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Choquard equations under confining external potentials

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Abstract. We consider the nonlinear Choquard equation

$$-\Delta u + Vu = \big(I_\alpha * |u|^p\big)|u|^{p-2}u \quad \text{in } \mathbb{R}^N$$

where $N \geq 1$, I_α is the Riesz potential integral operator of order $\alpha \in (0, N)$ and $p > 1$. If the potential $V \in C(\mathbb{R}^N; [0, +\infty))$ satisfies the confining condition

$$\liminf_{|x|\to+\infty} \frac{V(x)}{1 + |x|^{\frac{N+\alpha}{p}-N}} = +\infty,$$

and $\frac{1}{p} > \frac{N-2}{N+\alpha}$, we show the existence of a groundstate, of an infinite sequence of solutions of unbounded energy and, when $p \geq 2$ the existence of least energy nodal solution. The constructions are based on suitable weighted compact embedding theorems. The growth assumption is sharp in view of a Pohožaev identity that we establish.

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1. Introduction and main results

We are interested in the following class of Choquard equations

$$-\Delta u + Vu = \big(I_\alpha * |u|^p\big)|u|^{p-2}u \quad \text{in } \mathbb{R}^N \tag{$\mathcal{C}$}$$

in the Euclidean space $\mathbb{R}^N$ of dimension $N \geq 1$, where $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, N)$, which is defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \qquad \text{with } A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}}$$

where Γ denotes the classical Gamma function, and $p > 1$ is a given exponent.

When $N = 3$, $\alpha = 2$ and $p = 2$, the Eq. ($\mathcal{C}$) appears in several physical contexts, such as standing waves for the Hartree equation, the description by Pekar of the quantum physics of a polaron at rest [19], the description by Choquard of an electron trapped in its own hole [11] or the coupling of the Schrödinger equation under a classical Newtonian gravitational potential [6,9,10,15,20].

When the potential V is a positive constant function, groundstate solutions are known to exist [11,13,17] under the assumption that the exponent p satisfies the intercriticality condition

$$\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}. \tag{1.1}$$

Moreover, infinitely many geometrically distinct solutions can be constructed [13]. We refer the reader to the survey [18] for further discussion and references on the Choquard equation.

The goal of the present work is to examine how the presence of a confining potential V changes and possibly improves the situation. Our first result is that groundstates can exist in a *wider range of nonlinearities* when the external potential V is coercive enough.

Theorem 1.1. *Let $N \ge 1$, $\alpha \in (0, N)$, $p \in (1, +\infty)$ and $V \in C(\mathbb{R}^N; [0, +\infty))$. If*

$$\frac{1}{p} > \frac{N-2}{N+\alpha}$$

and if

$$\liminf_{|x| \to +\infty} \frac{V(x)}{1 + |x|^{\frac{N+\alpha}{p} - N}} = +\infty,$$

then the Choquard equation ($\mathcal{C}$) has a groundstate solution.

The solutions in Theorem 1.1 are groundstates in the sense that they minimize among nontrivial solutions the functional

$$J_p(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p;$$

solutions of the Choquard equation ($\mathcal{C}$) are formally critical points of the functional J_p.

If u is a groundstate, it can be observed that the function $|u|$ is also a groundstate, which by classical regularity theory is smooth and by the strong maximum principle does not vanish on $\mathbb{R}^N$. Therefore, the function u has constant sign over $\mathbb{R}^N$.

A striking feature of Theorem 1.1 is that the condition $p > \frac{N+\alpha}{N}$ in (1.1) can be loosened when V grows fast enough at infinity. In particular, if $V(x) = |x|^\beta$ with $\beta > 0$, one can take $p > \max\{\frac{N+\alpha}{N+\beta}, 1\}$. This growth assumption is sharp. Indeed, if $V(x) = |x|^\beta$ and $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N) \cap H^1_V(\mathbb{R}^N)$

solves the Choquard equation ($\mathcal{C}$) and then we have the Pohožaev identity (Theorem 5.1)

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2+\frac{N+\beta}{2}\int_{\mathbb{R}^N}V|u|^2=\frac{N+\alpha}{2p}\int_{\mathbb{R}^N}\bigl(I_\alpha*|u|^p\bigr)|u|^p,$$

provided that the integral on the right-hand side is finite, this implies that if $p\le\frac{N+\alpha}{N+\beta}$ the problem cannot have finite energy classical solutions. This condition cannot be satisfied when p does not satisfy the assumption of Theorem 1.1.

Theorem 1.1 can be thought as counterpart for the nonlocal Choquard equation of results for the nonlinear Schrödinger equation with a coercive potential [21]. Radial positive solutions for the Choquard equation ($\mathcal{C}$) had already been obtained in the quadratic case $p=2$ when the potential V is radial and radially increasing [3].

The core of the proof of Theorem 1.1 is to obtain the well-definiteness, the continuity and the compactness properties of the Riesz potential energy term in the definition of the functional J_p. This is done by combining a suitable Sobolev-type compact weighted embedding theorem together with the weighted estimates for fractional integrals of Stein and Weiss [22], which are a weighted counterpart of the more classical Hardy–Littlewood–Sobolev inequality.

We now turn on to the question whether the Choquard equation has, under the conditions of Theorem 1.1 more solutions. This is indeed the case and there are infinitely many solutions.

Theorem 1.2. *Let $N\ge 1$, $\alpha\in(0,N)$, $p\in(1,+\infty)$ and $V\in C(\mathbb{R}^N;[0,+\infty))$. If*

$$\frac{1}{p}>\frac{N-2}{N+\alpha}$$

and if

$$\liminf_{|x|\to+\infty}\frac{V(x)}{1+|x|^{\frac{N+\alpha}{p}-N}}=+\infty,$$

then the Choquard equation ($\mathcal{C}$) has an infinite sequence of solutions whose energies do not remain bounded.

The solutions are constructed with the fountain theorem [2] (see also [25],Theorem 3.6); thanks to the same weighted embedding and fractional integral estimates as in the proof of Theorem 1.1, the Palais–Smale condition for the functional J_p can be established by classical arguments.

Finally we investigate the question whether the Choquard equation ($\mathcal{C}$) has a least energy sign-changing solution, that is, a solution that changes sign and which minimizes the functional J_p among such solutions.

A natural way to construct such solutions is to minimize, as for the local semilinear elliptic problems [4,5,14,23], the functional on the *Nehari nodal set*:

$$\bigl\{u\in H^1(\mathbb{R}^N)\mid u^+\ne 0,\ u^-\ne 0,\ \langle J_p'(u),u^+\rangle=0\ \text{ and }\ \langle J_p'(u),u^-\rangle=0\bigr\},$$

where $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$. Such solutions of ($\mathcal{C}$) have been constructed when $V = 1$ and

$$\frac{N-2}{N+\alpha} < \frac{1}{p} \leq \frac{1}{2};$$

they were obtained by a new minimax principle and concentration-compactness method, and the minimization problem on the Nehari nodal set was observed to be degenerate when $p < 2$ [7,8].

Sign-changing solutions have been constructed for the Schrödinger–Poisson system in $\mathbb{R}^3$ in which a nonlocal nonlinearity appears with opposite sign [1,24].

Now, we are in a position to state our main results on the nodal solutions of equation ($\mathcal{C}$):

Theorem 1.3. *Let $N \geq 1$, $\alpha \in ((N-4)_+, N)$, $p \in [2, +\infty)$ and $V \in C(\mathbb{R}^N; [0, +\infty))$. If*

$$\frac{1}{p} > \frac{N-2}{N+\alpha}$$

and if

$$\liminf_{|x|\to+\infty} V(x) = +\infty,$$

then the Choquard equation ($\mathcal{C}$) has at least one least-energy sign-changing solution.

As before, the assumptions provide us with a functional with nice compactness properties. The situation is still more challenging than for a local semilinear elliptic equation on a bounded domain because some of the usual properties of the local nonlinear Schrödinger functional on negative and positive parts fail: in general $J_p(u) \neq J_p(u^+) + J_p(u^-)$, and $\langle J_p'(u), u^\pm\rangle \neq \langle J_p'(u^\pm), u^\pm\rangle$.

Theorem 1.3 was stated by Ye [27, Theorem 1.3]; it seems that his argument unfortunately overlooks the crucial question whether the proposed solution u does change sign, which is quite delicate when $p = 2$ (see the proof of Theorem 1.3 and [7]). We propose here a proof relying on the same compact embedding theorems as Theorems 1.1 and 1.2.

When $p < 2$, we prove that the energy functional does not achieve its minimum on the Nehari nodal set (see Proposition 4.4).

The remainder of this paper is organized as follows. In Sect. 2, we first prove a weighted embedding theorem, then show that the function J_p is of C^1 on the natural Sobolev space $H^1_V(\mathbb{R}^N)$ and satisfies the Palais–Smale condition. The proof of our main results will be postponed to the next two Sects. 3 and 4. In the last Sect. 5, we will establish Pohožaev identity responding to Eq. ($\mathcal{C}$), with which we can deduce some nonexistence results.

2. Function spaces and weighted embedding theorems

The linear part of the Choquard equation ($\mathcal{C}$) naturally induces the Euclidean norm

$$\|u\|_V := \left(\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2\right)^{\frac{1}{2}}.$$

We define $H^1_V(\mathbb{R}^N)$ as the Hilbert space obtained by completion of the set of smooth test functions $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_V$. We first establish some embedding theorem from $H^1_V(\mathbb{R}^N)$ into the weighted space $L^2(|x|^\gamma\,\mathrm{d}x;\mathbb{R}^N)$ which is defined for $\gamma \geq 0$ by

$$L^2(|x|^\gamma\,\mathrm{d}x;\mathbb{R}^N) := \left\{u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable and } \int_{\mathbb{R}^N} |x|^\gamma |u(x)|^2\,\mathrm{d}x < +\infty\right\}.$$

We begin by establishing the following embedding theorem.

Proposition 2.1. *Let $N \geq 1$ and $\gamma \in [0, +\infty)$. If $V \in C(\mathbb{R}^N; [0, +\infty))$ satisfies*

$$\liminf_{|x|\to+\infty} \frac{V(x)}{|x|^\gamma} > 0,$$

then there exists a constant $C > 0$ such that for every $u \in H^1_V(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |x|^\gamma |u(x)|^2\,\mathrm{d}x \leq C \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2.$$

If moreover,

$$\lim_{|x|\to+\infty} \frac{V(x)}{|x|^\gamma} = +\infty,$$

then the corresponding embedding is compact. In particular, the embedding $H^1_V(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for any q with $\frac{1}{q} \in (\frac{1}{2} - \frac{1}{N}, \frac{1}{2})$ if $\gamma = 0$.

Proof. Given $\lambda \in (0, +\infty)$ such that

$$\lambda < \liminf_{|x|\to+\infty} \frac{V(x)}{|x|^\gamma},$$

there exists $\kappa > 0$ sufficiently large so that if $x \in \mathbb{R}^N \setminus B(0, \frac{\kappa}{2})$, we have $V(x) \geq \lambda |x|^\gamma$. (Here and in the sequel, we use the notation $B(a, r)$ for the ball centered at a of radius r and in $\mathbb{R}^N$.) By integration, we have in particular,

$$\lambda \int_{\mathbb{R}^N \setminus B(0,\frac{\kappa}{2})} |x|^\gamma |u(x)|^2\,\mathrm{d}x \leq \int_{\mathbb{R}^N \setminus B(0,\frac{\kappa}{2})} V|u|^2. \tag{2.1}$$

We take a function $\varphi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$, $\varphi(x) = 1$ for every $x \in B(0, \frac{\kappa}{2})$ and $\varphi(x) = 0$ for every $x \in \mathbb{R}^N \setminus B(0, \kappa)$. Then, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^\gamma |u(x)|^2\,\mathrm{d}x &\leq \kappa^\gamma \int_{B(0,\kappa)} \varphi^2 |u|^2 + \int_{\mathbb{R}^N \setminus B(0,\frac{\kappa}{2})} |x|^\gamma |u(x)|^2\,\mathrm{d}x \\
&\leq C_1 \kappa^\gamma \int_{B(0,\kappa)} |\nabla(\varphi u)|^2 + \int_{\mathbb{R}^N \setminus B(0,\frac{\kappa}{2})} |x|^\gamma |u(x)|^2\,\mathrm{d}x \\
&\leq 2C_1 \kappa^\gamma \int_{B(0,\kappa)} |\nabla u|^2 + 2C_1 \kappa^\gamma \int_{B(0,\kappa)\setminus B(0,\frac{\kappa}{2})} |\nabla \varphi|^2 |u|^2
\end{aligned}$$

$$+\int_{\mathbb{R}^N\setminus B(0,\frac{\kappa}{2})} |x|^\gamma |u(x)|^2 \,\mathrm{d}x$$
$$\leq 2C_1\kappa^\gamma \int_{\mathbb{R}^N} |\nabla u|^2 + \left(2C_1\kappa^\gamma \frac{\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^N)}}{(\kappa/2)^\gamma} + 1\right) \int_{\mathbb{R}^N\setminus B(0,\frac{\kappa}{2})} |x|^\gamma |u(x)|^2 \,\mathrm{d}x.$$

where the constant C_1 comes from the Poincaré inequality with Dirichlet boundary conditions on the ball $B(0,\kappa)$, which is independent of the function u. We now apply the estimate (2.1) to the second term to obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^\gamma |u(x)|^2 \,\mathrm{d}x &\leq 2C_1\kappa^\gamma \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{2^{\gamma+1}C_1\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^N)} + 1}{\lambda} \int_{\mathbb{R}^N} V|u|^2 \\ &\leq \max\left\{2C_1\kappa^\gamma, \frac{2^{\gamma+1}C_1\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^N)}+1}{\lambda}\right\} \int_{\mathbb{R}^N} |\nabla u|^2+V|u|^2,\end{aligned} \tag{2.2}$$

and the first part of the conclusion follows.

For the compactness, without loss of generality, let $(v_n)_{n\in\mathbb{N}}$ be a sequence such that $v_n \rightharpoonup 0$ weakly as $n\to\infty$ in $H^1_V(\mathbb{R}^N)$. In particular, the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1_V(\mathbb{R}^N)$. We are going to prove that $v_n \to 0$ strongly as $n\to\infty$ in $L^2(|x|^\gamma \,\mathrm{d}x;\mathbb{R}^N)$. By assumption, for every $\varepsilon>0$, there exists $R_1>0$, such that

$$\left(\sup_{|x|\geq R_1} \frac{|x|^\gamma}{V(x)}\right)\|v_n\|_V^2 \leq \varepsilon.$$

Since $\gamma\geq 0$, for any fixed $R>0$, the weighted space $L^2(|x|^\gamma \,\mathrm{d}x; B(0,R))$ is embedded into the classical Lebesgue space $L^2(B(0,R))$ defined on bounded domain $B(0,R)$. By the classical Sobolev embedding theorem, $v_n\to 0$ strongly in $L^2(B(0,R))$ as $n\to\infty$. Therefore, for fixed $R\geq R_1$, there exists $N_1>0$ such that

$$\int_{B(0,R)} |x|^\gamma |v_n(x)|^2 \,\mathrm{d}x \leq \varepsilon \quad \text{for each } n\geq N_1.$$

Then for $n\geq N_1$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^\gamma |v_n(x)|^2 \,\mathrm{d}x &= \int_{B(0,R)} |x|^\gamma |v_n(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N\setminus B(0,R)} |x|^\gamma |v_n(x)|^2 \,\mathrm{d}x \\ &\leq \int_{B(0,R)} |x|^\gamma |v_n(x)|^2 \,\mathrm{d}x \\ &\quad + \left(\sup_{|x|\geq R} \frac{|x|^\gamma}{V(x)}\right) \int_{\mathbb{R}^N\setminus B(0,R)} V(x)|v_n(x)|^2 \,\mathrm{d}x \\ &\leq \varepsilon + \left(\sup_{|x|\geq R} \frac{|x|^\gamma}{V(x)}\right)\|v_n\|_V^2 \leq 2\varepsilon.\end{aligned}$$

Finally, we interpolate to conclude our proof of the compact embedding $H^1_V \subset L^q(\mathbb{R}^N)$ with $\frac{1}{2}-\frac{1}{N} < \frac{1}{q} < \frac{1}{2}$ for the case $\gamma=0$. Take $\bar{q}=\frac{2N}{N-2}$ if

$N \geq 3$, or any $\bar{q} \in (q, +\infty)$ if $N = 1, 2$, there exists $b \in (0, 1)$ such that

$$\frac{1}{q} = \frac{b}{2} + \frac{1-b}{\bar{q}},$$

it follows that as $n \to \infty$,

$$\|v_n\|_{L^q} \leq \|v_n\|_{L^2}^b \|v_n\|_{L^{\bar{q}}}^{1-b} \leq C^{(1-b)} \|v_n\|_{L^2}^b \|v_n\|_V^{1-b} \to 0.$$

□

With the aid of the Stein–Weiss inequality [22], we show that the nonlocal Riesz potential energy term $\mathcal{G}_p$ of the functional J_p is well-defined and prove that the functional J_p is of class C^1 on the weighted Sobolev space $H^1_V(\mathbb{R}^N)$. Finally, thanks to the compact embedding result, we close this section by verifying that the functional J_p satisfies the Palais–Smale condition.

Proposition 2.2. *Let $N \geq 1$ and $\alpha \in (0, N)$. If $V \in C(\mathbb{R}^N; [0, +\infty))$ satisfies*

$$\liminf_{|x| \to +\infty} \frac{V(x)}{1 + |x|^{\frac{N+\alpha}{p} - N}} > 0,$$

then the mappings $u \in H^1_V(\mathbb{R}^N) \longmapsto I_{\alpha/2} * |u|^p \in L^2(\mathbb{R}^N)$ and

$$u \in H^1_V(\mathbb{R}^N) \longmapsto \big(I_\alpha * |u|^p\big)|u|^{p-2}u \in (H^1_V(\mathbb{R}^N))'$$

are continuous for $p > 1$ and $\frac{1}{p} > \frac{N-2}{N+\alpha}$.
If moreover

$$\liminf_{|x| \to +\infty} \frac{V(x)}{1 + |x|^{\frac{N+\alpha}{p} - N}} = +\infty,$$

the above mappings are weak to strong type, that is, they map weakly converging sequence to strongly converging sequence.

Here and in the sequel, X' denotes the topological dual space of the normed space X.

Proof of Proposition 2.2. In the case $p > \frac{N+\alpha}{N}$, the well-definiteness and the continuity follow from the continuous embedding $H^1_V(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$, the classical Sobolev embedding and the Hardy–Littlewood–Sobolev inequality as in the case where V is constant [16]. If moreover $\liminf_{|x| \to +\infty} V(x)/(1 + |x|^{\frac{N+\alpha}{p} - N}) = +\infty$, the embedding $H^1_V(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ is compact for every $q \in [2, +\infty)$ with $\frac{1}{q} > \frac{1}{2} - \frac{1}{N}$ by Proposition 2.1, and then the weak to strong continuity property follows.

We assume now that $p \leq \frac{N+\alpha}{N} < 2$. We first show that the nonlocal term $\mathcal{G}_p$ of the functional J_p is well defined on the space $H^1_V(\mathbb{R}^N)$. By the Stein–Weiss inequality [22], together with the semi-group identity for the Riesz potential $I_\alpha = I_{\alpha/2} * I_{\alpha/2}$ [12], we have, since $\frac{2}{p} > 1$,

$$\mathcal{G}_p(u) := \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p = \int_{\mathbb{R}^N} |I_{\frac{\alpha}{2}} * |u|^p|^2 \leq C\left(\int_{\mathbb{R}^N} |x|^{\frac{N+\alpha}{p} - N} |u(x)|^2 \,\mathrm{d}x\right)^p.$$

In view of the above continuous embedding Proposition 2.1, the functional $\mathcal{G}_p$ is well defined on $H^1_V(\mathbb{R}^N)$. By Proposition 2.1 again, the superposition operator

$$u \in L^2\big(|x|^{\frac{N+\alpha}{p}-N}\,\mathrm{d}x;\mathbb{R}^N\big) \longmapsto |u|^p \in L^{\frac{2}{p}}\big(|x|^{\frac{N+\alpha}{p}-N}\,\mathrm{d}x;\mathbb{R}^N\big) \tag{2.3}$$

is continuous. Taking into account the Stein–Weiss inequality [22] again, the Riesz potential integral operator

$$f \in L^{\frac{2}{p}}\big(|x|^{\frac{N+\alpha}{p}-N}\,\mathrm{d}x;\mathbb{R}^N\big) \longmapsto I_{\alpha/2} * f \in L^2(\mathbb{R}^N) \tag{2.4}$$

is a continuous linear operator. Thus the conclusion follows and the stronger conclusion follows directly from the compact embedding Proposition 2.1. In fact, suppose that $u_n \rightharpoonup u$ weakly in $H^1_V(\mathbb{R}^N)$, by compactness, we know that, as $n \to \infty$

$$u_n \to u \text{ strongly in } L^2\big(|x|^{\frac{N+\alpha}{p}-N}\,\mathrm{d}x;\mathbb{R}^N\big),$$

thus, up to a subsequence, $u_n \to u$ almost everywhere in $\mathbb{R}^N$. From the continuity of the map defined by (2.3), we have that, as $n \to \infty$

$$\begin{aligned} &|u_n|^p \to |u|^p \text{ strongly in } L^{\frac{2}{p}}\big(|x|^{\frac{N+\alpha}{p}-N}\,\mathrm{d}x;\mathbb{R}^N\big),\\ &\quad\text{and}\quad |u_n|^{p-2}u_n \to |u|^{p-2}u \text{ strongly in } L^{\frac{2}{p-1}}(|x|^{\frac{N+\alpha}{p}-N}\,\mathrm{d}x;\mathbb{R}^N). \end{aligned}$$

By the Stein–Weiss inequality [22], we deduce that, as $n \to \infty$

$$I_\alpha * |u_n|^p \to I_\alpha * |u|^p \text{ strongly in } L^{\frac{2}{2-p}}(|x|^{-\frac{N+\alpha-pN}{2-p}}\,\mathrm{d}x;\mathbb{R}^N),$$

thus

$$\begin{aligned} &\big(I_\alpha * |u_n|^p\big)|u_n|^{p-2}u_n \to \big(I_\alpha * |u|^p\big)|u|^{p-2}u\\ &\quad\text{strongly in}\quad L^2\big(|x|^{N-\frac{N+\alpha}{p}}\,\mathrm{d}x;\mathbb{R}^N\big) = \Big(L^2\big(|x|^{\frac{N+\alpha}{p}-N}\,\mathrm{d}x;\mathbb{R}^N\big)\Big)'. \end{aligned}$$

By the continuous embedding results again, we have

$$\big(I_\alpha * |u_n|^p\big)|u_n|^{p-2}u_n \to \big(I_\alpha * |u|^p\big)|u|^{p-2}u \text{ strongly in } \big(H^1_V(\mathbb{R}^N)\big)'.$$

□

The compact embedding theorems imply straightforwardly that the functional J_p is well-defined and satisfies the Palais–Smale condition.

Lemma 2.3. *Let $N \geq 1$, $\alpha \in (0,N)$ and $p > 1$. If $\frac{1}{p} > \frac{N-2}{N+\alpha}$ and if*

$$\liminf_{|x|\to+\infty} \frac{V(x)}{1+|x|^{\frac{N+\alpha}{p}-N}} = +\infty,$$

then the functional J_p is of class C^1 on $H^1_V(\mathbb{R}^N)$ and satisfies the Palais–Smale condition, that is, any sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1_V(\mathbb{R}^N)$ such that $(J_p(u_n))_{n\in\mathbb{N}}$ is bounded, and $J_p'(u_n) \to 0$ strongly in $(H^1_V(\mathbb{R}^N))'$ as $n \to \infty$ has a subsequence that converges strongly in $H^1_V(\mathbb{R}^N)$.

Proof. To prove that the functional J_p is of continuously differentiable, we only need to consider the nonlocal term $\mathcal{G}_p$ of J_p, that is,

$$\mathcal{G}_p(u) = \int_{\mathbb{R}^N} \bigl(I_\alpha * |u|^p\bigr)|u|^p = \int_{\mathbb{R}^N} \bigl|I_{\alpha/2} * |u|^p\bigr|^2.$$

By Proposition 2.2, the functional $\mathcal{G}_p$ is continuous on $H^1_V(\mathbb{R}^N)$ and then the functional J_p is also continuous. For the continuous differentiability, we observe that by Proposition 2.2 again the map $\mathcal{G}_p$ is Gâteaux-differentiable on $H^1_V(\mathbb{R}^N)$ and hence it is continuously Fréchet differentiable on that space [25, Proposition 1.3] and the first part of the conclusion follows.

Suppose now that $(u_n)_{n\in\mathbb{N}}$ is a Palais–Smale sequence for the functional J_p, that is, as $n \to \infty$

$$(J_p(u_n))_{n\in\mathbb{N}} \text{ is bounded} \qquad \text{and} \qquad J_p'(u_n) \to 0 \text{ strongly in } \bigl(H^1_V(\mathbb{R}^N)\bigr)'.$$

First, we observe that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in the space $H^1_V(\mathbb{R}^N)$, because

$$\left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|_V^2 = J_p(u_n) - \frac{1}{2p}\langle J_p'(u_n), u_n\rangle = J_p(u_n) + o(\|u_n\|_V).$$

Up to a subsequence, we can assume that the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly to some function $u \in H^1_V(\mathbb{R}^N)$. By Proposition 2.2, we have $\mathcal{G}_p'(u_n) \to \mathcal{G}_p'(u)$ as $n \to \infty$ strongly in $(H^1_V(\mathbb{R}^N))'$ — that is, the map $\mathcal{G}_p'$ is weak to strong type. It follows then that, as $n \to \infty$,

$$\|u_n - u\|_V^2 = \langle J_p'(u_n) - J_p'(u), u_n - u\rangle + \frac{1}{2p}\langle \mathcal{G}_p'(u_n) - \mathcal{G}_p'(u), (u_n - u)\rangle \to 0,$$

which concludes the proof. □

3. Ground states and multiplicity solutions

We first give a proof of Theorem 1.1 by minimization of the Sobolev quotient and then prove the multiplicity result Theorem 1.2 by the fountain theorem at the end of this section.

Proof of Theorem 1.1. We are going to find a minimizer $u \in H^1_V(\mathbb{R}^N)$ for the infimum θ_p, defined by

$$\theta_p := \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \mid u \in \mathrm{H}^1_\mathrm{V}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \bigl(I_\alpha * |u|^p\bigr)|u|^p = 1\right\};$$

once this will be done a nontrivial solution v of equation ($\mathcal{C}$) will be obtained after a rescaling, more precisely, by taking $v = \theta_p^{1/(2p-2)} u$.

Let $(u_n)_{n\in\mathbb{N}}$ in $H^1_V(\mathbb{R}^N)$ be a minimizing sequence for θ_p, that is,

$$\|u_n\|_V^2 \to \theta_p \quad \text{and} \quad \int_{\mathbb{R}^N} \bigl(I_\alpha * |u_n|^p\bigr)|u_n|^p \,\mathrm{d}x = 1.$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_V(\mathbb{R}^N)$, we can assume without loss of generality that $u_n \rightharpoonup u$ weak in $H^1_V(\mathbb{R}^N)$ as $n \to \infty$. By the weakly lower semi-continuity of the norm, we know that

$$\|u\|_V \leq \liminf_{n\to\infty} \|u_n\|_V. \tag{3.1}$$

On the other hand, by Proposition 2.2, we deduce that

$$I_{\alpha/2} * |u_n|^p \to I_{\alpha/2} * |u|^p \text{ strongly in } L^2(\mathbb{R}^N),$$

and thus, as $n \to \infty$,

$$\mathcal{G}_p(u_n) = \int_{\mathbb{R}^N} \left|I_{\alpha/2} * |u_n|^p\right|^2 \to \int_{\mathbb{R}^N} \left|I_{\alpha/2} * |u|^p\right|^2 = \mathcal{G}_p(u).$$

Therefore,

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, \mathrm{d}x = 1,$$

which leads to $u \neq 0$ and $\|u\|_V^2 \geq \theta_p$ by the definition of θ_p. This, together with the inequality (3.1), implies that $\|u\|_V^2 = \theta_p$. Therefore, u is a minimizer for θ_p. □

Remark 3.1. In fact, the nontrivial solution obtained above is a positive solution with least energy, that is, a groundstate, see [25].

In the remainder of this section, we prove Theorem 1.2 on the multiplicity results by the Bartsch's fountain theorem [2]. For the convenience of the reader we recall its statement following [25, Theorem 3.6].

Theorem 3.2. (Fountain Theorem, Bartsch, 1993) *Let X be a Banach space, let G be a group acting isometrically on $X = \overline{\oplus_{j\in\mathbb{N}} X_j}$. Let $Y_k = \oplus_{j=0}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Assume that for every $k \in \mathbb{N}$, Y_k is invariant under the action of G and that if $U \subset Y_k$ is a bounded neighbourhood of 0 and if $f : \partial U \to Y_{k-1}$ is continuous, then there exists $x \in \partial U$ such that $f(x) = 0$. If $J \in C^1(X)$ is invariant under the action of G and satisfies the Palais–Smale condition, and if there exists $\rho_k > r_k > 0$, such that for every $k \in \mathbb{N}$,*

$$\sup_{\substack{u\in Y_k\\ \|u\|=\rho_k}} J_p(u) \leq 0 \tag{A_1}$$

and

$$\lim_{k\to\infty} \inf_{\substack{u\in Z_k\\ \|u\|=r_k}} J_p(u) = +\infty, \tag{A_2}$$

then the functional J has an unbounded sequence of critical values.

Proof of Theorem 1.2. We consider the action of the group $\mathbb{Z}/2\mathbb{Z} = \{-1, 1\}$ on the space $H^1_V(\mathbb{R}^N)$ defined for $g \in \{-1, 1\}$ and $u \in H^1_V(\mathbb{R}^N)$ by multiplication. This action is continuous and isometric: for every $g \in \{-1, 1\}$ and $u \in H^1_V(\mathbb{R}^N)$, $\|gu\|_V = \|u\|_V$. The functional J_p is invariant under this action of the group $\{-1, 1\}$ since it is an even functional. Moreover, by the Borsuk–Ulam theorem [25, Theorem D.17], every continuous odd map $f : \partial U \to \mathbb{R}^{k-1}$

has a zero, where $k \geq 2$ and U is an open bounded symmetric neighborhood of 0 in $\mathbb{R}^k$.

We choose an orthonormal basis $(e_j)_{j\geq 0}$ of $H^1_V(\mathbb{R}^N)$ and define $X_j := \mathbb{R}e_j$. To apply the fountain theorem, we still need to find suitable r_k and ρ_k. Since the unit sphere in the finite dimensional linear subspace Y_k is a compact set, the continuous functional $\mathcal{G}_p$ achieves a positive minimum σ_k on that set. On the finite-dimensional space Y_k, for any $u \in Y_k$ with $\|u\|_V = \rho_k$,

$$J_p(u) = \frac{1}{2}\|u\|_V^2 - \frac{1}{2p}\|u\|_V^{2p}\, \mathcal{G}_p\Big(\frac{u}{\|u\|_V}\Big) \leq \frac{1}{2}\rho_k^2 - \frac{\sigma_k}{2p}\rho_k^{2p}.$$

Thus the condition (A_1) follows for sufficiently large ρ_k since $p > 1$.

We now turn to (A_2). We define

$$\beta_k := \sup\Big\{\|I_{\alpha/2} * |u|^p\|_{L^2} \mid u \in Z_k \text{ and } \|u\|_V = 1\Big\}.$$

We show that $\beta_k \to 0$ as $k \to \infty$ with minor modification following [25, Proof of Lemma 3.8]. We observe that $0 < \beta_{k+1} \leq \beta_k$, so that $\beta_k \to \beta \geq 0$, as $k \to \infty$. By the definition of β_k, we know that for every $k \geq 0$, there exists $u_k \in Z_k$ such that

$$\|u_k\|_V = 1 \quad \text{ and } \quad \big\|I_{\alpha/2} * |u_k|^p\big\|_{L^2} > \frac{\beta_k}{2}.$$

By definition of Z_k, we have $u_k \rightharpoonup 0$ weakly in $H^1_V(\mathbb{R}^N)$. Thus by the weak to strong convergence property of Proposition 2.2, we deduce that $I_{\alpha/2} * |u_k|^p \to I_{\alpha/2} * |u|^p$ as $k \to \infty$ strongly in $L^2(\mathbb{R}^N)$. Therefore $\beta = 0$. For every $u \in Z_k$,

$$J_p(u) \geq \frac{1}{2}\|u\|_V^2 - \frac{\beta_k^2}{2p}\|u\|_V^{2p}.$$

We set $r_k := 1/(\beta_k)^{1/(p-1)}$, then we have

$$J_p(u) \geq \left(\frac{1}{2} - \frac{1}{2p}\right)\frac{1}{\beta_k^{\frac{2}{p-1}}} \to +\infty \text{ as } k \to \infty.$$

Since the conditions (A_1) and (A_2) hold, it follows from the fountain theorem that J_p has an unbounded sequence of critical values. □

4. Existence of nodal solution with least energy

In this section, we shall prove the existence of nodal solutions by minimization method on the Nehari nodal set defined by

$$\mathcal{M}_p = \big\{u \in H^1_V(\mathbb{R}^N) \mid u^+ \neq 0 \neq u^- \text{ and } \langle J_p'(u), u^+\rangle = \langle J_p'(u), u^-\rangle = 0\big\}.$$

It is obvious that all the sign-changing solutions are contained in $\mathcal{M}_p$. We are going to study whether it is possible to obtain a least energy nodal solution by finding a minimizer for

$$c_p := \inf_{u\in\mathcal{M}_p} J_p(u).$$

The following lemma plays an essential role in showing the existence of the minimizer for c_p. The proof follows the strategy of [8, Proof of proposition 3.2], [27, Lemma 3.2].

Lemma 4.1. *Let $p > 2$. For any $u \in H^1_V(\mathbb{R}^N)$ with $u^\pm \neq 0$, there exists a unique pair $(\bar{t}, \bar{s}) \in (0, +\infty)^2$ such that $\bar{t}u^+ + \bar{s}u^- \in \mathcal{M}_p$ and if $u \in \mathcal{M}_p$, then $J_p(u) \geq J_p(tu^+ + su^-)$ for any $t \geq 0, s \geq 0$.*

Proof. We define the function $\Phi_p : [0, +\infty)^2 \to \mathbb{R}$ for each $s, t \in [0, +\infty)$ by

$$\Phi_p(t,s) := J_p(t^{\frac{1}{p}}u^+ + s^{\frac{1}{p}}u^-) = \frac{t^{\frac{2}{p}}}{2}\|u^+\|_V^2 + \frac{s^{\frac{2}{p}}}{2}\|u^-\|_V^2 - \frac{1}{2p}\int_{\mathbb{R}^N} |I_{\alpha/2} * (t|u^+|^p + s|u^-|^p)|^2, \tag{4.1}$$

where $u = u^+ + u^-$ with $u^\pm \neq 0$. The condition $t^{\frac{1}{p}}u^+ + s^{\frac{1}{p}}u^- \in \mathcal{M}_p$ is equivalent to $\nabla\Phi_p(t,s) = 0$ with $t > 0, s > 0$. It is sufficient to prove that there exists a unique critical point for the function Φ_p on the domain $(0, +\infty)^2$.

By the definition of Φ_p,

$$\Phi_p(t,s) \leq \frac{t^{\frac{2}{p}}}{2}\|u^+\|_V^2 + \frac{s^{\frac{2}{p}}}{2}\|u^-\|_V^2 - \frac{t^2}{2p}\int_{\mathbb{R}^N} \left|I_{\alpha/2} * |u^+|^p\right|^2 - \frac{s^2}{2p}\int_{\mathbb{R}^N} \left|I_{\alpha/2} * |u^-|^p\right|^2,$$

from which we can get that

$$\lim_{t^2+s^2 \to +\infty} \Phi_p(t,s) \leq \lim_{t^2+s^2 \to +\infty} \left(\frac{t^{\frac{2}{p}}}{2}\|u^+\|_V^2 - \frac{t^2}{2p}\int_{\mathbb{R}^N} \left|I_{\alpha/2} * |u^+|^p\right|^2 + \frac{s^{\frac{2}{p}}}{2}\|u^-\|_V^2 - \frac{s^2}{2p}\int_{\mathbb{R}^N} \left|I_{\alpha/2} * |u^-|^p\right|^2\right) = -\infty.$$

Therefore, Φ_p must have at least one global maximum point on $[0, \infty) \times [0, \infty)$.

Since the quadratic form

$$(t,s) \mapsto \int_{\mathbb{R}^N} \left|I_{\alpha/2} * (t|u^+|^p + s|u^-|^p)\right|^2$$

is positive definite, the function Φ_p is strictly concave. In particular, any critical point is a maximum point and there is at most one maximum point.

The conclusion follows provided that we can rule out that this maximum point is on the boundary of $[0, +\infty)^2$. Suppose that $(t_0, 0)$ with $t_0 \geq 0$ is the global maximum point of Φ_p, then $\frac{\partial \Phi_p(t_0,0)}{\partial t} \leq 0$. However, a direct computation shows that

$$\frac{\partial \Phi_p(t_0,s)}{\partial s}\Big|_{s=0} = +\infty,$$

Similarly, Φ_p can not achieve its global maximum on $(0, s)$ for any $s \geq 0$. □

Proof of Theorem 1.3. when $p > 2$. For the case of $p > 2$, our proof, in fact, relies on the compact embedding: $H^1_V(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ with $\frac{1}{q} \in (\frac{1}{2} - \frac{1}{N}, \frac{1}{2}]$ and can be carried out into two steps. First, we show that $c_p > 0$ is attained by some minimizer $w \in \mathcal{M}_p$. Then, we prove the minimizer w for c_p is indeed a critical point of J_p, thus being a nodal solution of ($\mathcal{C}$).

Step 1 *The energy level $c_p > 0$ is achieved by some minimizer $w \in \mathcal{M}_p$.*

Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for c_p in $\mathcal{M}_p$, namely, $\lim_{n\to\infty} J_p(u_n) = c_p$. We first observe that

$$\Big(\frac{1}{2} - \frac{1}{2p}\Big)\|u_n\|^2 = J_p(u_n) - \frac{1}{2p}\langle J_p'(u_n), u_n\rangle = J_p(u_n) \to c_p, \tag{4.2}$$

from which we know that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H_V^1(\mathbb{R}^N)$ and so are the sequences $(u_n^{\pm})_{n\in\mathbb{N}}$. Passing to a subsequence, there exist $u^{\pm} \in H_V^1(\mathbb{R}^N)$ such that

$$u_n^{\pm} \rightharpoonup u^{\pm} \text{ weakly in } H_V^1(\mathbb{R}^N).$$

By the constraint $\langle J_p'(u_n), u_n^{\pm}\rangle = 0$, and by the Hardy–Littlewood–Sobolev inequality [[12],Theorem 4.3], which can be seen as a special case of the Stein–Weiss inequality [22], we deduce that

$$\begin{aligned} C_1\|u_n^{\pm}\|^2_{L^{\frac{2Np}{N+\alpha}}} \le \|u_n^{\pm}\|_V^2 &= \int_{\mathbb{R}^N} \big(I_\alpha * |u_n|^p\big)|u_n^{\pm}|^p \,\mathrm{d}x \\ &\le C_2\|u_n\|^p_{L^{\frac{2Np}{N+\alpha}}}\|u_n^{\pm}\|^p_{L^{\frac{2Np}{N+\alpha}}} \le C_3\|u_n^{\pm}\|^p_{L^{\frac{2Np}{N+\alpha}}}, \end{aligned} \tag{4.3}$$

which yields, since by our constraint again $u_n^{\pm} \neq 0$, that

$$\liminf_{n\to\infty} \|u_n^{\pm}\|_{L^{\frac{2Np}{N+\alpha}}} > 0. \tag{4.4}$$

Since the embedding $H_V^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ is compact, we have

$$u_n^{\pm} \to u^{\pm} \text{ strongly in } L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N),$$

and then we deduce from (4.4) that $u^{\pm} \neq 0$. Next, by the Hardy–Littlewood–Sobolev inequality, we see that

$$\int_{\mathbb{R}^N} \big(I_\alpha * |u_n^{\pm}|^p\big)|u_n^{\pm}|^p \to \int_{\mathbb{R}^N} \big(I_\alpha * |u^{\pm}|^p\big)|u^{\pm}|^p, \tag{4.5}$$

and

$$\int_{\mathbb{R}^N} \big(I_\alpha * |u_n^{+}|^p\big)|u_n^{-}|^p \to \int_{\mathbb{R}^N} \big(I_\alpha * |u^{+}|^p\big)|u^{-}|^p. \tag{4.6}$$

Hence, by Lemma 4.1, there exists a unique pair (t_0, s_0) with $t_0, s_0 > 0$ such that $t_0u^+ + s_0u^- \in \mathcal{M}_p$. Moreover, we have

$$\begin{aligned} c_p \le J_p(t_0u^+ + s_0u^-) &\le \liminf_{n\to\infty} J_p(t_0u_n^+ + s_0u_n^-) \\ &\le \limsup_{n\to\infty} J_p(t_0u_n^+ + s_0u_n^-) \le \lim_{n\to\infty} J_p(u_n) = c_p. \end{aligned}$$

The second inequality above follows from the weakly lower semi-continuity of the norm and from (4.5) and (4.6). We conclude by setting $w = t_0u^+ + s_0u^-$.

Step 2 *$J_p'(w) = 0$.*

To complete this, we follow the idea of perturbing the functional in one direction [14]. This argument seems simpler than previous deformation arguments [1,24].

Suppose that w is not a critical point, then there exists a function $v \in C_c^\infty(\mathbb{R}^N)$ such that $\langle J_p'(w), v\rangle = -2$. Since J_p is continuously differentiable, there exists $\delta > 0$ small enough such that

$$\langle J_p'(tu^+ + su^- + \epsilon v), v\rangle \le -1, \qquad \text{if } |t-t_0|+|s-s_0| \le \delta \text{ and } 0 \le \epsilon \le \delta. \tag{4.7}$$

We choose a continuous function $\eta : D \to [0, 1]$, where D being a bounded domain and is defined by

$$D := \bigl\{(t, s) \in \mathbb{R}^2 : |t - t_0| \le \delta, |s - s_0| \le \delta\bigr\},$$

such that

$$\eta(t, s) = \begin{cases} 1 & \text{if } |t - t_0| \le \frac{\delta}{4} \quad \text{and } \ |s - s_0| \le \frac{\delta}{4}, \\ 0 & \text{if } |t - t_0| \ge \frac{\delta}{2} \quad \text{or } \ |s - s_0| \ge \frac{\delta}{2}. \end{cases}$$

We define $Q \in C(D, H_V^1(\mathbb{R}^N))$ for $(t, s) \in D$ by

$$Q(t, s) = tu^+ + su^- + \delta\eta(t, s)v.$$

and $h : D \to \mathbb{R}^2$ for $(t, s) \in D$ as

$$h(t, s) := \bigl(\langle J_p'(Q(t, s)), Q(t, s)^+\rangle, \langle J_p'(Q(t, s)), Q(t, s)^-\rangle\bigr).$$

The map h is continuous because the map $u \mapsto u^+$ is continuous in $H_V^1(\mathbb{R}^N)$. If $|t - t_0| = \delta$, or $|s - s_0| = \delta$, then $\eta = 0$ by its definition, therefore $Q(t, s) = tu^+ + su^-$, which implies that $h(t, s) \ne (0, 0)$ by Lemma 4.1. As a consequence, the Brouwer topological degree $\deg(h, \mathrm{int}(D), 0)$ is well defined and $\deg(h, \mathrm{int}(D), 0) = 1$, thus there exists a pair $(t_1, s_1) \in \mathrm{int}(D)$ such that $h(t_1, s_1) = (0, 0)$. Thus $Q(t_1, s_1) \in \mathcal{M}_p$, and then, it follows from the definition of c_p that

$$J_p(Q(t_1, s_1)) \ge c_p. \tag{4.8}$$

On the other hand, from Eq. (4.7) we arrive at

$$\begin{aligned} J_p(Q(t_1, s_1)) &= J_p(t_1u^+ + s_1u^-) \\ &\quad + \int_0^1 \langle J_p'(t_1u^+ + s_1u^- + \rho\delta\eta(t_1, s_1)v), \delta\eta(t_1, s_1)v\rangle \,\mathrm{d}\rho \\ &\le J_p(t_1u^+ + s_1u^-) - \delta\eta(t_1, s_1). \end{aligned} \tag{4.9}$$

If $(t_1, s_1) \ne (t_0, s_0)$, we know from Lemma 4.1 that $J_p(t_1u^+ + s_1u^-) < J_p(t_0u^+ + s_0u^-) = c_p$, thus from inequality (4.9)

$$J_p(Q(t_1, s_1)) \le J_p(t_1u^+ + s_1u^-) < c_p.$$

If $(t_1, s_1) = (t_0, s_0)$, then $\eta(t_1, s_1) = 1$, follows from (4.9) we also have

$$J_p(Q(t_1, s_1)) \le c_p - \delta < c_p,$$

which contradicts inequality (4.8) in any case. □

We bring to the attention of the reader that the assumptions on the potential V are only used to ensure the compactness of the embedding $H_V^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$.

The case $p = 2$ is more complicated since we have neither a property similar to Lemma 4.1 nor an estimate like (4.3) to guarantee $u^\pm \ne 0$, where $u = u^+ + u^-$ is the weak limit of a minimizing sequence. To find a nodal solution with least energy for the quadratic case, we follow the idea of [7] of employing equation ($\mathcal{C}$) with $p > 2$ as a regularisation for the quadratic equation ($\mathcal{C}$) and then pass to the limit as $p \searrow 2$. We start our proof by showing that Nehari nodal set $\mathcal{M}_2$ is not empty.

Lemma 4.2. *One has $\mathcal{M}_2 \neq \emptyset$. In particular, $c_2 < +\infty$.*

Proof. We are going to construct a function $w \in H^1_V(\mathbb{R}^N)$ with $w^\pm \neq 0$ such that the following linear system admits a solution (t, s) with $t, s > 0$,

$$\begin{pmatrix} \int_{\mathbb{R}^N} (I_\alpha * |w^+|^2)|w^+|^2 & \int_{\mathbb{R}^N} (I_\alpha * |w^+|^2)|w^-|^2 \\ \int_{\mathbb{R}^N} (I_\alpha * |w^+|^2)|w^-|^2 & \int_{\mathbb{R}^N} (I_\alpha * |w^-|^2)|w^-|^2 \end{pmatrix} \begin{pmatrix} t^2 \\ s^2 \end{pmatrix} = \begin{pmatrix} \|w^+\|_V^2 \\ \|w^-\|_V^2 \end{pmatrix}. \tag{4.10}$$

The conclusion will then follow since $tw^+ + sw^- \in \mathcal{M}_2$. By Cramer's Rule, it is sufficient to find a function $w \in H^1_V(\mathbb{R}^N)$ with $w^\pm \neq 0$ such that

$$\frac{\int_{\mathbb{R}^N} (I_\alpha * |w^+|^2)|w^-|^2}{\int_{\mathbb{R}^N} (I_\alpha * |w^-|^2)|w^-|^2} < \frac{\|w^+\|_V^2}{\|w^-\|_V^2} < \frac{\int_{\mathbb{R}^N} (I_\alpha * |w^+|^2)|w^+|^2}{\int_{\mathbb{R}^N} (I_\alpha * |w^+|^2)|w^-|^2}. \tag{4.11}$$

Let $U \in C^1(\mathbb{R}^N)\backslash\{0\}$ such that $U \geq 0$ and $\operatorname{supp} U \subset B(0, 1)$. We choose $a_+, a_- \notin \operatorname{supp} U$ and we define

$$w_\sigma(x) := U(\tfrac{x-a_+}{\sigma}) - U(\tfrac{x-a_-}{\sigma}).$$

Since the function U has compact support, we know that $w^+_\sigma(x) = U(\frac{x-a_+}{\sigma})$ and $w^-_\sigma(x) = -U(\frac{x-a_-}{\sigma})$ for sufficiently small σ. To end the proof, we show that the estimate (4.11) holds as σ becomes small enough. In fact,

$$\begin{aligned} \|w^\pm_\sigma\|_V^2 &= \int_{\mathbb{R}^N} \sigma^{N-2}|\nabla U(x)|^2 + \sigma^N V(a_\pm + \sigma x)U^2(x)\,\mathrm{d}x \\ &= \sigma^{N-2}\Big(\int_{\mathbb{R}^N} |\nabla U|^2 + O(\sigma^2)\Big), \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * |w^\pm_\sigma|^2)|w^\pm_\sigma|^2 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{A_\alpha |U(\frac{y-a_\pm}{\sigma})|^2 |U(\frac{x-a_\pm}{\sigma})|^2}{|x-y|^{N-\alpha}}\,\mathrm{d}y\,\mathrm{d}x \\ &= \sigma^{N+\alpha}\int_{\mathbb{R}^N} (I_\alpha * |U|^2)|U|^2, \end{aligned}$$

and when $\sigma \leq |a_+ - a_-|/4$,

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * |w^+_\sigma|^2)|w^-_\sigma|^2 &= \sigma^{2N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{A_\alpha |U(y)|^2|U(x)|^2}{|(a_- + \sigma x) - (a_+ + \sigma y)|^{N-\alpha}}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \sigma^{2N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{2^{N-\alpha}A_\alpha |U(y)|^2|U(x)|^2}{|a_- - a_+|^{N-\alpha}}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

We observe that in (4.11), since $\alpha < N$, the left-hand side goes to 0 as $\sigma \to 0$, the middle term converges to a positive constant and the right-hand side diverges to $+\infty$. The inequality (4.11) holds for sufficiently small σ and thus the system (4.10) has a solution $(t, s) \in (0, +\infty)^2$, that is, $\mathcal{M}_2 \neq \emptyset$. □

Proof of Theorem 1.3 when $p = 2$. Let $(u_{p_n})_{n\in\mathbb{N}} \subset H^1_V(\mathbb{R}^N)$ be a sequence of least energy nodal solution for the Eq. ($\mathcal{C}$) with $\frac{1}{p_n} > \frac{N-2}{N+\alpha}$ and $p_n \searrow 2$ as $n \to \infty$. In particular, we have $J_{p_n}(u_{p_n}) = c_{p_n}$, and the function u_{p_n} satisfies the equation

$$-\Delta u_{p_n} + V u_{p_n} = \big(I_\alpha * |u_{p_n}|^{p_n}\big)|u_{p_n}|^{p_n-2}u_{p_n}.$$

We first show that $\|u_{p_n}\|_V$ is bounded both from below and above. In fact, by a direct computation, we see that for every $n \in \mathbb{N}$, by the Hardy–Littlewood–Sobolev and by the Sobolev inequality,

$$\|u_{p_n}\|_V^2 = \int_{\mathbb{R}^N} \Big(I_\alpha * |u_{p_n}|^{p_n}\Big)|u_{p_n}|^{p_n} \le C\bigg(\int_{\mathbb{R}^N} |u_{p_n}|^{\frac{2Np_n}{N+\alpha}}\bigg)^{\frac{N+\alpha}{N}} \le C_1\|u_{p_n}\|_V^{2p_n},$$

where the constant C_1 can be taken independently of p_n since $(p_n)_{n\in\mathbb{N}}$ remains bounded. It follows that

$$\liminf_{n\to\infty} \|u_{p_n}\|_V > 0. \tag{4.12}$$

On the other hand, thanks to Lemma 4.2 above, we can take $w \in \mathcal{M}_2$, and define $w_{p_n} = t^{1/p_n}w^+ + s^{1/p_n}w^-$, where $(t^{1/p_n}, s^{1/p_n})$ is given by Lemma 4.1. Then, $w_{p_n} \in \mathcal{M}_{p_n}$, and $J_{p_n}(w_{p_n}) \ge c_{p_n}$. Since $\Phi_p(t,s) \to -\infty$ as $(t,s) \to +\infty$ uniformly in p in bounded sets and $\Phi_p \to \Phi_2$ as $p \to 2$ uniformly over compact subsets of $[0,+\infty)^2$, we have $t^{1/p_n}, s^{1/p_n} \to 1$, and therefore $J_{p_n}(w_{p_n}) \to J_2(w)$. Since w is an arbitrary function in $\mathcal{M}_2$, we deduce that

$$\limsup_{n\to\infty} c_{p_n} \le c_2 < +\infty, \tag{4.13}$$

and thus

$$\|u_{p_n}\|_V^2 = \frac{1}{\frac{1}{2} - \frac{1}{2p_n}}\left(J_{p_n}(u_{p_n}) - \frac{1}{2p_n}\langle J'_{p_n}(u_{p_n}), u_{p_n}\rangle\right) = \frac{2p_n c_{p_n}}{p_n - 1} \le 4c_2 + o(1).$$

In particular, $(\|u_{p_n}\|_V)_{n\in\mathbb{N}}$ is bounded from above. It follows that there exists some function $u \in H^1_V(\mathbb{R}^N)$ such that $u_{p_n} \rightharpoonup u$ weakly in $H^1_V(\mathbb{R}^N)$ as $n \to \infty$. By the compactness of the embedding $H^1_V(\mathbb{R}^N) \hookrightarrow L^{\frac{4N}{N+\alpha}}(\mathbb{R}^N)$ (Proposition 2.1), we have

$$u_{p_n} \to u \text{ strongly in } L^{\frac{4N}{N+\alpha}}(\mathbb{R}^N) \quad \text{and} \quad u_{p_n} \to u \text{ almost everywhere in } \mathbb{R}^N,$$

so that $(u_{p_n})_{n\in\mathbb{N}}$ is bounded in $L^{\frac{4N}{N+\alpha}}(\mathbb{R}^N)$. Moreover, by interpolation through Hölder's inequality, we have

$$\|u_{p_n}\|_{L^{\frac{4N}{N+\alpha}}} \le \|u_{p_n}\|^{\lambda_n}_{L^{\frac{2Np_n}{N+\alpha}}}\|u_{p_n}\|^{1-\lambda_n}_{L^2} \le \|u_{p_n}\|^{\lambda_n}_{L^{\frac{2Np_n}{N+\alpha}}}\big(C\|u_{p_n}\|_V\big)^{1-\lambda_n}, \tag{4.14}$$

where $\lambda_n \in (0,1)$ satisfies that

$$\frac{1}{\frac{4N}{N+\alpha}} = \frac{\lambda_n}{\frac{2Np_n}{N+\alpha}} + \frac{1-\lambda_n}{2},$$

that is, $\lambda_n = \frac{N-\alpha}{Np_n-N-\alpha}\frac{p_n}{2} \to 1$ as $n \to \infty$ and the constant C can be chosen independently of u_{p_n}. Similarly, taking $q = \frac{2N}{N-2}$ for $N \geq 3$, and $\frac{4N}{N+\alpha} < q < +\infty$ for $N = 1, 2$, we have

$$\begin{aligned}\|u_{p_n}\|_{L^{\frac{2Np_n}{N+\alpha}}} &\leq \|u_{p_n}\|_{L^q}^{1-\mu_n}\|u_{p_n}\|^{\mu_n}_{L^{\frac{4N}{N+\alpha}}}\\ &\leq \big(C\|u_{p_n}\|_V\big)^{1-\mu_n}\|u_{p_n}\|^{\mu_n}_{L^{\frac{4N}{N+\alpha}}} \qquad (4.15)\end{aligned}$$

with

$$\mu_n = \frac{\frac{N+\alpha}{2Np_n} - \frac{1}{q}}{\frac{N+\alpha}{4N} - \frac{1}{q}} \to 1, \qquad \text{as } \ n \to \infty.$$

Taking limit on the both sides of (4.14) and (4.15) and combining the boundedness of $\|u_{p_n}\|_V$ from below and above, we obtain that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_{p_n}|^{\frac{2Np_n}{N+\alpha}} = \lim_{n\to\infty}\int_{\mathbb{R}^N} |u_{p_n}|^{\frac{4N}{N+\alpha}} = \int_{\mathbb{R}^N} |u|^{\frac{4N}{N+\alpha}}.$$

Thus, we get that (see for example [26, Proposition 4.2.6])

$$|u_{p_n}|^{p_n} \to |u|^2 \quad \text{ strongly in } \ L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N), \qquad (4.16)$$

which, together with the Stein–Weiss inequality [22], yields that

$$I_\alpha * |u_{p_n}|^{p_n} \to I_\alpha * |u|^2 \quad \text{ strongly in } \ L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N). \qquad (4.17)$$

Similarly to (4.3), we have

$$C_1\|u_{p_n}\|^2_{L^{\frac{4N}{N+\alpha}}} \leq \|u_{p_n}\|_V^2 = \int_{\mathbb{R}^N}\big(I_\alpha * |u_{p_n}|^{p_n}\big)|u_{p_n}|^{p_n} \leq C\big\||u_{p_n}|^{p_n}\big\|^2_{L^{\frac{2N}{N+\alpha}}},$$

which implies, by taking limit on both sides, that $\|u\|_{L^{\frac{4N}{N+\alpha}}} \geq C > 0$, that is $u \neq 0$.

For large n, we choose q as in (4.15), we employ the interpolation inequalities again, and we get

$$\|u_{p_n}\|_{L^{(p_n-1)\frac{4N}{N+\alpha}}} \leq \|u_{p_n}\|_{L^q}^{1-\lambda_n}\|u_{p_n}\|^{\lambda_n}_{L^{\frac{4N}{N+\alpha}}} \leq \big(C\|u_{p_n}\|_V\big)^{1-\lambda_n}\|u_{p_n}\|^{\lambda_n}_{L^{\frac{4N}{N+\alpha}}},$$

where

$$\lambda_n = \frac{\frac{N+\alpha}{4N(p_n-1)} - \frac{1}{q}}{\frac{N+\alpha}{4N} - \frac{1}{q}} \to 1,$$

that is $|u_{p_n}|^{p_n-2}u_{p_n}$ is bounded in $L^{\frac{4N}{N+\alpha}}(\mathbb{R}^N)$, it converges to u weakly in the space $L^{\frac{4N}{N+\alpha}}(\mathbb{R}^N)$ [26, Proposition 5.4.7]. Therefore,

$$\begin{aligned}\langle J_2'(u), \psi\rangle &= \int_{\mathbb{R}^N} \nabla u\cdot\nabla\psi + Vu\psi - (I_\alpha * |u|^2)u\psi\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N} \nabla u_{p_n}\cdot\nabla\psi + Vu_{p_n}\psi - (I_\alpha * |u_{p_n}|^{p_n})|u_{p_n}|^{p_n-2}u_{p_n}\psi\\ &= \lim_{n\to\infty}\langle J'_{p_n}(u_{p_n}), \psi\rangle = 0.\end{aligned}$$

which means that u is a weak solution of the quadratic Choquard equation ($\mathcal{C}$) since the function $\psi \in H^1_V(\mathbb{R}^N)$ is arbitrary. Moreover, we deduce from the convergences (4.16) and (4.17) that

$$\|u_{p_n}\|_V^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_{p_n}|^{p_n})|u_{p_n}|^{p_n} \to \int_{\mathbb{R}^N} (I_\alpha * |u|^2)|u|^2 = \|u\|_V^2, \tag{4.18}$$

and thus $u_{p_n} \to u$ strongly in $H^1_V(\mathbb{R}^N)$ as $n \to \infty$.

We are now in a position to finish our proof by showing that $u^\pm \neq 0$ and $J_2(u)$ is the least among all the nodal solutions of the quadratic Choquard equation. By (4.12) and (4.18), we have $u \neq 0$. Without loss of generality, let us assume by contradiction that $u^+ \neq 0$ and that $u^- = 0$. Set for each $n \in \mathbb{N}$

$$v_{p_n} := \frac{u^-_{p_n}}{\|u^-_{p_n}\|_V^{2/p_n}},$$

then from the equality $\langle J'_{p_n}(u_{p_n}), u^-_{p_n}\rangle = 0$, we get that for each $n \in \mathbb{N}$

$$\int_{\mathbb{R}^N} \bigl(I_\alpha * |u_{p_n}|^{p_n}\bigr)|v_{p_n}|^{p_n} = 1.$$

Since we have assumed that $u^- = 0$, we have $u^-_{p_n} \to 0$ strongly in $H^1_V(\mathbb{R}^N)$. By Young's inequality, we know that

$$\|v_{p_n}\|_V = \|u^-_{p_n}\|_V^{1-\frac{2}{p_n}} \leq \Bigl(1 - \frac{2}{p_n}\Bigr)\|u^-_{p_n}\|_V + \frac{2}{p_n},$$

which yields that the sequence $(v_{p_n})_{n\in\mathbb{N}}$ is bounded in $H^1_V(\mathbb{R}^N)$. The compactness of the embedding $H^1_V(\mathbb{R}^N) \hookrightarrow L^{\frac{4N}{N+\alpha}}(\mathbb{R}^N)$ (Proposition 2.1) implies in turn that $(v_{p_n})_{n\in\mathbb{N}}$ converges to some v strongly in $L^{\frac{4N}{N+\alpha}}(\mathbb{R}^N)$, which, together with (4.15) by replacing u_{p_n} with v_{p_n} and the boundness of $\|v_{p_n}\|_V$, implies that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \bigl||v_{p_n}|^{p_n}\bigr|^{\frac{2N}{N+\alpha}} < +\infty,$$

thus, it follows from [26, Proposition 5.4.7] again that

$$|v_{p_n}|^{p_n} \rightharpoonup |v|^2 \text{ in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Combining the strong convergence of (4.17), we deduce that

$$\int_{\mathbb{R}^N} \bigl(I_\alpha * |u|^2\bigr)|v|^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} \bigl(I_\alpha * |u_{p_n}|^{p_n}\bigr)|v_{p_n}|^{p_n} = 1. \tag{4.19}$$

On the other hand, by the definition of v_{p_n} and by the strong convergence of $(u^+_{p_n})_{n\in\mathbb{N}}$ to u in $H^1(\mathbb{R}^N)$, we have $uv = 0$ almost everywhere on $\mathbb{R}^N$. Since u is a nontrivial nonnegative weak solution to the Choquard equation, it is a classical solution (following [16, Theorem 3]) and thus, by the classical strong maximum principle for second order elliptic operators, $u > 0$ everywhere on $\mathbb{R}^N$ and thus $v = 0$, which is a contradiction with (4.19).

In particular, we have $u \in \mathcal{M}_2$, and thus $J_2(u) \geq c_2$. On the other hand, by (4.13) and the strong convergence of $(u_{p_n})_{n\in\mathbb{N}}$ we have

$$\begin{aligned} J_2(u) &= \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \\ &= \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2p_n}\right)\int_{\mathbb{R}^N} |\nabla u_{p_n}|^2 + V|u_{p_n}|^2 \\ &= \limsup_{n\to\infty} J_{p_n}(u_{p_n}) = \limsup_{n\to\infty} c_{p_n} \leq c_2; \end{aligned}$$

this concludes the proof. □

Remark 4.3. In fact, in the case of $p \geq 2$, we have $c_p > c_{0,p}$ where $c_{0,p}$ is the energy level of the groundstates, since any groundstate solution should have constant sign. However, the question of whether or not or when the estimate $c_p > 2c_{0,p}$ holds is open; in the case of constant potential this estimate was crucial for the compactness.

Finally, we prove that the energy level c_p is degenerate when $p < 2$.

Proposition 4.4. *For $p < 2$, we have $c_p = c_{0,p}$. Then the energy functional J_p does not achieve its minimum on the Nehari nodal set.*

Proof. We observe that if $u \in \mathcal{N}_p$, then $|u| \in \mathcal{N}_p$, where $\mathcal{N}_p$ denote the Nehari manifold, that is,

$$\mathcal{N}_p := \left\{u \in H^1_V(\mathbb{R}^N) : u \neq 0 \text{ and } \langle J_p'(u), u\rangle = 0\right\}.$$

With this notation, we know that $c_{0,p} = \inf_{u\in\mathcal{N}_p} J_p(u)$ (see Remark 4.3 and [25]). Since $\mathcal{M}_p \subset \mathcal{N}_p$, thus we get that $c_p \geq c_{0,p}$. In fact, we shall show the reverse inequality holds. By a density argument, it follows that

$$c_{0,p} = \inf\{J_p(u) : u \in \mathcal{N}_p \cap C^1_c(\mathbb{R}^N) \text{ and } u \geq 0 \text{ on } \mathbb{R}^N\}.$$

Let $u \in \mathcal{N}_p \cap C^1_c(\mathbb{R}^N)$ and $u \geq 0$ on $\mathbb{R}^N$. We choose a point $a \notin \operatorname{supp} u$ and a function $\psi \in C^1_c(\mathbb{R}^N)\backslash\{0\}$ such that $\psi \geq 0$ and we define as in [8] for each $\sigma > 0$ the function $u_\sigma : \mathbb{R}^N \to \mathbb{R}$ by

$$u_\sigma(x) = u(x) - \sigma^{\frac{2}{2-p}}\psi(\tfrac{x-a}{\sigma}).$$

Then, $u_\sigma^+ = u$ for sufficiently small σ. By a direct computation, $tu_\sigma^+ + su_\sigma^- \in \mathcal{M}_p$ if and only if

$$\left\{\begin{array}{l} (t^{2-p} - t^p)\int_{\mathbb{R}^N}\bigl(I_\alpha * |u|^p\bigr)|u|^p = s^p\sigma^{N+\frac{2p}{2-p}}K_\sigma, \\ s^{2-p}\int_{\mathbb{R}^N}|\nabla\psi(y)|^2 + \sigma^2 V(a+\sigma y)\psi^2(y)\,\mathrm{d}y \\ \quad = t^p K_\sigma + s^p\sigma^{\alpha+\frac{2p}{2-p}}\int_{\mathbb{R}^N}\bigl(I_\alpha * |\psi|^p\bigr)|\psi|^p \end{array}\right. \tag{4.20}$$

where $K_\sigma = \int_{\mathbb{R}^N}\bigl(I_\alpha * |u|^p\bigr)(a+\sigma y)|\psi(y)|^p dy$.

Observe that thet system (4.20) has a unique solution when $\sigma = 0$. By the implicit function theorem, for $\sigma > 0$ small enough there exists a pair $(t_\sigma, s_\sigma) \in (0, +\infty)^2$ that solves the system (4.20) and

$$\lim_{\sigma\to 0} t_\sigma = 1, \qquad \lim_{\sigma\to 0} s_\sigma = \left(\frac{\bigl(I_\alpha * |u|^p\bigr)(a)\int_{\mathbb{R}^N}|\psi|^p}{\int_{\mathbb{R}^N}|\nabla\psi|^2}\right)^{\frac{1}{2-p}}.$$

Since $\frac{4}{2-p} + N - 2 > 0$, we have $u_\sigma^- \to 0$ in $H_V^1(\mathbb{R}^N)$ as $\sigma \to 0$, and thus $t_\sigma u_\sigma^+ + s_\sigma u_\sigma^- \to u$ in $H_V^1(\mathbb{R}^N)$. Hence, we have that

$$c_p \leq J_p(t_\sigma u_\sigma^+ + s_\sigma u_\sigma^-) \to J_p(u), \qquad \text{as } \sigma \to 0,$$

which implies that $c_p \leq c_{0,p}$ since $u \in \mathcal{N}_p$ is arbitrary.

We assume that $u \in \mathcal{M}_p$ minimizes the functional J_p on the Nehari nodal set $\mathcal{M}_p$. Since $c_{0,p} = c_p$, thus u also minimizes J_p on the Nehari manifold $\mathcal{N}_p$. By regularity theory for the Choquard equation and by the strong maximum principle, either $u > 0$ or $u < 0$, which contradicts with $u \in \mathcal{M}_p$. □

5. Pohožaev identity

This section is devoted to the proof of a Pohožaev identity for the Choquard equation ($\mathcal{C}$).

Theorem 5.1. (Pohožaev identity) *Let $N \geq 3$, $V \in C^1(\mathbb{R}^N, [0, +\infty))$. If the function $u \in W^{2,2}_{\text{loc}}(\mathbb{R}^N) \cap H_V^1(\mathbb{R}^N)$ is a solution to the Choquard equation ($\mathcal{C}$) such that*

$$\int_{\mathbb{R}^N} |x \cdot \nabla V(x)|\, |u(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p < +\infty,$$

then

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} \big(NV(x) + x \cdot \nabla V(x)\big)|u(x)|^2 \,\mathrm{d}x = \frac{N+\alpha}{2p} \int \big(I_\alpha * |u|^p\big)|u|^p. \tag{5.1}$$

Here, $\eta \cdot \zeta$ denotes the canonical scalar product of vectors $\eta, \zeta \in \mathbb{R}^N$.

Proof of Theorem 5.1. We take $\varphi \in C_c^1(\mathbb{R}^N)$ such that $\varphi = 1$ on $B(0,1)$. Since the function φ has compact support, we can define a function $v_\lambda \in H_V^1(\mathbb{R}^N)$ for $\lambda \in (0, +\infty)$ by

$$v_\lambda(x) := \varphi(\lambda x)\, x \cdot \nabla u(x).$$

By testing the Choquard equation ($\mathcal{C}$) against the function v_λ, we have

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda + V u v_\lambda = \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^{p-2} u v_\lambda.$$

We compute the square term for $\lambda > 0$. By the definition of v_λ, the chain rule and by the Gauss integral formula, we get that

$$\begin{aligned}
\int_{\mathbb{R}^N} V u v_\lambda &= \int_{\mathbb{R}^N} V(x) u(x) \varphi(\lambda x)\, x \cdot \nabla u(x) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} V(x) \varphi(\lambda x)\, x \cdot \nabla\big(\tfrac{1}{2}|u|^2\big)(x) \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^N} \big(NV(x)\varphi(\lambda x) + V(x)\,(\lambda x) \cdot \nabla \varphi(\lambda x) \\
&\qquad + x \cdot \nabla V(x) \varphi(\lambda x)\big) \frac{|u(x)|^2}{2} \,\mathrm{d}x.
\end{aligned} \tag{5.2}$$

In view of the various boundedness assumptions, Lebesgue's dominated convergence theorem applies and gives us

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N} V(x)uv_\lambda = -\frac{1}{2}\int_{\mathbb{R}^N}\big(NV(x)+x\cdot\nabla V(x)\big)|u(x)|^2\,\mathrm{d}x$$

In view of the assumption $u\in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$, we can perform an integration by parts

$$\begin{aligned}\int_{\mathbb{R}^N}\nabla u\cdot\nabla v_\lambda &= \int_{\mathbb{R}^N}\varphi(\lambda x)\Big(|\nabla u(x)|^2 + x\cdot\nabla\big(\tfrac{|\nabla u|^2}{2}\big)(x)\Big)\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}\big(\lambda\nabla u(x)\cdot\nabla\varphi(\lambda x)\big)\big(x\cdot\nabla u(x)\big)\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^N}\big((N-2)\varphi(\lambda x)+\lambda x\cdot\nabla\varphi(\lambda x)\big)\frac{|\nabla u(x)|^2}{2}\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}\big(\nabla u(x)\cdot\nabla\varphi(\lambda x)\big)\big(\lambda x\cdot\nabla u(x)\big)\,\mathrm{d}x\end{aligned}$$

Since $|(\eta\cdot\zeta)(\eta\cdot\xi)|\le|\eta|^2|\zeta||\xi|$ for any $\zeta,\xi,\eta\in\mathbb{R}^N$, we have for each $x\in\mathbb{R}^N$

$$\begin{aligned}\big|\big(\nabla u(x)\cdot\nabla\varphi(\lambda x)\big)\big(\lambda x\cdot\nabla u(x)\big)\big| &\le |\nabla u(x)|^2|\lambda x||\nabla\varphi(\lambda x)|\\ &\le |\nabla u(x)|^2\sup_{z\in\mathbb{R}^N}|z|\,|\nabla\varphi(z)|.\end{aligned}$$

By Lebesgue's dominated convergence theorem again, we have, since $u\in H^1_V(\mathbb{R}^N)$,

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\nabla u\cdot\nabla v_\lambda = -\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2.$$

Finally, by symmetry and integration by parts

$$\begin{aligned}&\int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^{p-2}uv_\lambda\\ &\quad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y)|u(y)|^p\varphi(\lambda x)\,x\cdot\nabla\Big(\tfrac{|u|^p}{p}\Big)(x)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y)\Big(|u(y)|^p\varphi(\lambda x)\,x\cdot\nabla\Big(\tfrac{|u|^p}{p}\Big)(x)\\ &\qquad\qquad+|u(x)|^p\varphi(\lambda y)y\cdot\nabla\Big(\tfrac{|u|^p}{p}\Big)(y)\Big)\,\mathrm{d}x\,\mathrm{d}y\\ &=-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y)|u(y)|^p\Big(N\varphi(\lambda x)+\lambda x\cdot\nabla\varphi(\lambda x)\Big)\frac{|u(x)|^p}{p}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\frac{N-\alpha}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y)|u(y)|^p\frac{(x-y)\cdot(x\varphi(\lambda x)-y\varphi(\lambda y))}{|x-y|^2}|u(x)|^p\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

For any $\lambda > 0$ and $x, y \in \mathbb{R}^N$,

$$\left|\frac{(x-y)\cdot(x\varphi(\lambda x) - y\varphi(\lambda y))}{|x-y|^2}\right| = \left|\frac{(\lambda x-\lambda y)\cdot(\lambda x\varphi(\lambda x) - \lambda y\varphi(\lambda y))}{|\lambda x-\lambda y|^2}\right| \leq \sup_{z,w\in\mathbb{R}^{\mathbb{N}}}\left|\frac{(w-z)\cdot(w\varphi(w) - z\varphi(z))}{|w-z|^2}\right| < +\infty. \tag{5.3}$$

We can thus apply Lebesgue's dominated convergence theorem to conclude that

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^{p-2}uv_\lambda = -\frac{N+\alpha}{2p}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^p.$$

Hence, the identity (5.1) holds. □

Remark 5.2. The Pohožaev identity implies some nonexistence results for the Choquard equation ($\mathcal{C}$). In general, if

$$\text{either} \qquad \big(2V(x) + x\cdot\nabla V(x)\big)\big(N-2-\tfrac{N+\alpha}{p}\big) \geq 0, \tag{5.4}$$

$$\text{or} \qquad \big((N-\tfrac{N+\alpha}{p})V(x) + x\cdot\nabla V(x)\big)\big(N-2-\tfrac{N+\alpha}{p}\big) \geq 0, \tag{5.5}$$

then the Choquard equation ($\mathcal{C}$) has no nontrivial solutions satisfying the regularity and boundedness assumptions of Theorem 5.1. In particular, if $V(x) = |x|^\beta$ is homogeneous, then the Choquard equation ($\mathcal{C}$) has no such solution if $p \in (1, \max\{1, \frac{N+\alpha}{N+\beta}\}] \cup [\frac{N+\alpha}{N-2}, +\infty)$.

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