, \partial_t v\rangle + \langle \partial_\xi \psi_k^\varepsilon \frac{d\xi}{dt}, v\rangle \\ &= \lambda_k^\varepsilon(\xi)\, v_k + \langle \psi_k^\varepsilon, \mathcal{F}^\varepsilon[U^\varepsilon]\rangle + \langle \psi_k^\varepsilon, \mathcal{Q}^\varepsilon[v,\xi]\rangle + \frac{d\xi}{dt}\left(\langle \partial_\xi \psi_k^\varepsilon, v\rangle - \langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon\rangle\right). \end{aligned}$$

Now let us set

$$E_k(s,t) := \exp\left(\int_s^t \lambda_k^\varepsilon(\xi(\sigma))d\sigma\right).$$

Since $v_1 \equiv 0$, by Duhamel's formula, we have the following expression for the coefficients v_k, $k \geq 2$

$$\begin{aligned} v_k(t) &= v_k(0)\, E_k(0,t) \\ &\quad + \int_0^t \left\{ \langle \psi_k^\varepsilon, \mathcal{F}^\varepsilon[U^\varepsilon] \rangle + \frac{d\xi}{dt}\big(\langle \partial_\xi \psi_k^\varepsilon, v \rangle - \langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon \rangle \big) \right\} E_k(s,t) ds \\ &\quad + \int_0^t \langle \psi_k^\varepsilon, \mathcal{Q}^\varepsilon[v,\xi] \rangle E_k(s,t) ds. \end{aligned}$$

Hence, by using the assumptions (2.2), (2.6) and (2.14), we deduce

$$\begin{aligned} |v_k(t)| &\leq |v_k(0)|\, E_k(0,t) \\ &\quad + C \int_0^t \left\{ \Omega^\varepsilon(\xi) + \left| \frac{d\xi}{dt} \right| \left(k^{\frac{\alpha}{2}} |v|_{L^2} + \frac{1}{\sqrt{\varepsilon}} \right) \right\} E_k(s,t) ds \\ &\quad + \int_0^t |\langle \psi_k^\varepsilon, \mathcal{Q}^\varepsilon[v,\xi] \rangle| E_k(s,t) ds. \end{aligned} \tag{2.17}$$

Recalling the equation for $\xi(t)$ in system (2.12), we have

$$\left| \frac{d\xi}{dt} \right| \leq C\, \Omega^\varepsilon(\xi) \left(1 + |\partial_\xi \psi_1^\varepsilon|_{L^2}^2 + |v|_{L^2}^2 \right) + |\rho^\varepsilon[\xi, v]|,$$

where we used $|\theta^\varepsilon(\xi)| \leq C\Omega^\varepsilon(\xi)$ and Young's inequality. Going further, using the assumptions (2.5) and (2.14) in the definition of ρ^ε (2.11), we obtain

$$\begin{aligned} |\rho^\varepsilon[\xi, v]| &\leq \left| \frac{C|v|_{L^2}^2}{1 - \langle \partial_\xi \psi_1^\varepsilon(\cdot;\xi), v \rangle} \right| + C\, \Omega^\varepsilon(\xi) \left| \frac{\langle \partial_\xi \psi_1^\varepsilon(\cdot;\xi), v \rangle^2}{1 - \langle \partial_\xi \psi_1^\varepsilon(\cdot;\xi), v \rangle} \right| \\ &\leq C \frac{(1 + \Omega^\varepsilon(\xi))}{|1 - \langle \partial_\xi \psi_1^\varepsilon(\cdot;\xi), v \rangle|} |v|_{L^2}^2, \end{aligned}$$

leading to

$$\left| \frac{d\xi}{dt} \right| \leq C\, \Omega^\varepsilon(\xi) \left(1 + |v|_{L^2}^2 \right) + C|v|_{L^2}^2 \leq C \left(\Omega^\varepsilon(\xi) + |v|_{L^2}^2 \right), \tag{2.18}$$

for any $t > 0$ such that

$$|v|_{L^2} \leq \frac{1}{2|\partial_\xi \psi_1^\varepsilon|_{L^2}}, \tag{2.19}$$

and using the boundedness of $\Omega^\varepsilon(\xi)$. As concerning the last term in (2.17), because of (2.5), (2.6) and (2.14) we have

$$|\langle \psi_k^\varepsilon, \mathcal{Q}^\varepsilon[v,\xi] \rangle| \leq C |\partial_x \psi_k^\varepsilon|_{L^\infty} |v|_{L^2}^2 \leq C \sqrt{|\lambda_k^\varepsilon|}\, |v|_{L^2}^2 \leq C k^{\frac{\alpha}{2}} |v|_{L^2}^2,$$

and (2.17) becomes

$$|v_k(t)| \leq |v_k(0)|\, E_k(0,t) + C \int_0^t \left\{ \left(k^{\frac{\alpha}{2}} + \frac{1}{\sqrt{\varepsilon}} \right) \left(\Omega^\varepsilon(\xi) + |v|_{L^2}^2 \right) \right\} E_k(s,t) ds,$$

for ε small and for any $t > 0$ such that

$$|v|_{L^2} \leq 1. \tag{2.20}$$

We estimate the integral as follows

$$\int_0^t E_k(s,t)\,ds \le \frac{1}{Ck^\alpha}\left(1-e^{-Ck^\alpha t}\right) \le \frac{1}{Ck^\alpha},$$

where we used $0 \le E_k(s,t) \le \exp\left(-Ck^\alpha(t-s)\right)$. It follows that

$$|v_k(t)| \le |v_k(0)|e^{-Ck^\alpha t} + C\left(\frac{1}{k^{\frac{\alpha}{2}}}+\frac{1}{\sqrt{\varepsilon}k^\alpha}\right)\left(|\Omega^\varepsilon|_{L^\infty} + \sup_{s\in[0,t]}|v|^2_{L^2}(s)\right).$$

We claim that

$$|v|^2_{L^2} \le C\sum_{k\ge 2}|v_k(t)|^2.$$

Indeed, for any $n\in\mathbb{N}$ define $v^{(n)} := \sum_{k=2}^n v_k(t)\varphi_k^\varepsilon(x;\xi)$

$$\left|v^{(n)}\right|^2_{L^2} \le C\sum_{k=2}^n |v_k(t)|^2\,|\varphi_k^\varepsilon(\cdot,\xi)|^2_{L^2} \le C\sum_{k=2}^n |v_k(t)|^2,$$

and our claim follows by passing to the limit $n\to+\infty$. We thus obtain

$$\begin{aligned}
|v|^2_{L^2} &\le C\sum_{k\ge 2}\left(|v_k(0)|e^{-Ck^\alpha t} + \left(\frac{1}{k^{\frac{\alpha}{2}}}+\frac{1}{\sqrt{\varepsilon}k^\alpha}\right)\left(|\Omega^\varepsilon|_{L^\infty}+\sup_{s\in[0,t]}|v|^2_{L^2}(s)\right)\right)^2\\
&\le C\sum_{k\ge 2}|v_k(0)|^2e^{-Ck^\alpha t}\\
&\quad + C\left(|\Omega^\varepsilon|_{L^\infty} + \sup_{s\in[0,t]}|v|^2_{L^2}(s)\right)^2\sum_{k\ge 2}\left(\frac{1}{k^\alpha}+\frac{1}{\varepsilon k^{2\alpha}}\right)\\
&\le C\sum_{k\ge 2}|v_k(0)|^2e^{-Ck^\alpha t} + \frac{C}{\varepsilon}\left(|\Omega^\varepsilon|_{L^\infty} + \sup_{s\in[0,t]}|v|^2_{L^2}(s)\right)^2,
\end{aligned}$$

for ε small and because of the convergence of the series, being $\alpha > 1$. We use (2.14) to estimate $|v_k(0)| \le |\psi_k^\varepsilon|_{L^2}|v_0|_{L^2} \le C|v_0|_{L^2}$. Therefore, for $0\le t\le t_0$, $|v|_{L^2} \le 2|v_0|_{L^2}$, while for $t\ge t_0$

$$\sum_{k\ge 2}|v_k(0)|^2e^{-Ck^\alpha t} \le C|v_0|^2_{L^2},$$

again for the convergence of the series (for $\alpha > 0$). Finally, for any $t > 0$, we end up with

$$|v|_{L^2} \le C|v_0|_{L^2} + \frac{C}{\sqrt{\varepsilon}}\left(|\Omega^\varepsilon|_{L^\infty} + \sup_{s\in[0,t]}|v|^2_{L^2}(s)\right),$$

because $\sqrt{a^2+b^2+c^2} \le a+b+c$, for nonnegative values a, b, c.

Thus, setting

$$N(t) := \sup_{s\in[0,t]}|v|_{L^2}(s),$$

we infer

$$N(t) \le \frac{C}{\sqrt{\varepsilon}} N^2(t) + C\left(|v_0|_{L^2} + \frac{1}{\sqrt{\varepsilon}} |\Omega^\varepsilon|_{L^\infty} \right)$$
$$=: AN^2(t) + B.$$

Hence, as soon as

$$4AB = \frac{4C^2}{\sqrt{\varepsilon}} \left(|v_0|_{L^2} + \frac{1}{\sqrt{\varepsilon}} |\Omega^\varepsilon|_{L^\infty} \right) \le 1,$$

which is satisfied for ε small enough in view of (2.15), we end up with

$$N(t) \le \frac{2B}{1 + \sqrt{4AB}} \le 2B,$$

that is

$$|v|_{L^2} \le C\left(|v_0|_{L^2} + \frac{1}{\sqrt{\varepsilon}} |\Omega^\varepsilon|_{L^\infty} \right). \tag{2.21}$$

Therefore, for ε sufficiently small, (2.19) and (2.20), and hence (2.21), are valid for any $t > 0$. Substituting (2.21) in (2.18), since $|\Omega^\varepsilon|_{L^\infty} = o(\varepsilon)$ we then obtain the second estimate in (2.16) and the proof is complete. □

Remark 2.2. We underline that (2.16) states that both the L^2-norm of the perturbation and the speed of the shock layer are small with respect to ε, provided Ω^ε and $|v_0|_{L^2}$ to be small in ε (as required in (2.15)). Actually, in the next section we will prove that, in the case of the viscous Burgers equation, it holds

$$|\Omega^\varepsilon|_{L^\infty} \le Ce^{-c/\varepsilon},$$

so that, also requiring $|v_0|_{L^2}$ to be exponentially small, we prove the *exponentially slow motion* for the solutions to (1.2).

3. Application to the Burgers equation

In this section, we apply the strategy presented in the previous section to the case of the Burgers equation (1.2); then, in this case the nonlinear differential operator $\mathcal{F}^\varepsilon$ reads

$$\mathcal{F}^\varepsilon[u] := \varepsilon\, \partial_x^2 u - \partial_x \left(\tfrac{1}{2} u^2 \right).$$

The family of approximate steady states. In order to follow the strategy introduced in the previous section, the first step is the construction of the one-parameter family of approximate steady states $\{U^\varepsilon(x;\xi)\}_{\xi \in J}$, satisfying the assumption (2.2). There are many possible choices to construct the generic element of such a family; here we use the same construction of [16], which consists in matching in a continuous way two exact steady states in the intervals $(-\ell, \xi)$ and (ξ, ℓ) satisfying, respectively, the left and the right boundary conditions together with the request $U^\varepsilon(\xi;\xi) = 0$. Precisely,

$$U^\varepsilon(x;\xi) = \begin{cases} U^\varepsilon_-(x;\xi) & \text{in} \quad (-\ell, \xi) \\ U^\varepsilon_+(x;\xi) & \text{in} \quad (\xi, \ell), \end{cases}$$

where $U_-^\varepsilon(x;\xi)$ solves

$$\varepsilon\partial_x^2 U_-^\varepsilon - U_-^\varepsilon \partial_x U_-^\varepsilon = 0, \quad U_-^\varepsilon(-\ell;\xi) = u_*, \; U_-^\varepsilon(\xi;\xi) = 0,$$

and similarly for U_+^ε in (ξ,ℓ). In particular, by separating variables we get an implicit expression for $U_\pm^\varepsilon$, given by

$$\int_0^{U_\pm^\varepsilon} \frac{ds}{\kappa_\pm - s^2/2} = \frac{x-\xi}{\varepsilon}.$$

Therefore, U^ε is defined as

$$U^\varepsilon(x;\xi) = \begin{cases} \kappa_- \tanh\left(\kappa_-(\xi - x)/2\varepsilon\right) & \text{in} \quad (-\ell,\xi) \\ \kappa_+ \tanh\left(\kappa_+(\xi - x)/2\varepsilon\right) & \text{in} \quad (\xi,\ell), \end{cases} \tag{3.1}$$

where $\kappa_\pm$ are chosen so that the boundary conditions are satisfied, and are implicitly defined by

$$\frac{2\varepsilon}{\kappa_\pm} \tanh^{-1}\left(\frac{\mp u_*}{\kappa_\pm}\right) \pm \ell = \xi,$$

or equivalently

$$\kappa_+ \tanh\left(\frac{\kappa_+}{2\varepsilon}(\ell - \xi)\right) = \kappa_- \tanh\left(\frac{\kappa_-}{2\varepsilon}(\ell + \xi)\right) = u_*. \tag{3.2}$$

It follows that $\kappa_+ = \kappa_-$ if and only if $\xi = 0$, and

$$\lim_{\xi\to\ell^-}(\kappa_- - \kappa_+) = -\infty, \qquad \lim_{\xi\to-\ell^+}(\kappa_- - \kappa_+) = +\infty.$$

We also note that, with such a construction, the parameter ξ represents the location of the unique internal shock layer, and the function U^ε is a continuous function with a jump in the first derivative located in $x = \xi$; hence, when computing $\mathcal{F}^\varepsilon[U^\varepsilon]$ in order to obtain an explicit expression for Ω^ε, we get

$$\mathcal{F}^\varepsilon[U^\varepsilon] = [\![\partial_x U^\varepsilon]\!]_{x=\xi}\, \delta_{x=\xi},$$

where δ is the Dirac delta distribution located in $x = \xi$.

By the explicit expression of U^ε given in (3.1), we obtain

$$[\![\partial_x U^\varepsilon]\!]_{x=\xi} = \frac{1}{2\varepsilon}(\kappa_-^2 - \kappa_+^2).$$

Let us now set $\kappa_\pm := u_*(1 + h_\pm)$; from (3.2) we have

$$u_* = \kappa_\pm \tanh\left(\frac{\kappa_\pm}{2\varepsilon}(\ell \mp \xi)\right) \tag{3.3}$$

and we deduce

$$\tanh\left(\frac{u_*}{2\varepsilon}(\ell \mp \xi)(1 + h_\pm)\right) = \frac{1}{1 + h_\pm}.$$

From (3.3) we can state that the values $h_\pm$ are both positive and then

$$\tanh\left(\frac{u_*}{2\varepsilon}(\ell \mp \xi)\right) \leq \frac{1}{1 + h_\pm},$$

that gives the asymptotic representation

$$h_\pm \leq \frac{1}{\tanh\left(u_*(\ell \mp \xi)/2\varepsilon\right)} - 1 = \frac{2}{e^{u_*(\ell\mp\xi)/\varepsilon} - 1} \leq Ce^{-u_*(\ell\mp\xi)/\varepsilon}. \tag{3.4}$$

Hence, the behavior as $\varepsilon \to 0$ of $[\![\partial_x U^\varepsilon]\!]_{x=\xi}$ is encoded in the difference $\kappa_- - \kappa_+$, and since we have

$$\kappa_- - \kappa_+ = u_*(h_- - h_+)$$

we end up with

$$|[\![\partial_x U^\varepsilon]\!]_{x=\xi}| \leq \frac{C}{\varepsilon} e^{-u_*(\ell - |\xi|)/\varepsilon}. \tag{3.5}$$

Hence, the error Ω^ε defined in (2.2) is exponentially small for $\varepsilon \to 0$, uniformly in any compact subset of $(-\ell, \ell)$; moreover, it is zero when $\xi = 0$, that corresponds to the equilibrium location of the shock.

Linearization and spectral analysis. The second step of our strategy is the linearization of the original Eq. (1.2) around an element of the family $\{U^\varepsilon(x;\xi)\}_{\xi \in J}$; then, we look for a solution of the form

$$u(x,t) = U^\varepsilon(x;\xi(t)) + v(x,t),$$

where the perturbation $v \in L^2(I)$ and the parameter $\xi(t)$ has to be determined. The parameter ξ, describing the position of the internal shock layer, depends on time, and by following its dynamics we mean to describe the time dependent evolution of the layered solution. Hence, the idea is to reduce the dynamics of the PDE (1.2) into a one-dimensional dynamics for the parameter ξ.

The Eq. (2.4) in the case of Burgers equation (1.2) reads

$$\partial_t v = \mathcal{L}^\varepsilon_\xi v + \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] - \partial_\xi U^\varepsilon(\cdot;\xi)\,\frac{d\xi}{dt} - v\partial_x v, \tag{3.6}$$

where

$$\mathcal{L}^\varepsilon_\xi v := \varepsilon \partial_x^2 v - \partial_x(U^\varepsilon v)$$

is the linearized operator arising from the linearization around U^ε.

Observe that in this case, $\mathcal{Q}^\varepsilon[v,\xi] := -\partial_x\left(\frac{1}{2}v^2\right)$, so that the assumption (2.5) is clearly satisfied.

In order to exploit spectral properties of the linearized operator $\mathcal{L}^\varepsilon_\xi$, we are thus interested in studying the eigenvalue problem

$$\varepsilon \partial_x^2 v - \partial_x(U^\varepsilon v) = \lambda v, \qquad v(\pm\ell) = 0. \tag{3.7}$$

We first notice that, introducing the operator

$$\mathcal{M}^\varepsilon_\xi v := \varepsilon^2 \partial_x^2 v - b^\varepsilon v, \quad \text{where} \quad b^\varepsilon := \left(\frac{1}{2}\,U^\varepsilon\right)^2 + \frac{\varepsilon}{2}\partial_x U^\varepsilon,$$

then $\varphi(x;\xi)$ is an eigenfunction for (3.7) relative to the eigenvalue λ if and only if the function $\phi(x;\xi)$, defined as

$$\phi(x;\xi) = \exp\left(-\frac{1}{2\varepsilon}\int_0^x U^\varepsilon(y;\xi)\,dy\right)\varphi(x;\xi)$$

is an eigenfunction for $\mathcal{M}^\varepsilon_\xi$ of eigenvalue $\mu := \varepsilon\lambda$. Since $\mathcal{M}^\varepsilon_\xi$ is self-adjoint, we can state that the spectrum of $\mathcal{L}^\varepsilon_\xi$ is composed by a decreasing sequence of real eigenvalues $\{\lambda^\varepsilon_k(\xi)\}_{k\in\mathbb{N}}$ converging to $-\infty$; also, a straightforward computation (see [16]) shows that

$$\sigma(\mathcal{L}^\varepsilon_\xi) \subset (-\infty, 0).$$

Going further, the following Proposition can be proved by applying [16], Proposition 4.1 and Proposition 4.4.

Proposition 3.1. *Let $\{\lambda_k^\varepsilon\}$ be the sequence of eigenvalues of $\mathcal{L}_\xi^\varepsilon$. Then there exist positive constants c_1, c_2 such that*

$$-e^{-c_1/\varepsilon} \leq \lambda_1^\varepsilon < 0, \qquad \textit{and} \qquad \lambda_k^\varepsilon \leq -c_2/\varepsilon \quad \textit{for any } k \geq 2.$$

Let now $\varphi_k^\varepsilon(\cdot;\xi)$ and $\psi_k^\varepsilon(\cdot;\xi)$ be the eigenfunctions of $\mathcal{L}_\xi^\varepsilon$ and its adjoint

$$\mathcal{L}_\xi^{\varepsilon,*} v := \varepsilon \partial_x^2 v + U^\varepsilon(\cdot;\xi)\partial_x v,$$

respectively. Proposition 3.1 states that the first eigenvalue of the linearized operator is exponentially small in ε. In the case of viscous Burgers equation, the equation for ξ becomes

$$\frac{d\xi}{dt} = \frac{\langle \psi_1^\varepsilon(\cdot;\xi), \mathcal{F}^\varepsilon[U^\varepsilon(\cdot;\xi)] - \partial_x(\frac{1}{2}v^2)\rangle}{\langle \psi_1^\varepsilon(\cdot;\xi), \partial_\xi U^\varepsilon(\cdot;\xi)\rangle - \langle \partial_\xi \psi_1^\varepsilon(\cdot;\xi), v\rangle}. \tag{3.8}$$

Observe that, since $U^\varepsilon(\cdot;0)$ is the exact stationary solution, then

$$\mathcal{F}[U^\varepsilon(\cdot;\xi)] = \mathcal{F}[U^\varepsilon(\cdot;\xi)] - \mathcal{F}[U^\varepsilon(\cdot;0)] \approx \mathcal{L}_\xi^\varepsilon \partial_\xi U^\varepsilon(\cdot;0)\,\xi.$$

The fact that $\mathcal{L}_\xi^\varepsilon(\partial_\xi U^\varepsilon)$ is uniformly small suggests that the first eigenfunction ψ_1^ε is proportional to $\partial_\xi U^\varepsilon$ (at least for small ε), so that we can renormalize the first adjoint eigenfunction in such a way

$$\langle \psi_1^\varepsilon(\cdot;\xi), \partial_\xi U^\varepsilon(\cdot;\xi)\rangle = 1, \quad \forall\, \xi \in I.$$

We thus obtain the following ODE for the variable $\xi(t)$

$$\frac{d\xi}{dt} = \theta^\varepsilon(\xi)\big(1 + \langle \partial_\xi \psi_1^\varepsilon, v\rangle\big) + \rho^\varepsilon[\xi, v], \tag{3.9}$$

where θ^ε and ρ^ε are defined as in (2.11). Heuristically, and for small perturbation v, the speed of the parameter ξ is given by $\theta^\varepsilon(\xi)$, the leading order term in the equation. In particular, by the very definition it follows that for any $\delta > 0$, we have

$$|\theta^\varepsilon(\xi)| \leq C\,\Omega^\varepsilon(\xi) \sim e^{-c/\varepsilon}, \qquad \xi \in (-\ell + \delta, \ell - \delta),$$

suggesting that the speed rate of convergence of the shock layer location towards its equilibrium is indeed exponentially small in ε.

Applying Theorem 2.1 in the case of Burgers equation, we can say that if the L^2-norm of the initial perturbation v_0 is exponentially small, then the L^2-norm of the perturbation v and the speed of the shock layer location are exponentially small for all $t > 0$, namely

$$|v|_{L^2} + \left|\frac{d\xi}{dt}\right| \leq Ce^{-c/\varepsilon}, \tag{3.10}$$

for all $t \geq 0$. Since, for $\varepsilon \to 0$, the estimate on the L^2-norm of the perturbation given in (3.10) holds globally in time, it can be used in the equation for $\xi(t)$ in order to decouple the system (2.12). Indeed, $\xi(t)$ solves an equation of the form

$$\frac{d\xi}{dt} = \theta^\varepsilon(\xi)(1 + r) + \rho^\varepsilon,$$

where

$$|r| \le C|v|_{L^2} \qquad \text{and} \qquad |\rho^\varepsilon| \le C(1 + |\Omega^\varepsilon|_{L^\infty})|v|^2_{L^2}.$$

Hence, for small ε and initial data v_0 such that $|v_0|_{L^2} \le e^{-c/\varepsilon}$, the dynamics of ξ is described by the ODE

$$\frac{d\xi}{dt} = \theta^\varepsilon(\xi).$$

Using a standard method of separation of variables, it follows that the function ξ satisfies

$$|\xi| \le |\xi_0| e^{-\beta^\varepsilon t}, \qquad -\beta^\varepsilon \sim \theta^{\varepsilon\prime}(0), \tag{3.11}$$

for $\beta^\varepsilon > 0$ and $\beta^\varepsilon \to 0$ as $\varepsilon \to 0$.

Estimate (3.11) shows the exponentially slow motion of the shock layer: indeed, recalling that $U^\varepsilon(\cdot;0)$ is the exact steady state of the problem, (3.11) states that the speed rate of convergence of $\xi(t)$ towards its equilibrium position is much slower as ε becomes smaller, since $\beta^\varepsilon \sim e^{-c/\varepsilon}$ as $\varepsilon \to 0$ (for more details, see [16, Example 3.4]).

In terms of the original solution to (1.2), recalling the decomposition

$$u(x,t) = v(x,t) + U^\varepsilon(x;\xi(t)),$$

estimates (3.10)–(3.11) show that the solution u is converging to $U^\varepsilon(\cdot;0)$ for large times; moreover, the speed rate of such convergence is dictated by the speed rate of convergence of $\xi(t)$ towards 0, hence it is smaller as ε becomes smaller, leading to a metastable behavior.

Asymptotic for the eigenfunctions. We here mean to exploit some properties of the eigenfunctions ψ_k^ε in order to heuristically justify the assumptions (2.14) we used in the proof of Theorem 2.1. To this aim, we consider the eigenvalue problem

$$\mathcal{L}_\xi^{\varepsilon,*} v = \lambda v, \qquad v(\pm\ell) = 0,$$

where

$$\mathcal{L}_\xi^{\varepsilon,*} v := \varepsilon\partial_x^2 v + U^\varepsilon(\cdot;\xi)\partial_x v.$$

We expect that, for ε small, the eigenfunctions ψ_k^ε of $\mathcal{L}_\xi^{\varepsilon,*}$ are close to the eigenfunctions of the operator $\mathcal{L}_\xi^{0,*}$, defined as

$$\mathcal{L}_\xi^{0,*} v := \varepsilon\partial_x^2 v + U^0(\cdot;\xi)\partial_x v, \qquad U^0(x;\xi) := u_*\chi(-\ell,\xi) - u_*\chi(\xi,\ell).$$

Hence, $\psi_1^\varepsilon \approx \psi_1^0$, where ψ_1^0 is the eigenfunction of the operator $\mathcal{L}_\xi^{0,*}$ relative to the eigenvalue $\lambda = 0$, that is ψ_1^0 solves

$$\begin{cases} \varepsilon\partial_x^2\psi_1^0 + u_*\partial_x\psi_1^0 = 0, \\ \psi_1^0(-\ell) = 0, \end{cases} \text{in } (-\ell,\xi), \qquad \begin{cases} \varepsilon\partial_x^2\psi_1^0 - u_*\partial_x\psi_1^0 = 0, \\ \psi_1^0(\ell) = 0, \end{cases} \text{in } (\xi,\ell),$$

together with the request $[\![\psi_1^0]\!]_{x=\xi} = 0$. We thus obtain obtain the representation formula

$$\psi_1^\varepsilon(x) \approx \psi_1^0(x) := \begin{cases} (1 - e^{u_*(\xi-\ell)/\varepsilon})(1 - e^{-u_*(x+\ell)/\varepsilon}) & x \le \xi, \\ (1 - e^{-u_*(\xi+\ell)/\varepsilon})(1 - e^{u_*(x-\ell)/\varepsilon}) & x > \xi. \end{cases}$$

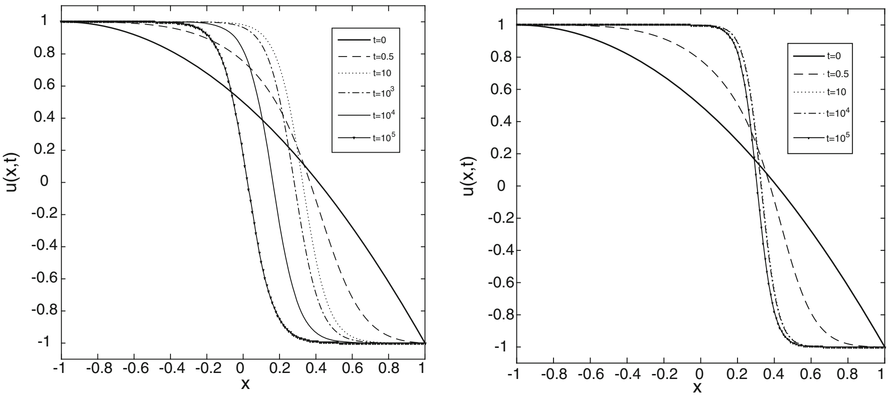

FIGURE 2. The solutions to (1.2) with $\varepsilon = 0.07$ and $\varepsilon = 0.045$, respectively. The initial datum is $u_0(x) = -\frac{1}{2}x^2 - x + \frac{1}{2}$

For any $k \geq 2$, the eigenfunctions ψ_k^ε are approximately given by the solutions to

$$\begin{cases} \varepsilon\partial_x^2\psi_k^0 \pm u_*\partial_x\psi_k^0 = \lambda_k^\varepsilon\psi_k^0, \\ \psi_k^0(\mp\ell) = 0, \end{cases}$$

to be considered in the intervals $(-\ell, \xi)$ and (ξ, ℓ) respectively, together with the jump condition $[\![\psi_k^0]\!]_{x=\xi} = 0$.

It turns out that

$$\psi_k^0(x) = \begin{cases} c_1 e^{-u_*(x+\ell)/2\varepsilon} \sin\left(\sqrt{-4\varepsilon\lambda_k^\varepsilon - u_*^2}(x+\ell)/2\varepsilon\right) & x \leq \xi \\ c_2 e^{u_*(x-\ell)/2\varepsilon} \sin\left(\sqrt{-4\varepsilon\lambda_k^\varepsilon - u_*^2}(x-\ell)/2\varepsilon\right) & x > \xi, \end{cases}$$

where the constants are determined by the condition $[\![\psi_k^0]\!]_{x=\xi} = 0$

$$c_1 = e^{u_*(\xi-\ell)/2\varepsilon} \sin\left(\sqrt{-4\varepsilon\lambda_k^\varepsilon - u_*^2}(\xi-\ell)/2\varepsilon\right),$$

$$c_2 = e^{-u_*(\xi+\ell)/2\varepsilon} \sin\left(\sqrt{-4\varepsilon\lambda_k^\varepsilon - u_*^2}(\xi+\ell)/2\varepsilon\right).$$

We underline that these expressions make sense as soon as $4\varepsilon|\lambda_k^\varepsilon| > u_*^2$, which holds true because of the distribution of the eigenvalues given in Proposition 3.1.

A straightforward computation shows that the functions ψ_k^0 satisfy the assumptions (2.14).

Numerical solutions. We conclude this paper by showing some numerical simulations describing the metastable dynamics of the solutions to (1.2). As already seen in the Introduction (see Fig. 1), the time taken for the solution to reach the stable steady state increases as $\varepsilon \to 0$.

In Fig. 2 we describe the evolution of the concave initial datum $u_0(x) = -\frac{1}{2}x^2 - x + \frac{1}{2}$; as the initial position of the shock is $\xi_0 > 0$, the function $\xi(t)$ is

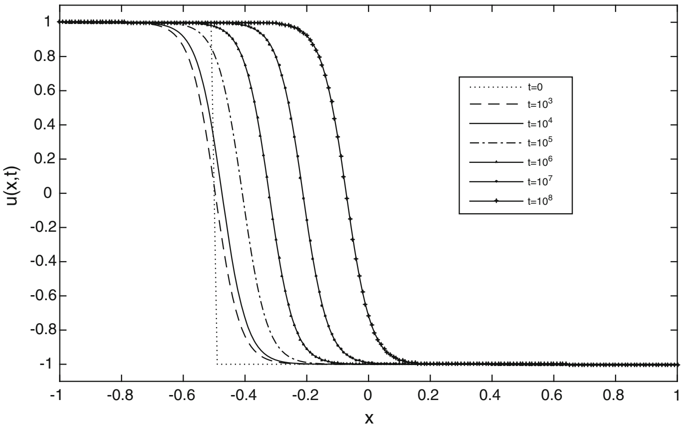

FIGURE 3. The solutions to (1.2) with $\varepsilon = 0.04$ and discontinuous initial datum

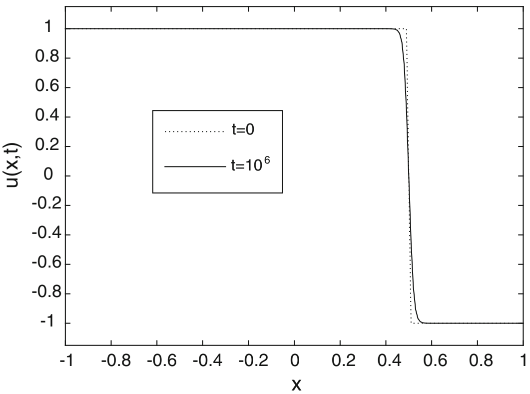

FIGURE 4. The solutions to (1.2) with $\varepsilon = 0.01$ and discontinuous initial datum

decreasing in time. Also in this case, for times of order $\mathcal{O}(1)$ we observe the formation of a single internal layers located in ξ_0; subsequently, such layer starts to drift towards its equilibrium but the time taken for the time-dependent solution to reach the asymptotic limit is longer as $\varepsilon \to 0$.

In Fig. 3 we plot the solution to (1.2) with $\varepsilon = 0.04$ and discontinuous initial datum $u_0(x) = \chi_{(-1,-0.5)} - \chi_{(-0.5,1)}$, that is a stationary solution to (1.3); in addition to the well-known regularizing effect of the Laplacian, we can see that the time of convergence is in this case greater than $t = 10^8$.

Finally, in Fig. 4, we can see that the evolution of the solution is extremely slow, provided ε to be very small; indeed, for times of order $t = 10^6$, the position of the shock is still very close to the position of the shock in the initial datum.

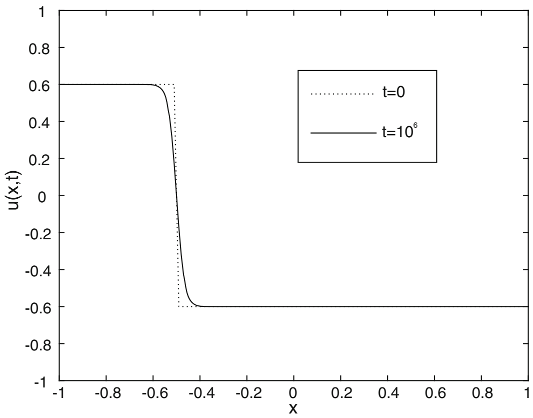

FIGURE 5. The solutions to (1.2) with $\varepsilon = 0.01$ and discontinuous initial datum. The boundary conditions are $u_{\pm} = \mp 0.6$

A similar example is given in Fig. 5, where we changed both the boundary conditions and the initial datum. Again, we can see that the shock layer, after a long time, is almost still very close to the shock layer of the initial datum.

These examples also show that the limits $\varepsilon \to 0$ and $t \to +\infty$ are not interchangeable. Indeed, if we first consider the limit as $t \to +\infty$ and then the limit $\varepsilon \to 0$, we obtain a function with a jump discontinuity in 0; by interchanging the limits we obtain a jump function with a shock layer located far away from 0.

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