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Nonlinear Differential Equations and Applications NoDEA

Regularity results for non-autonomous functionals with $L \log L$-growth and Orlicz Sobolev coefficients

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Abstract. We study the regularity properties of local minimizers of non-autonomous convex integral functionals of the type

$$\mathcal{F}(u, \Omega) = \int_\Omega f(x, Du)\; dx,$$

when the integrand f has almost linear growth with respect to the gradient variable and the dependence on the x-variable is controlled by a function which belongs to a suitable Orlicz Sobolev space.

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1. Introduction

We consider integral functionals of the form

$$\mathcal{F}(u, \Omega) = \int_\Omega f(x, Du)\; dx, \tag{1.1}$$

for a mapping $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $n > 2$, $N \geq 1$ and Ω a bounded open set in $\mathbb{R}^n$. Here the integrand $f : (x, \xi) \in \Omega \times \mathbb{R}^{n \times N} \to [0, +\infty)$ is strictly convex with respect to the variable $\xi \in \mathbb{R}^{n \times N}$.

The aim of this paper is to establish a higher differentiability result of minimizers of (1.1) with an integrand $\xi \to f(x, \xi) \in C^2(\mathbb{R}^{n \times N})$ with almost linear growth with respect to the gradient variable and with Orlicz Sobolev

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dependence with respect to the x-variable. More precisely, we shall assume that there exist positive constants c_1, c_2, c_3, c_4, l and ν such that

$$c_1|\xi|\log(\mathrm{e}+|\xi|) \leq f(x,\xi) \leq c_2|\xi|\log(\mathrm{e}+|\xi|) + c_3, \tag{F1}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{n\times N}$.
Concerning the dependence on the x-variable, we suppose that there exists a function $k \in L^n \log^n L$ such that

$$|D_x f(x,\xi)| \leq |k(x)||\xi|\log(\mathrm{e}+|\xi|), \tag{F2}$$

and

$$|D_x D_\xi f(x,\xi)| \leq |k(x)|\log(\mathrm{e}+|\xi|); \tag{F3}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{n\times N}$.
The model case we have in mind is

$$\mathcal{F}(u,\Omega) = \int_\Omega a(x)|Du|\log(\mathrm{e}+|Du|)\,dx,$$

with $a(x) \in WL^n \log^n L(\Omega)$ (for the definition of the Orlicz Sobolev spaces see Sect. 2.1 below). Note that, by the embedding in the Orlicz Sobolev spaces (see Theorem 2.3 in Sect. 2.1), the function $a(x)$ has a logarithmic modulus of continuity. Indeed, we have

$$|a(x)-a(y)| \leq \frac{c}{\log^{\frac{1}{n}}(\mathrm{e}+\frac{1}{|x-y|})}\|Da\|_{L^n log^n L(\Omega)}.$$

Further, we assume that

$$\langle D_{\xi\xi} f(x,\xi)\eta,\eta\rangle \leq l\frac{\log(\mathrm{e}+|\xi|)}{|\xi|}|\eta|^2, \tag{F4}$$

for a.e. $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^{n\times N}$. Finally, we remark that, thanks to the C^2 regularity of f with respect to the variable ξ, the strict convexity assumption on the integrand f is equivalent to the following ellipticity condition on the matrix $D_{\xi\xi} f$

$$\langle D_{\xi\xi} f(x,\xi)\eta,\eta\rangle \geq \nu(1+|\xi|)^{-1}|\eta|^2, \tag{F5}$$

for a.e. $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^{n\times N}$.

Regularity properties of minimizers of integral functionals of the type (1.1) when the integrand $f(x,\xi)$ depends on the x-variable through a Hölder continuous function and satisfies p-growth conditions with respect to ξ variable, have been widely investigated and are by now classical (we refer the interest reader to [31] and references therein).

On the other hand, the study of the regularity has been carried on under weaker assumptions with respect to x variable and we refer to [22] for the case of continuous dependence on the x-variable through a modulus of continuity not necessarily Hölder, to [6] for the case of VMO coefficients, for the case of Sobolev coefficients to [11,13,14,27,30,39–41] (when the integrand satisfies the so called standard growth conditions) and to [19] (when the integrand satisfies the non standard growth conditions), to [3,12,33] for the case of fractional Sobolev coefficients and to [29] for parabolic systems.

However, all the above mentioned papers refer to functionals that grow as a power of the gradient with exponent $p > 1$. Here, we are interested in functionals with integrands which are not too far from being linear in $|\xi|$, that is

$$\lim_{|\xi|\to+\infty} \frac{|f(x,\xi)|}{|\xi|} = \infty, \qquad \lim_{|\xi|\to+\infty} \frac{|f(x,\xi)|}{|\xi|^p} = 0 \qquad \forall p > 1. \tag{1.2}$$

Note that functionals with nearly linear growth have features in common with ones satisfying non standard growth since, by virtue of (1.2), we have that

$$c|\xi| \leq f(x,\xi) \leq C(1 + |\xi|^p), \qquad \forall p > 1.$$

We recall that the theory of regularity of minimizers in this framework of non-standard growths was started by Marcellini [34,35], see also Acerbi and Fusco [1], Boccardo et al. [5], Fusco and Sbordone [25,26], and later widely investigated by many authors and in different settings of applicability (see e.g. [4,7,18]). It is well known that the dependence of the integrand on the x-variable can give substantial difficulties since the Lavrentiev phenomenon may appear (see [20]). For more details and references on this subject we refer to [36].

Actually, many regularity results have been established for integrals with nearly linear growth in case they do not depend on the x variable. The first result in this direction is due to Greco et al. [32], where they proved the higher integrability of the minimizers of functionals in the scale of Orlicz spaces. After that, Fuchs and Seregin [24] proved the $C^{1,\gamma}$-partial regularity for minimizers when the dimension $n \leq 4$. Such result has been extended to any dimension n by Esposito and Mingione [21] and later on the full $C^{1,\gamma}$-regularity has been established in [23,37]. We recall that all the mentioned papers concern the autonomous case. Recently, regularity results have been established for integrals of this type also in the case of variable exponent (see [28,38]).
It is well known that, dealing with functionals with non standard growth conditions, the presence of the x-variable in the integrand can give substantial difficulties since the Lavrentiev phenomenon may appear. However, we mention that in [8,42] the higher differentiability of local minimizers of $\mathcal{F}(u,\Omega)$ has been established under a Hölder continuous dependence of the integrand with respect to the x-variable.

The aim of this paper is to establish regularity results and more precisely higher differentiability results for minimizers of non-autonomous functionals of the type (1.1) when the integrand function f has a nonlinearity into the gradient variable which is slowly increasing at infinity and, respect to x-variable, is assumed weakly differentiable with a summability assumption on the weak gradient that implies only the continuity, with a logarithmic modulus of continuity. More precisely, we shall proved the following

Theorem 1.1. *Let* $f\colon \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$ *satisfy the assumptions* (F1)–(F5). *If* $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *is a local minimizer of the functional* (1.1), *then*

$$\frac{Du}{\sqrt{1+|Du|}} \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

Furthermore, there exists a radius $R_0 = R_0(n, N, l, \nu, \|k\|_{L^n \log^n L})$ *such that, whenever* $B_{2R} \subset B_{R_0} \subset \Omega$*, we have the Caccioppoli type inequality*

$$\int_{B_{\frac{R}{4}}} \frac{|D^2u|^2}{1+|Du|} \leq \frac{c}{R^2} \int_{B_R} (1 + |Du| \log(\mathrm{e} + |Du|)) \ dx, \tag{1.3}$$

where $c = c(n, l, \nu)$*.*

The proof of previous Theorem is achieved combining an a priori estimate with a suitable approximation argument. In the proof of the a priori estimate, we take advantage from previous results for minimizers of functionals with almost linear growth that allow us to use, in the Euler Lagrange system associated to $\mathcal{F}(u, \Omega)$, test functions that are proportional to the second derivatives of the minimizers.
The core of the proof consists in establishing estimates that, in the right hand side, contain only the energy of the functionals and that, for this reason, are preserved in passing to the limit. In order to reabsorb terms with critical summability we use the Hölder inequality in Orlicz–Zygmund spaces and the absolute continuity of the integral. Next, the approximation is achieved by constructing a sequence of functionals that grow almost linearly with respect to the gradient variable and which are C^∞-smooth with respect to the x-variable.

We'd like to point out that Theorem 1.1 holds true also for functionals with integrand $f(x, \xi) \approx a(x)|\xi| \log^\alpha(e + |\xi|)$ with $\alpha > 0$. Obviously, in this case, the Sobolev Orlicz assumption on the coefficient will be $L^n \log^{n\alpha} L$. This assumption becomes weaker when α is less than 1 and this, roughly speaking, means that when the anisotropy of the functional is small we need less regularity with respect to the x-variable.

In the following theorem we want to show that for a priori bounded minimizers the higher differentiability result proved in Theorem 1.1 allows us to obtain the following higher integrability result for the gradient Du:

Theorem 1.2. *Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *be a local minimizer of the functional* (1.1) *under the assumptions (F1)-(F5). Then*

$$Du \in L^3_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times N}).$$

Furthermore, there exists a radius $R_0 = R_0(n, l, \nu)$ *such that, whenever* $B_R \subset B_{8R} \subset B_{R_0} \subset \Omega$ *we have the following inequality*

$$\int_{B_R} |Du|^3 \leq \frac{c\|u\|^2_{L^\infty(2R)}}{R^2} \int_{B_{8R}} (1 + |Du| \log(\mathrm{e} + |Du|)) \ dx, \tag{1.4}$$

for a constant $c = c(n, l, \nu)$

It is well known that in the vectorial setting the local boundedness of the minimizers can not be expected, as shown by the counterexamples by De Giorgi and Šverák and Yan. On the other hand, also in the vectorial setting, the local boundedness of the minimizers can be obtained under suitable structure assumptions on the integrand (see e.g. [15]). It is also known that the local boundedness of the minimizers in the context of functionals with non standard growth is the starting point to the investigation of higher regularity. Here, inspired by [9], the main tool is an interpolation type inequality which is the suitable modification to our context of Lemma 10 in [9].

2. Preliminaries

In this section we recall some standard definitions, a lemma that we shall need to establish our main results and at last the definition and a embedding theorem of Orlicz spaces.

We shall adopt the usual convention and denote by c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses. All the norms we use on $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{\mathbb{R}^{n\times N}}$ will be the standard Euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices ξ, $\eta \in \mathbb{R}^{n\times N}$ we write $\langle \xi, \eta\rangle := \mathrm{trace}(\xi^T\eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi\rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

For a C^2 function $f\colon \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$, we write

$$D_\xi f(x,\xi)[\eta] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(x, \xi + t\eta)$$

$$\text{and} \qquad D_{\xi\xi} f(x,\xi)[\eta,\eta] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} f(x, \xi + t\eta)$$

for ξ, $\eta \in \mathbb{R}^{n\times N}$.
In what follows, $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n :\ |y-x| < r\}$ will denote the ball centered at x of radius r. We shall omit the dependence on the center when no confusion arises.
Let us recall the definition of local minimizer.

Definition 2.1. A mapping $u \in \mathrm{W}^{1,L\log L}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local f–minimizer if

$$\int_{\mathrm{supp}\varphi} f(x, Du)\, dx \le \int_{\mathrm{supp}\varphi} f(x, Du + D\varphi)\, dx$$

for any $O \subset \Omega$ and any $\varphi \in \mathrm{C}^\infty_0(O, \mathbb{R}^N)$.

The following lemma finds an important application in the so called hole-filling method. Its proof can be found for example in [31, Lemma 6.1] .

Lemma 2.2. *Let $h : [r, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \vartheta < 1$, $A, B \ge 0$ and $\beta > 0$. Assume that*

$$h(s) \le \vartheta h(t) + \frac{A}{(t-s)^\beta} + B,$$

for all $r \le s < t \le R_0$. Then

$$h(r) \le \frac{cA}{(R_0 - r)^\beta} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

2.1. Orlicz Spaces

We need to recall some basic properties of Orlicz spaces (for more details we refer to [2]).

Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function, that is $\Phi(0) = 0$, Φ is increasing and convex. If Ω is a open subset of $\mathbb{R}^n$, we define the Orlicz space $L^\Phi(\Omega)$ generated by the Young function Φ as the set of the measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx < \infty,$$

for some $\lambda > 0$. This space is equipped with the Luxemburg norm

$$\|u\|_{L^\Phi(\Omega)} = \inf\left\{\lambda > 0 \ : \ \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \leqslant 1\right\}.$$

We define the space $WL^\Phi(\Omega)$ as the set

$$WL^\Phi(\Omega) = \left\{u \in W^{1,1}(\Omega) : |\nabla u| \in L^\Phi(\Omega)\right\}.$$

The Zygmund space $L^p \log^\alpha L(\Omega)$, for $1 \leqslant p < \infty$, $\alpha \in \mathbb{R}$ ($\alpha \geqslant 0$ for $p = 1$), is defined as the Orlicz space $L^\Phi(\Omega)$ when the Young function Φ is given by

$$\Phi(t) \simeq t^p \log^\alpha(e + t) \quad \text{for every } t \geq t_0 \geq 0. \tag{2.1}$$

Therefore, a measurable function u on Ω belongs to $L^p \log^\alpha L(\Omega)$ if

$$\int_\Omega |u|^p \log^\alpha(e + |u|)\, dx < \infty.$$

For $\alpha = 0$ we have the ordinary Lebesgue spaces. We will need to use the following Hölder type inequality for Zygmund spaces

$$\|u_1 \cdots u_k\|_{L \log^\alpha L} \leqslant C\, \|u_1\|_{L^{p_1} \log^{\alpha_1} L} \cdots \|u_k\|_{L^{p_k} \log^{\alpha_k} L}, \tag{2.2}$$

where $p_i > 1$, $\alpha_i \in \mathbb{R}$, $u_i \in L^{p_i} \log^{\alpha_i} L$ for $i = 1, \ldots, k$, and

$$1 = \frac{1}{p_1} + \cdots + \frac{1}{p_k}, \qquad \alpha = \frac{\alpha_1}{p_1} + \cdots + \frac{\alpha_k}{p_k}.$$

The following inclusions hold

$$L^p \log^\beta L(\Omega) \subset L^p(\Omega) \subset L^p \log^\alpha L(\Omega)$$

with continuous embeddings if $\alpha < 0 < \beta$.

We define the Orlicz Sobolev space $WL^p \log^\alpha L(\Omega)$ as the set

$$WL^p \log^\alpha L(\Omega) = \left\{u \in W^{1,1}(\Omega) : |\nabla u| \in L^p \log^\alpha L(\Omega)\right\}.$$

Next, we recall an embedding Theorem in the Orlicz–Sobolev setting (see for example [10]).

Theorem 2.3. *Let $h \in W^{1,1}(\Omega)$ be a function such that $|Dh| \in L^n log^\sigma L(\Omega)$, some $\sigma > n - 1$. Then $h \in C^0(\Omega)$ and*

$$|h(x) - h(y)| \leq \frac{c_n}{\log(\mathrm{e} + \frac{1}{|x-y|})^{\frac{\sigma-n+1}{n}}} \|Dh\|_{L^n log^\sigma L(\Omega)}$$

3. Proof of the Theorems 1.1

The proof of Theorem 1.1 is achieved combining a suitable a priori estimate for the second derivatives of the minimizers of the functional with an approximation argument.

Proof of Theorem 1.1. **Step 1. The a priori estimate**

Recall that local minimizers u of the functional (1.1) are solutions of the corresponding Euler Lagrange system

$$\int_\Omega \langle D_\xi f(x, Du), D\varphi\rangle\ dx = 0,$$

for every $\varphi \in C_0^\infty(\Omega)$. Let us fix a ball $B_R \subset \Omega$ and arbitrary radii $\frac{R}{2} < s < t < R$. Let us consider a cut off function $\rho \in C_0^\infty(B_t)$ such that $\rho = 1$ on B_s and $|\nabla\rho| \le \frac{c}{t-s}$. Using $\varphi = D_i(\rho^2 D_i u)$ as test function in the Euler Lagrange system, we get

$$\int_\Omega \langle D_\xi f(x, Du), D(D_i(\rho^2 D_i u))\rangle\ dx = \int_\Omega \langle D_\xi f(x, Du), D_i D(\rho^2 D_i u)\rangle\ dx = 0,$$

which, by a simple integration by parts, is equivalent to

$$\int_\Omega \langle D_i\left(D_\xi f(x, Du)\right), D(\rho^2 D_i u)\rangle\ dx = 0. \tag{3.1}$$

This equality can be written as

$$\begin{aligned} I = &\int_\Omega \langle D_{\xi\xi} f(x, Du)\, D_i Du, \rho^2 D_i Du\rangle\ dx \\ = &-\int_\Omega \langle D_x(D_\xi f(x, Du)), \rho^2 D_i Du\rangle\ dx \\ &-\int_\Omega \langle D_{\xi\xi} f(x, Du)\, D_i Du, 2\rho D\rho D_i u\rangle\ dx \\ &-\int_\Omega \langle D_x(D_\xi f(x, Du)), 2\rho D\rho D_i u)\rangle\ dx \\ = &-II - III - IV \\ \le &\ |II| + |III| + |IV|. \end{aligned} \tag{3.2}$$

In view of the inequality (F5), the bilinear form $(\xi, \eta) \mapsto \langle D_{\xi\xi} f(x, Du)\,\xi, \eta\rangle$ defines for each x an inner product on $\mathbb{R}^{n\times N}$, and so by use of Cauchy–Schwarz' inequality we have

$$\begin{aligned} &\langle D_{\xi\xi} f(x, Du)\, \rho D_i Du, D\rho D_i u\rangle \\ &\quad \le \langle D_{\xi\xi} f(x, Du)\, \rho D_i Du, \rho D_i Du\rangle^{\frac12} \langle D_{\xi\xi} f(x, Du)\, D\rho D_i u, D\rho D_i u\rangle^{\frac12} \end{aligned}$$

on Ω. Hence invoking the Young's inequality we arrive at

$$
\begin{aligned}
|III| &\leq \varepsilon \int_\Omega \langle D_{\xi\xi} f(x, Du)\, D_i Du, \rho^2 D_i Du\rangle \; dx \\
&\quad + c(\varepsilon) \int_\Omega |\langle D_{\xi\xi} f(x, Du)\, D\rho D_i u, D\rho D_i u\rangle| \; dx \\
&\leq \varepsilon \int_\Omega \langle D_{\xi\xi} f(x, Du)\, D_i Du, \rho^2 D_i Du\rangle \; dx \\
&\quad + c(\varepsilon, l) \int_\Omega |D\rho|^2 \, |Du| \log(\mathrm{e} + |Du|) \, dx
\end{aligned} \tag{3.3}
$$

where $0 < \varepsilon < 1$ and we also used the assumption (F4). Inserting estimates (3.3) into (3.2) we obtain

$$
\begin{aligned}
I &= \int_\Omega \langle D_{\xi\xi} f(x, Du)\, D_i Du, \rho^2 D_i Du\rangle \; dx \\
&\leq \varepsilon \int_\Omega \langle D_{\xi\xi} f(x, Du)\, D_i Du, \rho^2 D_i Du\rangle \; dx \\
&\quad + c(\varepsilon, l) \int_\Omega |D\rho|^2 \, |Du| \log(\mathrm{e} + |Du|) \, dx \\
&\quad + |II| + |IV|.
\end{aligned}
$$

Then, reabsorbing the first integral in the right side of the previous estimate by left hand side we get

$$
\begin{aligned}
(1-\varepsilon) I &= (1-\varepsilon) \int_\Omega \langle D_{\xi\xi} f(x, Du)\, D_i Du, \rho^2 D_i Du\rangle \; dx \\
&\leq |II| + c(\varepsilon, l) \int_\Omega |D\rho|^2 \, |Du| \log(\mathrm{e} + |Du|) + |IV|.
\end{aligned} \tag{3.4}
$$

The ellipticity condition (F5) yields that

$$
\nu \int_\Omega \rho^2 \frac{|D^2 u|^2}{1 + |Du|} \leq I. \tag{3.5}
$$

Using the assumption (F3), we obtain

$$
|II| \leq \int_\Omega \rho^2 k(x) \log(\mathrm{e} + |Du|) |D^2 u| \; dx
$$

and

$$
|IV| \leq \int_\Omega k(x) \log(\mathrm{e} + |Du|)\, 2\rho \, |D\rho| \, |Du| \; dx.
$$

Hence, applying the Young's inequality, it follows that

$$
|II| \leq \frac{\nu}{4} \int_\Omega \rho^2 \frac{|D^2 u|^2}{1 + |Du|} \; dx + c(\nu) \int_\Omega \rho^2 k^2(x) (1 + |Du|) \log^2(\mathrm{e} + |Du|) \; dx, \tag{3.6}
$$

and

$$|IV| \le c\int_\Omega \rho^2 k^2(x)|Du|\log^2(\mathrm{e}+|Du|) + c\int_\Omega |D\rho|^2\,|Du|\,dx$$
$$\le c\int_\Omega \rho^2 k^2(x)(1+|Du|)\log^2(\mathrm{e}+|Du|)\,dx$$
$$+ c\int_\Omega |D\rho|^2\,|Du|\log(\mathrm{e}+|Du|)\,dx. \tag{3.7}$$

Inserting estimates (3.5), (3.6) and (3.7) into (3.4), we obtain

$$(1-\varepsilon)\nu\int_\Omega \rho^2\frac{|D^2u|^2}{1+|Du|} \le \frac{\nu}{4}\int_\Omega \rho^2\frac{|D^2u|^2}{1+|Du|}\,dx$$
$$+ c(\nu)\int_\Omega \rho^2 k^2(x)(1+|Du|)\log^2(\mathrm{e}+|Du|)\,dx$$
$$+ c(\varepsilon,l)\int_\Omega |D\rho|^2\,|Du|\,\log(\mathrm{e}+|Du|)\,dx.$$

Choosing $\varepsilon = \frac{1}{4}$ and reabsorbing the first integral in the right side of the previous estimate by left hand side, we obtain

$$\frac{\nu}{2}\int_\Omega \rho^2\frac{|D^2u|^2}{1+|Du|} \le c(\nu)\int_\Omega \rho^2 k^2(x)(1+|Du|)\log^2(\mathrm{e}+|Du|)\,dx$$
$$+ c(l)\int_\Omega |D\rho|^2\,|Du|\,\log(\mathrm{e}+|Du|)\,dx. \tag{3.8}$$

One can easily check that

$$[D(\rho\sqrt{1+|Du|})]^2 \le c\rho^2\frac{|D^2u|^2}{1+|Du|} + c|D\rho|^2(1+|Du|). \tag{3.9}$$

Integrating previous estimate over Ω and recalling (3.8) we get

$$\int_\Omega \left|D(\rho\sqrt{1+|Du|})\right|^2\,dx \le c\int_\Omega \rho^2 k^2(x)(1+|Du|)\log^2(\mathrm{e}+|Du|)\,dx$$
$$+ c\int_\Omega |D\rho|^2\,(1+|Du|)\log(\mathrm{e}+|Du|)\,dx. \tag{3.10}$$

Now, taking in account that

$$\int_\Omega |D\rho|^2\,(1+|Du|)\log(\mathrm{e}+|Du|)\,dx$$
$$= \int_{|Du|\le 1} |D\rho|^2\,(1+|Du|)\log(\mathrm{e}+|Du|)\,dx$$
$$+ \int_{|Du|>1} |D\rho|^2\,(1+|Du|)\log(\mathrm{e}+|Du|)\,dx$$
$$\le 2\int_\Omega |D\rho|^2\,\log(\mathrm{e}+1)\,dx + 2\int_\Omega |D\rho|^2\,|Du|\log(\mathrm{e}+|Du|)\,dx \tag{3.11}$$

and using Sobolev embedding in the left hand side and properties of ρ, from (3.10) we obtain

$$\begin{aligned}&\left(\int_{B_s}\left(\sqrt{1+|Du|}\right)^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\\&\le c\int_{B_t}k^2(x)(1+|Du|)\log^2(\mathrm{e}+|Du|)\,dx\\&\quad+\frac{c}{(t-s)^2}|B_t\backslash B_s|+\frac{c}{(t-s)^2}\int_{B_t\backslash B_s}|Du|\log(\mathrm{e}+|Du|)\,dx\\&=J_1+J_2+J_3.\end{aligned}\tag{3.12}$$

We estimate J_1 using the Hölder type inequality for Zygmund spaces (2.2) with $p_1=\frac{n}{2}$, $p_2=\frac{n}{n-2}$ and $\alpha=0$, $\alpha_1=n$, $\alpha_2=-\frac{2n}{n-2}$ and we get

$$\begin{aligned}J_1&\le c\left(\int_{B_t}k^n(x)\log^n(\mathrm{e}+|k|)\,dx\right)^{\frac{2}{n}}\\&\quad\times\left(\int_{B_t}(1+|Du|)^{\frac{n}{n-2}}\frac{\log^{\frac{2n}{n-2}}(\mathrm{e}+|Du|)}{\log^{\frac{2n}{n-2}}(\mathrm{e}+(1+|Du|)\log^2(\mathrm{e}+|Du|))}\,dx\right)^{\frac{n-2}{n}}\\&\le c\left(\int_{B_t}k^n(x)\log^n(\mathrm{e}+|k|)\,dx\right)^{\frac{2}{n}}\left(\int_{B_t}(1+|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}}.\end{aligned}\tag{3.13}$$

Inserting estimates (3.13) in (3.12), we obtain

$$\begin{aligned}&\left(\int_{B_s}(1+|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\\&\le c\left(\int_{B_t}k^n(x)\log^n(\mathrm{e}+|k|)\,dx\right)^{\frac{2}{n}}\left(\int_{B_t}(1+|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\\&\quad+\frac{c}{(t-s)^2}|B_R|+\frac{c}{(t-s)^2}\int_{B_R}|Du|\log(\mathrm{e}+|Du|)\,dx,\end{aligned}\tag{3.14}$$

where $c=c(n,l,\nu)$.
By the absolute continuity of the integral we can choose R_0 such that

$$c\left(\int_{B_{R_0}}k^n(x)\log^n(\mathrm{e}+|k|)\,dx\right)^{\frac{2}{n}}\le\frac{1}{2}\tag{3.15}$$

so that if $R<R_0$, estimate (3.14) becomes

$$\begin{aligned}&\left(\int_{B_s}(1+|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\\&\le\frac{1}{2}\left(\int_{B_t}(1+|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\\&\quad+\frac{c|B_R|}{(t-s)^2}+\frac{c}{(t-s)^2}\int_{B_R}|Du|\log(\mathrm{e}+|Du|)\,dx.\end{aligned}$$

Since the previous inequality is valid for every radii $\frac{R}{2} < s < t < R$, we can use the iteration Lemma 2.2 with $h(s) = \left(\int_{B_s}(1+|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}}$ and $\theta = \frac{1}{2}$, thus obtaining

$$\left(\int_{B_{\frac{R}{2}}}(1+|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \leq R^{n-2} + \frac{c}{R^2}\int_{B_R}|Du|\log(\mathrm{e}+|Du|)\,dx. \tag{3.16}$$

In view of (3.16) and of the estimates (3.11) and (3.13), by the arbitrariness of the ball $B_R \subset B_{R_0}$, estimate (3.10) can be written as follows

$$\int_{B_{\frac{R}{4}}}\frac{|D^2u|^2}{1+|Du|} \leq \frac{c}{R^2}\int_{B_R}(1+|Du|\log(\mathrm{e}+|Du|))\,dx,$$

where we used also (3.15).

Step 2. The approximation

Fix a compact set $\Omega' \subset \Omega$, and for a smooth kernel $\phi \in \mathrm{C}^\infty_c(B_1(0))$ with $\phi \geq 0$ and $\int_{B_1(0)}\phi = 1$, let us consider the corresponding family of mollifiers $(\phi_\varepsilon)_{\varepsilon>0}$ and put

$$k_\varepsilon = k * \phi_\varepsilon$$

and

$$f_\varepsilon(x,\xi) := f(x,\xi) * \phi_\varepsilon = \int_{B_1}\phi(\omega)f(x+\varepsilon\omega,\xi)\,\mathrm{d}\omega \tag{3.17}$$

on Ω', for each positive $\varepsilon < \mathrm{dist}\,(\Omega',\Omega)$. Note that

$$D_\xi f_\varepsilon(x,\xi) := D_\xi f(x,\xi) * \phi_\varepsilon = \int_{B_1}\phi(\omega)D_\xi f(x+\varepsilon\omega,\xi)\,\mathrm{d}\omega.$$

One can easily check that the assumption (F1) implies

$$c_1|\xi|\log(\mathrm{e}+|\xi|) \leq f_\varepsilon(x,\xi) \leq c_2|\xi|\log(\mathrm{e}+|\xi|) + c_3. \tag{A1}$$

By virtue of assumptions (F2) and (F3), we have that

$$|D_x f_\varepsilon(x,\xi)| \leq |k_\varepsilon(x)||\xi|\log(\mathrm{e}+|\xi|), \tag{A2}$$

and

$$|D_x D_\xi f_\varepsilon(x,\xi)| \leq |k_\varepsilon(x)|\log(\mathrm{e}+|\xi|), \tag{A3}$$

for almost every $x \in \Omega$ and for all $\xi \in \mathbb{R}^{n\times N}$.

Let u be a local minimizer of the functional (1.1) and let fix a ball $B_R \Subset \Omega'$. Let us denote by u_ε the unique minimizer of the functional

$$\mathcal{F}_\varepsilon(v,B_R) := \int_{B_R} f_\varepsilon(x,Dv)\,dx$$

under the boundary condition

$$v = u \quad \text{on } \partial B_R.$$

Since f_ε satisfies the assumptions of Theorem 1.1 in [8], u_ε admits second derivatives and we are legitimate to apply the a priori estimate proved in the previous step

$$\int_{B_{\frac{r}{4}}} \frac{|D^2u_\varepsilon|^2}{1+|Du_\varepsilon|}\, dx \le \frac{c}{r^2}\int_{B_r}(1+|Du_\varepsilon|\log(\mathrm{e}+|Du_\varepsilon|))\, dx, \tag{3.18}$$

for every B_r such that $B_r \subset B_R$. Therefore, by (A1) and by minimality of u_ε we have

$$\begin{aligned}\int_{B_{\frac{r}{4}}} \frac{|D^2u_\varepsilon|^2}{1+|Du_\varepsilon|}\, dx &\le \frac{c}{r^2}\int_{B_r}(1+f_\varepsilon(x,Du_\varepsilon))\, dx\\ &\le \frac{c}{r^2}\int_{B_r}(1+f_\varepsilon(x,Du))\, dx\\ &= \frac{c}{r^2}\int_{B_r}(f_\varepsilon(x,Du)-f(x,Du))\, dx\\ &\quad + \frac{c}{r^2}\int_{B_r} f(x,Du)\, dx + cr^{n-2}. \end{aligned}\tag{3.19}$$

Since f_ε converges to f strongly in L^1, it is bounded in L^1 and then we deduce that $D(\sqrt{1+|Du_\varepsilon|})$ has $L^2(B_{\frac{r}{4}})$ norm bounded independently of ε where B_r is any ball strictly contained in B_R. Therefore, there exists a not relabeled subsequence $\sqrt{1+|Du_\varepsilon|}$ such that

$$\sqrt{1+|Du_\varepsilon|} \rightharpoonup w \quad \text{weakly in } \mathrm{W}^{1,2}(B_r),$$

and so $\sqrt{1+|Du_\varepsilon|}$ converges to w strongly in $L^p(B_r)$, $\forall p < 2^*$ and so $L^p_{\mathrm{loc}}(B_R)$, $\forall p < 2^*$. In particular, by a simple covering argument,

$$\sqrt{1+|Du_\varepsilon|} \quad \to \quad w \qquad \text{a.e. in } B_R. \tag{3.20}$$

We also have, as a consequence, that there exists v such that Du_ε converges to Dv strongly in $L^p(B_{\frac{r}{4}}), \forall p < 2^*$ and a.e. in B_R.

Our next aim is to show that v is a local minimizer of $\mathcal{F}(u, B_R)$. The lower semicontinuity of the functional together with the minimality of u_ε allow us to conclude that, for every $\rho < R$

$$\begin{aligned}\int_{B_\rho} f(x,Dv)\, dx &\le \lim_{\varepsilon\to 0}\int_{B_\rho} f(x,Du_\varepsilon)\, dx \le \lim_{\varepsilon\to 0}\int_{B_R} f(x,Du_\varepsilon)\, dx\\ &= \lim_{\varepsilon\to 0}\int_{B_R}[f(x,Du_\varepsilon)-f_\varepsilon(x,Du_\varepsilon)+f_\varepsilon(x,Du_\varepsilon)]\, dx\\ &\le \lim_{\varepsilon\to 0}\left[\int_{B_R} f(x,Du_\varepsilon)-f_\varepsilon(x,Du_\varepsilon)\, dx\right]\\ &\quad + \lim_{\varepsilon\to 0}\int_{B_R} f_\varepsilon(x,Du)\, dx\\ &= \lim_{\varepsilon\to 0}\left[\int_{B_R} f(x,Du_\varepsilon) - \int_{B_R}\int_{B_1}\phi(\omega)f(x+\varepsilon\omega,Du_\varepsilon)\,\mathrm{d}\omega\, dx\right]\end{aligned}$$

$$+\lim_{\varepsilon\to 0}\left[\int_{B_R} f_\varepsilon(x,Du)-f(x,Du)\ dx\right]+\int_{B_R} f(x,Du)\ dx$$
$$=\lim_{\varepsilon\to 0}A_\varepsilon+\lim_{\varepsilon\to 0}B_\varepsilon+\int_{B_R} f(x,Du)\ dx \tag{3.21}$$

Now, using that $\int_{B_1}\phi(\omega)\,\mathrm{d}\omega=1$ we obtain

$$A_\varepsilon=\int_{B_R}\left(\int_{B_1}[f(x,Du_\varepsilon)-f(x+\varepsilon\omega,Du_\varepsilon)]\phi(\omega)\,\mathrm{d}\omega\right)\,dx \tag{3.22}$$

therefore, by the assumption (F2), we have

$$\begin{aligned}|A_\varepsilon| &\le \int_{B_R}\left(\int_{B_1}|f(x,Du_\varepsilon)-f(x+\varepsilon\omega,Du_\varepsilon)|\,\phi(\omega)\,\mathrm{d}\omega\right)\ dx\\ &=\varepsilon\int_{B_R}\left(\int_{B_1}|(k(x)+k(x+\varepsilon\omega))|Du_\varepsilon|\log(\mathrm{e}+|Du_\varepsilon||\,\phi(\omega)\,\mathrm{d}\omega\right)\ dx\\ &=\varepsilon\int_{B_R}|k(x)+k_\varepsilon(x)||Du_\varepsilon|\log(\mathrm{e}+|Du_\varepsilon|)\ dx\\ &\le\varepsilon\left(\int_{B_R}|k|^n+|k_\varepsilon|^n\right)^{\frac{1}{n}}\left(\int_{B_R}|Du_\varepsilon|^{\frac{n}{n-1}}\log^{\frac{n}{n-1}}(\mathrm{e}+|Du_\varepsilon|)\right)^{\frac{n-1}{n}}\\ &\le c\varepsilon,\end{aligned}$$

where we used also that Du_ε is bounded in $L^{\frac{n}{n-2}}$.
Therefore, as an immediate consequence, we obtain that

$$A_\varepsilon\to 0\quad\text{as}\quad\varepsilon\to 0. \tag{3.23}$$

On the other hand, since f_ε is the convolution of f also

$$B_\varepsilon\to 0\quad\text{as}\quad\varepsilon\to 0. \tag{3.24}$$

Combining (3.23) and (3.24) with (3.21), we have that

$$\int_{B_\rho} f(x,Dv)\ dx\le\int_{B_R} f(x,Du)\ dx,\quad\forall\rho<R.$$

Taking the limit as $\rho\to R$ we get

$$\int_{B_R} f(x,Dv)\ dx\le\int_{B_R} f(x,Du)\ dx\le\int_{B_R} f(x,Dv)\ dx,$$

which implies that

$$u\equiv v\quad\text{a.e. in } B_R$$

since the minimizer is unique thanks to the strictly convexity of f.

By passing to the limit as $\varepsilon \searrow 0$ in (3.18), thanks to Fatou's Lemma, we finally get

$$\begin{aligned}\int_{B_{\frac{r}{4}}} \frac{|D^2u|^2}{1+|Du|}\, dx &\le \lim_{\varepsilon\searrow 0} \int_{B_{\frac{r}{4}}} \frac{|D^2u_\varepsilon|^2}{1+|Du_\varepsilon|}\, dx \\ &\le \frac{c}{r^2} \lim_{\varepsilon\searrow 0} \int_{B_r} (1+|Du_\varepsilon|\log(\mathrm{e}+|Du_\varepsilon|))\, dx \\ &= \frac{c}{r^2} \int_{B_r} (1+|Du|\log(\mathrm{e}+|Du|))\, dx\end{aligned}$$

and this concludes the proof. □

4. Proof of Theorem 1.2

In this section we deal with a priori bounded minimizers of the integral functional (1.1). The main tool used in the next proof is a suitable interpolation type inequality.

Proof of Theorem 1.2. Let R_0 be the radius we found in Theorem 1.1, fix concentric balls $B_R \subset B_{8R} \subset B_{R_0}$ and a cut-off function $\rho \in C^1_c(B_{2R})$. Integration by parts yields

$$\begin{aligned}\int_\Omega \rho^2|Du|^3\, dx &= \int_\Omega \langle \rho^2|Du|Du, Du\rangle\, dx \\ &= -\int_\Omega D(\rho^2|Du|Du)\cdot u\, dx \\ &\le 2\int_\Omega \rho^2|u||Du||D^2u|\, dx \\ &\quad + \int_\Omega 2\rho|\nabla\rho||u||Du|^2\, dx = I_1 + I_2. \end{aligned} \tag{4.1}$$

Since u is locally bounded in Ω, we have

$$I_1 \le \|u\|_{L^\infty(B_{2R})} \int_\Omega \rho^2|Du| \frac{|D^2u|}{\sqrt{1+|Du|}} \sqrt{1+|Du|}\, dx.$$

By using the Young's inequality and thanks to Theorem 1.1, from the previous inequality we obtain

$$\begin{aligned}I_1 &\le c\|u\|^2_{L^\infty(B_{2R})} \int_\Omega \rho^2 \frac{|D^2u|^2}{1+|Du|}\, dx + \frac{1}{4}\int_\Omega \rho^2|Du|^2(1+|Du|)\, dx \\ &\le c\|u\|^2_{L^\infty(B_{2R})} \int_\Omega \rho^2 \frac{|D^2u|^2}{1+|Du|}\, dx + \frac{1}{4}\int_\Omega \rho^2|Du|^3\, dx + cR^n. \end{aligned} \tag{4.2}$$

Now, we use again the Young's inequality to estimate the integral I_2 and we have

$$
\begin{aligned}
I_2 &\le \frac{1}{4}\int_\Omega \rho^2|Du|^3\,dx + c\int_\Omega |u|^2|\nabla\rho|^2|Du|\,dx \\
&\le \frac{1}{4}\int_\Omega \rho^2|Du|^3\,dx + \frac{c\|u\|^2_{L^\infty(B_{2R})}}{R^2}\int_{B_{2R}}|Du|\,dx. \qquad (4.3)
\end{aligned}
$$

Hence, inserting (4.2) and (4.3) in (4.1), we get

$$
\begin{aligned}
\int_\Omega \rho^2|Du|^3\,dx \le \frac{1}{2}\int_\Omega \rho^2|Du|^3\,dx
&+ c\|u\|^2_{L^\infty(B_{2R})}\int_\Omega \rho^2\frac{|D^2u|^2}{1+|Du|}\,dx \\
&+ \frac{c\|u\|^2_{L^\infty(B_{2R})}}{R^2}\int_{B_{2R}}|Du|\,dx. \qquad (4.4)
\end{aligned}
$$

Reabsorbing the first integral in the right hand side by the left hand side and using the estimate (1.3) in Theorem 1.1, we conclude

$$
\int_{B_R}|Du|^3\,dx \le \frac{c\|u\|^2_{L^\infty(B_{2R})}}{R^2}\int_{B_{8R}}(1+|Du|\log(\mathrm{e}+|Du|)\,dx,
$$

where $c = c(n,l,\nu)$. □

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