Nonlinear Differ. Equ. Appl. (2017) 24:10

1021-9722/17/020001-53
published online February 8, 2017
DOI 10.1007/s00030-017-0434-1

Nonlinear Differential Equations and Applications NoDEA

The Cauchy problem for the nonlinear damped wave equation with slowly decaying data

Masahiro Ikeda, Takahisa Inui and Yuta Wakasugi

Abstract. We study the Cauchy problem for the nonlinear damped wave equation and establish the large data local well-posedness and small data global well-posedness with slowly decaying initial data. We also prove that the asymptotic profile of the global solution is given by a solution of the corresponding parabolic problem, which shows that the solution of the damped wave equation has the diffusion phenomena. Moreover, we show blow-up of solution and give the estimate of the lifespan for a subcritical nonlinearity. In particular, we determine the critical exponent for any space dimension.

Mathematics Subject Classification. 35L71, 35A01, 35B40, 35B44.

Keywords. Nonlinear damped wave equation, Slowly decaying data, Global well-posedness, Asymptotic behavior, Blow-up, Estimates of lifespan.

Contents

1. Introduction

In this paper, we study the Cauchy problem for the nonlinear damped wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = \mathcal{N}(u), & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x),\ \partial_t u(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \in \mathbb{N}$ and u is a real-valued unknown function, $\mathcal{N}(u)$ is a power type nonlinearity, (u_0, u_1) are given data, and $\varepsilon > 0$ is a positive parameter, which describes the amplitude of the initial data.

Our purpose is to establish the large data local well-posedness and the small data global well-posedness for the Cauchy problem (1.1) with slowly decaying initial data, that is, we treat the initial data not belonging to $L^1(\mathbb{R}^n)$ in general. Moreover, we investigate the asymptotic behavior of the global solution and the estimate of the lifespan from both above and below for subcritical nonlinearities.

The Eq. (1.1) is firstly derived by Oliver Heaviside as the telegrapher's equation, which describes the current and voltage in an electrical circuit with resistance and inductance. Cattaneo [1] also introduced the Eq. (1.1) as a modified heat conduction equation which equips the finite propagation speed property. The Eq. (1.1) also has several background related to biology and stochastic models such as genetics, population dynamics [3,7] and correlated random walk [6,29].

The local and global well-posedness, asymptotic behavior of global-in-time solutions and blow-up of local-in-time solutions have been widely studied for a long time. Since a pioneer work by Matsumura [38], it has been well known that solutions of the damped wave equation behaves like that of the heat equation as time tends to infinity. Namely, he established L^p-L^q estimates of the linear damped wave equation (Eq. (1.1) with $\mathcal{N}(u) = 0$), whose decay rates are the same as those of the linear heat equation $v_t - \Delta v = 0$ (see also Racke [57] for more general setting). After that, the so-called diffusion phenomena was found by Hsiao and Liu [17] for hyperbolic conservation laws with damping. Namely, they showed that the asymptotic profile of the solution is given by the heat kernel (see also [35,44,45,67]). Later on, Nishihara [46], Marcati and Nishihara [37], Hosono and Ogawa [16], and Narazaki [40] derived more precise L^p-L^q estimates for the linear damped wave equation and applied them to semilinear equations to obtain global solutions. Also, the diffusion phenomena for abstract damped wave equations were studied by [2,21,23,52,58,59].

For the nonlinear damped wave equation with the absorbing nonlinearity $\mathcal{N}(u) = -|u|^{p-1}u$, Kawashima, Nakao and Ono [32] refined Matsumura's L^p-L^q estimates and applied them to the global well-posedness for the Eq. (1.1) with arbitrary initial data $(u_0, u_1) \in H^1 \times L^2$. Based on this result, Karch [30] showed the diffusion phenomena when $p > 1 + \frac{4}{n}$ and $n \le 3$. After that, Hayashi, Kaikina and Naumkin [12], Ikehata, Nishihara and Zhao [24] and Nishihara [48] treated the case $p > 1 + \frac{2}{n}$ and $n \le 4$ if the initial data belongs to $(H^1 \cap L^1) \times (L^2 \cap L^1)$. Also, Hayashi, Kaikina and Naumkin [10–13], Hayashi and Naumkin [14] and Hamza [8] studied the asymptotic profile of solutions for critical and subcritical nonlinearities $1 < p \le 1 + \frac{2}{n}$.

The nonlinear damped wave equation with the source term $\mathcal{N}(u) = |u|^p$ or $|u|^{p-1}u$ has been widely studied. In this case, Levine [34] showed that the solution in general blows up in finite time for large initial data. Therefore, to obtain the global existence of solutions, we need some smallness condition for the initial data. Nakao and Ono [39] studied the case $\mathcal{N}(u) = |u|^{p-1}u$ with $p \ge 1 + \frac{4}{n}$ and proved the global existence of solutions by the method of modified potential well. Li and Zhou [36] found that, when $n \le 2$, the critical exponent of (1.1) is given by $p = 1 + \frac{2}{n}$, that is, the local-in-time solution can be extended time-globally if $p > 1 + \frac{2}{n}$ and the initial data is sufficiently small, while the finite time blow-up occurs if $p \le 1 + \frac{2}{n}$ and the initial data has positive integral value. The number $1 + \frac{2}{n}$ is well known as Fujita's critical exponent named after his seminal work [4], which is the threshold between the global existence and the blow-up of solutions to the semilinear heat equation. Also, in [36], the optimal upper estimate of the lifespan for blow-up solutions was also given (see also [47] for the case $n = 3$, [20] for $n \ge 4$ and $p < 1 + \frac{2}{n}$ and the first author and Ogawa [19] for $n \ge 4$, $p = 1 + \frac{2}{n}$). Later on, Todorova and Yordanov [65] and Zhang [68] determined the critical exponent as $p = 1 + \frac{2}{n}$ for all space dimensions. Moreover, Ono [55,56] derived L^m-decay of solutions for $1 \le m \le 2n/(n-2)_+$. The results of [36] and [65] require that the initial data belongs to $H^1 \times L^2$ and has the compact support. Ikehata, Miyaoka and Nakatake [22], Ikehata and Tanizawa [26] and Hayashi, Kaikina and Naumkin [9] removed the compactness assumption and proved the global existence of solutions for the initial data belonging to L^1. Moreover, Nakao and Ono [39], Ikehata and Ohta [25] and Narazaki and Nishihara [43] studied the global well-posedness for slowly decaying initial data not belonging to L^1. In particular, in [25], small data global existence is proved when the nonlinearity is $\mathcal{N}(u) = |u|^{p-1}u$ with $p > 1 + \frac{2r}{n}$ for $n \le 6$ and $(H^1 \cap L^r) \times (L^2 \cap L^r)$-data, where r satisfies $r \in [1, 2]$ if $n = 1, 2$ and $r \in [\frac{\sqrt{n^2+16n}-n}{4}, \min\{2, \frac{n}{n-2}\}]$ if $3 \le n \le 6$. Finite time blow-up of local solutions was also obtained for any $n \ge 1$ and $1 < p < \frac{2r}{n}$. However, the above global well-posedness results are restricted to $n \le 6$ and there are no results for higher dimensional cases. Also, Narazaki [42] considered the slowly decaying data belonging to modulation spaces and proved the global existence when the nonlinearity has integer power.

Concerning the asymptotic profile of global solutions, Gallay and Raugel [5] determined the asymptotic expansion up to the second order when $n =$

1 and the initial data belongs to the weighted Sobolev space $H^{1,1} \times H^{0,1}$ (see Sect. 1.2 for the definition). Using the expansion of solutions to the heat equation, Kawakami and Ueda [31] extended it to the case $n \le 3$. Hayashi, Kaikina and Naumkin [9] obtained the first order asymptotics for all $n \ge 1$ and the initial data belonging to $(H^{s,0} \cap H^{0,\alpha}) \times (H^{s-1,0} \cap H^{0,\alpha})$ with $\alpha > \frac{n}{2}$ (particularly, belonging to L^1). Recently, Takeda [63,64] determined the higher order asymptotic expansion of global solutions. Narazaki and Nishihara [43] studied the case of slowly decaying data and proved that if $n \le 3$ and the data behaves like $(1+|x|)^{-kn}$ with $0 < k \le 1$, then, the asymptotic profile of the global solution is given by $G(t,x) * (1+|x|)^{-kn}$, where G is the Gaussian and $*$ denotes the convolution with respect to spatial variables.

Related to the Eq. (1.1), systems of nonlinear damped wave equation were studied and the critical exponent and the asymptotic behavior of solutions were investigated (see [15,41,42,49–51,53,54,61,62]).

In the present paper, we establish the large data local well-posedness and the small data global well-posedness for the nonlinear damped wave equation (1.1) with slowly decaying initial data. Our global well-posedness results extend those of [25,43] to all space dimensions, and generalize that of [9] to slowly decaying initial data. Moreover, we study the asymptotic profile of the global solution. This also extends those of [43] to all space dimensions. Considering the asymptotic behavior of solutions in weighted norms, we further extended the result of [9] to the asymptotics in L^m-norm with $m \le 2$. Finally, we give an almost optimal lifespan estimate from both above and below. This is also an extension of [20,36,47], in which L^1-data were treated.

1.1. Main results

We say that $u \in L^\infty(0,T;L^2(\mathbb{R}^n))$ is a mild solution of (1.1) if u satisfies the integral equation

$$u(t) = (\partial_t + 1)\,\mathcal{D}(t)\varepsilon u_0 + \mathcal{D}(t)\varepsilon u_1 + \int_0^t \mathcal{D}(t-\tau)\mathcal{N}(u(\tau))\,d\tau.$$

in $L^\infty(0,T;L^2(\mathbb{R}^n))$, where $\mathcal{D}(t)$ is the solution operator of the damped wave equation defined in (1.8) below.

We assume that there exists $p > 1$ such that the nonlinear term $\mathcal{N}(u)$ satisfies $\mathcal{N} \in C^{p_0}(\mathbb{R})$ with some integer $p_0 \in [0,p]$ and

$$\begin{cases} \mathcal{N}^{(l)}(0) = 0, \\ \left|\mathcal{N}^{(l)}(u) - \mathcal{N}^{(l)}(v)\right| \le C|u-v|(|u|+|v|)^{p-l-1} \end{cases} \quad (l = 0, \dots, p_0) \tag{1.2}$$

with some constant $C > 0$.

Theorem 1.1. (Local well-posedness for large data) *Let $n \in \mathbb{N}$ and let $s \ge 0$ be $0 \le [s] \le p_0 - 1$. When $n = 1$, we also assume that $0 \le s < 1$. Let $r \in [1,2]$ and s, p satisfy*

$$\begin{aligned}
&\min\left\{1+\frac{r}{2}, 1+\frac{r}{n}\right\} \le p < \infty && \text{if } 1 \le n \le 2s,\\
&1+\frac{r}{n} \le p \le \min\left\{1+\frac{2}{n-2s}, \frac{2n}{r(n-2s)}\right\}, && \text{if } 2s < n, \quad n \ge 2,\\
&1+\frac{r}{2} \le p \le \frac{1}{1-2s}, && \text{if } 2s < n, \quad n = 1.
\end{aligned}$$

We take an initial data from

$$u_0 \in H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n), \quad u_1 \in H^{s-1,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n),$$

where $\alpha > n\left(\frac{1}{r} - \frac{1}{2}\right)$. *Then, for any* $\varepsilon > 0$, *there exists* $T = T(\varepsilon) \in (0, \infty]$ *such that the Cauchy problem* (1.1) *admits a unique local mild solution* $u \in C([0,T); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n))$, *with continuous dependence on the data. Moreover, if* $T < \infty$, u *satisfies*

$$\liminf_{t \to T} \|u(t)\|_{H^{s,0} \cap H^{0,\alpha}} = \infty.$$

Theorem 1.2. (Global well-posedness for small data) *In addition to the assumption in Theorem* 1.1, *we assume that* $r \in [1,2]$ *and* s, p *satisfy*

$$\begin{aligned}
&1+\frac{2r}{n} < p < \infty && \text{if } 1 \le n \le 2s,\\
&1+\frac{2r}{n} < p \le \min\left\{1+\frac{2}{n-2s}, \frac{2n}{r(n-2s)}\right\} && \text{if } 2s < n, \quad n \ge 2,\\
&1+2r < p \le \frac{1}{1-2s} && \text{if } 2s < n, \quad n = 1
\end{aligned}$$

(when $r \in (1,2]$, *we may take* $p \ge 1 + \frac{2r}{n}$*). Then, there exists a constant* $\varepsilon_0 = \varepsilon_0(n, p, r, s, \alpha, \|u_0\|_{H^{s,0}\cap H^{0,\alpha}}, \|u_1\|_{H^{s-1,0}\cap H^{0,\alpha}}) > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$, *the Cauchy problem* (1.1) *admits a unique global mild solution* $u \in C([0,\infty); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n))$, *with continuous dependence on the data. Moreover, the solution* u *satisfies the decay estimates*

$$\begin{aligned}
\|u(t)\|_{L^2} &\le C\varepsilon \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)},\\
\||\nabla|^s u(t)\|_{L^2} &\le C\varepsilon \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}},\\
\||\cdot|^\alpha u(t)\|_{L^2} &\le C\varepsilon \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}}.
\end{aligned}$$

Next, we study the asymptotic behavior of the global solutions. To state our result, we denote $G(t,x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right)$ and $\mathcal{G}(t)\phi = G(t) * \phi$.

Theorem 1.3. (Asymptotic behavior of global solutions) *Under the assumption of Theorem* 1.2, *we assume that* $p > 1 + \frac{2r}{n}$ *if* $r \in (1,2]$. *Let* m *be*

$$r \le m \quad \text{and} \quad m \begin{cases} \le \frac{2n}{n-2s} & \left(s < \frac{n}{2}\right),\\ < \infty & \left(s = \frac{n}{2}\right),\\ \le \infty & \left(s > \frac{n}{2}\right). \end{cases} \tag{1.3}$$

Then, the the global solution u of (1.1) constructed in Theorem 1.2 satisfies the following asymptotic behavior: When $r > 1$, for any $\varsigma > 0$, there exists a constant $C_\varsigma > 0$ such that

$$\begin{aligned}\|u(t) - \varepsilon\mathcal{G}(t)(u_0 + u_1)\|_{L^m} \\ \le C_\varsigma \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{m}\right)-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}\end{aligned} \tag{1.4}$$

holds for $t \ge 1$; When $r = 1$, for any $\varsigma > 0$, there exists a constant $C_\varsigma > 0$ such that

$$\|u(t) - \theta G(t)\|_{L^m} \le C_\varsigma \langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-\min\{\frac{\alpha}{2}-\frac{n}{4},\frac{1}{2},\frac{n}{2}(p-1)-1\}+\varsigma} \tag{1.5}$$

holds for $t \ge 1$, where

$$\theta = \varepsilon\int_{\mathbb{R}^n}(u_0 + u_1)(x)dx + \int_0^\infty\int_{\mathbb{R}^n}\mathcal{N}(u)dxdt. \tag{1.6}$$

Remark 1.4. (i) The case $r = 1, m \ge 2$ was studied by Hayashi et al. [9], though in this paper, we refine their argument (see Sects. 3.3, 3.4).

(ii) When $r > 1$, as we will see in the proof, the nonlinear term $\mathcal{N}(u)$ has better spatial integrability than the linear part. Hence, the nonlinear term decays faster as time tends to infinity and does not affect the asymptotic profile.

(iii) When $n \le 3$ and the initial data behaves like $\langle x\rangle^{-k}$ as $|x| \to \infty$ with some $0 < k \le 1$, a similar asymptotic behavior was obtained by Narazaki and Nishihara [43]. If we take an initial data satisfying the assumption of Theorem 1.3 and $\lim_{|x|\to\infty}|x|^{kn}(u_0 + u_1)(x) = c_1$ with some constants $c_1 \ne 0$ and $1/2 < k \le 1$ (note that $u_0, u_1 \in L^2$ requires $1/2 < k$), then Theorem 1.3 implies that $u(t)$ behaves as $\varepsilon\mathcal{G}(t)(c_1(1 + |x|^2)^{-\frac{kn}{2}})$ as $t \to \infty$. In this sense Theorem 1.3 generalizes the result of [43] to all $n \ge 1$ and more general initial data, while the class of the data and the solution is slightly different.

In the critical or subcritical case, we have the estimate of the lifespan from below. We define the lifespan of the solution of (1.1) by

$$\begin{aligned}T(\varepsilon) := \sup\{T \in (0,\infty);\ &\text{there exists a mild solution} \\ &u \in C([0,T); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n))\}.\end{aligned}$$

Theorem 1.5. (Lower bound of the lifespan) *In addition to the assumption in Theorem 1.1, we assume that*

$$r \in [1,2], \quad \min\left\{1 + \frac{r}{2}, 1 + \frac{r}{n}\right\} \le p < 1 + \frac{2r}{n} \tag{Case 1}$$

or

$$r = 1, \quad p = 1 + \frac{2r}{n}. \tag{Case 2}$$

Then, there exists $\varepsilon_1 = \varepsilon_1(n,p,r,s,\alpha,\|u_0\|_{H^{s,0}\cap H^{0,\alpha}},\|u_1\|_{H^{s-1,0}\cap H^{0,\alpha}}) > 0$ *such that for any* $\varepsilon \in (0,\varepsilon_1]$*, the lifespan* $T = T(\varepsilon)$ *of the solution is estimated as*

$$T(\varepsilon) \geq \begin{cases} C\varepsilon^{-1/\omega} & \text{in Case 1,} \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{in Case 2,} \end{cases}$$

where $\omega = \frac{1}{p-1} - \frac{n}{2r}$ *and* $C > 0$ *is a positive constant independent of* $\varepsilon \in (0,\varepsilon_1]$.

Remark 1.6. The estimate in Case 2 was proved by the first author and Ogawa [19].

Finally, we prove a blow-up result in the subcritical case with the nonlinearity $\mathcal{N}(u) = \pm|u|^p$.

Theorem 1.7. (Upper bound of the lifespan) *In addition to the assumptions in Theorem 1.1, we assume that*

$$\mathcal{N}(u) = \pm|u|^p \quad \text{and} \quad p < 1 + \frac{2r}{n} \tag{1.7}$$

and α *satisfies*

$$n\left(\frac{1}{r} - \frac{1}{2}\right) < \alpha < \frac{2}{p-1} - \frac{n}{2}.$$

Moreover, we take the initial data

$$u_0 \in H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n), \quad u_1 \in H^{s-1,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n)$$

fulfilling

$$\pm(u_0(x) + u_1(x)) \geq \begin{cases} |x|^{-\lambda} & \text{if} \quad |x| > 1, \\ 0 & \text{if} \quad |x| \leq 1 \end{cases}$$

(double-sign corresponds to (1.7)) with some λ *satisfying* $\frac{n}{2} + \alpha < \lambda < \frac{2}{p-1}$*. Then, there exists* $\varepsilon_2 = \varepsilon_2(n,p,r,s,\alpha,\lambda) > 0$ *such that for any* $\varepsilon \in (0,\varepsilon_2]$*, the lifespan* $T = T(\varepsilon)$ *of the solution is estimated as*

$$T(\varepsilon) \leq C\varepsilon^{-1/\kappa},$$

where $\kappa = \frac{1}{p-1} - \frac{\lambda}{2}$ *and* C *is a positive constant independent of* $\varepsilon \in (0,\varepsilon_2]$.

Remark 1.8. (i) The case $r = 1$ and $p \leq 1 + \frac{2r}{n}$ was studied by Todorova and Yordanov [65], Zhang [68], Li and Zhou [36], Nishihara [47], the first author and Ogawa [19], and the first author and the third author [20]. In particular, Theorem 1.5 shows that when $1 + \frac{1}{n} \leq p < 1 + \frac{2}{n}$, the lifespan $T(\varepsilon)$ is estimated as $T(\varepsilon) \sim \varepsilon^{-1/\omega}$ with $\omega = \frac{1}{p-1} - \frac{n}{2}$.

(ii) When $r > 1$, $p = 1 + \frac{2r}{n}$ and the initial data belong to $L^r(\mathbb{R}^n)$ but not $H^{0,\alpha}(\mathbb{R}^n)$, it is still an open problem whether the local solution blows up or not.

(iii) When $\mathcal{N}(u) = |u|^{p-1}u$, the blow-up of the solution was proved by Ikehata and Ohta [25], while estimates of lifespan were not obtained. Theorems 1.5 and 1.7 give an almost optimal estimate of lifespan for $\mathcal{N}(u) = \pm|u|^p$.

$r\backslash p$	$\min\left\{1+\frac{r}{2}, 1+\frac{r}{n}\right\}\ldots$	$1+\frac{2r}{n}$	$\ldots 1+\frac{2}{n-2s}$
1	SDBU [65] $C^{-1}\varepsilon^{-1/\omega} \le T(\varepsilon) \le C\varepsilon^{-1/\kappa}$ [20]	SDBU [68] $e^{C\varepsilon^{-(p-1)}} \le T(\varepsilon) \le e^{C\varepsilon^{-p}}$ [19]	SDGE [9] $T(\varepsilon)=\infty$
$r>1$	SDBU [25] $C^{-1}\varepsilon^{-1/\omega} \le T(\varepsilon) \le C\varepsilon^{-1/\kappa}$	SDGE $T(\varepsilon)=\infty$	SDGE $T(\varepsilon)=\infty$

Table 1. SDBU: small data blow-up, SDGE: small data global existence

Our results are summarized in Table 1.

Our strategy for proving Theorems 1.1 and 1.2 are based on that of Hayashi, Kaikina and Naumkin [9]. However, we have to refine their estimates to fit the slowly decaying data and solutions. The main ingredient is the estimate of the fundamental solution $\mathcal{D}(t)$ (see (1.8) for the definition) of the linear problem, which are given in Lemma 2.1. To prove these estimates, we use a gain of one derivative coming from the high frequency part of the kernel $L(t,\xi)$ of $\mathcal{D}(t)$ (see (1.10)). Combining these estimates and nonlinear estimates with the contraction mapping principle, we prove the existence of solutions.

To prove Theorem 1.3, we first show that the solution u of the damped wave equation (1.1) is approximated by the solution of the linear heat equation with the homogeneous term $\mathcal{N}(u)$ (see Proposition 3.1). After that, we investigate the precise asymptotic behavior of solutions to the inhomogeneous linear heat equation (see Proposition 3.4).

For the upper estimates of the lifespan, we employ a test function method developed by Zhang [68], while this is based on a contradiction argument and not directly applicable to obtain the lifespan estimate. To avoid the contradiction argument, we use the ideas by Kuiper [33], Sun [60] and [20] to obtain an almost optimal estimate of the lifespan.

The rest of the paper is organized as follows. In Sect. 2, we give a proof of Theorems 1.1 and 1.2. Section 3 is devoted to the proof of Theorem 1.3. Theorems 1.5 and 1.7 will be proved in Sect. 4. Finally, we collect some useful lemmas in “Appendix”.

1.2. Notations

For the reader’s convenience, we collect the notations used throughout this paper. The letter C indicates a generic constant, which may change from line to line. Let $\langle x\rangle = (1+|x|^2)^{1/2}$. Sometimes we use $a \vee b := \max\{a,b\}$.

For functions $f=f(x):\mathbb{R}^n\to\mathbb{R}$ and $\phi=\phi(\xi):\mathbb{R}^n\to\mathbb{R}$, the Fourier transform and the inverse Fourier transform are defined by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx,$$
$$\mathcal{F}^{-1}[\phi](x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(\xi) e^{ix\xi} d\xi,$$

respectively.

Let $L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$ and $H^{s,\alpha}(\mathbb{R}^n)$ $(s, \alpha \ge 0)$ be the usual Lebesgue and the weighted Sobolev spaces, respectively, equipped with the norms defined by

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} \quad (1 \le p < \infty), \quad \|f\|_{L^\infty} = \operatorname{ess\,sup} |f|.$$
$$\|f\|_{H^{s,\alpha}} = \|\langle x \rangle^\alpha \langle \nabla \rangle^s f\|_{L^2},$$

where $\langle \nabla \rangle^s f = \mathcal{F}^{-1}[\langle \xi \rangle^s \hat{f}]$. For an interval I and a Banach space X, we define $C^r(I;X)$ as the space of r-times continuously differentiable mapping from I to X with respect to the topology in X.

We denote by $\mathcal{D}(t)$ and $\mathcal{G}(t)$ the solution operator of the linear damped wave and linear heat equations, respectively, that is,

$$\mathcal{D}(t) := e^{-\frac{t}{2}} \mathcal{F}^{-1} L(t,\xi) \mathcal{F}, \tag{1.8}$$

$$\mathcal{G}(t) := \mathcal{F}^{-1} e^{-t|\xi|^2} \mathcal{F}, \tag{1.9}$$

where

$$L(t,\xi) := \begin{cases} \dfrac{\sinh(t\sqrt{1/4 - |\xi|^2})}{\sqrt{1/4 - |\xi|^2}} & \text{if } |\xi| < 1/2, \\ \dfrac{\sin(t\sqrt{|\xi|^2 - 1/4})}{\sqrt{|\xi|^2 - 1/4}} & \text{if } |\xi| > 1/2. \end{cases} \tag{1.10}$$

Also, we use

$$\tilde{\mathcal{D}}(t) := (\partial_t + 1)\, \mathcal{D}(t). \tag{1.11}$$

Throughout this paper, we always use s, r, α as real numbers satisfying $s \ge 0$, $r \in [1,2]$, $\alpha > n\left(\frac{1}{r} - \frac{1}{2}\right)$, respectively. Also, for a real number s, we denote by $[s]$ the integer part of s. For $T \in (0,\infty]$, we define

$$\begin{aligned} \|\phi\|_{X(T)} := \sup_{0<t<T} \Big[&\langle t \rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \|\phi(t)\|_{L^2} + \langle t \rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{s}{2}} \||\nabla|^s \phi(t)\|_{L^2} \\ &+ \langle t \rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\alpha}{2}} \||\cdot|^\alpha \phi(t)\|_{L^2} \Big], \end{aligned} \tag{1.12}$$

$$\begin{aligned} \|\psi\|_{Y(T)} := \sup_{0<t<T} \Big[&\langle t \rangle^{\eta} \||\nabla|^{[s]} \psi(t)\|_{L^\rho} + \sup_{\gamma \in [\sigma_1,\sigma_2]} \langle t \rangle^{\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\gamma}\right)} \|\psi(t)\|_{L^\gamma} \\ &+ \langle t \rangle^{\zeta} \|\langle \cdot \rangle^\alpha \psi(t)\|_{L^q} \Big], \end{aligned} \tag{1.13}$$

where the parameters are defined in Table 2. Here $(n-2s)_+$ denotes $0 \vee (n-2s)$.

	$n \geq 2$	$n = 1$
$\eta :=$	$\frac{\mu}{2}(p-1) + \frac{s}{2} + \frac{np}{2}\left(\frac{1}{r} - \frac{1}{2}\right)$	$\frac{1}{2}\left(\frac{p}{r} - \frac{1}{2}\right)$
$\mu :=$	$\frac{n}{2} - \frac{1}{p-1}$	–
$\zeta :=$	$\frac{n}{2r}(p-1) - \frac{1}{2} + \frac{n}{2}\left(\frac{1}{r} - \frac{1}{2}\right) - \frac{\alpha}{2}$	$\frac{1}{2r}(p-1) - \frac{1}{4} + \frac{1}{2}\left(\frac{1}{r} - \frac{1}{2}\right) - \frac{\alpha}{2}$
$q :=$	$\frac{2n}{n+2}$	1
$\rho :=$	$\frac{2n}{n+2-2(s-[s])}$	2
$\sigma_1 :=$	$\max\left\{1, \frac{nr}{n+r}\right\}$	1
$\sigma_2 :=$	$\min\left\{2, \frac{2n}{p(n-2s)_+}\right\}$	2

TABLE 2. Definition of parameters

2. Local and global existence of solutions

2.1. Preliminary estimates

Lemma 2.1. *Let* $\gamma, \nu \in [1,2]$, $\beta \geq 0$, $s_1 \geq s_2 \geq 0$. *Then, we have*

$$\||\nabla|^{s_1}\mathcal{D}(t)\psi\|_{L^2} \leq C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{s_1-s_2}{2}}\||\nabla|^{s_2}\psi\|_{L^\gamma} + Ce^{-\frac{t}{4}}\||\nabla|^{s_1}\langle\nabla\rangle^{-1}\psi\|_{L^2}, \quad (2.1)$$

$$\||\cdot|^{\beta}\mathcal{D}(t)\psi\|_{L^2} \leq C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\frac{\beta}{2}}\|\psi\|_{L^\gamma} + C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\nu}-\frac{1}{2}\right)}\||\cdot|^{\beta}\psi\|_{L^\nu} + Ce^{-\frac{t}{4}}\|\langle\cdot\rangle^{\beta}\langle\nabla\rangle^{-1}\psi\|_{L^2}. \quad (2.2)$$

We also have the continuity of $\mathcal{D}(t)$ *with respect to* $H^{s_1,0} \cap H^{0,\beta}$*-norm:*

$$\lim_{t_1 \to t_2} \|\mathcal{D}(t_1)\psi - \mathcal{D}(t_2)\psi\|_{H^{s_1,0}\cap H^{0,\beta}} = 0. \quad (2.3)$$

When $\gamma = 1$, the estimates (2.1) and (2.2) were proved by [9]. Here we give a generalization of it to $\gamma \in [1,2]$ with a slightly simpler proof.

Proof. By the definition of $L(t,\xi)$, it is easy to see that

$$e^{-\frac{t}{2}}\||\xi|^{s}L(t,\xi)\|_{L^\gamma(|\xi|\leq 1)} \leq C\langle t\rangle^{-\frac{s}{2}-\frac{n}{2\gamma}}$$

and

$$\|\langle\xi\rangle L(t,\xi)\|_{L^\infty(|\xi|\geq 1)} \leq C.$$

Therefore, applying the Plancherel theorem and the Hölder inequality, we have

$$\||\nabla|^{s_1}\mathcal{D}(t)\psi\|_{L^2} = \||\xi|^{s_1}e^{-\frac{t}{2}}L(t,\xi)\hat{\psi}\|_{L^2}$$
$$\leq Ce^{-\frac{t}{2}}\||\xi|^{s_1-s_2}L(t,\xi)\|_{L^{\frac{2\gamma}{2-\gamma}}(|\xi|\leq 1)}\||\xi|^{s_2}\hat{\psi}\|_{L^{\frac{\gamma}{\gamma-1}}(|\xi|\leq 1)}$$

$$+ Ce^{-\frac{t}{2}} \|\langle\xi\rangle L(t,\xi)\|_{L^\infty(|\xi|\geq 1)} \||\xi|^{s_1}\langle\xi\rangle^{-1}\hat{\psi}\|_{L^2(|\xi|\geq 1)}$$
$$\leq C\langle t\rangle^{-\frac{s_1-s_2}{2}-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)} \||\nabla|^{s_2}\psi\|_{L^\gamma} + Ce^{-\frac{t}{4}} \||\nabla|^{s_1}\langle\nabla\rangle^{-1}\psi\|_{L^2},$$

which gives (2.1).

Next, we prove (2.2). Let $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function satisfying $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$. We put

$$K(t,x) := e^{-\frac{t}{2}} \mathcal{F}^{-1}\left(L(t,\xi)\chi(\xi)\right).$$

The Hausdorff–Young inequality implies

$$\begin{aligned}
&\||\cdot|^\beta \mathcal{F}^{-1}\left(e^{-\frac{t}{2}} L(t,\xi)\chi(\xi)\hat{\psi}\right)\|_{L^2} \\
&= \left\| |x|^\beta \int_{\mathbb{R}^n} K(t,x-y)\psi(y)dy \right\|_{L^2} \\
&\leq C \left\| \int_{\mathbb{R}^n} \left(|x-y|^\beta |K(t,x-y)| + |K(t,x-y)||y|^\beta\right)\psi(y)dy \right\|_{L^2} \\
&\leq C\||\cdot|^\beta K(t)\|_{L^{\frac{2\gamma}{3\gamma-2}}} \|\psi\|_{L^\gamma} + C\|K(t)\|_{L^{\frac{2\nu}{3\nu-2}}} \||\cdot|^\beta\psi\|_{L^\nu}. \qquad (2.4)
\end{aligned}$$

Now we prove

$$\||\cdot|^\beta K(t)\|_{L^k} \leq C\langle t\rangle^{\frac{\beta}{2}-\frac{n}{2}\left(1-\frac{1}{k}\right)} \qquad (2.5)$$

for $t > 0$ and $k \in [1,\infty)$. First, we divide the proof into the cases $0 < t < 1$ and $t \geq 1$. For the case $0 < t < 1$, we easily prove $\||\cdot|^\beta K(t)\|_{L^k} \leq C$. Indeed, noting $|x|^{\beta k} \leq \langle x\rangle^{2Nk-(n+1)}$ for sufficiently large integer $N \in \mathbb{N}$, we have

$$\begin{aligned}
\||\cdot|^\beta K(t)\|_{L^k}^k &\leq C\int_{\mathbb{R}^n} \langle x\rangle^{2Nk-(n+1)} |K(t,x)|^k dx \\
&= C\int_{\mathbb{R}^n} \langle x\rangle^{-(n+1)} \left| \int_{|\xi|\leq 1/4} e^{ix\xi} e^{-\frac{t}{2}} \langle\nabla\rangle^{2N}(L(t,\xi)\chi(\xi))d\xi \right|^k dx \\
&\leq C.
\end{aligned}$$

For the case $t \geq 1$, we change the variables as $\sqrt{t}\xi = \eta$ and $x = \sqrt{t}y$ to obtain

$$\begin{aligned}
\||\cdot|^\beta K(t)\|_{L^k}^k &= \int_{\mathbb{R}^n} |x|^{\beta k} \left| \int_{\mathbb{R}^n} e^{ix\xi} e^{-\frac{t}{2}} L(t,\xi)\chi(\xi)d\xi \right|^k dx \\
&= t^{-\frac{n}{2}k} \int_{\mathbb{R}^n} |x|^{\beta k} \left| \int_{\mathbb{R}^n} e^{ix\eta/\sqrt{t}} e^{-\frac{t}{2}} L(t,\eta/\sqrt{t})\chi(\eta/\sqrt{t})d\eta \right|^k dx \\
&= t^{\frac{\beta}{2}k-\frac{n}{2}(k-1)} \int_{\mathbb{R}^n} |y|^{\beta k} \left| \int_{\mathbb{R}^n} e^{iy\eta} e^{-\frac{t}{2}} L(t,\eta/\sqrt{t})\chi(\eta/\sqrt{t})d\eta \right|^k dy.
\end{aligned}$$

As before, using $|y|^{\beta k} \leq \langle y\rangle^{2Nk-(n+1)}$ with sufficiently large integer $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\||\cdot|^{\beta}K(t)\|_{L^k}^k\\
&\le Ct^{\frac{\beta}{2}k-\frac{n}{2}(k-1)}\\
&\quad\times\int_{\mathbb{R}^n}\langle y\rangle^{-(n+1)}\left|\int_{\mathbb{R}^n}e^{iy\eta}\langle\nabla_\eta\rangle^{2N}\left(e^{-\frac{t}{2}}L(t,\eta/\sqrt{t})\chi(\eta/\sqrt{t})\right)d\eta\right|^k dy.
\end{aligned}$$

By the definition of $L(t,\xi)$ (see (1.10)), we can easily see that for $t\ge 1$

$$\left|\langle\nabla_\eta\rangle^{2N}\left(e^{-\frac{t}{2}}L(t,\eta/\sqrt{t})\chi(\eta/\sqrt{t})\right)\right|\le Ce^{-\frac{|\eta|^2}{2}}.$$

Thus, we obtain the desired estimate (2.5).

Applying (2.5)–(2.4), we have

$$\begin{aligned}
&\left\||\cdot|^{\beta}\mathcal{F}^{-1}\left(e^{-\frac{t}{2}}L(t,\xi)\chi(\xi)\hat{\psi}\right)\right\|_{L^2}\\
&\quad\le C\langle t\rangle^{\frac{\beta}{2}-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)}\|\psi\|_{L^\gamma}+C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\nu}-\frac{1}{2}\right)}\||\cdot|^{\beta}\psi\|_{L^\nu}.
\end{aligned}$$

Hence, it suffices to show that

$$\left\||\cdot|^{\beta}\mathcal{F}^{-1}\left(e^{-\frac{t}{2}}L(t,\xi)(1-\chi(\xi))\hat{\psi}\right)\right\|_{L^2}\le Ce^{-\frac{t}{4}}\|\langle\cdot\rangle^{\beta}\langle\nabla\rangle^{-1}\psi\|_{L^2}. \tag{2.6}$$

To prove this, we calculate the fractional derivative. Let $\omega:=\beta-[\beta]$. Making use of Lemma A.1, we see that

$$\begin{aligned}
|\partial_j|^{\omega}(\phi\psi)(x)&=C\int_{\mathbb{R}}(\phi(x)\psi(x)-\phi(x+y)\psi(x+y))\frac{dy_j}{|y_j|^{1+\omega}}\\
&=\phi(x)|\partial_j|^{\omega}\psi(x)+C\int_{\mathbb{R}}(\phi(x)-\phi(x+y))\psi(x+y)\frac{dy_j}{|y_j|^{1+\omega}}
\end{aligned}$$

holds, where $y:=(0,\dots,y_j,\dots,0)$. Then, using the Plancherel theorem and the Leibniz rule, we have

$$\begin{aligned}
\||\nabla|^{\beta}(\phi\psi)\|_{L^2}&\le C\sum_{j=1}^{n}\||\partial_j|^{\omega}|\partial_j|^{[\beta]}(\phi\psi)\|_{L^2}\\
&\le C\sum_{j=1}^{n}\sum_{k=0}^{[\beta]}\left\||\partial_j|^{\omega}\left((\partial_j^{[\beta]-k}\phi)\partial_j^k\psi\right)\right\|_{L^2},
\end{aligned}$$

and hence,

$$\begin{aligned}
&\||\nabla|^{\beta}(\phi\psi)\|_{L^2}\\
&\le C\sum_{j=1}^{n}\sum_{k=0}^{[\beta]}\left\|\left(\partial_j^{[\beta]-k}\phi\right)|\partial_j|^{k+\omega}\psi\right\|_{L^2}\\
&\quad+C\sum_{j=1}^{n}\sum_{k=0}^{[\beta]}\left\|\int_{\mathbb{R}}\left(\partial_j^{[\beta]-k}\phi(\cdot)-\partial_j^{[\beta]-k}\phi(\cdot+y)\right)\partial_j^k\psi(\cdot+y)\frac{dy_j}{|y_j|^{1+\omega}}\right\|_{L^2}\\
&\le C\left(\sum_{j=1}^{n}\sum_{k=0}^{[\beta]+1}\|\partial_j^k\phi\|_{L^\infty}\right)\|\langle\nabla\rangle^{\beta}\psi\|_{L^2}.
\end{aligned}$$

We apply the above estimate to the left-hand side of (2.6) with $\psi = \langle\xi\rangle^{-1}\hat{\psi}$ and $\phi = \langle\xi\rangle e^{-\frac{t}{2}} L(t,\xi)(1-\chi(\xi))$. Since

$$\|\partial_j^k e^{-\frac{t}{2}}\langle\xi\rangle L(t,\xi)(1-\chi(\xi))\|_{L^\infty} \le Ce^{-\frac{t}{4}}$$

for any $k \in \mathbb{N}$, we obtain

$$\left\| |\cdot|^\beta \mathcal{F}^{-1}\left(e^{-\frac{t}{2}} L(t,\xi)(1-\chi(\xi))\hat{\psi}\right)\right\|_{L^2} \le Ce^{-\frac{t}{4}}\|\langle\nabla\rangle^\beta \langle\xi\rangle^{-1}\hat{\psi}\|_{L^2}.$$

Finally, the bounds of $L(t,\xi)$ proved above and the continuity of $L(t,\xi)$ with respect to t show (2.3). This completes the proof. □

Similarly, we can prove the following lemma.

Lemma 2.2. *Let $\gamma, \nu \in [1,2]$, $\beta \ge 0$, $s_1 \ge s_2 \ge 0$. Then, we have*

$$\begin{aligned} \||\nabla|^{s_1}\tilde{\mathcal{D}}(t)\psi\|_{L^2} &\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{s_1-s_2}{2}}\||\nabla|^{s_2}\psi\|_{L^\gamma} \\ &\quad + Ce^{-\frac{t}{4}}\||\nabla|^{s_1}\psi\|_{L^2}, \qquad (2.7)\\ \||\cdot|^\beta\tilde{\mathcal{D}}(t)\psi\|_{L^2} &\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\frac{\beta}{2}}\|\psi\|_{L^\gamma} + C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\nu}-\frac{1}{2}\right)}\||\cdot|^\beta\psi\|_{L^\nu} \\ &\quad + Ce^{-\frac{t}{4}}\|\langle\cdot\rangle^\beta\psi\|_{L^2}. \qquad (2.8)\end{aligned}$$

We also have the continuity of $\tilde{\mathcal{D}}(t)$ with respect to $H^{s_1,0}\cap H^{0,\beta}$-norm:

$$\lim_{t_1\to t_2}\|\tilde{\mathcal{D}}(t_1)\psi - \tilde{\mathcal{D}}(t_2)\psi\|_{H^{s_1,0}\cap H^{0,\beta}} = 0. \qquad (2.9)$$

Let $\|\cdot\|_{X(T)}$ be defined by (1.12). In order to prove Theorems 1.1 and 1.2, we frequently use the following interpolation inequalities.

Lemma 2.3. *Let $s \ge 0$, $r \in [1,2]$, $\alpha > n(\frac{1}{r}-\frac{1}{2})$ and let $\|\cdot\|_{X(T)}$ be defined by (1.12). Then, the following interpolation inequalities hold:*

(i) We have

$$\sup_{0<t<T}\left(\sup_{0\le s'\le s}\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{s'}{2}}\||\nabla|^{s'}\phi(t)\|_{L^2} + \sup_{0\le\beta\le\alpha}\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\beta}{2}}\||\cdot|^\beta\phi(t)\|_{L^2}\right) \le C\|\phi\|_{X(T)}.$$

(ii) We have

$$\sup_{0<t<T}\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{\gamma}\right)}\|\phi(t)\|_{L^\gamma} \le C\|\phi\|_{X(T)} \qquad (2.10)$$

for

$$\begin{cases} r \le \gamma \le \infty & \text{if} \quad 1 \le n < 2s,\\ r \le \gamma < \infty & \text{if} \quad n = 2s,\\ r \le \gamma \le \frac{2n}{n-2s} & \text{if} \quad 2s < n.\end{cases}$$

Proof. (i) The Plancherel theorem and the Hölder inequality imply

$$\begin{aligned}
&\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{s'}{2}}\||\nabla|^{s'}\phi(t)\|_{L^2}\\
&\quad=\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{s'}{2}}\||\xi|^{s'}\hat{\phi}(t)\|_{L^2}\\
&\quad\le\left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}\|\hat{\phi}(t)\|_{L^2}\right)^{1-\frac{s'}{s}}\left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{s}{2}}\||\xi|^{s}\hat{\phi}(t)\|_{L^2}\right)^{\frac{s'}{s}}\\
&\quad\le\|\phi\|_{X(T)}.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\beta}{2}}\||\cdot|^{\beta}\phi(t)\|_{L^2}\\
&\quad\le\left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}\|\phi(t)\|_{L^2}\right)^{1-\frac{\beta}{\alpha}}\left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\alpha}{2}}\||\cdot|^{\alpha}\phi(t)\|_{L^2}\right)^{\frac{\beta}{\alpha}}\\
&\quad\le\|\phi\|_{X(T)}.
\end{aligned}$$

(ii) If γ satisfies $2\le\gamma<\infty$ and the condition in (2.10), we apply the Sobolev inequality and obtain

$$\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{\gamma}\right)}\|\phi(t)\|_{L^\gamma}\le C\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{\gamma}\right)}\||\nabla|^{s'}\phi(t)\|_{L^2}\tag{2.11}$$

with $s'=n(\frac{1}{2}-\frac{1}{\gamma})$. Clearly, $s'\in[0,s]$ holds under the condition in (2.10) and hence, (i) implies

$$\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{\gamma}\right)}\||\nabla|^{s'}\phi(t)\|_{L^2}=\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{s'}{2}}\||\nabla|^{s'}\phi(t)\|_{L^2}\le C\|\phi\|_{X(T)}.$$

When $2s>n$ and $\gamma=\infty$, instead of (2.11), using (2.12) below with $\varphi=\hat{\phi}$, $\gamma=1$ and $\beta=s$, we have

$$\begin{aligned}
\langle t\rangle^{\frac{n}{2r}}\|\phi(t)\|_{L^\infty}&\le C\langle t\rangle^{\frac{n}{2r}}\|\hat{\phi}(t)\|_{L^1}\\
&\le C\left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{s}{2}}\||\nabla|^{s}\phi(t)\|_{L^2}\right)^{\frac{n}{2s}}\left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}\|\phi(t)\|_{L^2}\right)^{1-\frac{n}{2s}}\\
&\le C\|\phi\|_{X(T)}.
\end{aligned}$$

On the other hand, to prove (2.10) in the case $r\le\gamma\le 2$, we first claim that

$$\|\varphi\|_{L^\gamma}\le C\|\varphi\|_{L^2}^{1-\frac{n}{\beta}\frac{2-\gamma}{2\gamma}}\||\cdot|^{\beta}\varphi\|_{L^2}^{\frac{n}{\beta}\frac{2-\gamma}{2\gamma}}\tag{2.12}$$

with β satisfying $\beta>n(\frac{1}{\gamma}-\frac{1}{2})$ and for $\varphi\in H^{0,\beta}$. Indeed, let θ be determined later and we calculate

$$\begin{aligned}
\|\varphi\|_{L^\gamma}^{\gamma}&=\int_{\mathbb{R}^n}(\theta^2+|x|^2)^{-\frac{\beta\gamma}{2}}(\theta^2+|x|^2)^{\frac{\beta\gamma}{2}}|\varphi(x)|^{\gamma}dx\\
&\le\left(\int_{\mathbb{R}^n}(\theta^2+|x|^2)^{\beta}|\varphi(x)|^2dx\right)^{\frac{\gamma}{2}}\left(\int_{\mathbb{R}^n}(\theta^2+|x|^2)^{-\frac{\beta\gamma}{2-\gamma}}dx\right)^{\frac{2-\gamma}{2}}.
\end{aligned}$$

Since $\frac{\beta\gamma}{2-\gamma}>\frac{n}{2}$, we have

$$\int_{\mathbb{R}^n}(\theta^2+|x|^2)^{-\frac{\beta\gamma}{2-\gamma}}dx\le C\theta^{-\frac{2\beta\gamma}{2-\gamma}+n}$$

and hence,

$$\|\varphi\|_{L^\gamma} \le C\theta^{-\beta+\frac{n(2-\gamma)}{2\gamma}} \left(\int_{\mathbb{R}^n} (\theta^2 + |x|^2)^\beta |\phi(x)|^2\, dx\right)^{\frac{1}{2}}$$
$$\le C\theta^{\frac{n(2-\gamma)}{2\gamma}} \|\varphi\|_{L^2} + C\theta^{-\beta+\frac{n(2-\gamma)}{2\gamma}} \||\cdot|^\beta \varphi\|_{L^2}.$$

Taking $\theta = \|\varphi\|_{L^2}^{-\frac{1}{\beta}} \||\cdot|^\beta\varphi\|_{L^2}^{\frac{1}{\beta}}$, we have (2.12).

From (2.12), for $r \le \gamma \le 2$, letting β satisfy $n(\frac{1}{\gamma} - \frac{1}{2}) < \beta \le \alpha$, we obtain

$$\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{\gamma}\right)} \|\phi(t)\|_{L^\gamma}$$
$$\le \left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \|\phi(t)\|_{L^2}\right)^{1-\frac{n}{\beta}\frac{2-r}{2r}} \left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\beta}{2}} \||\cdot|^\beta \phi(t)\|_{L^2}\right)^{\frac{n}{\beta}\frac{2-r}{2r}}$$

and hence, (i) gives

$$\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{\gamma}\right)} \|\phi(t)\|_{L^\gamma} \le C\|\phi\|_{X(T)}.$$

Thus, we finish the proof. □

2.2. Proof of Theorems 1.1 and 1.2 in higher dimensional cases

We start with the estimate of the Duhamel term. Let $\|\cdot\|_{Y(T)}$ be defined by (1.13).

Lemma 2.4. *Under the assumption in Theorem 1.1, we have*

$$\left\|\int_0^t \mathcal{D}(t-\tau)\psi(\tau)d\tau\right\|_{X(T)} \le C\int_0^T \|\psi\|_{Y(\tau)} d\tau \tag{2.13}$$

for $0 < T \le 1$. *Moreover, we have*

$$\left\|\int_0^t \mathcal{D}(t-\tau)\psi(\tau)d\tau\right\|_{X(T)}$$
$$\le C\|\psi\|_{Y(T)} \begin{cases} 1 & \text{if} \quad p > 1 + \frac{2r}{n}, \\ \log(2+T) & \text{if} \quad p = 1 + \frac{2r}{n}, \\ \langle T\rangle^{1-\frac{n}{2r}(p-1)} & \text{if} \quad p < 1 + \frac{2r}{n} \end{cases} \tag{2.14}$$

for $r = 1$, $0 < T < \infty$, *and*

$$\left\|\int_0^t \mathcal{D}(t-\tau)\psi(\tau)d\tau\right\|_{X(T)}$$
$$\le C\|\psi\|_{Y(T)} \begin{cases} 1 & \text{if} \quad p \ge 1 + \frac{2r}{n}, \\ \langle T\rangle^{1-\frac{n}{2r}(p-1)} & \text{if} \quad p < 1 + \frac{2r}{n} \end{cases} \tag{2.15}$$

for $r \in (1, 2]$, $0 < T < \infty$. *Here we may take* $T = \infty$ *if* $p > 1 + \frac{2}{n}, r = 1$ *or* $p \ge 1 + \frac{2r}{n}, r \in (1, 2]$.

Proof. The estimate (2.13) is easily proved by looking at the proof of (2.14) and (2.15) carefully, and we may omit it.

Step 1 Estimate of $\||\nabla|^s \int_0^t \mathcal{D}(t-\tau)\psi(\tau)\,d\tau\|_{L^2}$. We have

$$\begin{aligned}
&\left\||\nabla|^s \int_0^t \mathcal{D}(t-\tau)\psi(\tau)\,d\tau\right\|_{L^2}\\
&\le \int_0^{\frac{t}{2}} \||\nabla|^s\mathcal{D}(t-\tau)\psi(\tau)\|_{L^2}\,d\tau + \int_{\frac{t}{2}}^t \||\nabla|^s\mathcal{D}(t-\tau)\psi(\tau)\|_{L^2}\,d\tau\\
&=: I + II.
\end{aligned}$$

For II, we apply Lemma 2.1 with $s_1 = s, s_2 = [s], \gamma = \rho$ (see Table 2 for notations) and have

$$\begin{aligned}
II &\le C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{s-[s]}{2}-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)}\||\nabla|^{[s]}\psi(\tau)\|_{L^\rho}\,d\tau\\
&\quad + C\int_{\frac{t}{2}}^t e^{-\frac{t-\tau}{4}}\||\nabla|^s\langle\nabla\rangle^{-1}\psi(\tau)\|_{L^2}\,d\tau.
\end{aligned}$$

The Sobolev inequality implies

$$\||\nabla|^s\langle\nabla\rangle^{-1}\psi(\tau)\|_{L^2} \le C\||\nabla|^{[s]}\psi(\tau)\|_{L^\rho} \tag{2.16}$$

and hence,

$$\begin{aligned}
II &\le C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)-\frac{s-[s]}{2}}\||\nabla|^{[s]}\psi(\tau)\|_{L^\rho}\,d\tau\\
&\le C\|\psi\|_{Y(T)}\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)-\frac{s-[s]}{2}}\langle\tau\rangle^{-\eta}\,d\tau\\
&\le C\langle t\rangle^{-\frac{s}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+1-\frac{n}{2r}(p-1)}\|\psi\|_{Y(T)}.
\end{aligned}$$

Next, we estimate I. Applying Lemma 2.1 with $s_1 = s, s_2 = 0$ and $\gamma \in [\sigma_1, \sigma_2]$ (see Table 2 for notations) determined later, we have

$$\begin{aligned}
I &\le C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{s}{2}}\|\psi(\tau)\|_{L^\gamma}\,d\tau\\
&\quad + C\int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{4}}\||\nabla|^s\langle\nabla\rangle^{-1}\psi(\tau)\|_{L^2}\,d\tau\\
&=: I_1 + I_2.
\end{aligned}$$

We calculate I_2. The Sobolev embedding and the definition of $Y(T)$-norm imply

$$\begin{aligned}
I_2 &\le C\int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{4}}\||\nabla|^{[s]}\psi(\tau)\|_{L^\rho}\,d\tau\\
&\le C\|\psi\|_{Y(T)}\int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{4}}\langle\tau\rangle^{-\eta}\,d\tau\\
&\le Ce^{-\frac{t}{8}}\|\psi\|_{Y(T)}.
\end{aligned}$$

We divide the estimate of I_1 into three cases.

Case 1 When $p < 1 + \frac{2r}{n}$, taking $\gamma = r$, and noting $-\frac{n}{2r}(p-1) > -1$ and $r \in [\sigma_1, \sigma_2]$, we see that

$$\begin{aligned} I_1 &\le C\|\psi\|_{Y(T)} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}} \langle\tau\rangle^{-\frac{n}{2r}(p-1)}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+1-\frac{n}{2r}(p-1)}. \end{aligned}$$

Case 2 When $p > 1 + \frac{2r}{n}$, taking $\gamma = r$ and noting $-\frac{n}{2r}(p-1) < -1$, we infer that

$$\begin{aligned} I_1 &\le C\|\psi\|_{Y(T)} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}} \langle\tau\rangle^{-\frac{n}{2r}(p-1)}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}}. \end{aligned}$$

Case 3-1 When $p = 1 + \frac{2r}{n}$ and $r > 1$, we take $\gamma = \sigma_1$ and have $-\frac{n}{2}(\frac{p}{r} - \frac{1}{\sigma_1}) = -\frac{n}{2}(\frac{1}{r} - \frac{1}{\sigma_1}) - 1 > -1$, since $r > \sigma_1$. Thus, we obtain

$$\begin{aligned} I_1 &\le C\|\psi\|_{Y(T)} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)-\frac{s}{2}} \langle\tau\rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_1}\right)}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)-\frac{s}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{\sigma_1}\right)} \\ &= \|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}}. \end{aligned}$$

Case 3-2 When $p = 1 + \frac{2r}{n}$ and $r = 1$, taking $\gamma = r$, we see that

$$\begin{aligned} I_1 &\le C\|\psi\|_{Y(T)} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}} \langle\tau\rangle^{-\frac{n}{2r}(p-1)}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}} \log(1+t). \end{aligned}$$

Step 2 Estimate of $\| \int_0^t \mathcal{D}(t-\tau)\psi(\tau)\, d\tau\|_{L^2}$. We have

$$\begin{aligned} &\left\| \int_0^t \mathcal{D}(t-\tau)\psi(\tau)\, d\tau \right\|_{L^2} \\ &\le C \int_0^{\frac{t}{2}} \|\mathcal{D}(t-\tau)\psi(\tau)\|_{L^2}\, d\tau + C \int_{\frac{t}{2}}^t \|\mathcal{D}(t-\tau)\psi(\tau)\|_{L^2}\, d\tau \\ &=: \mathit{III} + \mathit{IV}. \end{aligned}$$

For IV, we apply Lemma 2.1 with $s_1 = s_2 = 0$ and $\gamma = q$, where q is defined in Sect. 1.2, and obtain

$$\begin{aligned} \mathit{IV} &\le C \int_{\frac{t}{2}}^t \left[\langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \|\psi(\tau)\|_{L^q} + e^{-\frac{t-\tau}{4}} \|\langle\nabla\rangle^{-1}\psi(\tau)\|_{L^2} \right] d\tau \\ &\le C \int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \|\psi(\tau)\|_{L^q}\, d\tau. \end{aligned}$$

Here we have used the Sobolev embedding $\|\langle\nabla\rangle^{-1}\psi(\tau)\|_{L^2} \le C\|\psi(\tau)\|_{L^q}$. Since $q = \frac{2n}{n+2} \in [\sigma_1, \sigma_2]$ and $-\frac{n}{2}(\frac{1}{q} - \frac{1}{2}) = -\frac{1}{2}$, we have

$$\begin{aligned} I\!V &\le C\|\psi\|_{Y(T)} \int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \langle\tau\rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{q}\right)}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+1-\frac{n}{2r}(p-1)}. \end{aligned}$$

Let us estimate the term $I\!I\!I$. Applying Lemma 2.1 with $s_1 = s_2 = 0$ and $\gamma \in [1, 2]$ determined later, we demonstrate

$$\begin{aligned} I\!I\!I &\le C \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)} \|\psi(\tau)\|_{L^\gamma}\, d\tau + C\int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{4}} \|\langle\nabla\rangle^{-1}\psi(\tau)\|_{L^2}\, d\tau \\ &=: I\!I\!I_1 + I\!I\!I_2. \end{aligned}$$

From $q \in [\sigma_1, \sigma_2]$, the Sobolev embedding and the definition of $Y(T)$-norm imply

$$\begin{aligned} I\!I\!I_2 &\le C\int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{4}} \|\psi(\tau)\|_{L^q}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{4}} \langle\tau\rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{q}\right)}\, d\tau \\ &\le Ce^{-\frac{t}{8}}\|\psi\|_{Y(T)}. \end{aligned}$$

Similarly to the estimate of I_1, we divide the estimate of $I\!I\!I_1$ into three cases.
Case 1 When $p < 1 + \frac{2r}{n}$, we take $\gamma = r$ to obtain

$$\begin{aligned} I\!I\!I_1 &\le C\|\psi\|_{Y(T)} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \langle\tau\rangle^{-\frac{n}{2r}(p-1)}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+1-\frac{n}{2r}(p-1)}, \end{aligned}$$

where we have used $-\frac{n}{2r}(p-1) > -1$.
Case 2 When $p > 1 + \frac{2r}{n}$, taking $\gamma = r$, we see that

$$\begin{aligned} I\!I\!I_1 &\le C\|\psi\|_{Y(T)} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \langle\tau\rangle^{-\frac{n}{2r}(p-1)}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}, \end{aligned}$$

since $-\frac{n}{2r}(p-1) < -1$.
Case 3-1 When $p = 1 + \frac{2r}{n}$ and $r > 1$, we let $\gamma = \sigma_1$ to obtain

$$\begin{aligned} I\!I\!I_1 &\le C\|\psi\|_{Y(T)} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)} \langle\tau\rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_1}\right)}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}. \end{aligned}$$

Here we have used $-\frac{n}{2}(\frac{p}{r} - \frac{1}{\sigma_1}) = -\frac{n}{2}(\frac{1}{r} - \frac{1}{\sigma_1}) - 1 > -1$.

Case 3-2 When $p = 1 + \frac{2r}{n}$ and $r = 1$, taking $\gamma = r$ gives

$$\begin{aligned} III_1 &\le C\|\psi\|_{Y(T)} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \langle\tau\rangle^{-\frac{n}{2r}(p-1)}\, d\tau \\ &\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \log(1+t), \end{aligned}$$

since $-\frac{n}{2r}(p-1) = -1$.

Step 3 Estimate of $\| |\cdot|^\alpha \int_0^t \mathcal{D}(t-\tau)\psi(\tau)\, d\tau\|_{L^2}$. We apply Lemma 2.1 with $\beta = \alpha, \nu = q$ and $\gamma \in [\sigma_1, \sigma_2]$ determined later, and obtain

$$\begin{aligned} \left\| |\cdot|^\alpha \int_0^t \mathcal{D}(t-\tau)\psi(\tau)\, d\tau \right\|_{L^2} &\le C \int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\frac{\alpha}{2}} \|\psi(\tau)\|_{L^\gamma}\, d\tau \\ &\quad + C \int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \| |\cdot|^\alpha \psi(\tau)\|_{L^q}\, d\tau \\ &\quad + C \int_0^t e^{-\frac{t-\tau}{4}} \|\langle\cdot\rangle^\alpha \langle\nabla\rangle^{-1}\psi(\tau)\|_{L^2}\, d\tau \\ &=: V_1 + V_2 + V_3. \end{aligned}$$

In order to estimate V_3, we employ Lemma A.2 and deduce that

$$V_3 \le C \int_0^t e^{-\frac{t-\tau}{4}} \|\langle\cdot\rangle^\alpha \psi(\tau)\|_{L^q}\, d\tau.$$

Therefore, the estimates of V_2, V_3 reduce to that of $\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \|\langle\cdot\rangle^\alpha \psi(\tau)\|_{L^q}\, d\tau$. Noting $-\frac{n}{2}(\frac{1}{q}-\frac{1}{2}) = -\frac{1}{2}$, we have

$$\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \|\langle\cdot\rangle^\alpha \psi(\tau)\|_{L^q}\, d\tau \le C\|\psi\|_{Y(T)} \int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \langle\tau\rangle^{-\zeta}\, d\tau,$$

where ζ is defined in Sect. 1.2. We compute

$$\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \langle\tau\rangle^{-\zeta}\, d\tau \le C \begin{cases} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}+1-\frac{n}{2r}(p-1)} & (\zeta < 1), \\ \langle t\rangle^{-\frac{1}{2}} \log(1+t) & (\zeta = 1), \\ \langle t\rangle^{-\frac{1}{2}} & (\zeta > 1) \end{cases}$$

and note that $\zeta < 1$ holds if $p < 1 + \frac{2r}{n}$. Therefore, we may summarize them as

$$\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \langle\tau\rangle^{-\zeta}\, d\tau \le C \begin{cases} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}+1-\frac{n}{2r}(p-1)} & (p < 1 + \frac{2r}{n}), \\ \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}} & (p \ge 1 + \frac{2r}{n}) \end{cases}$$

and hence, we have

$$V_2 + V_3 \le C\|\psi\|_{Y(T)} \begin{cases} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}+1-\frac{n}{2r}(p-1)} & (p < 1 + \frac{2r}{n}), \\ \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}} & (p \ge 1 + \frac{2r}{n}). \end{cases}$$

For V_1, as before, we divide the estimate into three cases.

Case 1 When $p < 1 + \frac{2r}{n}$, by taking $\gamma = r$, the definition of $Y(T)$-norm leads to

$$V_1 \le C\|\psi\|_{Y(T)} \int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}} \langle \tau\rangle^{-\frac{n}{2r}(p-1)}\, d\tau$$
$$\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}+1-\frac{n}{2r}(p-1)},$$

since $-\frac{n}{2r}(p-1) > -1$ and $-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2} > -1$.
Case 2 When $p > 1 + \frac{2r}{n}$, we choose $\gamma = r$ and have

$$V_1 \le C\|\psi\|_{Y(T)} \int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}} \langle \tau\rangle^{-\frac{n}{2r}(p-1)}\, d\tau$$
$$\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}},$$

since $-\frac{n}{2r}(p-1) < -1$ and $-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2} > -1$.
Case 3-1 When $p = 1 + \frac{2r}{n}$ and $r > 1$, letting $\gamma = \sigma_1$, we see that

$$V_1 \le C\|\psi\|_{Y(T)} \int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)+\frac{\alpha}{2}} \langle \tau\rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_1}\right)}\, d\tau$$
$$\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)+\frac{\alpha}{2}-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_1}\right)+1}$$
$$= C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}}.$$

Here we have used that $-\frac{n}{2}(\frac{p}{r}-\frac{1}{\sigma_1}) > -1$ and $-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)+\frac{\alpha}{2} > -1$.
Case 3-2 When $p = 1 + \frac{2r}{n}$ and $r = 1$, taking $\gamma = r$ implies

$$V_1 \le C\|\psi\|_{Y(T)} \int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}} \langle \tau\rangle^{-\frac{n}{2r}(p-1)}\, d\tau$$
$$\le C\|\psi\|_{Y(T)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\alpha}{2}} \log(1+t),$$

since $-\frac{n}{2r}(p-1) = -1$. Summing up all the estimates above, we reach (2.14) and (2.15). □

Lemma 2.5. *Under the assumptions in Theorem 1.1, we have*

$$\|\mathcal{N}(u)\|_{Y(T)} \le C\|u\|_{X(T)}^p,$$
$$\|\mathcal{N}(u)-\mathcal{N}(v)\|_{Y(T)} \le C\|u-v\|_{X(T)} \left(\|u\|_{X(T)}+\|v\|_{X(T)}\right)^{p-1}.$$

Proof. First, we consider the term $\|\langle\cdot\rangle^\alpha \mathcal{N}(u)\|_{L^q}$. By the assumption (1.2) and the Hölder inequality with $\frac{1}{q} = \frac{1}{2} + \frac{1}{n}$, we have

$$\|\langle\cdot\rangle^\alpha \mathcal{N}(u)\|_{L^q} \le C\|\langle\cdot\rangle^\alpha u\|_{L^2} \|u\|_{L^{n(p-1)}}^{p-1}$$

and hence,

$$\langle t\rangle^\zeta \|\langle\cdot\rangle^\alpha \mathcal{N}(u)\|_{L^q} \le C\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\alpha}{2}} \|\langle\cdot\rangle^\alpha u\|_{L^2} \left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{n(p-1)}\right)} \|u\|_{L^{n(p-1)}}\right)^{p-1}.$$

From the assumption of Theorem 1.1, $r \le n(p-1)$ is valid and $n(p-1) \le \frac{2n}{n-2s}$ also holds when $2s < n$. Therefore, we apply Lemma 2.3 to obtain

$$\langle t\rangle^{\zeta}\|\langle\cdot\rangle^{\alpha}\mathcal{N}(u)\|_{L^q}\le C\|u\|_{X(T)}^p.$$

Next, we estimate $\|\mathcal{N}(u)\|_{L^\gamma}$ for $\gamma\in[\sigma_1,\sigma_2]$. By the assumption (1.2), we see that

$$\langle t\rangle^{\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\gamma}\right)}\|\mathcal{N}(u)\|_{L^\gamma}\le C\left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p\gamma}\right)}\|u\|_{L^{p\gamma}}\right)^p.$$

Since $p\ge 1+\frac{r}{n}$ and $\gamma\ge\sigma_1\ge\frac{nr}{n+r}$, we have

$$r=\left(1+\frac{r}{n}\right)\frac{nr}{n+r}\le p\gamma.$$

Also, when $2s<n$, the assumption $\sigma_2\le\frac{2n}{p(n-2s)}$ leads to

$$p\gamma\le p\sigma_2\le\frac{2n}{n-2s}.$$

Therefore, $p\gamma\in[r,\frac{2n}{n-2s}]$ is valid and we apply Lemma 2.3 to obtain

$$\langle t\rangle^{\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\gamma}\right)}\|\mathcal{N}(u)\|_{L^\gamma}\le C\|u\|_{X(T)}^p. \tag{2.17}$$

Finally, we estimate $\||\nabla|^{[s]}\mathcal{N}(u)\|_{L^\rho}$. The case $[s]=0$ reduces to (2.17), because $\rho\in[\sigma_1,\sigma_2]$. When $[s]\ge1$, by the Faà di Bruno formula, for $\nu\in\mathbb{Z}_{\ge0}$ with $|\nu|\ge1$ we have

$$\partial^\nu\mathcal{N}(u)=\sum_{l=1}^{|\nu|}\mathcal{N}^{(l)}(u)\sum_{\substack{|\nu_1|\ge1,\dots,|\nu_l|\ge1\\ \nu_1+\cdots+\nu_l=\nu}}C^l_{\nu_1,\dots,\nu_l}\partial^{\nu_1}u\cdots\partial^{\nu_l}u.$$

Using this with $|\mathcal{N}^{(l)}(u)|\le C|u|^{p-l}=|u|^{p-[s]}|u|^{[s]-l}$ and the Hölder inequality, we see that

$$\begin{aligned}&\||\nabla|^{[s]}\mathcal{N}(u)\|_{L^\rho}\\ &\quad\le C\||u|^{p-[s]}\|_{L^{q_0}}\sum_k\||\nabla|^{k_1}u\|_{L^{q_1(k)}}\cdots\||\nabla|^{k_{[s]}}u\|_{L^{q_{[s]}(k)}},\end{aligned} \tag{2.18}$$

where the sum is taken over $k=(k_1,\dots,k_{[s]})\in\mathbb{Z}_{\ge0}^{[s]}$ satisfying $|k|=k_1+\cdots k_{[s]}=[s]$, and $q_1(k),\dots,q_{[s]}(k)$ satisfy

$$\frac{1}{\rho}=\frac{1}{q_0}+\frac{1}{q_1(k)}+\cdots+\frac{1}{q_{[s]}(k)} \tag{2.19}$$

and are defined in the following way.

For each fixed $k=(k_1,\dots,k_{[s]})$, let us choose $s_1,\dots,s_{[s]}$ so that

$$\max\{0,k_j-\mu\}\le s_j<k_j+\frac{1}{p-1},\quad\sum_{j=1}^{[s]}s_j=s-\mu, \tag{2.20}$$

where $\mu=\frac{n}{2}-\frac{1}{p-1}$. This is always possible. Indeed, first, it is obvious that $\max\{0,k_j-\mu\}<k_j+\frac{1}{p-1}$ and the interval $[\max\{0,k_j-\mu\},k_j+\frac{1}{p-1})$ is not empty. Next, we demonstrate that

$$\sum_{j=1}^{[s]} \max\{0, k_j - \mu\} \le s - \mu < \sum_{j=1}^{[s]} \left(k_j + \frac{1}{p-1} \right). \tag{2.21}$$

To prove this, with a direct calculation we have

$$\sum_{j=1}^{[s]} \left(k_j + \frac{1}{p-1} \right) = [s] + \frac{[s]}{p-1} > s - \mu,$$

since $n \ge 2$. Also, when $[s] = 1$, it is trivial that $\max\{0, [s] - \mu\} \le s - \mu$, since the assumption that $p \le 1 + \frac{2}{n-2s}$ if $2s < n$ implies $s \ge \mu$. When $[s] \ge 2$, noting that $\mu < 0$ leads to $[s] < p < 1 + \frac{2}{n} \le 2$, we may assume $\mu \ge 0$. Therefore, we have

$$\sum_{j=1}^{[s]} \max\{0, k_j - \mu\} = 0 \le s - \mu$$

if $k_j \le \mu$ for all $j = 1, \dots, [s]$ and

$$\sum_{j=1}^{[s]} \max\{0, k_j - \mu\} \le (k_i - \mu) + \sum_{j \ne i} k_j = [s] - \mu \le s - \mu$$

if $k_i > \mu$ for some $i \in \{1, \dots, [s]\}$. Thus, we prove (2.21) and we can actually find s_j $(j = 1, \dots, [s])$ satisfying (2.20).

From these s_j, we define

$$\frac{1}{q_j(k)} = \frac{1}{2} - \frac{\mu + s_j - k_j}{n}, \quad \frac{1}{q_0} = \left(\frac{1}{2} - \frac{\mu}{n} \right) (p - [s]).$$

Then, a straightforward calculation shows (2.19). Moreover, by the property (2.20) and the assumption $p \ge 1 + \frac{r}{n}$, we have $2 \le q_j(k) < \infty$ for $j = 1, \dots, [s]$ and $r \le n(p-1) = q_0(p-[s]) < \infty$, respectively. Also, we remark that $\mu + s_j \le s$ holds for all $j = 1, \dots, [s]$ due to $s_j \ge 0$ and $\sum_{j=1}^{[s]} s_j = s - \mu$. Hence, we apply the Sobolev embedding and Lemma 2.3 to (2.18) and obtain

$$\begin{aligned}
\langle t \rangle^{\eta} \| |\nabla|^{[s]} \mathcal{N}(u) \|_{L^{\rho}} &\le C \langle t \rangle^{\eta} \|u\|_{L^{q_0(p-[s])}}^{p-[s]} \sum_k \| |\nabla|^{\mu + s_1} u\|_{L^2} \cdots \| |\nabla|^{\mu + s_{[s]}} u \|_{L^2} \\
&\le C \left(\langle t \rangle^{\frac{n}{2}\left(\frac{1}{r} - \frac{1}{n(p-1)}\right)} \|u\|_{L^{n(p-1)}} \right)^{p-[s]} \\
&\quad \times \sum_k \prod_{j=1}^{[s]} \langle t \rangle^{\frac{n}{2}\left(\frac{1}{r} - \frac{1}{2}\right) + \frac{\mu + s_j}{2}} \| |\nabla|^{\mu + s_j} u \|_{L^2} \\
&\le C \|u\|_{X(T)}^p.
\end{aligned}$$

In the same manner, with the assumption (1.2), we can prove the estimate for $\|\mathcal{N}(u) - \mathcal{N}(v)\|_{Y(T)}$. Indeed, for example, we demonstrate

$$\begin{aligned}
\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^{\gamma}} &\le C \|u - v\|_{L^{p\gamma}} \| (|u| + |v|)^{p-1} \|_{L^{\frac{p\gamma}{p-1}}} \\
&\le C \|u - v\|_{L^{p\gamma}} \left(\|u\|_{L^{p\gamma}} + \|v\|_{L^{p\gamma}} \right)^{p-1}.
\end{aligned}$$

Hence, we find that

$$\begin{aligned}
&\langle t\rangle^{\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\gamma}\right)}\|\mathcal{N}(u)-\mathcal{N}(v)\|_{L^\gamma}\\
&\leq C\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p\gamma}\right)}\|u-v\|_{L^{p\gamma}}\left(\langle t\rangle^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p\gamma}\right)}\left(\|u\|_{L^{p\gamma}}+\|v\|_{L^{p\gamma}}\right)\right)^{p-1}\\
&\leq C\|u-v\|_{X(T)}\left(\|u\|_{X(T)}+\|v\|_{X(T)}\right)^{p-1}.
\end{aligned}$$

The other terms can be estimated in a similar way. □

Now we are in the position to prove Theorem 1.1 when $n \geq 2$.

Proof of Theorem 1.1 when $n \geq 2$. We apply the contraction mapping principle in

$$X_\varepsilon(T) = \left\{v \in L^\infty(0,T;H^{s,0}\cap H^{0,\alpha}(\mathbb{R}^n));\|v\|_{X(T)} \leq C_0\varepsilon\right\}, \tag{2.22}$$

where $C_0 > 0$ is determined later. Also, we define a metric in $X_\varepsilon(T)$ by

$$d(u,v) := \|u-v\|_{X(T)}. \tag{2.23}$$

Then, clearly, $X_\varepsilon(T)$ becomes a complete metric space.

We define the mapping $\mathcal{M}$ by

$$\mathcal{M}v(t) = \tilde{\mathcal{D}}(t)\varepsilon u_0 + \mathcal{D}(t)\varepsilon u_1 + \int_0^t \mathcal{D}(t-\tau)\mathcal{N}(v(\tau))d\tau. \tag{2.24}$$

Let $0 < T < 1$. Then, by Lemmas 2.1, 2.2, 2.4 (2.13), and 2.5, we have

$$\|\mathcal{M}v\|_{X(T)} \leq C\varepsilon(\|u_0\|_{H^{s,0}\cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0}\cap H^{0,\alpha}}) + CT\|v\|_{X(T)}^p.$$

Taking a constant C_0 so that $C(\|u_0\|_{H^{s,0}\cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0}\cap H^{0,\alpha}}) \leq \frac{C_0}{2}$, we have

$$\|\mathcal{M}v\|_{X(T)} \leq \frac{C_0}{2}\varepsilon + CTC_0^p\varepsilon^p.$$

Letting T be sufficiently small so that $CTC_0^p\varepsilon^{p-1} \leq \frac{C_0}{2}$, we conclude $\|\mathcal{M}v\|_{X(T)} \leq C_0\varepsilon$. Thus, $\mathcal{M}$ maps $X_\varepsilon(T)$ to itself.

In a similar way, we have

$$\|\mathcal{M}(v)-\mathcal{M}(w)\|_{X(T)} \leq C\varepsilon^{p-1}T\|v-w\|_{X(T)}$$

for $v, w \in X_\varepsilon(T)$ and hence, taking T further small, we have

$$\|\mathcal{M}(v)-\mathcal{M}(w)\|_{X(T)} \leq \frac{1}{2}\|v-w\|_{X(T)}.$$

Therefore, $\mathcal{M}$ is a contraction mapping on $X_\varepsilon(T)$ and there is a unique fixed point u in $X_\varepsilon(T)$. By the definition of $\mathcal{M}$, u is a solution of the Cauchy problem (1.1).

Finally, we prove the continuity with respect to t of the solution u. For $t_1, t_2 \geq 0$, we have

$$u(t_1)-u(t_2)=(\mathcal{D}(t_1)-\mathcal{D}(t_2))u_1+(\tilde{\mathcal{D}}(t_1)-\tilde{\mathcal{D}}(t_2))u_0 \\ +\int_{t_2}^{t_1}\mathcal{D}(t_1-\tau)\mathcal{N}(u(\tau))\,d\tau \\ +\int_0^{t_2}(\mathcal{D}(t_1-\tau)-\mathcal{D}(t_2-\tau))\mathcal{N}(u(\tau))\,d\tau.$$

Applying (2.3), (2.9) and the Lebesgue convergence theorem with the bound (2.13), we can easily prove $\lim_{t_1\to t_2}\|u(t_1)-u(t_2)\|_{H^{s,0}\cap H^{0,\alpha}}=0$, which finishes the proof. □

Lemma 2.6. *Under the assumptions in Theorem 1.1, the mild solution of (1.1) is unique in the class $C([0,T);H^{s,0}(\mathbb{R}^n)\cap H^{0,\alpha}(\mathbb{R}^n))$.*

Proof. Let $T_0>0$ and let $u,v\in C([0,T_0);H^{s,0}(\mathbb{R}^n)\cap H^{0,\alpha}(\mathbb{R}^n))$ are mild solutions of (1.1) with the same initial data (u_0,u_1). We take arbitrary $0<T_1<\min\{1,T_0\}$ and fix it. Then, we have $\|u\|_{X(T_1)}+\|v\|_{X(T_1)}\le M$ with some $M>0$. By the first assertion of Lemma 2.4 and applying Lemma 2.5, we see that for $T\in[0,T_1]$

$$\|u-v\|_{X(T)}\le CM^{p-1}\int_0^T\|u-v\|_{X(\tau)}d\tau.$$

Hence, we apply the Gronwall inequality and obtain $\|u-v\|_{X(T)}\equiv 0$ for $T\in[0,T_1]$, namely $u\equiv v$ in $t\in[0,T_1]$. Applying the same argument starting at $(u(T_1),u_t(T_1))$ instead of (u_0,u_1), we have $u\equiv v$ on $[0,2T_1]$. Continuing this until reaching T_0, we have the uniqueness in $C([0,T_0);H^{s,0}(\mathbb{R}^n)\cap H^{0,\alpha}(\mathbb{R}^n))$. □

Finally, we mention about the continuity of the solution with respect to the initial data.

Lemma 2.7. (Lipschitz continuity of the solution map) *The solution map*

$$\left(H^{s,0}\cap H^{0,\alpha}\right)\times\left(H^{s-1,0}\cap H^{0,\alpha}\right)\to C([0,T);H^{s,0}\cap H^{0,\alpha});\quad (u_0,u_1)\mapsto u$$

is locally Lipschitz continuous, that is, for any $T_1<T$, we have

$$\|u(t)-v(t)\|_{H^{s,0}\cap H^{0,\alpha}}\le C\|u_0-v_0\|_{H^{s,0}\cap H^{0,\alpha}}+C\|u_1-v_1\|_{H^{s-1,0}\cap H^{0,\alpha}}$$

on $t\in[0,T_1]$, where u and v are solutions of (1.1) in $C([0,T);H^{s,0}\cap H^{0,\alpha})$ with the initial data (u_0,u_1) and (v_0,v_1), respectively.

Proof. Let u and v are solutions of (1.1) in $C([0,T);H^{s,0}\cap H^{0,\alpha})$ and fix $T_1<T$. Then, we have $\|u\|_{X(T_1)}+\|v\|_{X(T_1)}\le M$ with some constant $M>0$. Therefore, by Lemmas 2.1, 2.4 and 2.5, we have

$$\|u(t)-v(t)\|_{X(T_1)}\le C\|u_0-v_0\|_{H^{s,0}\cap H^{0,\alpha}}+\|u_1-v_1\|_{H^{s-1,0}\cap H^{0,\alpha}} \\ +CM^{p-1}\int_0^{T_1}\|u(\tau)-v(\tau)\|_{X(\tau)}\,d\tau.$$

The Gronwall inequality implies

$$\|u(t)-v(t)\|_{X(T_1)}\le C\|u_0-v_0\|_{H^{s,0}\cap H^{0,\alpha}}+C\|u_1-v_1\|_{H^{s-1,0}\cap H^{0,\alpha}},$$

which completes the proof. □

Proof of Theorem 1.2. We assume that $p > 1 + \frac{2r}{n}$, $r \in [1, 2]$ or $p = 1 + \frac{2r}{n}$, $r \in (1, 2]$ and consider the mapping $\mathcal{M}$ defined on (2.24) in the complete metric space

$$X_\varepsilon(\infty) = \left\{v \in L^\infty(0, \infty; H^{s,0} \cap H^{0,\alpha}(\mathbb{R}^n)); \|v\|_{X(\infty)} \le C_0\varepsilon\right\}$$

with the metric (2.23). Then, by Lemma 2.4, we have

$$\left\| \int_0^t \mathcal{D}(t-\tau)\mathcal{N}(u(\tau))d\tau \right\|_{X(\infty)} \le C\|\mathcal{N}(u)\|_{Y(\infty)}.$$

Hence, Lemma 2.5 implies

$$\|\mathcal{M}v\|_{X(\infty)} \le C\varepsilon(\|u_0\|_{H^{s,0}\cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0}\cap H^{0,\alpha}}) + C\|v\|_{X(\infty)}^p.$$

As before, taking a constant C_0 so that $C(\|u_0\|_{H^{s,0}\cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0}\cap H^{0,\alpha}}) \le \frac{C_0}{2}$, we see that

$$\|\mathcal{M}v\|_{X(\infty)} \le \frac{C_0}{2}\varepsilon + C\varepsilon^p.$$

Finally, taking ε sufficiently small, we conclude $\|\mathcal{M}v\|_{X(\infty)} \le C_0\varepsilon$ and hence, $\mathcal{M}$ maps $X_\varepsilon(\infty)$ to itself.

In a similar way, we have

$$\|\mathcal{M}(v) - \mathcal{M}(w)\|_{X(\infty)} \le C\varepsilon^{p-1}\|v - w\|_{X(\infty)}$$

for $v, w \in X_\varepsilon(\infty)$ and hence, taking ε further small, we conclude that $\mathcal{M}$ is a contraction mapping. Therefore, $\mathcal{M}$ has a unique fixed point u in $X_\varepsilon(\infty)$ and by the definition of $\mathcal{M}$, u is a mild solution of (1.1). In the same way as in the proof of Theorem 1.1, we deduce that u belongs to $C([0, \infty); H^{s,0}(\mathbb{R}^n) \cap H^{0,\alpha}(\mathbb{R}^n))$. The uniqueness and continuity of the solution have already proved in Lemmas 2.7 and 2.6. □

2.3. Proof of Theorems 1.1 and 1.2 in the one-dimensional case

Lemma 2.8. *Under the assumption in Theorem 1.1, we have*

$$\left\| \int_0^t \mathcal{D}(t-\tau)\psi(\tau)d\tau \right\|_{X(T)} \le C\int_0^T \|\psi\|_{Y(\tau)}d\tau$$

for $0 < T \le 1$. *Moreover, we have*

$$\left\| \int_0^t \mathcal{D}(t-\tau)\psi(\tau)d\tau \right\|_{X(T)} \le C\|\psi\|_{Y(T)} \begin{cases} 1 & \text{if} \quad p > 1 + 2r, \\ \log(2+T) & \text{if} \quad p = 1 + 2r, \\ \langle T \rangle^{1-\frac{1}{2r}(p-1)} & \text{if} \quad p < 1 + 2r \end{cases}$$

for $r = 1$, $0 < T < \infty$, *and*

$$\left\| \int_0^t \mathcal{D}(t-\tau)\psi(\tau)d\tau \right\|_{X(T)} \le C\|\psi\|_{Y(T)} \begin{cases} 1 & \text{if} \quad p \ge 1 + 2r, \\ \langle T \rangle^{1-\frac{1}{2r}(p-1)} & \text{if} \quad p < 1 + 2r \end{cases}$$

for $r \in (1, 2]$, $0 < T < \infty$.

Proof. The proof is almost the same as that of Lemma 2.4 and hence we present only the outline. The main difference arises in the Sobolev inequality, that is, we shall use

$$\||\nabla|^s\langle\nabla\rangle^{-1}\psi\|_{L^2} \le C\|\psi\|_{L^2}$$

instead of (2.16), and we use Lemma A.2 with $q = 1$, that is,

$$\|\langle\cdot\rangle^{\alpha}\langle\nabla\rangle^{-1}\psi\|_{L^2} \le C\|\langle\cdot\rangle^{\alpha}\psi\|_{L^1}.$$

We estimate

$$\begin{aligned}
&\left\||\nabla|^s\int_0^t \mathcal{D}(t-\tau)\psi(\tau)\,d\tau\right\|_{L^2}\\
&\quad\le C\int_0^{\frac{t}{2}} \||\nabla|^s\mathcal{D}(t-\tau)\psi(\tau)\|_{L^2}\,d\tau + C\int_{\frac{t}{2}}^t \||\nabla|^s\mathcal{D}(t-\tau)\psi(\tau)\|_{L^2}\,d\tau\\
&\quad=: I + II
\end{aligned}$$

and Lemma 2.1 with $s_1 = s, s_2 = 0$, $\gamma = 1$ implies

$$\begin{aligned}
II &\le C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac14-\frac{s}{2}}\|\psi(\tau)\|_{L^1}\,d\tau + C\int_{\frac{t}{2}}^t e^{-\frac{t-\tau}{4}}\|\psi(\tau)\|_{L^2}\,d\tau\\
&\le C\|\psi\|_{Y(T)}\langle t\rangle^{-\frac12\left(\frac1r-\frac12\right)-\frac{s}{2}-\frac{1}{2r}(p-1)+1}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I &\le C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac12\left(\frac1\gamma-\frac12\right)-\frac{s}{2}}\|\psi(\tau)\|_{L^\gamma}\,d\tau + C\int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{4}}\|\psi(\tau)\|_{L^2}\,d\tau\\
&\le C\|\psi\|_{Y(T)}\langle t\rangle^{-\frac12\left(\frac1r-\frac12\right)-\frac{s}{2}}\\
&\quad\times\begin{cases} 1 & \text{if}\quad p > 1+2r \text{ or } p = 1+2r, r > 1,\\ \log(2+t) & \text{if}\quad p = 1+2r, r = 1,\\ \langle t\rangle^{1-\frac{1}{2r}(p-1)} & \text{if}\quad p < 1+2r,\end{cases}
\end{aligned}$$

where $\gamma = 1$ if $p = 1+2r, r > 1$ and $\gamma = r$ otherwise. In the same way, we have

$$\begin{aligned}
&\left\|\int_0^t \mathcal{D}(t-\tau)\psi(\tau)\,d\tau\right\|_{L^2}\\
&\quad\le C\|\psi\|_{Y(T)}\langle t\rangle^{-\frac12\left(\frac1r-\frac12\right)}\\
&\qquad\times\begin{cases} 1 & \text{if}\quad p > 1+2r \text{ or } p = 1+2r,\quad r > 1,\\ \log(2+t) & \text{if}\quad p = 1+2r,\quad r = 1,\\ \langle t\rangle^{1-\frac{1}{2r}(p-1)} & \text{if}\quad p < 1+2r.\end{cases}
\end{aligned}$$

Finally, we estimate

$$\begin{aligned}
&\left\||\cdot|^{\alpha}\int_0^t \mathcal{D}(t-\tau)\psi(\tau)\,d\tau\right\|_{L^2}\\
&\quad\le C\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac1\gamma-\frac12\right)+\frac{\alpha}{2}}\|\psi(\tau)\|_{L^\gamma}\,d\tau + C\int_0^t \langle t-\tau\rangle^{-\frac14}\||\cdot|^{\alpha}\psi(\tau)\|_{L^1}\,d\tau
\end{aligned}$$

$$+ C\int_0^t e^{-\frac{t-\tau}{4}} \|\langle\cdot\rangle^{\alpha}\langle\nabla\rangle^{-1}\psi(\tau)\|_{L^2}\, d\tau$$
$$=: V_1 + V_2 + V_3.$$

Lemma A.2 leads to

$$V_3 \le C\int_0^t e^{-\frac{t-\tau}{4}} \|\langle\cdot\rangle^{\alpha}\psi(\tau)\|_{L^q}\, d\tau.$$

Therefore, the estimates of V_2, V_3 reduce to that of $\int_0^t \langle t-\tau\rangle^{-\frac14}\|\langle\cdot\rangle^{\alpha}\psi(\tau)\|_{L^1}\, d\tau$. We have

$$\int_0^t \langle t-\tau\rangle^{-\frac14}\|\langle\cdot\rangle^{\alpha}\psi(\tau)\|_{L^q}\, d\tau \le C\|\psi\|_{Y(T)}\int_0^t \langle t-\tau\rangle^{-\frac14}\langle\tau\rangle^{-\zeta}\, d\tau,$$

where ζ is defined in Table 2 (see Sect. 1.2). We compute

$$\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac1q-\frac12\right)}\langle\tau\rangle^{-\zeta}\, d\tau \le C\begin{cases}\langle t\rangle^{-\frac12\left(\frac1r-\frac12\right)+\frac{\alpha}{2}+1-\frac{n}{2r}(p-1)} & (\zeta<1),\\ \langle t\rangle^{-\frac14}\log(1+t) & (\zeta=1),\\ \langle t\rangle^{-\frac14} & (\zeta>1)\end{cases}$$

and note that $\zeta < 1$ holds if $p < 1 + 2r$. Therefore, we may summarize them as

$$\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac1q-\frac12\right)}\langle\tau\rangle^{-\zeta}\, d\tau \le C\begin{cases}\langle t\rangle^{-\frac{n}{2}\left(\frac1r-\frac12\right)+\frac{\alpha}{2}+1-\frac{n}{2r}(p-1)} & (p<1+\frac{2r}{n}),\\ \langle t\rangle^{-\frac{n}{2}\left(\frac1r-\frac12\right)+\frac{\alpha}{2}} & (p\ge 1+\frac{2r}{n})\end{cases}$$

and hence, we have

$$V_2+V_3 \le C\|\psi\|_{Y(T)}\begin{cases}\langle t\rangle^{-\frac{n}{2}\left(\frac1r-\frac12\right)+\frac{\alpha}{2}+1-\frac{n}{2r}(p-1)} & (p<1+\frac{2r}{n}),\\ \langle t\rangle^{-\frac{n}{2}\left(\frac1r-\frac12\right)+\frac{\alpha}{2}} & (p\ge 1+\frac{2r}{n}).\end{cases}$$

For V_1, taking $\gamma = 1$ if $p = 1 + 2r, r > 1$ and $\gamma = r$ otherwise, we see that

$$V_1 \le C\|\psi\|_{Y(T)}\langle t\rangle^{-\frac12\left(\frac1r-\frac12\right)+\frac{\alpha}{2}} \times\begin{cases}1 & \text{if } \ p>1+2r \text{ or } p=1+2r, \quad r>1,\\ \log(2+t) & \text{if } \ p=1+2r, \quad r=1,\\ \langle t\rangle^{1-\frac{1}{2r}(p-1)} & \text{if } \ p<1+2r.\end{cases}$$

This completes the proof. □

Lemma 2.9. *Under the assumptions in Theorem 1.1, we have*

$$\|\mathcal{N}(u)\|_{Y(T)} \le C\|u\|_{X(T)}^p,$$
$$\|\mathcal{N}(u)-\mathcal{N}(v)\|_{Y(T)} \le C\|u-v\|_{X(T)}\left(\|u\|_{X(T)}+\|v\|_{X(T)}\right)^{p-1}.$$

Proof. At first, we consider $\|\langle\cdot\rangle^{\alpha}\mathcal{N}(u)\|_{L^1}$. The Hölder inequality yields

$$\langle t\rangle^{\zeta}\|\langle\cdot\rangle^{\alpha}\mathcal{N}(u)\|_{L^1} \le C\langle t\rangle^{\frac12\left(\frac1r-\frac12\right)-\frac{\alpha}{2}}\|\langle\cdot\rangle^{\alpha}u\|_{L^2}\left(\langle t\rangle^{\frac12\left(\frac1r-\frac{1}{2(p-1)}\right)}\|u\|_{L^{2(p-1)}}\right)^{p-1}.$$

Since $1 + \frac{r}{2} \leq p$, we see that $2(p-1) \geq r$. Moreover, when $n > 2s$, the assumption $p \leq \frac{1}{1-2s}$ implies $2(p-1) \leq \frac{2n}{n-2s}$. Therefore, we apply Lemma 2.3 to obtain

$$\langle t\rangle^{\zeta}\|\langle\cdot\rangle^{\alpha}\mathcal{N}(u)\|_{L^1} \leq C\|u\|_{X(T)}^p.$$

Next, we estimate $\|\mathcal{N}(u)\|_{L^\gamma}$ with $\gamma \in [1, 2]$. We first obtain

$$\|\mathcal{N}(u)\|_{L^\gamma} \leq C\|u\|_{L^{p\gamma}}^p.$$

It follows from $p \geq 1 + \frac{r}{2}$ that $p\gamma \geq r$ for any $\gamma \in [1, 2]$. Also, when $n > 2s$, the assumption $p \leq \frac{1}{1-2s}$ ensures $p\gamma \leq \frac{2n}{n-2s}$. Hence, we apply Lemma 2.3 to derive

$$\langle t\rangle^{\frac{1}{2}\left(\frac{p}{r}-\frac{1}{\gamma}\right)}\|\mathcal{N}(u)\|_{L^\gamma} \leq C\left(\langle t\rangle^{\frac{1}{2}\left(\frac{1}{r}-\frac{1}{p\gamma}\right)}\|u\|_{L^{p\gamma}}\right)^p \leq C\|u\|_{X(T)}^p.$$

The above estimates shows the conclusion. □

Proof of Theorems 1.1 and 1.2 when $n = 1$. The proof of Theorems 1.1 and 1.2 is completely the same as the case $n \geq 2$ and we omit the detail. □

3. Asymptotic behavior of the global solution

3.1. Approximation by an inhomogeneous heat equation

In this section, we study the asymptotic behavior of the global solution. Let u be the global-in-time solution proved in the previous section, that is,

$$u(t) = \tilde{\mathcal{D}}(t)\varepsilon u_0 + \mathcal{D}(t)\varepsilon u_1 + \int_0^t \mathcal{D}(t-\tau)\mathcal{N}(u(\tau))\, d\tau.$$

First, we consider the solution of the inhomogeneous linear heat equation

$$(\partial_t - \Delta)v = \mathcal{N}(u) \tag{3.1}$$

with the initial data $v(0, x) = \varepsilon(u_0 + u_1)$, that is,

$$v(t) = \mathcal{G}(t)\left(\varepsilon u_0 + \varepsilon u_1\right) + \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\, d\tau,$$

where $\mathcal{G}(t)$ is defined in (1.9). We first prove that the asymptotic profile of u is given by v in $H^{s,0} \cap H^{0,\alpha}$-sense.

Proposition 3.1. *Under the assumptions of Theorem 1.2, we have*

$$\||\nabla|^s(u(t) - v(t))\|_{L^2} \leq C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}-\min\{1,\frac{n}{2r}(p-1)-\frac{1}{2}\}}, \tag{3.2}$$

$$\|u(t) - v(t)\|_{L^2} \leq C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\min\{1,\frac{n}{2r}(p-1)-\frac{1}{2}\}}, \tag{3.3}$$

$$\||\cdot|^\beta(u(t) - v(t))\|_{L^2} \leq C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}-\min\{1,\frac{n}{2r}(p-1)-\min\{\frac{n}{4},\frac{1}{2}\}\}}, \tag{3.4}$$

where β *is an arbitrary number satisfying* $n(\frac{1}{r} - \frac{1}{2}) < \beta \leq \alpha$.

3.2. Preliminary estimates

In order to prove Proposition 3.1, we prepare the following lemma.

Lemma 3.2. *Let $\gamma, \nu \in [1,2]$, $\beta \geq 0$ and $s_1 \geq s_2 \geq 0$. Then, we have*

$$\begin{aligned} \||\nabla|^{s_1}(\mathcal{D}(t)\psi - \mathcal{G}(t)\psi)\|_{L^2} &\leq C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{s_1-s_2}{2}-1}\||\nabla|^{s_2}\psi\|_{L^\gamma} \\ &\quad + e^{-\frac{t}{4}}\||\nabla|^{s_1}\langle\nabla\rangle^{-1}\psi\|_{L^2}, \\ \||\cdot|^{\beta}(\mathcal{D}(t)\psi - \mathcal{G}(t)\psi)\|_{L^2} &\leq C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\frac{\beta}{2}-1}\|\psi\|_{L^\gamma} \\ &\quad + \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\nu}-\frac{1}{2}\right)-1}\||\cdot|^{\beta}\psi\|_{L^\nu} \\ &\quad + e^{-\frac{t}{4}}\|\langle\cdot\rangle^{\beta}\langle\nabla\rangle^{-1}\psi\|_{L^2} \end{aligned}$$

for any $t \geq 1$.

Proof. Since

$$\||\xi|^{s}(e^{-\frac{t}{2}}L(t,\xi) - e^{-t|\xi|^2})\|_{L^\gamma(|\xi|\leq 1)} \leq C\langle t\rangle^{-\frac{s}{2}-\frac{n}{2\gamma}-1}$$

and

$$\|\langle\xi\rangle L(t,\xi)\|_{L^\infty(|\xi|>1)} \leq C, \quad \|\langle\xi\rangle e^{-t|\xi|^2}\|_{L^\infty(|\xi|>1)} \leq Ce^{-t/4},$$

we have

$$\begin{aligned} &\||\nabla|^{s_1}(\mathcal{D}(t)\psi - \mathcal{G}(t)\psi)\|_{L^2} \\ &= \||\xi|^{s_1}(e^{-\frac{t}{2}}L(t,\xi) - e^{-t|\xi|^2})\hat{\psi}\|_{L^2} \\ &\leq C\||\xi|^{s_1-s_2}(e^{-\frac{t}{2}}L(t,\xi) - e^{-t|\xi|^2})\|_{L^{\frac{2\gamma}{2-\gamma}}(|\xi|\leq 1)}\||\xi|^{s_2}\hat{\psi}\|_{L^{\frac{\gamma}{\gamma-1}}(|\xi|\leq 1)} \\ &\quad + Ce^{-\frac{t}{2}}\|\langle\xi\rangle L(t,\xi)\|_{L^\infty(|\xi|>1)}\||\xi|^{s_1}\langle\xi\rangle^{-1}\hat{\psi}\|_{L^2(|\xi|>1)} \\ &\quad + Ce^{-\frac{t}{4}}\|\langle\xi\rangle e^{-\frac{t}{4}|\xi|^2}\|_{L^\infty(|\xi|>1)}\||\xi|^{s_1}\langle\xi\rangle^{-1}\hat{\psi}\|_{L^2(|\xi|>1)} \\ &\leq C\langle t\rangle^{-\frac{s_1-s_2}{2}-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-1}\||\nabla|^{s_2}\psi\|_{L^\gamma} + Ce^{-\frac{t}{4}}\||\nabla|^{s_1}\langle\nabla\rangle^{-1}\psi\|_{L^2}, \end{aligned}$$

which implies the first assertion.

To prove the second estimate, we take a cut-off function $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$ satisfying $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$. We put

$$\bar{K}(t,x) := \mathcal{F}^{-1}\chi(\xi)\left(e^{-\frac{t}{2}}L(t,\xi) - e^{-t|\xi|^2}\right).$$

Then, in the same way to (2.4), we see that

$$\begin{aligned} &\left\||\cdot|^{\beta}\mathcal{F}^{-1}\chi(\xi)\left(e^{-\frac{t}{2}}L(t,\xi) - e^{-t|\xi|^2}\right)\hat{\psi}\right\|_{L^2} \\ &\quad \leq C\||\cdot|^{\beta}\bar{K}(t)\|_{L^{\frac{2\gamma}{3\gamma-2}}}\|\psi\|_{L^\gamma} + C\|\bar{K}(t)\|_{L^{\frac{2\nu}{3\nu-2}}}\||\cdot|^{\beta}\psi\|_{L^\nu}. \end{aligned}$$

Moreover, in a similar way to (2.5) with

$$e^{-\frac{t}{2}}L(t,\xi) - e^{-t|\xi|^2} = e^{-t|\xi|^2}O(|\xi|^2)$$

as $|\xi| \to 0$, we can prove

$$\||\cdot|^{\beta}\bar{K}(t)\|_{L^k} \leq C\langle t\rangle^{\frac{\beta}{2}-\frac{n}{2}\left(1-\frac{1}{k}\right)-1},$$

which implies

$$\left\| |\cdot|^\beta \mathcal{F}^{-1}\chi(\xi)\left(e^{-\frac{t}{2}}L(t,\xi)-e^{-t|\xi|^2}\right)\hat{\psi}\right\|_{L^2} \\ \le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\frac{\beta}{2}-1}\|\psi\|_{L^\gamma}+C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\nu}-\frac{1}{2}\right)-1}\||\cdot|^\beta\psi\|_{L^\nu}.$$

We also easily obtain

$$\||\cdot|^\beta\mathcal{F}^{-1}\left(e^{-t|\xi|^2}(1-\chi(\xi))\hat{\psi}\right)\|_{L^2}\le Ce^{-\frac{t}{4}}\|\langle\cdot\rangle^\beta\langle\nabla\rangle^{-1}\psi\|_{L^2}.$$

This and (2.6) lead to

$$\left\||\cdot|^\beta\mathcal{F}^{-1}\left(e^{-\frac{t}{2}}L(t,\xi)-e^{-t|\xi|^2}\right)(1-\chi(\xi))\hat{\psi}\right\|_{L^2}\le Ce^{-\frac{t}{4}}\left\|\langle\cdot\rangle^\beta\langle\nabla\rangle^{-1}\psi\right\|_{L^2}.$$

From this, we reach the conclusion. □

In the same way, we have the following.

Lemma 3.3. *Let* $\gamma,\nu\in[1,2]$, $\beta\ge 0$, $s_1\ge s_2\ge 0$. *Then, we have*

$$\||\nabla|^{s_1}(\tilde{\mathcal{D}}(t)-\mathcal{G}(t))\psi\|_{L^2}\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{s_1-s_2}{2}-1}\||\nabla|^{s_2}\psi\|_{L^\gamma} \\ +Ce^{-t/4}\||\nabla|^{s_1}\psi\|_{L^2}, \tag{3.5}$$

$$\||\cdot|^\beta(\tilde{\mathcal{D}}(t)-\mathcal{G}(t))\psi\|_{L^2}\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\frac{\beta}{2}-1}\|\psi\|_{L^\gamma} \\ +C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\nu}-\frac{1}{2}\right)-1}\||\cdot|^\beta\psi\|_{L^\nu} \\ +Ce^{-\frac{t}{4}}\|\langle\cdot\rangle^\beta\psi\|_{L^2} \tag{3.6}$$

for $t\ge 1$.

3.3. Proof of Proposition 3.1

For simplicity, we treat only the case $n\ge 2$. The proof of the case $n=1$ is similar. Let us estimate the nonlinear part of $u-v$. We divide

$$\left\||\nabla|^s\int_0^t(\mathcal{D}(t-\tau)\mathcal{N}(u(\tau))-\mathcal{G}(t-\tau)\mathcal{N}(u(\tau)))\,d\tau\right\|_{L^2}\le I+II,$$

where

$$I=\int_0^{t/2}\||\nabla|^s\left(\mathcal{D}(t-\tau)-\mathcal{G}(t-\tau)\right)\mathcal{N}(u(\tau))\|_{L^2}\,d\tau$$

and

$$II=\int_{t/2}^t\||\nabla|^s\left(\mathcal{D}(t-\tau)-\mathcal{G}(t-\tau)\right)\mathcal{N}(u(\tau))\|_{L^2}\,d\tau.$$

We claim that each term has the desired decay rate.

First, we estimate II. Applying Lemma 3.2 with $s_1=s$, $s_2=[s]$ and $\gamma=\rho$, we have

$$
\begin{aligned}
II &\le C\int_{\frac{t}{2}}^{t} \||\nabla|^s(\mathcal{D}(t-\tau)-\mathcal{G}(t-\tau))\mathcal{N}(u(\tau))\|_{L^2}\,d\tau\\
&\le C\int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)-\frac{s-[s]}{2}-1}\||\nabla|^{[s]}\mathcal{N}(u(\tau))\|_{L^\rho}\,d\tau\\
&\quad + C\int_{\frac{t}{2}}^{t} e^{-\frac{t-\tau}{4}}\||\nabla|^s\langle\nabla\rangle^{-1}\mathcal{N}(u(\tau))\|_{L^2}\,d\tau.
\end{aligned}
$$

Using the Sobolev inequality (2.16), we see that

$$
\begin{aligned}
II &\le C\int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)-\frac{s-[s]}{2}-1}\||\nabla|^{[s]}\mathcal{N}(u(\tau))\|_{L^\rho}\,d\tau\\
&\le C\|\mathcal{N}(u)\|_{Y(\infty)}\int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)-\frac{s-[s]}{2}-1}\langle\tau\rangle^{-\eta}\,d\tau\\
&\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\frac{1}{2}-\frac{n}{2r}(p-1)}.
\end{aligned}
\tag{3.7}
$$

Next, we give an estimate of I. Applying Lemma 3.2 with $s_1 = s$, $s_2 = 0$ and $\gamma \in [1, 2]$ determined later, we have

$$
\begin{aligned}
I &\le C\int_{0}^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{s}{2}-1}\|\mathcal{N}(u(\tau))\|_{L^\gamma}\,d\tau\\
&\quad + C\int_{0}^{\frac{t}{2}} e^{-\frac{t-\tau}{4}}\||\nabla|^s\langle\nabla\rangle^{-1}\mathcal{N}(u(\tau))\|_{L^2}\,d\tau\\
&=: I_1 + I_2.
\end{aligned}
$$

In the same way as the proof of Lemma 2.4, it is easy to see that $I_2 \le Ce^{-\frac{t}{8}}\|\mathcal{N}(u)\|_{Y(\infty)}$. Also, for I_1, taking $\gamma = r$ if $p > 1 + \frac{2r}{n}$ and $\gamma = \sigma_1$ if $p = 1 + \frac{2r}{n}$, $r > 1$, respectively, we can prove

$$
I_1 \le C\|\mathcal{N}(u)\|_{Y(\infty)}\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}-1}. \tag{3.8}
$$

Combining the estimates (3.7) and (3.8), we conclude

$$
\begin{aligned}
&\left\||\nabla|^s\int_0^t \left(\mathcal{D}(t-\tau)\mathcal{N}(u(\tau))-\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\right)\,d\tau\right\|_{L^2}\\
&\quad \le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}-\min\{1,\frac{n}{2r}(p-1)-\frac{1}{2}\}}.
\end{aligned}
$$

The estimate of the linear part of $u - v$ is obvious from Lemmas 3.2 and 3.3. Thus, we obtain the first assertion (3.2).

Secondly, we prove (3.3). As before, we divide the nonlinear term of $u-v$ as

$$
\begin{aligned}
&\left\|\int_0^t (\mathcal{D}(t-\tau)-\mathcal{G}(t-\tau))\mathcal{N}(u(\tau))\,d\tau\right\|_{L^2}\\
&\quad \le C\int_0^{\frac{t}{2}} \|(\mathcal{D}(t-\tau)-\mathcal{G}(t-\tau))\mathcal{N}(u(\tau))\|_{L^2}\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ C \int_{\frac{t}{2}}^{t} \|(\mathcal{D}(t-\tau) - \mathcal{G}(t-\tau))\mathcal{N}(u(\tau))\|_{L^2}\, d\tau \\
&=: III + IV.
\end{aligned}
$$

Making use of Lemma 3.2 with $s_1 = s_2 = 0$ and $\gamma = q$, and then using the Sobolev embedding $\|\langle\nabla\rangle^{-1}\psi\|_{L^2} \le C\|\psi\|_{L^q}$, we see that

$$
\begin{aligned}
IV &\le C \int_{\frac{t}{2}}^{t} \left[\langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-1} \|\mathcal{N}(u(\tau))\|_{L^q} + e^{-\frac{t-\tau}{4}} \|\langle\nabla\rangle^{-1}\mathcal{N}(u(\tau))\|_{L^2} \right] d\tau \\
&\le C \int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-1} \|\mathcal{N}(u(\tau))\|_{L^q}\, d\tau.
\end{aligned}
$$

Noting $-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right) - 1 = -\frac{3}{2} < -1$, we further estimate

$$
\begin{aligned}
IV &\le C\|\mathcal{N}(u)\|_{Y(\infty)} \int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-1} \langle\tau\rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{q}\right)}\, d\tau \\
&\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{1}{2}-\frac{n}{2r}(p-1)}.
\end{aligned}
\tag{3.9}
$$

On the other hand, applying Lemma 3.2 with $s_1 = s_2 = 0$ and $\gamma \in [1, 2]$ determined later, we deduce

$$
\begin{aligned}
III &\le C \int_{0}^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-1} \|\mathcal{N}(u(\tau))\|_{L^\gamma}\, d\tau \\
&\quad + C \int_{0}^{\frac{t}{2}} e^{-\frac{t-\tau}{4}} \|\langle\nabla\rangle^{-1}\mathcal{N}(u(\tau))\|_{L^2}\, d\tau \\
&=: III_1 + III_2.
\end{aligned}
$$

In the same manner as the proof of Lemma 2.4, we see that the term III_2 is bounded by $e^{-\frac{t}{8}}\|\mathcal{N}(u)\|_{Y(\infty)}$. Furthermore, taking $\gamma = r$ if $p > 1 + \frac{2r}{n}$ and $\gamma = \sigma_1$ if $p = 1 + \frac{2r}{n}$, $r > 1$, respectively, we have

$$
III_1 \le C\|\mathcal{N}(u)\|_{Y(\infty)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-1}. \tag{3.10}
$$

Summing up the estimates (3.9) and (3.10), we conclude

$$
\left\| \int_0^t (\mathcal{D}(t-\tau) - \mathcal{G}(t-\tau))\mathcal{N}(u(\tau))\, d\tau \right\|_{L^2} \le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\min\{1, \frac{n}{2r}(p-1)-\frac{1}{2}\}}.
$$

The linear part of $u - v$ is easily estimated from Lemmas 3.2 and 3.3 and we reach the second assertion (3.3).

Finally, we prove (3.4). We take β satisfying $n(\frac{1}{r}-\frac{1}{2}) < \beta \le \alpha$. Then, by the interpolation

$$
\|\langle\cdot\rangle^{\beta}\psi\|_{L^q} \le \|\langle\cdot\rangle^{\alpha}\psi\|_{L^q}^{\frac{\beta}{\alpha}} \|\psi\|_{L^q}^{1-\frac{\beta}{\alpha}},
$$

we deduce that the solution u satisfies

$$\begin{aligned}
&\langle t\rangle^{\zeta_\beta}\|\langle\cdot\rangle^\beta\mathcal{N}(u(t))\|_{L^q}\\
&\le C\left(\langle t\rangle^{\zeta}\|\langle\cdot\rangle^\alpha\mathcal{N}(u(t))\|_{L^q}\right)^{\frac{\beta}{\alpha}}\left(\langle t\rangle^{\frac{n}{2}\left(\frac{p}{r}-\frac{1}{q}\right)}\|\mathcal{N}(u(t))\|_{L^q}\right)^{1-\frac{\beta}{\alpha}}\\
&\le C
\end{aligned}$$

with $\zeta_\beta = \frac{n}{2r}(p-1) - \frac{1}{2} + \frac{n}{2}(\frac{1}{r} - \frac{1}{2}) - \frac{\beta}{2}$. From Lemma 3.2 with $\nu = q$ and $\gamma \in [1, 2]$ determined later, we obtain

$$\begin{aligned}
&\left\| |\cdot|^\beta \int_0^t (\mathcal{D}(t-\tau) - \mathcal{G}(t-\tau))\mathcal{N}(u(\tau))\,d\tau \right\|_{L^2}\\
&\le C\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\frac{\beta}{2}-1}\|\mathcal{N}(u(\tau))\|_{L^\gamma}\,d\tau\\
&\quad + C\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-1}\||\cdot|^\beta\mathcal{N}(u(\tau))\|_{L^q}\,d\tau\\
&\quad + C\int_0^t e^{-\frac{t-\tau}{4}}\|\langle\cdot\rangle^\beta\langle\nabla\rangle^{-1}\mathcal{N}(u(\tau))\|_{L^2}\,d\tau\\
&=: V_1 + V_2 + V_3.
\end{aligned}$$

Lemma A.2 implies

$$V_3 \le C\int_0^t e^{-\frac{t-\tau}{4}}\|\langle\cdot\rangle^\beta\mathcal{N}(u(\tau))\|_{L^q}\,d\tau$$

and hence, we have

$$V_2 + V_3 \le C\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-1}\|\langle\cdot\rangle^\beta\mathcal{N}(u(\tau))\|_{L^q}\,d\tau.$$

Noting $-\frac{n}{2}(\frac{1}{q} - \frac{1}{2}) - 1 = -\frac{3}{2}$, we proceed the estimate as

$$\begin{aligned}
V_2 + V_3 &\le C\|\mathcal{N}(u)\|_{Y(\infty)}\int_0^t \langle t-\tau\rangle^{-\frac{3}{2}}\langle\tau\rangle^{-\zeta_\beta}\,d\tau\\
&\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}-\frac{n}{2r}(p-1)+\frac{1}{2}}\\
&\quad + C\begin{cases}\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}-\frac{n}{2r}(p-1)} & (\zeta_\beta<1),\\ \langle t\rangle^{-\frac{3}{2}}\log(1+t) & (\zeta_\beta=1),\\ \langle t\rangle^{-\frac{3}{2}} & (\zeta_\beta>1).\end{cases}
\end{aligned}$$

For V_1, as in the proof of Lemma 2.4, taking $\gamma = r$ if $p > 1 + \frac{2r}{n}$ and $\gamma = \sigma_1$ if $p = 1 + \frac{2r}{n}$, $r > 1$, respectively, we can see that

$$V_1 \le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}-1}.$$

Consequently, we obtain

$$\left\| |\cdot|^\beta \int_0^t (\mathcal{D}(t-\tau) - \mathcal{G}(t-\tau))\mathcal{N}(u(\tau))\,d\tau \right\|_{L^2} \\ \le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}-\min\{1,\frac{n}{2r}(p-1)-\frac{1}{2}\}},$$

which shows the third assertion (3.4) and finishes the proof of Proposition 3.1.

3.4. Proof of Theorem 1.3

By virtue of Proposition 3.1, the proof of Theorem 1.3 reduces to the analysis of the asymptotic behavior of solutions to the inhomogeneous heat equation (3.1) with the initial data $\varepsilon(u_0+u_1)$. In this subsection, we prove the following:

Proposition 3.4. *Under the assumption of Theorem 1.3, Let β satisfy $n(\frac{1}{r}-\frac{1}{2}) < \beta \le \alpha$. Then, the solution v to the Eq. (3.1) with the initial data $\varepsilon(u_0+u_1)$ satisfies the following asymptotic behavior: When $r>1$, for any $\varsigma > 0$, there exists $C_\varsigma > 0$ such that*

$$\||\nabla|^s(v(t) - \varepsilon\mathcal{G}(t)(u_0+u_1))\|_{L^2} \\ \le C_\varsigma\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}, \tag{3.11}$$

$$\|v(t) - \varepsilon\mathcal{G}(t)(u_0+u_1)\|_{L^2} \\ \le C_\varsigma\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}, \tag{3.12}$$

$$\||\cdot|^\beta(v(t) - \varepsilon\mathcal{G}(t)(u_0+u_1))\|_{L^2} \\ \le C_\varsigma\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma} \tag{3.13}$$

for $t \ge 1$. When $r=1$, for any $\varsigma>0$, there exists $C_\varsigma > 0$ such that

$$\|v(t) - \theta G(t)\|_{L^m} \le C_\varsigma\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-\min\{\frac{\alpha}{2}-\frac{n}{4},\frac{1}{2},\frac{n}{2}(p-1)-1\}+\varsigma} \tag{3.14}$$

for $t \ge 1$.

To prove this proposition, we first prepare the following lemma.

Lemma 3.5. *Let $s_1 \ge s_2 \ge 0$, $\beta \ge 0$ and $\gamma,\nu \in [1,2]$. Then, we have*

$$\||\nabla|^{s_1}\mathcal{G}(t)\psi\|_{L^2} \le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{s_1-s_2}{2}}\||\nabla|^{s_2}\psi\|_{L^\gamma},$$

$$\||\cdot|^\beta\mathcal{G}(t)\psi\|_{L^2} \le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\gamma}-\frac{1}{2}\right)+\frac{\beta}{2}}\|\psi\|_{L^\gamma} + C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\nu}-\frac{1}{2}\right)}\||\cdot|^\nu\psi\|_{L^\nu}$$

for $t \ge 1$.

Proof. The first assertion is trivial. Noting $\||\cdot|^\beta G(t)\|_{L^k} \le Ct^{-\frac{n}{2}\left(1-\frac{1}{k}\right)+\frac{\beta}{2}}$, we can easily prove the second one in the same way as Lemma 2.1. □

Proof of Proposition 3.4. For simplicity, we treat only higher dimensional cases $n \ge 2$. The one-dimensional case can be proved in a similar way. At first we shall consider the case $r>1$. In this case, it suffices to estimate

$$v(t) - \varepsilon\mathcal{G}(u_0+u_1) = \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\,d\tau.$$

Let us start with

$$\begin{aligned}\left\| |\nabla|^s \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\,d\tau \right\|_{L^2} &\le C \int_0^{\frac{t}{2}} \||\nabla|^s \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\|_{L^2}\,d\tau \\ &\quad + C\int_{\frac{t}{2}}^t \||\nabla|^s \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\|_{L^2}\,d\tau \\ &=: I + II.\end{aligned}$$

Applying Lemma 3.5 with $s_1 = s$, $s_2 = 0$ and $\gamma = \sigma_1$, we have

$$\begin{aligned} I &\le C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)-\frac{s}{2}} \|\mathcal{N}(u(\tau))\|_{L^{\sigma_1}}\,d\tau \\ &\le C\|\mathcal{N}(u)\|_{Y(\infty)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)-\frac{s}{2}} \int_0^{\frac{t}{2}} \langle \tau\rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_1}\right)}\,d\tau \\ &\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)-\frac{s}{2}-\min\{0,\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_1}\right)-1\}+\varsigma} \\ &\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}. \end{aligned}$$

For II, we apply Lemma 3.5 with $s_1 = s-[s]$, $s_2 = [s]$ and $\gamma = \rho$ to obtain

$$\begin{aligned} II &\le C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)-\frac{s-[s]}{2}} \||\nabla|^{[s]}\mathcal{N}(u(\tau))\|_{L^\rho}\,d\tau \\ &\le C\|\mathcal{N}(u)\|_{Y(\infty)}\langle t\rangle^{-\eta} \int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)-\frac{s-[s]}{2}}\,d\tau \\ &\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}-\frac{n}{2r}(p-1)+1}. \end{aligned}$$

Summing up the above estimates, we have (3.11).

Next, we prove (3.12). First we have

$$\begin{aligned}\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\,d\tau \right\|_{L^2} &\le C \int_0^{\frac{t}{2}} \|\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\|_{L^2}\,d\tau \\ &\quad + C\int_{\frac{t}{2}}^t \|\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\|_{L^2}\,d\tau \\ &=: III + IV.\end{aligned}$$

Lemma 3.5 with $s_1 = s_2 = 0$ and $\gamma = \sigma_1$ leads to

$$\begin{aligned} III &\le C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)} \|\mathcal{N}(u(\tau))\|_{L^{\sigma_1}}\,d\tau \\ &\le C\|\mathcal{N}(u)\|_{Y(\infty)} \langle t\rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)} \int_0^{\frac{t}{2}} \langle \tau\rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_1}\right)}\,d\tau \\ &\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}. \end{aligned}$$

Also, taking $s_1 = s_2 = 0$ and $\gamma = \sigma_2$ in Lemma 3.5, we see that

$$\begin{aligned} IV &\le C \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_2}-\frac{1}{2}\right)} \|\mathcal{N}(u(\tau))\|_{L^{\sigma_2}}\, d\tau \\ &\le C \|\mathcal{N}(u)\|_{Y(\infty)} \langle t \rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_2}\right)} \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_2}-\frac{1}{2}\right)}\, d\tau \\ &\le C \langle t \rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{n}{2r}(p-1)+1+\varsigma}, \end{aligned}$$

since $-\frac{n}{2}(\frac{1}{\sigma_2}-\frac{1}{2}) > -1$. Combining the estimates of *III* and *IV*, we reach the estimate (3.12).

Finally, we give a proof of (3.13).

$$\begin{aligned} \int_0^t \||\cdot|^\beta \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\|_{L^2}\, d\tau &\le C \int_0^{\frac{t}{2}} \||\cdot|^\beta \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\|_{L^2}\, d\tau \\ &\quad + C \int_{\frac{t}{2}}^{t} \||\cdot|^\beta \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\|_{L^2}\, d\tau \\ &=: V + VI. \end{aligned}$$

For V, applying Lemma 3.5 with $\gamma = \sigma_1$ and $\nu = q$, we have

$$\begin{aligned} V &\le C \int_0^{\frac{t}{2}} \langle t-\tau \rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)+\frac{\beta}{2}} \|\mathcal{N}(u)\|_{L^{\sigma_1}}\, d\tau \\ &\quad + C \int_0^{\frac{t}{2}} \langle t-\tau \rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \||\cdot|^\beta \mathcal{N}(u)\|_{L^q}\, d\tau \\ &\le C \|\mathcal{N}(u)\|_{Y(\infty)} \langle t \rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{2}\right)+\frac{\beta}{2}} \int_0^{\frac{t}{2}} \langle \tau \rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_1}\right)}\, d\tau \\ &\quad + C \|\mathcal{N}(u)\|_{Y(\infty)} \langle t \rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \int_0^{\frac{t}{2}} \langle \tau \rangle^{-\zeta_\beta}\, d\tau \\ &\le C \langle t \rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}. \end{aligned}$$

For VI, letting $\gamma = \sigma_2$ and $\nu = q$ in Lemma 3.5, we see that

$$\begin{aligned} VI &\le C \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_2}-\frac{1}{2}\right)+\frac{\beta}{2}} \|\mathcal{N}(u(\tau))\|_{L^{\sigma_2}}\, d\tau \\ &\quad + C \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \||\cdot|^\beta \mathcal{N}(u(\tau))\|_{L^q}\, d\tau \\ &\le C \|\mathcal{N}(u)\|_{Y(\infty)} \langle t \rangle^{-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{\sigma_2}\right)} \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{n}{2}\left(\frac{1}{\sigma_2}-\frac{1}{2}\right)+\frac{\beta}{2}}\, d\tau \\ &\quad + C \|\mathcal{N}(u)\|_{Y(\infty)} \langle t \rangle^{-\zeta_\beta} \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\, d\tau \\ &\le C \langle t \rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}-\frac{n}{2r}(p-1)+1}. \end{aligned}$$

Summing up the above estimates, we have (3.13) and complete the proof when $r > 1$.

Next, we give a proof of the case $r = 1$. We put

$$\theta = \theta_1 + \theta_2, \quad \theta_1 = \varepsilon \int_{\mathbb{R}^n} (u_0 + u_1)\, dx, \quad \theta_2 = \int_0^\infty \int_{\mathbb{R}^n} \mathcal{N}(u(\tau))\, dx d\tau.$$

We claim that for $m \in [1, \infty]$, it follows that

$$\|\varepsilon \mathcal{G}(t)(u_0 + u_1) - \theta_1 G(t)\|_{L^m} = o(t^{-\frac{n}{2}\left(1-\frac{1}{m}\right)}) \tag{3.15}$$

as $t \to \infty$ for $u_0 + u_1 \in L^1$ and

$$\begin{aligned} &\|\varepsilon \mathcal{G}(t)(u_0 + u_1) - \theta_1 G(t)\|_{L^m} \\ &\quad \le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-\min\{\frac{1}{2},\frac{\delta}{2}\}} \|\langle\cdot\rangle^\delta (u_0 + u_1)\|_{L^1} \end{aligned} \tag{3.16}$$

for $t \ge 1$ and $u_0 + u_1 \in H^{0,\alpha}$, where δ is an arbitrary number satisfying $0 < \delta < \alpha - \frac{n}{2}$. Here we also note that $\|\langle\cdot\rangle^\delta \phi\|_{L^1} \le C\|\phi\|_{H^{0,\alpha}}$. To prove (3.15), we first consider the case $m = 1$. We write $\phi = \varepsilon(u_0 + u_1)$ and have

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (G(t, x - y) - G(t, x))\phi(y)\, dy \right| dx \\ &\quad \le C \int_{\mathbb{R}^n} \left| \int_{|y|<\varrho t^{1/2}} (G(t, x - y) - G(t, x))\phi(y)\, dy \right| dx \\ &\qquad + C \int_{\mathbb{R}^n} \left| \int_{|y|>\varrho t^{1/2}} (G(t, x - y) - G(t, x))\phi(y)\, dy \right| dx =: I + II, \end{aligned}$$

where $\varrho > 0$ is an arbitrary small number. For I, we use the mean value theorem

$$|G(t, x - y) - G(t, x)| \le 2(4\pi t)^{-\frac{n}{2}} t^{-\frac{1}{2}} |y| \int_0^1 \frac{|x - ay|}{\sqrt{t}} e^{-\frac{|x-ay|^2}{4t}}\, da.$$

To prove (3.15), noting that $t^{-\frac{1}{2}}|y| \le \varrho$. Thus, we have $I \le C\varrho$. For II, we easily estimate

$$\begin{aligned} II &\le C \int_{|y|>\varrho t^{1/2}} \left(\int_{\mathbb{R}^n} |G(t, x - y) - G(t, x)|\, dx \right) |\phi(y)|\, dy \\ &\le C \int_{|y|>\varrho t^{\frac{1}{2}}} |\phi(y)|\, dy. \end{aligned}$$

Taking t sufficiently large, we have $II \le \varrho$. Consequently, we obtain $I + II \le C\varrho$ for sufficiently large t. Since ϱ is arbitrary, this proves (3.15). Similarly, we can easily prove (3.15) in the case $m = \infty$, and then, by the interpolation, we can obtain (3.15) for all $m \in [1, \infty]$.

To prove (3.16), noting $|y| \le |y|^{\min\{\delta,1\}} t^{\frac{1}{2}(1-\min\{\delta,1\})}$ on the integral region of I with $\varrho = 1$, and then, by the Fubini theorem, we see that

$$I \le Ct^{-\frac{1}{2}+\frac{1}{2}(1-\min\{\delta,1\})} \int_{\mathbb{R}^n} |y|^{\min\{\delta,1\}} |\phi(y)|\, dy \le Ct^{-\min\{\frac{1}{2},\frac{\delta}{2}\}} \|\langle\cdot\rangle^\delta \phi\|_{L^1}.$$

On the other hand, the term II is easily estimated as

$$\begin{aligned} II &\le C \int_{|y|>t^{1/2}} \left(\int_{\mathbb{R}^n} |G(t,x-y) - G(t,x)|\, dx \right) |\phi(y)|\, dy \\ &\le Ct^{-\frac{\delta}{2}} \int_{\mathbb{R}^n} |y|^{\delta} |\phi(y)|\, dy. \end{aligned}$$

Thus, we obtain (3.16) when $m = 1$. In a similar way, we can prove (3.16) when $m = \infty$ and hence, the interpolation gives (3.16) for all $m \in [1, \infty]$.

Next, we claim that

$$\begin{aligned} &\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\, d\tau - \theta_2 G(t) \right\|_{L^m} \\ &\quad \le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-\min\{\frac{1}{2},\frac{n}{2}(p-1)-1\}+\varsigma} \end{aligned} \tag{3.17}$$

for all $t \ge 1$ and m satisfying (1.3). Indeed, we first divide the left-hand side as

$$\begin{aligned} &\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\, d\tau - \theta_2 G(t) \right\|_{L^m} \\ &\le C \int_0^{\frac{t}{2}} \left\| \mathcal{G}(t-\tau)\mathcal{N}(u(\tau)) - G(t-\tau) \left(\int_{\mathbb{R}^n} \mathcal{N}(u(\tau))\, dx \right) \right\|_{L^m} d\tau \\ &\quad + C \int_0^{\frac{t}{2}} \left\| (G(t-\tau) - G(t)) \left(\int_{\mathbb{R}^n} \mathcal{N}(u(\tau))\, dx \right) \right\|_{L^m} d\tau \\ &\quad + C \int_{\frac{t}{2}}^t \|\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\|_{L^m}\, d\tau \\ &\quad + C \int_{\frac{t}{2}}^{\infty} \left\| G(t) \left(\int_{\mathbb{R}^n} \mathcal{N}(u(\tau))\, dx \right) \right\|_{L^m} d\tau \\ &=: III_m + IV_m + V_m + VI_m. \end{aligned}$$

Let us start with VI_m. Since $\|G(t)\|_{L^m} = Ct^{-\frac{n}{2}(1-\frac{1}{m})}$ and $\|\mathcal{N}(u(\tau))\|_{L^1} \le C\langle\tau\rangle^{-\frac{n}{2}(p-1)}$, it is easy to see that

$$VI_m \le Ct^{-\frac{n}{2}(1-\frac{1}{m})-\frac{n}{2}(p-1)+1}.$$

Secondly, we estimate V_m. For simplicity, we only consider the case $n > 2s$. When $m = \frac{2n}{n-2s}$, the Sobolev inequality gives

$$\begin{aligned} V_{\frac{2n}{n-2s}} &\le C \int_{\frac{t}{2}}^t \||\nabla|^s \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\|_{L^2}\, d\tau \\ &\le C \int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)-\frac{s-[s]}{2}} \||\nabla|^{[s]}\mathcal{N}(u(\tau))\|_{L^\rho}\, d\tau \\ &\le C \int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right)-\frac{s-[s]}{2}} \langle\tau\rangle^{-\eta}\, d\tau \\ &\le C\langle t\rangle^{-\frac{n}{2}(1-\frac{1}{m})-\frac{n}{2}(p-1)+1}. \end{aligned}$$

Also, by $\|\mathcal{N}(u(\tau))\|_{L^1} \leq C\langle\tau\rangle^{-\frac{n}{2}(p-1)}$, we immediately obtain $V_1 \leq C\langle t\rangle^{-\frac{n}{2}(p-1)+1}$. Interpolating these two cases, we conclude that

$$V_m \leq C\langle t\rangle^{-\frac{n}{2}(1-\frac{1}{m})-\frac{n}{2}(p-1)+1}$$

for all $m \in [1, \frac{2n}{n-2s}]$. The case $n \leq 2s$ can be also proved by using a similar interpolation as in the proof of Lemma 2.3 (ii).

Next, we consider IV_m. From $\|\mathcal{N}(u(\tau))\|_{L^1} \leq C\langle\tau\rangle^{-\frac{n}{2}(p-1)}$ and the fundamental theorem of calculus

$$G(t-\tau,x)-G(t,x) = -\tau\int_0^1 (\partial_t G)(t-a\tau,x)\,da,$$

we deduce that

$$\begin{aligned} IV_m &\leq C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-1}\langle\tau\rangle^{1-\frac{n}{2}(p-1)}\,d\tau \\ &\leq C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)}\begin{cases}\langle t\rangle^{-1} & (p>1+\frac{4}{n}),\\ \langle t\rangle^{-1}\log\langle t\rangle & (p=1+\frac{4}{n}),\\ \langle t\rangle^{-\frac{n}{2}(p-1)+1} & (p<1+\frac{4}{n}).\end{cases}\end{aligned}$$

Finally, we estimate III_m. Applying (3.16), we have

$$III_m \leq C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-\min\{\frac{1}{2},\frac{\delta}{2}\}}\|\langle\cdot\rangle^{\delta}\mathcal{N}(u)\|_{L^1}\,d\tau.$$

Taking $\delta=\beta$ $(n=2), \delta=1$ $(n\geq 3)$, we conclude

$$III_m \leq C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-\min\{\frac{1}{2},\frac{n}{2}(p-1)-1\}+\varsigma}.$$

Putting together the estimates above, we reach (3.17). □

From Propositions 3.1 and 3.4, we finish up the proof of Theorem 1.3.

Proof of Theorem 1.3. We start with the case $r>1$. Let m be a real number satisfying (1.3). First, we assume that $m\geq 2$. In this case, the Sobolev embedding theorem implies $\|\psi\|_{L^m} \leq \||\nabla|^{s_m}\psi\|_{L^2}$ with $s_m = n(\frac{1}{2}-\frac{1}{m}) \in [0,s]$. This and the interpolation inequality

$$\||\nabla|^{s_m}\psi\|_{L^2} \leq \||\nabla|^{s}\psi\|_{L^2}^{\frac{s_m}{s}}\|\psi\|_{L^2}^{1-\frac{s_m}{s}}$$

lead to

$$\begin{aligned}&\|u(t)-\varepsilon\mathcal{G}(t)(u_0+u_1)\|_{L^m}\\ &\quad\leq \||\nabla|^s(u(t)-\varepsilon\mathcal{G}(t)(u_0+u_1))\|_{L^2}^{\frac{s_m}{s}}\|u(t)-\varepsilon\mathcal{G}(t)(u_0+u_1)\|_{L^2}^{1-\frac{s_m}{s}}.\end{aligned}$$

Using Propositions 3.1 and 3.4, we calculate

$$\begin{aligned}&\||\nabla|^s(u(t)-\varepsilon\mathcal{G}(t)(u_0+u_1))\|_{L^2}\\ &\leq C\||\nabla|^s(u(t)-v(t))\|_{L^2}+\||\nabla|^s(v(t)-\varepsilon\mathcal{G}(t)(u_0+u_1))\|_{L^2}\\ &\leq C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}-\min\{1,\frac{n}{2r}(p-1)-\frac{1}{2}\}}\end{aligned}$$

$$
\begin{aligned}
&+ C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}\\
&\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}.
\end{aligned}
$$

Similarly, we can see that

$$
\|u(t)-\varepsilon\mathcal{G}(t)(u_0+u_1)\|_{L^2} \le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}.
$$

These two estimates yield

$$
\begin{aligned}
\|u(t)-\varepsilon\mathcal{G}(t)(u_0+u_1)\|_{L^m} &\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s_m}{2}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}\\
&= C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{m}\right)-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma},
\end{aligned}
$$

which shows the desired estimate.

When $m \in [r,2]$, by the interpolation, we have

$$
\|\psi\|_{L^m} \le \|\psi\|_{L^r}^{\frac{r(2-m)}{m(2-r)}}\|\psi\|_{L^2}^{\frac{2(m-r)}{m(2-r)}} \le \|\langle\cdot\rangle^{\beta}\psi\|_{L^2}^{\frac{r(2-m)}{m(2-r)}}\|\psi\|_{L^2}^{\frac{2(m-r)}{m(2-r)}}
$$

with β satisfying $n(\frac{1}{r}-\frac{1}{2}) < \beta \le \alpha$ and $(\beta - n(\frac{1}{r}-\frac{1}{2}))\frac{r(2-m)}{m(2-r)} < \varsigma$. Thus, it suffices to estimate the right-hand side with $\psi = u(t)-\varepsilon\mathcal{G}(t)(u_0+u_1)$. In the same way as before, we can deduce that

$$
\|\langle\cdot\rangle^{\beta}(u(t)-\varepsilon\mathcal{G}(t)(u_0+u_1))\|_{L^2} \le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}
$$

and hence, we have

$$
\begin{aligned}
&\|u(t)-\varepsilon\mathcal{G}(t)(u_0+u_1)\|_{L^m}\\
&\le C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\beta}{2}\frac{r(2-m)}{m(2-r)}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}\\
&= C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{m}\right)+\left(\frac{\beta}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)\right)\frac{r(2-m)}{m(2-r)}-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\varsigma}\\
&= C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{m}\right)-\min\{\frac{n}{2}\left(1-\frac{1}{r}\right),\frac{1}{2},\frac{n}{2r}(p-1)-1\}+\frac{3\varsigma}{2}}.
\end{aligned}
$$

Rewriting $\frac{3\varsigma}{2}$ as ς, we complete the proof.

When $r=1$, as before, we first divide

$$
\|u(t)-\theta G(t)\|_{L^m} \le \|u(t)-v(t)\|_{L^m} + \|v(t)-\theta G(t)\|_{L^m}.
$$

Let β satisfy $\frac{n}{2} < \beta \le \alpha$ and $(\beta-\frac{n}{2})(\frac{1}{m}-\frac{1}{2}) < \varsigma$. Then, the first term is estimated as

$$
\begin{aligned}
&\|u(t)-v(t)\|_{L^m}\\
&\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)}\begin{cases}\langle t\rangle^{-\min\{1,\frac{n}{2}(p-1)-\frac{1}{2}\}} & (m\ge 2),\\ \langle t\rangle^{\left(\beta-\frac{n}{2}\right)\left(\frac{1}{m}-\frac{1}{2}\right)-\min\{1,\frac{n}{2}(p-1)-\frac{1}{2}\}} & (1\le m<2)\end{cases}\\
&\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-\min\{1,\frac{n}{2}(p-1)-\frac{1}{2}\}+\varsigma}.
\end{aligned}
$$

For the second term, we apply Proposition 3.4 and comparing the decay rate leads to the conclusion. □

4. Blow-up and estimates of the lifespan

In this section, we give a proof of Theorems 1.5 and 1.7.

4.1. Estimates of the lifespan from below

At first, we prove Theorem 1.5.

Proof of Theorem 1.5. We first consider Case 1, namely, we assume $r \in [1, 2]$ and $\min\{1 + \frac{r}{2}, 1 + \frac{r}{n}\} \le p < 1 + \frac{2r}{n}$.

If $T(\varepsilon) = \infty$, then the assertion of the theorem is obviously true. Hence, in what follows we assume $T(\varepsilon) < \infty$. Then, Theorem 1.1 shows that

$$\lim_{t \to T(\varepsilon)} \|u(t)\|_{H^{s,0} \cap H^{0,\alpha}} = \infty. \tag{4.1}$$

On the other hand, by Lemmas 2.4 and 2.5, we have

$$\begin{aligned} \left\| \int_0^t \mathcal{D}(t-\tau)\mathcal{N}(u(\tau))\, d\tau \right\|_{X(T)} &\le C \|\mathcal{N}(u)\|_{Y(T)} \langle T \rangle^{1 - \frac{n}{2r}(p-1)} \\ &\le C \|u\|_{X(T)}^p \langle T \rangle^{1 - \frac{n}{2r}(p-1)}. \end{aligned}$$

This implies

$$\|u\|_{X(T)} \le \varepsilon C_0 I_0 + C_1 \|u\|_{X(T)}^p \langle T \rangle^{1 - \frac{n}{2r}(p-1)} \tag{4.2}$$

with some constant $C_0, C_1 > 0$, where $I_0 = \|u_0\|_{H^{s,0} \cap H^{0,\alpha}} + \|u_1\|_{H^{s-1,0} \cap H^{0,\alpha}}$. Now, by (4.1) and the continuity of $\|u\|_{X(T)}$ with respect to T, there exists the smallest time $\widetilde{T}(\varepsilon)$ such that $\|u\|_{X(\widetilde{T}(\varepsilon))} = 2C_0 I_0 \varepsilon$ holds. Then, letting $T = \widetilde{T}(\varepsilon)$ in the above inequality gives

$$2C_0 I_0 \varepsilon \le C_0 I_0 \varepsilon + C_1 (2C_0 I_0 \varepsilon)^p \langle \widetilde{T}(\varepsilon) \rangle^{1 - \frac{n}{2r}(p-1)}.$$

We rewrite it as

$$C_0 I_0 \varepsilon^{-(p-1)} \le C_1 (2C_0 I_0)^p \langle \widetilde{T}(\varepsilon) \rangle^{1 - \frac{n}{2r}(p-1)},$$

which implies

$$\langle \widetilde{T}(\varepsilon) \rangle \ge C \varepsilon^{-1/\omega}$$

with $\omega = \frac{1}{p-1} - \frac{n}{2r}$. Thus, taking ε sufficiently small, we easily see that $\langle \widetilde{T}(\varepsilon) \rangle \ge 1$, which enables us to replace $\langle \widetilde{T}(\varepsilon) \rangle$ by $\widetilde{T}(\varepsilon)$ in the above estimate. This and $\widetilde{T}(\varepsilon) < T(\varepsilon)$ imply the desired estimate.

In Case 2, that is, when $r = 1$ and $p = 1 + \frac{2}{n}$, the first author and Ogawa [19] have already proved the conclusion of Theorem 1.5 by a slightly different argument.

Instead of (4.2), we obtain

$$\|u\|_{X(T)} \le \varepsilon C_0 I_0 + C_1 \|u\|_{X(T)}^p \log \langle T \rangle.$$

In the same manner as above, we can see that

$$\langle \widetilde{T}(\varepsilon) \rangle \ge \exp\left(C \varepsilon^{-(p-1)} \right),$$

which gives the desired conclusion. □

4.2. Estimates of the lifespan from above

In this subsection, we give a proof of Theorem 1.7. The proof is based on the test function method introduced by Zhang [68] and refined to estimate the lifespan by Kuiper [33], Sun [60] and [20] in which initial data belonging to L^1 are treated. Here we further adapt their method to fit initial data not belonging to L^1 by the argument in [18].

Before proving Theorem 1.7, we introduce the definition of weak solutions of (1.1). Let $T > 0$ and $u_0, u_1 \in L^1_{loc}(\mathbb{R}^n)$. We say that a function $u \in L^p_{loc}([0,T) \times \mathbb{R}^n)$ is a weak solution of (1.1) on the interval $[0,T)$ if the identity

$$\int_{[0,T)\times\mathbb{R}^n} u\,(\psi_{tt} - \Delta\psi - \psi_t)\,dxdt$$
$$= \varepsilon \int_{\mathbb{R}^n} ((u_0+u_1)\psi(0,x) - u_0\psi_t(0,x))\,dx + \int_{[0,T)\times\mathbb{R}^n} \mathcal{N}(u)\psi\,dxdt$$

is valid for any $\psi \in C_0^\infty([0,T)\times\mathbb{R}^n)$. We also define the lifespan of the weak solution:

$$T_w(\varepsilon) := \sup\{T \in (0,\infty); \text{there exists a weak solution } u \text{ on } [0,T)\}.$$

Proposition 4.1. *Under the assumptions of Theorem 1.7, we have*

$$T_w(\varepsilon) \le C\varepsilon^{-1/\kappa}$$

for $\varepsilon \in (0,1]$, *where* $\kappa = \frac{1}{p-1} - \frac{\lambda}{2}$ *and* λ *satisfies* $\frac{n}{2} + \alpha < \lambda < \frac{2}{p-1}$.

Proof of Proposition 4.1. We assume that $\mathcal{N}(u) = |u|^p$. The case $\mathcal{N}(u) = -|u|^p$ reduces to the case above by considering $-u$.

First, we may assume that $T_w(\varepsilon) \ge 4$. Because, if $T_w(\varepsilon) \le 4$, then we immediately obtain $T_w(\varepsilon) \le 4\varepsilon^{-1/\kappa}$ for any $\varepsilon \in (0,1]$.

Let $\eta = \eta(t) \in C_0^\infty([0,\infty))$ be a test function satisfying

$$0 \le \eta(t) \le 1, \quad \eta(t) = \begin{cases} 1 & (0 \le t \le 1/2), \\ 0 & (t \ge 1) \end{cases}$$

and let $\phi(x) := \eta(|x|)$. Then, we have (see for example, [50])

$$|\eta^{(j)}(t)| \le C\eta(t)^{1/p}\ (j=1,2), \quad |\Delta\phi(x)| \le C\phi(x)^{1/p}. \tag{4.3}$$

Let $\tau \in [1, T_w(\varepsilon))$ and $R \in [2,\infty)$ be parameters. We define $\eta_\tau(t) = \eta(\frac{t}{\tau})$, $\phi_R(x) = \phi(\frac{x}{R})$ and

$$\psi = \psi_{\tau,R}(t,x) = \eta_\tau(t)\phi_R(x).$$

We also put

$$I_{\tau,R} := \int_{[0,\tau)\times B_R} |u|^p \psi_{\tau,R}\,dxdt, \quad J_R := \varepsilon\int_{B_R} (u_0+u_1)\phi_R(x)\,dx,$$

where $B_R = \{x \in \mathbb{R}^n; |x| < R\}$. Taking $\psi = \psi_{\tau,R}$ in the definition of the weak solution, we have the identity

$$
\begin{aligned}
I_{\tau,R} + J_R &= \int_{[0,\tau)\times B_R} u\partial_t^2\psi_{\tau,R}\,dxdt - \int_{[0,\tau)\times B_R} u\Delta\psi_{\tau,R}\,dxdt \\
&\quad - \int_{[0,\tau)\times B_R} u\partial_t\psi_{\tau,R}\,dxdt \\
&=: K_1 + K_2 + K_3.
\end{aligned}
$$

By the Hölder inequality and (4.3), we have

$$
\begin{aligned}
K_1 &\le C\tau^{-2}\int_{[0,T)\times B_R} |u||\eta''(t/\tau)||\phi_R(x)|\,dxdt \\
&\le C\tau^{-2}\int_{[0,T)\times B_R} |u|\eta_\tau(t)^{1/p}|\phi_R(x)|\,dxdt \\
&\le C\tau^{-2} I_{\tau,R}^{1/p}\left(\int_{\frac{\tau}{2}}^{\tau}\int_{B_R}\phi_R\,dxdt\right)^{1/p'} \\
&\le C\tau^{-2+1/p'}R^{n/p'}I_{\tau,R}^{1/p},
\end{aligned}
\tag{4.4}
$$

where $p' = \frac{p}{p-1}$. In a similar way, we see that

$$
K_2 \le C\tau^{1/p'}R^{-2+n/p'}I_{\tau,R}^{1/p} \tag{4.5}
$$

and

$$
K_3 \le C\tau^{-1+1/p'}R^{n/p'}I_{\tau,R}^{1/p}. \tag{4.6}
$$

It follows from (4.4)–(4.6) that

$$
I_{\tau,R} + J_R \le C\left(\tau^{1/p'}R^{-2+n/p'} + \tau^{-1+1/p'}R^{n/p'}\right)I_{\tau,R}^{1/p}.
$$

By the Young inequality, the right-hand side is bounded by

$$
\frac{1}{2}I_{\tau,R} + C(\tau R^{-2p'+n} + \tau^{-p'+1}R^n).
$$

Therefore, we have

$$
J_R \le C\tau R^{-2p'+n} + \tau^{-p'+1}R^n.
$$

From the assumption on the initial data, we see that

$$
\begin{aligned}
J_R &\ge \varepsilon\int_{|x|>1}(u_0(x)+u_1(x))\phi\left(\frac{x}{R}\right)\,dx \\
&\ge \varepsilon\int_{|x|>1}|x|^{-\lambda}\phi\left(\frac{x}{R}\right)\,dx \\
&\ge \varepsilon R^{-\lambda+n}\int_{|y|>1/R}|y|^{-\lambda}\phi(y)\,dy \\
&\ge \varepsilon R^{-\lambda+n}\int_{|y|>1/2}|y|^{-\lambda}\phi(y)\,dy,
\end{aligned}
$$

since $R \ge 2$. Thus, we have

$$
\varepsilon \le C\tau R^{-2p'+\lambda} + C\tau^{-p'+1}R^{\lambda}.
$$

By taking $R = \tau^{1/2}$, we obtain $\varepsilon \le C\tau^{-(\frac{1}{p-1}-\frac{\lambda}{2})}$, which implies

$$\tau \le C\varepsilon^{-1/\kappa},$$

with $\kappa = \frac{1}{p-1} - \frac{\lambda}{2}$. Since τ is arbitrary in $[1, T_w(\varepsilon))$, we conclude

$$T_w(\varepsilon) \le C\varepsilon^{-1/\kappa}.$$

This finishes the proof. □

Proposition 4.2. *Under the assumptions of Theorem 1.1, the mild solution u on $[0,T)$ is also a weak solution on $[0,T)$.*

Proof. Let $\psi \in C_0^\infty([0,T)\times\mathbb{R}^n)$ be a test function and we assume $\operatorname{supp}\psi \in [0,T_1)\times\mathbb{R}^n$, where $T_1 < T$. Then, Theorem 1.1 or Lemma 2.5 yields $\mathcal{N}(u(\tau)) \in L^\infty(0,T_1;L^q(\mathbb{R}^n))$. Let $F_j \in C_0^\infty([0,\infty)\times\mathbb{R}^n)$ $(j=1,2,\ldots)$ be a sequence such that

$$\lim_{j\to\infty} \|F_j - \mathcal{N}(u)\|_{L^\infty(0,T_1;L^q(\mathbb{R}^n))} = 0.$$

We also set

$$v_j := \int_0^t \mathcal{D}(t-\tau)F_j(\tau)\,d\tau, \quad u^{nl} := \int_0^t \mathcal{D}(t-\tau)\mathcal{N}(u(\tau))\,d\tau.$$

Then, taking $\psi = F_j - \mathcal{N}(u)$ in the proof of Lemma 2.4, we deduce

$$\|v_j - u^{nl}\|_{L^\infty(0,T_1;L^2(\mathbb{R}^n))} \le C\|F_j - \mathcal{N}(u)\|_{L^\infty(0,T_1;L^q(\mathbb{R}^n))} \to 0 \quad (j\to\infty)$$

(indeed, in the proof of Lemma 2.4, take $\gamma = q$ and use the Sobolev embedding theorem). Therefore, we have

$$\lim_{j\to\infty}\int_0^{T_1}\int_{\mathbb{R}^n} v_j(\partial_t^2 - \Delta - \partial_t)\psi\,dxdt = \int_0^{T_1}\int_{\mathbb{R}^n} u^{nl}(\partial_t^2 - \Delta - \partial_t)\psi\,dxdt.$$

On the other hand, since F_j is smooth and compactly supported with respect to x, so is v_j and hence, using integration by parts we easily compute

$$\int_0^{T_1}\int_{\mathbb{R}^n} v_j(\partial_t^2 - \Delta - \partial_t)\psi\,dxdt = \int_0^{T_1}\int_{\mathbb{R}^n} F_j\psi\,dxdt.$$

Taking the limit $j\to\infty$ in the right-hand side and noting $F_j \to \mathcal{N}(u)$ in $L^\infty(0,T_1;L^q(\mathbb{R}^n))$, we have

$$\lim_{j\to\infty}\int_0^{T_1}\int_{\mathbb{R}^n} v_j(\partial_t^2 - \Delta - \partial_t)\psi\,dxdt = \int_0^{T_1}\int_{\mathbb{R}^n} \mathcal{N}(u)\psi\,dxdt.$$

Thus, we conclude

$$\int_0^{T_1}\int_{\mathbb{R}^n} u^{nl}(\partial_t^2 - \Delta - \partial_t)\psi\,dxdt = \int_0^{T_1}\int_{\mathbb{R}^n} \mathcal{N}(u)\psi\,dxdt.$$

In a similar way, approximating u_0, u_1 by smooth functions and using the integration by parts, we find that

$$\begin{aligned}\varepsilon \int_0^{T_1} \int_{\mathbb{R}^n} \left(\tilde{\mathcal{D}}(t)u_0 + \mathcal{D}(t)u_1\right) \left(\partial_t^2 - \Delta - \partial_t\right) \psi \, dxdt \\ = \varepsilon \int_{\mathbb{R}^n} \left((u_0 + u_1)\psi(0, x) - u_0\psi_t(0, x)\right) \, dx.\end{aligned}$$

This completes the proof. □

Now, we are in the position to prove Theorem 1.7.

Proof of Theorem 1.7. By Proposition 4.2, we see that a mild solution is also a weak solution. Therefore, we have $T(\varepsilon) \le T_w(\varepsilon)$. Hence, Proposition 4.1 implies the desired estimate. □

Appendix A. Some auxiliary estimates

Lemma A.1. *Let $\omega \in (0, 1)$. Then, there exists a constant $C = C(\omega) > 0$ such that we have*

$$|\partial_j|^\omega \phi(x) = C \int_{\mathbb{R}} (\phi(x-y) - \phi(x)) \frac{dy_j}{|y_j|^{1+\omega}},$$

where $y = (0, \dots, y_j, \dots, 0)$. In particular, it follows that

$$\begin{aligned}|\partial_j|^\omega (\phi\psi)(x) &= C \int_{\mathbb{R}} (\phi(x-y)\psi(x-y) - \phi(x)\psi(x)) \frac{dy_j}{|y_j|^{1+\omega}} \\ &= \phi(x)|\partial_j|^\omega \psi(x) + C \int_{\mathbb{R}} (\phi(x-y) - \phi(x))\psi(x-y) \frac{dy_j}{|y_j|^{1+\omega}}.\end{aligned}$$

Proof. Recalling the identity

$$\mathcal{F}^{-1}[\phi](x-y) = \mathcal{F}^{-1}[e^{iy_j\xi_j}\phi](x),$$

we have

$$\begin{aligned}\int_{\mathbb{R}} (\phi(x-y) - \phi(x)) \frac{dy_j}{|y_j|^{1+\omega}} &= \int_{\mathbb{R}} \mathcal{F}^{-1}\left[\left(e^{iy_j\xi_j} - 1\right)\hat{\phi}\right](x) \frac{dy_j}{|y_j|^{1+\omega}} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{ix\xi} \left(e^{iy_j\xi_j} - 1\right) \hat{\phi} d\xi \frac{dy_j}{|y_j|^{1+\omega}} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi} T(\xi_j) \hat{\phi} \, d\xi,\end{aligned}$$

where

$$T(\xi_j) = \int_{\mathbb{R}} \left(e^{iy_j\xi_j} - 1\right) \frac{dy_j}{|y_j|^{1+\omega}} = 2|\xi_j|^\omega \int_0^\infty (\cos y_j - 1) \frac{dy_j}{y_j^{1+\omega}} = C|\xi_j|^\omega.$$

Therefore, we conclude

$$\int_{\mathbb{R}} (\phi(x-y) - \phi(x)) \frac{dy_j}{|y_j|^{1+\omega}} = \frac{C}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi} |\xi_j|^\omega \hat{\phi} \, d\xi = C|\partial_j|^\omega \phi(x).$$

□

Lemma A.2. (Sobolev-type inequality) *Let $\beta \geq 0$. Then, we have*

$$\|\langle\cdot\rangle^{\beta}\langle\nabla\rangle^{-1}\psi\|_{L^2} \leq C\|\langle\cdot\rangle^{\beta}\psi\|_{L^q},$$

where q is defined in Sect. 1.2.

Proof. By the Plancherel theorem, it suffices to estimate $\|\langle\nabla\rangle^{\beta}\langle\xi\rangle^{-1}\hat{\psi}\|_{L^2}$. First, we note that

$$\|\langle\nabla\rangle^{\beta}\langle\xi\rangle^{-1}\hat{\psi}\|_{L^2} \leq C\|\langle\xi\rangle^{-1}\hat{\psi}\|_{L^2} + C\||\nabla|^{\beta}\langle\xi\rangle^{-1}\hat{\psi}\|_{L^2}.$$

The Sobolev embedding implies $\|\langle\nabla\rangle^{-1}\psi\|_{L^2} \leq C\|\psi\|_{L^q}$ with $q = \max\{1, \frac{2n}{n+2}\}$, which gives the estimate for the first term of the right-hand side. Let us estimate the second term. When $\beta \in \mathbb{Z}$, the Leibniz rule and the Sobolev embedding and noting that $|\partial_j^k\langle\xi\rangle^{-1}| \leq C\langle\xi\rangle^{-1}$ immediately imply $\||\nabla|^{\beta}\langle\xi\rangle^{-1}\hat{\psi}\|_{L^2} \leq C\|\langle x\rangle^{\beta}\psi\|_{L^q}$. When $\beta \notin \mathbb{Z}$, letting $\omega = \beta - [\beta] > 0$ and using Lemma A.1, we have

$$\begin{aligned}
&\||\nabla|^{\beta}(\langle\xi\rangle^{-1}\hat{\psi})\|_{L^2} \\
&\quad \leq C\sum_{j=1}^{n}\sum_{k=0}^{[\beta]}\left\|\left(\partial_j^{[\beta]-k}\langle\xi\rangle^{-1}\right)|\partial_j|^{k+\omega}\hat{\psi}\right\|_{L^2} \\
&\qquad + C\sum_{j=1}^{n}\sum_{k=0}^{[\beta]}\left\|\int_{\mathbb{R}}\left(\partial_j^{[\beta]-k}\langle\xi\rangle^{-1} - \partial_j^{[\beta]-k}\langle\xi+\eta\rangle^{-1}\right)\partial_j^k\hat{\psi}(\xi+\eta)\frac{d\eta_j}{|\eta_j|^{1+\omega}}\right\|_{L^2},
\end{aligned} \tag{A.1}$$

where $\eta = (0, \ldots, \eta_j, \ldots, 0)$. Then, a straight forward calculation shows

$$\begin{aligned}
\left\|\left(\partial_j^{[\beta]-k}\langle\xi\rangle^{-1}\right)|\partial_j|^{k+\omega}\hat{\psi}\right\|_{L^2} &\leq C\|\langle\xi\rangle^{-1}|\partial_j|^{k+\omega}\hat{\psi}\|_{L^2} \\
&\leq C\||x_j|^{k+\omega}\psi\|_{L^q},
\end{aligned} \tag{A.2}$$

where we have used the Sobolev embedding again. Moreover, we claim that

$$\begin{aligned}
&\left\|\int_{\mathbb{R}}\left(\partial_j^{[\beta]-k}\langle\xi\rangle^{-1} - \partial_j^{[\beta]-k}\langle\xi+\eta\rangle^{-1}\right)\partial_j^k\hat{\psi}(\xi+\eta)\frac{d\eta_j}{|\eta_j|^{1+\omega}}\right\|_{L^2} \\
&\quad \leq C\||x_j|^{k}\psi\|_{L^q}.
\end{aligned} \tag{A.3}$$

Indeed, we first note that the left-hand side is bounded by

$$\begin{aligned}
&\left\|\int_{|\eta_j|\leq 1}\left(\partial_j^{[\beta]-k}\langle\xi\rangle^{-1} - \partial_j^{[\beta]-k}\langle\xi+\eta\rangle^{-1}\right)\partial_j^k\hat{\psi}(\xi+\eta)\frac{d\eta_j}{|\eta_j|^{1+\omega}}\right\|_{L^2} \\
&\quad + \left\|\int_{|\eta_j|>1}\left(\partial_j^{[\beta]-k}\langle\xi\rangle^{-1} - \partial_j^{[\beta]-k}\langle\xi+\eta\rangle^{-1}\right)\partial_j^k\hat{\psi}(\xi+\eta)\frac{d\eta_j}{|\eta_j|^{1+\omega}}\right\|_{L^2} \\
&=: I + II.
\end{aligned}$$

For I, the fundamental theorem of calculus implies

$$\begin{aligned} I &\le C\int_{|\eta_j|\le 1}\left\|\int_0^1 \frac{d}{da}\partial_j^{[\beta]-k}\langle\xi+a\eta\rangle^{-1}\,da\cdot\partial_j^k\hat{\psi}(\xi+\eta)\right\|_{L^2}\frac{d\eta_j}{|\eta_j|^{1+\omega}} \\ &\le C\int_{|\eta_j|\le 1}\left\|\int_0^1 \partial_j^{[\beta]-k+1}\langle\xi+a\eta\rangle^{-1}\,da\cdot\partial_j^k\hat{\psi}(\xi+\eta)\right\|_{L^2}\frac{d\eta_j}{|\eta_j|^{\omega}} \\ &= C\int_{|\eta_j|\le 1}\left\|\int_0^1 \partial_j^{[\beta]-k+1}\langle\xi+a\eta\rangle^{-1}\,da\cdot\partial_j^k\hat{\psi}(\xi)\right\|_{L^2}\frac{d\eta_j}{|\eta_j|^{\omega}} \\ &\le C\int_{|\eta_j|\le 1}\left\|\langle\xi\rangle^{-1}\partial_j^k\hat{\psi}(\xi)\right\|_{L^2}\frac{d\eta_j}{|\eta_j|^{\omega}} \\ &\le C\left\|\langle\xi\rangle^{-1}\partial_j^k\hat{\psi}(\xi)\right\|_{L^2}. \end{aligned}$$

This and the Sobolev embedding lead to $I \le C\||x_j|^k\psi\|_{L^q}$. For II, we first have

$$\begin{aligned} II &\le C\int_{|\eta_j|>1}\left\|\left(\partial_j^{[\beta]-k}\langle\xi\rangle^{-1}-\partial_j^{[\beta]-k}\langle\xi+\eta\rangle^{-1}\right)\partial_j^k\hat{\psi}(\xi+\eta)\right\|_{L^2}\frac{d\eta_j}{|\eta_j|^{1+\omega}} \\ &\le C\int_{|\eta_j|>1}\left\|\partial_j^{[\beta]-k}\langle\xi-\eta\rangle^{-1}\partial_j^k\hat{\psi}(\xi)\right\|_{L^2(|\xi-\eta|\le|\xi|/2)}\frac{d\eta_j}{|\eta_j|^{1+\omega}} \\ &\quad + C\int_{|\eta_j|>1}\left\|\partial_j^{[\beta]-k}\langle\xi-\eta\rangle^{-1}\partial_j^k\hat{\psi}(\xi)\right\|_{L^2(|\xi-\eta|>|\xi|/2)}\frac{d\eta_j}{|\eta_j|^{1+\omega}} \\ &\quad + C\int_{|\eta_j|>1}\left\|\partial_j^{[\beta]-k}\langle\xi+\eta\rangle^{-1}\partial_j^k\hat{\psi}(\xi+\eta)\right\|_{L^2(\mathbb{R}^n)}\frac{d\eta_j}{|\eta_j|^{1+\omega}} \\ &=: II_1+II_2+II_3. \end{aligned}$$

The third term II_3 is easily estimated as

$$II_3 \le C\left\|\langle\xi\rangle^{-1}\partial_j^k\hat{\psi}(\xi)\right\|_{L^2} \le C\||x_j|^k\psi\|_{L^q}. \tag{A.4}$$

For II_2, we note that

$$\left|\partial_j^{[\beta]-k}\langle\xi-\eta\rangle^{-1}\right| \le C\langle\xi-\eta\rangle^{-1} \le C\langle\xi\rangle^{-1}$$

holds for $|\xi-\eta|>|\xi|/2$ and hence,

$$\begin{aligned} II_2 &\le C\int_{|\eta_j|>1}\left\|\langle\xi\rangle^{-1}\partial_j^k\hat{\psi}(\xi)\right\|_{L^2(|\xi-\eta|>|\xi|/2)}\frac{d\eta_j}{|\eta_j|^{1+\omega}} \\ &\le C\left\|\langle\xi\rangle^{-1}\partial_j^k\hat{\psi}(\xi)\right\|_{L^2(\mathbb{R}^n)} \\ &\le C\||x_j|^k\psi\|_{L^q}. \end{aligned} \tag{A.5}$$

Finally, for II_1, we first remark that $|\eta| \ge |\xi| - |\xi-\eta| \ge \frac{1}{2}|\xi|$ holds. Noting again that $|\partial_j^{[\beta]-k}\langle\xi-\eta\rangle^{-1}| \le C\langle\xi-\eta\rangle^{-1}$ and applying the Hölder inequality, we see that

$$II_1 \le C\int_{|\eta_j|>1}\|\langle\xi-\eta\rangle^{-1}\|_{L^n(|\xi-\eta|\le|\xi|/2)}\left\|\partial_j^k\hat{\psi}(\xi)\right\|_{L^{q'}(|\xi-\eta|\le|\xi|/2)}\frac{d\eta_j}{|\eta_j|^{1+\omega}} \tag{A.6}$$

for $n \geq 2$, where q' is the conjugate of q, and

$$II_1 \leq C \int_{|\eta_j|>1} \left\| \langle \xi - \eta \rangle^{-1} \right\|_{L^2(|\xi-\eta|\leq|\xi|/2)} \left\| \partial_j^k \hat{\psi}(\xi) \right\|_{L^\infty(|\xi-\eta|\leq|\xi|/2)} \frac{d\eta_j}{|\eta_j|^{1+\omega}}$$

for $n = 1$. In what follows, we only consider the case $n \geq 2$, because the case $n = 1$ is similar. Using $|\eta_j| = |\eta| > \frac{1}{2}|\xi|$, we have

$$\begin{aligned} \left\| \langle \xi - \eta \rangle^{-1} \right\|_{L^n(|\xi-\eta|\leq|\xi|/2)} &\leq \left\| \langle \xi - \eta \rangle^{-1} \right\|_{L^n(|\xi-\eta|\leq|\eta_j|)} \\ &= \left\| \langle \Xi \rangle^{-1} \right\|_{L^n(|\Xi|\leq|\eta_j|)} \\ &\leq C(\log |\eta_j| + 1)^{\frac{1}{n}}. \end{aligned}$$

Also, we immediately obtain

$$\left\| \partial_j^k \hat{\psi}(\xi) \right\|_{L^{q'}(|\xi-\eta|\leq|\xi|/2)} \leq C \left\| \partial_j^k \hat{\psi}(\xi) \right\|_{L^{q'}(\mathbb{R}^n)} \leq C \left\| |x_j|^k \psi \right\|_{L^q(\mathbb{R}^n)}.$$

Combining them with (A.6), we conclude

$$II_1 \leq C \left\| |x_j|^k \psi \right\|_{L^q} \int_{|\eta_j|>1} (\log |\eta_j| + 1)^{\frac{1}{n}} \frac{d\eta_j}{|\eta_j|^{1+\omega}} \leq C \left\| |x_j|^k \psi \right\|_{L^q}. \tag{A.7}$$

By (A.4), (A.5) and (A.7), we have $II \leq C \| |x_j|^k \psi \|_{L^q}$. Consequently, we obtain

$$\begin{aligned} \| \langle \cdot \rangle^\beta \langle \nabla \rangle^{-1} \psi \|_{L^2} &\leq C \|\psi\|_{L^q} + C \sum_{j=1}^{n} \sum_{k=0}^{[\beta]} \left(\| |x_j|^k \psi \|_{L^q} + \| |x_j|^{k+\omega} \psi \|_{L^q} \right) \\ &\leq C \| \langle x \rangle^\beta \psi \|_{L^q}, \end{aligned}$$

which shows the assertion. □

Acknowledgements

The authors are deeply grateful to Professor Mitsuru Sugimoto for useful suggestions and comments. They also thank an anonymous referee for the helpful suggestions. This work was supported in part by Grant-in-Aid for JSPS Fellows 15J01600 (Y. Wakasugi), 15J02570 (T. Inui), 14J01884 (M. Ikeda), and Grant-in-Aid for Young Scientists (B) 15K17571 (M. Ikeda) of Japan Society for the Promotion of Science.

References

[1] Cattaneo, C.R.: Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée. C. R. **247**, 431–433 (1958)

[2] Chill, R., Haraux, A.: An optimal estimate for the difference of solutions of two abstract evolution equations. J. Diff. Equ. **193**, 385–395 (2003)

[3] Dunbar, S.R., Othmer, H.G.: On a nonlinear hyperbolic equation describing transmission lines, cell movement, and branching random walks. In: Othmer, H.G. (eds.) Nonlinear Oscillations in Biology and Chemistry, Lecture Notes in Biomathematics, vol. 66. Springer, New York (1986)

[4] Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sec. I **13**, 109–124 (1966)

[5] Gallay, Th, Raugel, G.: Scaling variables and asymptotic expansions in damped wave equations. J. Diff. Equ. **150**, 42–97 (1998)

[6] Goldstein, S.: On diffusion by discontinuous movements and the telegraph equation. Q. J. Mech. Appl. Math. **4**, 129–156 (1951)

[7] Hadeler, K.P.: Reaction transport systems in biological modelling. In: Capasso, V., et al. (eds.) Mathematics Inspired by Biology, pp. 95–150. CIME Lectures 1997, Lecture Notes in Mathematics 1714. Springer (1999)

[8] Hamza, M.: Asymptotically self-similar solutions of the damped wave equation. Nonlinear Anal. **73**, 2897–2916 (2010)

[9] Hayashi, N., Kaikina, E.I., Naumkin, P.I.: Damped wave equation with super critical nonlinearities. Diff. Integral Equ. **17**, 637–652 (2004)

[10] Hayashi, N., Kaikina, E.I., Naumkin, P.I.: Damped wave equation in the subcritical case. J. Diff. Equ. **207**, 161–194 (2004)

[11] Hayashi, N., Kaikina, E.I., Naumkin, P.I.: Damped wave equation with a critical nonlinearity. Trans. Am. Math. Soc. **358**, 1165–1185 (2006)

[12] Hayashi, N., Kaikina, E.I., Naumkin, P.I.: Asymptotics for nonlinear damped wave equations with large initial data. Sib. Élektron. Mat. Izv. **4**, 249–277 (2007)

[13] Hayashi, N., Kaikina, E.I., Naumkin, P.I.: On the critical nonlinear damped wave equation with large initial data. J. Math. Anal. Appl. **334**, 1400–1425 (2007)

[14] Hayashi, N., Naumkin, P.I.: Damped wave equation with a critical nonlinearity in higher space dimensions, preprint

[15] Hayashi, N., Naumkin, P.I., Tominaga, M.: Remark on a weakly coupled system of nonlinear damped wave equations. J. Math. Anal. Appl. **428**, 490–501 (2015)

[16] Hosono, T., Ogawa, T.: Large time behavior and L^p-L^q estimate of solutions of 2-dimensional nonlinear damped wave equations. J. Diff. Equ. **203**, 82–118 (2004)

[17] Hsiao, L., Liu, T.-P.: Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping. Commun. Math. Phys. **43**, 599–605 (1992)

[18] Ikeda, M., Inui, T.: Small data blow-up of L^2 or H^1-solution for the semilinear Schrödinger equation without gauge invariance. J. Evol. Equ. **15**, 571–581 (2015)

[19] Ikeda, M., Ogawa, T.: Lifespan of solutions to the damped wave equation with a critical nonlinearity. J. Diff. Equ. **261**, 1880–1903 (2016)

[20] Ikeda, M., Wakasugi, Y.: A note on the lifespan of solutions to the semilinear damped wave equation. Proc. Am. Math. Soc. **143**, 163–171 (2015)

[21] Ikehata, R.: Diffusion phenomenon for linear dissipative wave equations in an exterior domain. J. Diff. Equ. **186**, 633–651 (2002)

[22] Ikehata, R., Miyaoka, Y., Nakatake, T.: Decay estimates of solutions for dissipative wave equations in $\mathbf{R}^N$ with lower power nonlinearities. J. Math. Soc. Japan **56**, 365–373 (2004)

[23] Ikehata, R., Nishihara, K.: Diffusion phenomenon for second order linear evolution equations. Studia Math. **158**, 153–161 (2003)

[24] Ikehata, R., Nishihara, K., Zhao, H.: Global asymptotics of solutions to the Cauchy problem for the damped wave equation with absorption. J. Diff. Equ. **226**, 1–29 (2006)

[25] Ikehata, R., Ohta, M.: Critical exponents for semilinear dissipative wave equations in $\mathbf{R}^N$. J. Math. Anal. Appl. **269**, 87–97 (2002)

[26] Ikehata, R., Tanizawa, K.: Global existence of solutions for semilinear damped wave equations in $\mathbf{R}^N$ with noncompactly supported initial data. Nonlinear Anal. **61**, 1189–1208 (2005)

[27] Ikehata, R., Todorova, G., Yordanov, B.: Optimal decay rate of the energy for wave equations with critical potential. J. Math. Soc. Japan **65**, 183–236 (2013)

[28] Ikehata, R., Todorova, G., Yordanov, B.: Wave equations with strong damping in Hilbert spaces. J. Diff. Equ. **254**, 3352–3368 (2013)

[29] Kac, M.: A stochastic model related to the telegrapher's equation. Rocky Mount. J. Math. **4**, 497–509 (1974)

[30] Karch, G.: Selfsimilar profiles in large time asymptotics of solutions to damped wave equations. Studia Math. **143**, 175–197 (2000)

[31] Kawakami, T., Ueda, Y.: Asymptotic profiles to the solutions for a nonlinear damped wave equation. Diff. Integral Equ. **26**, 781–814 (2013)

[32] Kawashima, S., Nakao, M., Ono, K.: On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term. J. Math. Soc. Japan **47**, 617–653 (1995)

[33] Kuiper, H.: Life span of nonnegative solutions to certain quasilinear parabolic Cauchy problems. Electron. J. Diff. Equ. **2003**, 1–11 (2003)

[34] Levine, H.A.: Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. Trans. Am. Math. Soc. **192**, 1–21 (1974)

[35] Li, T.-T.: Nonlinear heat conduction with finite speed of propagation. In: Proceedings of the China–Japan Symposium on Reaction Diffusion Equations and Their Applications to Computational Aspects, World Scientific, Singapore, (1997)

[36] Li, T.-T., Zhou, Y.: Breakdown of solutions to $\Box u + u_t = |u|^{1+\alpha}$. Discrete Contin. Dyn. Syst. **1**, 503–520 (1995)

[37] Marcati, P., Nishihara, K.: The L^p-L^q estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media. J. Diff. Equ. **191**, 445–469 (2003)

[38] Matsumura, A.: On the asymptotic behavior of solutions of semi-linear wave equations. Publ. Res. Inst. Math. Sci. **12**, 169–189 (1976)

[39] Nakao, M., Ono, K.: Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations. Math. Z. **214**, 325–342 (1993)

[40] Narazaki, T.: L^p-L^q estimates for damped wave equations and their applications to semi-linear problem. J. Math. Soc. Japan **56**, 585–626 (2004)

[41] Narazaki, T.: Global solutions to the Cauchy problem for the weakly coupled system of damped wave equations. Discrete Contin. Dyn. Syst. Suppl. 592–601 (2009)

[42] Narazaki, T.: Global solutions to the Cauchy problem for a system of damped wave equations. Diff. Integral Equ. **24**, 569–600 (2011)

[43] Narazaki, T., Nishihara, K.: Asymptotic behavior of solutions for the damped wave equation with slowly decaying data. J. Math. Anal. Appl. **338**, 803–819 (2008)

[44] Nishihara, K.: Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping. J. Diff. Equ. **131**, 171–188 (1996)

[45] Nishihara, K.: Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping. J. Diff. Equ. **137**, 384–395 (1997)

[46] Nishihara, K.: $L^p - L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application. Math. Z. **244**, 631–649 (2003)

[47] Nishihara, K.: L^p-L^q estimates for the 3-D damped wave equation and their application to the semilinear problem. Semin. Notes Math. Sci., Ibaraki Univ. **6**, 69–83 (2003)

[48] Nishihara, K.: Global asymptotics for the damped wave equation with absorption in higher dimensional space. J. Math. Soc. Japan **58**, 805–836 (2006)

[49] Nishihara, K.: Asymptotic behavior of solutions for a system of semilinear heat equations and the corresponding damped wave system. Osaka J. Math. **49**, 331–348 (2012)

[50] Nishihara, K., Wakasugi, Y.: Critical exponent for the Cauchy problem to the weakly coupled damped wave system. Nonlinear Anal. **108**, 249–259 (2014)

[51] Nishihara, K., Wakasugi, Y.: Global existence of solutions for a weakly coupled system of semilinear damped wave equations. J. Diff. Equ. **259**, 4172–4201 (2015)

[52] Nishiyama, H.: Remarks on the asymptotic behavior of the solution to damped wave equations. J. Diff. Equ. **261**, 3893–3940 (2016)

[53] Ogawa, T., Takeda, H.: Global existence of solutions for a system of nonlinear damped wave equations. Diff. Integral Equ. **23**, 635–657 (2010)

[54] Ogawa, T., Takeda, H.: Large time behavior of solutions for a system of nonlinear damped wave equations. J. Diff. Equ. **251**, 3090–3113 (2011)

[55] Ono, K.: Global existence and asymptotic behavior of small solutions for semilinear dissipative wave equations. Discrete Contin. Dyn. Syst. **9**, 651–662 (2003)

[56] Ono, K.: Global solvability and L^p decay for the semilinear dissipative wave equations in four and five dimensions. Funkcial. Ekvac. **49**, 215–233 (2006)

[57] Racke, R.: Non-homogeneous non-linear damped wave equations in unbounded domains. Math. Methods Appl. Sci. **13**, 481–491 (1990)

[58] Radu, P., Todorova, G., Yordanov, B.: Diffusion phenomenon in Hilbert spaces and applications. J. Diff. Equ. **250**, 4200–4218 (2011)

[59] Radu, P., Todorova, G., Yordanov, B.: The generalized diffusion phenomenon and applications. SIAM J. Math. Anal. **48**, 174–203 (2016)

[60] Sun, F.: Life span of blow-up solutions for higher-order semilinear parabolic equations. Electron. J. Diff. Equ. **2010**, 1–9 (2010)

[61] Sun, F., Wang, M.: Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping. Nonlinear Anal. **66**, 2889–2910 (2007)

[62] Takeda, H.: Global existence and nonexistence of solutions for a system of nonlinear damped wave equations. J. Math. Anal. Appl. **360**, 631–650 (2009)

[63] Takeda, H.: Higher-order expansion of solutions for a damped wave equation. Asymptot. Anal. **94**, 1–31 (2015)

[64] Takeda, H.: Large time behavior of solutions for a nonlinear damped wave equation. Commun. Pure Appl. Anal. **15**, 41–55 (2016)

[65] Todorova, G., Yordanov, B.: Critical exponent for a nonlinear wave equation with damping. J. Diff. Equ. **174**, 464–489 (2001)

[66] Todorova, G., Yordanov, B.: Weighted L^2-estimates for dissipative wave equations with variable coefficients. J. Diff. Equ. **246**, 4497–4518 (2009)

[67] Yang, H., Milani, A.: On the diffusion phenomenon of quasilinear hyperbolic waves. Bull. Sci. Math. **124**, 415–433 (2000)

[68] Zhang, Q.S.: A blow-up result for a nonlinear wave equation with damping: the critical case. C. R. Acad. Sci. Paris Sér. I Math. **333**, 109–114 (2001)

Masahiro Ikeda and Takahisa Inui
Department of Mathematics, Graduate School of Science
Kyoto University
Kyoto 606-8502
Japan
e-mail: mikeda@math.kyoto-u.ac.jp

Takahisa Inui
e-mail: inui@math.kyoto-u.ac.jp

Yuta Wakasugi
Department of Engineering for Production and Environment,
Graduate School of Science and Engineering
Ehime University
Matsuyama
Ehime 790-8577
Japan
e-mail: wakasugi.yuta.vi@ehime-u.ac.jp

Received: 4 October 2016.
Accepted: 24 January 2017.