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A class of history-dependent variational-hemivariational inequalities

Mircea Sofonea and Stanisław Migórski

Abstract. We consider a new class of variational-hemivariational inequalities which arise in the study of quasistatic models of contact. The novelty lies in the special structure of these inequalities, since each inequality of the class involve unilateral constraints, a history-dependent operator and two nondifferentiable functionals, of which at least one is convex. We prove an existence and uniqueness result of the solution. The proof is based on arguments on elliptic variational-hemivariational inequalities obtained in our previous work [23], combined with a fixed point result obtained in [30]. Then, we prove a convergence result which shows the continuous dependence of the solution with respect to the data. Finally, we present a quasistatic frictionless problem for viscoelastic materials in which the contact is modeled with normal compliance and finite penetration and the elasticity operator is associated to a history-dependent Von Mises convex. We prove that the variational formulation of the problem cast in the abstract setting of history-dependent quasivariational inequalities, with a convenient choice of spaces and operators. Then we apply our general results in order to prove the unique weak solvability of the contact problem and its continuous dependence on the data.

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Keywords. Variational-hemivariational inequality, Clarke subdifferential, history-dependent operator, viscoelastic material, Von Mises convex, frictionless contact, normal compliance, unilateral constraint, weak solution.

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1. Introduction

Variational and hemivariational inequalities play an important role in the study of both the qualitative and numerical analysis of various boundary value problems and mathematical models arising in Contact Mechanics. The theory of variational inequalities started in early sixties, as illustrated in the works [1–3,9,10,17,18,26,33] and the references therein. It intensively uses arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. In contrast, the theory of hemivariational inequalities is based on properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions which may be nonconvex. Its development started in early eighties. Main references in the field include [4,5,14,21,24,27]. Applications of the variational and hemivariational inequalities in Contact Mechanics in can be found in [7,8,12,13,15,17,19,21,26–29], among others. Variational-hemivariational inequalities represent a special class of inequalities, in which both convex and nonconvex functions are involved. Interest in their study is motivated by various problems in Mechanics, as shown in [11,22,24,25].

History-dependent operators represent an important class of operators defined on vector-valued function spaces. They arise both in functional analysis, theory of differential equations and partial differential equations, and in Contact Mechanics, as well. Some simple examples in analysis are the integral operator and the Volterra-type operators. In Contact Mechanics, history-dependent operators could arise both in the constitutive law of the material and in the frictional contact conditions. The memory term in the viscoelastic consitutive laws, the total slip, the total slip rate and the accumulated penetration represent simple examples of history-dependent operators, among others. For all these reasons, the need to consider variational and hemivariational inequalities involving history-dependent operators was widely accepted.

A first step in this direction was made in [32]. There, a class of quasivariational inequalities involving history-dependent operators was considered, an abstract existence and uniqueness result was proved and it was completed with a regularity result. One of the novelties of the above mentioned paper arises in the fact that the inequalities were formulated on the unbounded interval of time $[0, +\infty)$. In addition, the results obtained there were used in the study of a large number of quasistatic frictional of frictionless contact problems. The numerical analysis of the quasivariational inequalities introduced in [32], including the unique solvability of discrete schemes and error estimates, was provided in [16]. An extension of the existence and uniquenss result in [32] was recently obtained in [34].

On the other hand, the first existence and uniqueness result in the study of subdifferential inclusions with history-dependent operators was obtained in [20]. The proof was based on arguments on pseudomonotone operators and fixed point. This result was used then to prove the unique solvability of a class of history-dependent hemivariational inequalities.

The study of variational-hemivariational inequalities with history-dependent operators is currently emerging. Thus, a first example was considered and studied in [22], related to the unique solvability of a contact model for viscoleastic materials. The novelty of this model arises from the new boundary condition used, which describes both the instantaneous and the memory effects of the foundation. The analysis of the quasivariational inequalities introduced in [22] was continued in [31], where a continuous dependence result was proved. There, numerical schemes to solve the inequalities were also presented and error estimates were derived.

The history-dependent hemivariational inequalities studied in [20] as well as the history-dependent variational-hemivariational inequalities considered in [22,31] were formulated in the particular case of Sobolev spaces associated to a bounded domain and to specific operators like the trace operator, for instance. In contrast, the history-dependent variational inequalities considered in [16,32,34] are formulated in the framework of abstract Hilbert spaces. So is the case of the elliptic variational-hemivariational inequalitities studied recently in [23]. This situation rises the need to extend the results obtained in [20,22,31] to an abstract class of history-dependent variational-hemivariational inequalities and to provide the history-dependent version of the results obtained in [23]. The aim of this paper is to fill this gap by obtaining such kind of abstract results.

The rest of the paper is organized as follows. In Sect. 2 we review some preliminary material on nonlinear analysis. In Sect. 3 we introduce the class of history-variational-hemivariational inequalities to be studied, then we state and prove an abstract existence and uniqueness result, Theorem 5. The proof is based on arguments on elliptic variational-hemivariational inequalities obtained in our previous work [23], combined with a fixed point result obtained in [30]. In Sect. 4 we study the continuous dependence of the solution with respect to the data and establish a convergence result. Next, in Sect. 5, we introduce a frictionless contact problem in which the material behavior is modeled with a viscoelastic constitutive law with long memory and the contact conditions are with normal compliance and unilateral constraints. We list the assumptions on the problem data and derive the weak formulation of the problem, which is in a form of a variational-hemivariational inequality for the displacement field. Finally, in Sect. 6 we apply our abstract results in the analysis of this contact problem.

2. Preliminaries

In this section we present some notation, definitions and preliminary results used later in this paper. More details on the material presented below can be found in the books [4–6,21,24,33].

First, we precise that all linear spaces used in this paper are assumed to be real. For a normed space X we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing of X and X^*. Sometimes, when no confusion could arise, we simply write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{X^*\times X}$. The symbol 2^{X^*} is used to represent the family of all subsets of X^*.

Recall that an operator $A\colon X \to X^*$ is called pseudomonotone, if it is bounded and $u_n \to u$ weakly in X together with $\limsup \langle Au_n, u_n - u\rangle_{X^*\times X} \le 0$ imply

$$\langle Au, u - v\rangle_{X^*\times X} \le \liminf \langle Au_n, u_n - v\rangle_{X^*\times X}$$

for all $v \in X$. A function $\varphi\colon K \subset X \to \mathbb{R}$ is lower semicontinuous (l.s.c, for short), if for any sequence $\{x_n\} \subset K$ and any $x \in K$, $x_n \to x$ in X implies $\varphi(x) \le \liminf \varphi(x_n)$. For a convex function φ, the mapping $\partial\varphi\colon X \to 2^{X^*}$ defined by

$$\partial\varphi(x) = \{\, x^* \in X^* \mid \varphi(v) - \varphi(x) \ge \langle x^*, v - x\rangle_{X^*\times X} \ \text{ for all } \ v \in X\}$$

is called the subdifferential of φ. An element $x^* \in \partial\varphi(x)$ (if any) is called a subgradient of φ in x.

Let $h\colon X \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of h at $x \in X$ in the direction $v \in X$, denoted by $h^0(x; v)$, is defined by

$$h^0(x; v) = \limsup_{y\to x,\ \lambda\downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

The generalized gradient (subdifferential) of h at x, denoted by $\partial h(x)$, is a subset of the dual space X^* given by

$$\partial h(x) = \{\, \zeta \in X^* \mid h^0(x; v) \ge \langle \zeta, v\rangle_{X^*\times X} \ \text{ for all } \ v \in X\}.$$

We recall in what follows the following existence and uniqueness result for elliptic variational-hemivariational inequalities.

Theorem 1. *Let X be a reflexive Banach space and assume that the following conditions are satisfied.*

$$K \text{ is a nonempty, closed and convex subset of } X. \tag{2.1}$$

$$\begin{cases} A\colon X \to X^* \text{ is an operator such that} \\ \text{(a) } A \textit{ is pseudomonotone and there exist} \\ \qquad \alpha_A > 0, \beta_A, \gamma_A \in \mathbb{R} \textit{ and } u_0 \in K \textit{ such that} \\ \qquad \langle Av, v - u_0\rangle \ge \alpha_A \|v\|_X^2 - \beta_A\|v\|_X - \gamma_A \quad \textit{for all } v \in X. \\ \text{(b) } A \textit{ is strongly monotone, i.e., there exists } m_A > 0 \quad \textit{such that} \\ \qquad\quad \langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A\|v_1 - v_2\|_X^2 \quad \textit{for all } v_1, v_2 \in X. \end{cases} \tag{2.2}$$

$$\begin{cases} \phi\colon K \times K \to \mathbb{R} \textit{ is a function such that} \\ \text{(a)} \quad \phi(u, \cdot)\colon K \to \mathbb{R} \textit{ is convex and l.s.c. on } \ K, \quad \textit{for all } u \in K. \\ \text{(b)} \quad \textit{there exists } \alpha_\phi > 0 \textit{ such that} \\ \qquad\quad \phi(u_1, v_2) - \phi(u_1, v_1) + \phi(u_2, v_1) - \phi(u_2, v_2) \\ \qquad\qquad \le \alpha_\phi \|u_1 - u_2\|_X \|v_1 - v_2\|_X \quad \textit{for all } u_1,\, u_2,\, v_1, v_2 \in K. \end{cases} \tag{2.3}$$

$$\begin{cases} j\colon X \to \mathbb{R} \textit{ is a function such that} \\ \text{(a)} \quad j \quad \textit{is locally Lipschitz;} \\ \text{(b)} \quad \|\partial j(v)\|_{X^*} \le c_0 + c_1\|v\|_X \quad \textit{for all} \quad v \in X \quad \textit{with } c_0, c_1 \ge 0. \\ \text{(c)} \quad \textit{there exists} \quad \alpha_j > 0 \quad \textit{such that} \\ \qquad\quad j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \le \alpha_j \|v_1 - v_2\|_X^2 \\ \qquad\qquad \textit{for all} \quad v_1, v_2 \in X. \end{cases} \tag{2.4}$$

$$\alpha_\phi + \alpha_j < m_A. \tag{2.5}$$

$$\alpha_j < \alpha_A. \tag{2.6}$$

$$f \in X^*. \tag{2.7}$$

Then, there exists a unique element $u \in K$ such that

$$\langle Au, v-u\rangle + \phi(u,v) - \phi(u,u) + j^0(u; v-u) \geq \langle f, v-u\rangle \quad \text{for all } v \in K. \tag{2.8}$$

Theorem 1 represents the first ingredient we use to prove the unique solvability of the history-dependent variational-hemivariational inequalities we consider in this paper. Its proof was obtained in our recent paper [23] and was carried out in several steps, based on a surjectivity result for multivalued pseudomonotone operators and the Banach fixed point argument. We complete the statement of Theorem 1 with the remark that for a locally Lipschitz function $j\colon X \to \mathbb{R}$, the hypothesis (2.4)(c) is equivalent to the condition

$$\langle \partial j(v_1) - \partial j(v_2), v_1 - v_2\rangle_{X^*\times X} \geq -\alpha_j \, \|v_1 - v_2\|_X^2 \quad \text{for all } \ v_1, v_2 \in X. \tag{2.9}$$

The latter is the so-called relaxed monotonicity condition and it was extensively used in the literature, see for instance [21] and the references therein. Examples of nonconvex functions which satisfy condition (2.4) can be found in [22]. Note also that if $j\colon X \to \mathbb{R}$ is a convex function, then (2.4)(c) or (2.9), equivalently, are satisfied with $\alpha_j = 0$, due to the monotonicity of the (convex) subdifferential. For $\alpha_j = 0$, the condition (2.6) is also automatically satisfied.

We now move to some preliminaries on spaces of continuous functions and related operators which we need in the rest of the paper. First, we recall that we use $\mathbb{N}$ for the set of positive integers and $\mathbb{R}_+$ for the set of nonnegative real numbers, i.e., $\mathbb{R}_+ = [0, +\infty)$. For a normed space X we use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X, and $C^1(\mathbb{R}_+; X)$ for the space of continuous differentiable functions defined on $\mathbb{R}_+$ with values in X. For a subset $K \subset X$ we still use the symbols $C(\mathbb{R}_+; K)$ and $C^1(\mathbb{R}_+; K)$ for the set of continuous and continuously differentiable functions defined on $\mathbb{R}_+$ with values on K, respectively. It is well known that, if X is a Banach space, then $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e., a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Moreover, the convergence of a sequence $\{x_k\}_k$ to the element x, in the space $C(\mathbb{R}_+; X)$ can be described as follows

$$\begin{cases} x_k \to x \quad \text{in } C(\mathbb{R}_+; X) \quad \text{as } k \to \infty \quad \text{if and only if} \\ \max\limits_{r\in[0,n]} \|x_k(r) - x(r)\|_X \to 0 \quad \text{as } k \to \infty, \quad \text{for all } n \in \mathbb{N}. \end{cases} \tag{2.10}$$

In other words, the sequence $\{x_k\}_k$ converges to the element x in the space $C(\mathbb{R}_+; X)$ if and only if it converges to x in the space $C([0,n]; X)$ for all $n \in \mathbb{N}$, $C([0,n]; X)$ being the space of continuous functions defined on the compact interval $[0,n]$ with values in X, endowed with its canonical norm.

Consider now two normed spaces X and Y. An operator $\mathcal{S}\colon C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ is called a history-dependent operator, if it satisfies the following condition

$$\begin{cases} \text{for any } \ n \in \mathbb{N} \text{ there exists } \ s_n > 0 \text{ such that} \\ \qquad \|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_Y \le s_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \ \text{for all } u_1,\, u_2 \in C(\mathbb{R}_+; X), \quad \text{for all } t \in [0, n]. \end{cases} \tag{2.11}$$

Examples of such kind of operators can be found in [20,32]. Here, we restrict ourselves to recall that the class of history-dependent operators includes Volterra-type operators and, more generaly, various integral-type operators.

We now recall the following fixed point result.

Theorem 2. *Let $(X, \|\cdot\|_X)$ be a Banach space and let $\Lambda\colon C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be a nonlinear operator. Assume that there exists $k \in \mathbb{N}$ with the following property: for all $n \in \mathbb{N}$ there exist two constants $c_n \ge 0$ and $d_n \in [0, 1)$ such that*

$$\|(\Lambda u)(t) - (\Lambda v)(t)\|_X^k \le c_n \int_0^t \|u(s) - v(s)\|_X^k \, ds + d_n \, \|u(t) - v(t)\|_X^k$$

for all u, $v \in C(\mathbb{R}_+; X)$ and for all $t \in [0, n]$. Then the operator Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.

Theorem 2 represents the second ingredient we use to prove the unique solvability of the history-dependent variational-hemivariational inequalities we consider in this paper. Its proof could be found in [30].

We end this section with some results we shall use in the last part of the paper, in the study of a frictionless contact problem. Let $d \in \mathbb{N}$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. Recall that the canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i),\, \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\, \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. Here and below in this paper the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used.

For a tensor $\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d$ we use the notation $\boldsymbol{\tau}^D$ for its deviatoric part, defined by $\boldsymbol{\tau}^D = \boldsymbol{\tau} - \frac{1}{d}\,(tr\,\boldsymbol{\tau})\,\boldsymbol{I}_d$ where $\boldsymbol{I}_d \in \mathbb{S}^d$ is the identity tensor and $tr\,\boldsymbol{\tau} = \tau_{ii}$. The Von Mises convex is a nonempty, convex and closed set in the space of symmetric tensors of the second order. It is defined by

$$M(\kappa) = \{\ \boldsymbol{\tau} \in \mathbb{S}^d \mid \|\boldsymbol{\tau}^D\| \le \kappa\}, \tag{2.12}$$

for a given $\kappa \ge 0$. It is used in the literature as ingredient in the structure of some nonlinear elastic, viscoelastic and viscoplastic constitutive laws, respectively. It is also intensively used in the theory of plasticity, as explained in [12,33], for instance. Note also that, on occasion, the Von Mises convex (2.12) is defined with a bound κ which depends on the mechanical process under consideration.

In the present paper we shall use the following result.

Proposition 3. *Let κ_1, $\kappa_2 > 0$ and let $M(\kappa_i)$ be the Von Mises convex sets defined by (2.12) for $\kappa = \kappa_i$, $i = 1, 2$. Denote by $P_{M(\kappa_i)}\colon \mathbb{S}^d \to M(\kappa_i)$ the projections operators on the convex $M(\kappa_i)$, for $i = 1, 2$. Then, the following inequality holds*

$$\|P_{M(\kappa_1)}\boldsymbol{\tau} - P_{M(\kappa_2)}\boldsymbol{\tau}\| \le |\kappa_1 - \kappa_2| \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^d. \tag{2.13}$$

A proof of Proposition 3 as well as more details related to the Von Mises convex can be found in [33].

3. An existence and uniqueness result

Let X be a reflexive Banach space and let Y be a normed space. Let K be a subset of X and let $A\colon X \to X^*$, $\mathcal{S}\colon C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ be given operators. Consider also a function $\varphi\colon Y \times K \times K \to \mathbb{R}$, a locally Lipschitz function $j\colon X \to \mathbb{R}$ and a function $f\colon \mathbb{R}_+ \to X^*$. We associate with these data the following problem.

Problem 4. *Find a function $u \in C(\mathbb{R}_+; K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds*

$$\begin{aligned} &\langle Au(t), v - u(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), u(t)) \\ &\quad + j^0(u(t); v - u(t)) \ge \langle f(t), v - u(t)\rangle \quad \text{for all } v \in K. \end{aligned} \tag{3.1}$$

In the study of Problem 4, besides the assumptions on K, A and j already introduced in Sect. 2, we consider the following hypotheses.

$$\left\{\begin{array}{l} \mathcal{S}\colon C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y) \ \text{ is a history-dependent operator,} \\ \qquad \text{i.e., it satisfies condition (2.11).} \end{array}\right. \tag{3.2}$$

$$\left\{\begin{array}{l} \varphi\colon Y \times K \times K \to \mathbb{R} \quad \text{is a function such that} \\ \quad \text{(a) } \varphi(y, u, \cdot)\colon K \to \mathbb{R} \text{ is convex and l.s.c. on } K, \quad \text{for all } y \in Y,\, u \in K. \\ \quad \text{(b) there exists } \alpha_\varphi > 0 \text{ and } \beta_\varphi > 0 \quad \text{such that} \\ \qquad \varphi(y_1, u_1, v_2) - \varphi(y_1, u_1, v_1) + \varphi(y_2, u_2, v_1) - \varphi(y_2, u_2, v_2) \\ \qquad\quad \le \alpha_\varphi \|u_1 - u_2\|_X \, \|v_1 - v_2\|_X + \beta_\varphi \|y_1 - y_2\|_Y \, \|v_1 - v_2\|_X \\ \qquad\qquad \text{for all } y_1, y_2 \in Y,\ u_1, u_2, v_1, v_2 \in K. \end{array}\right. \tag{3.3}$$

$$\alpha_\varphi + \alpha_j < m_A. \tag{3.4}$$

$$f \in C(\mathbb{R}_+; X^*). \tag{3.5}$$

Concerning the above assumptions we have the following comments. First, we stress that the function φ is assumed to be convex with respect to its third argument while the function j is locally Lipschitz in the second argument and could be nonconvex. For this reason, inequality (3.1) represents, in fact, a *variational-hemivariational inequality*. In addition, the function φ in (3.1) depends on the operator $\mathcal{S}$, assumed to be history-dependent. To combine these two ingredients, we refer to Problem 4 as a *history-dependent variational-hemivariational inequality*. In the study of this problem we have the following existence and uniqueness result.

Theorem 5. *Let X be a reflexive Banach space, Y a normed space, and assume that* (2.1)*,* (2.2)*,* (2.4)*,* (2.6) *and* (3.2)*–*(3.5) *hold. Then, Problem* 4 *has a unique solution $u \in C(\mathbb{R}_+; K)$.*

The proof of Theorem 5 is based on a fixed point argument and will be established in several steps. We assume in what follows that (2.1), (2.2), (2.4), (2.6) and (3.2)–(3.5) hold. In the first step, let $\eta \in C(\mathbb{R}_+; X)$ be fixed and denote by $y_\eta \in C(\mathbb{R}_+; Y)$ the function given by

$$y_\eta(t) = (\mathcal{S}\eta)(t) \quad \text{for all } \ t \in \mathbb{R}_+. \tag{3.6}$$

We consider the intermediate problem of finding a function $u_\eta\colon \mathbb{R}_+ \to K$ such that for all $t \in \mathbb{R}_+$, the following inequality holds

$$\begin{aligned} &\langle A(u_\eta(t)), v - u_\eta(t)\rangle + \varphi(y_\eta(t), u_\eta(t), v) - \varphi(y_\eta(t), u_\eta(t), u_\eta(t)) \\ &\quad + j^0(u(t); v - u(t)) \geq \langle f(t), v - u(t)\rangle \quad \text{for all } \ v \in K. \end{aligned} \tag{3.7}$$

The unique solvability of this problem is provided by the following result.

Lemma 6. *There exists a unique function $u_\eta \in C(\mathbb{R}_+; K)$ which solves the inequality* (3.7)*, for all $t \in \mathbb{R}_+$.*

Proof. Let $t \in \mathbb{R}_+$ be fixed and denote by $\phi\colon K \times K \to \mathbb{R}$ the function given by

$$\phi(u, v) = \varphi(y_\eta(t), u, v) \quad \text{for all } \ u, \, v \in K. \tag{3.8}$$

The function ϕ depends on η and t but, for simplicity, we do not indicate it explicitly. Then, using assumption (3.3), it is obvious to see that the function ϕ satisfies condition (2.3) with constant $\alpha_\phi = \alpha_\varphi$. Therefore, from (3.4), we deduce that the smallness condition (2.5) is satisfied. Thus, assumptions (2.1), (2.2), (2.4) and (2.6) allows to apply Theorem 1 in order to obtain that there exists a unique element $u_\eta(t) \in K$ which verifies (3.7).

Let us show that the map $t \mapsto u_\eta(t)\colon \mathbb{R}_+ \to K$ is continuous. To this end, consider $t_1, \, t_2 \in \mathbb{R}_+$ and, for the sake of simplicity in writing, denote $u_\eta(t_i) = u_i$, $y_\eta(t_i) = y_i$, $f(t_i) = f_i$ for $i = 1, \, 2$. Using (3.7), we obtain

$$\begin{aligned} &\langle Au_1, v - u_1\rangle + \varphi(y_1, u_1, v) - \varphi(y_1, u_1, u_1) \\ &\quad + j^0(u_1; v - u_1) \geq \langle f_1, v - u_1\rangle \quad \text{for all } \ v \in K, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\langle Au_2, v - u_2\rangle + \varphi(y_2, u_2, v) - \varphi(y_2, u_2, u_1) \\ &\quad + j^0(u_2; v - u_2) \geq \langle f_2, v - u_2\rangle \quad \text{for all } \ v \in K. \end{aligned} \tag{3.10}$$

We take $v = u_2$ in (3.9) and $v = u_1$ in (3.10), then we add the resulting inequalities to find that

$$\begin{aligned} &\langle Au_1 - Au_2, u_1 - u_2\rangle \\ &\quad \leq \varphi(y_1, u_1, u_2) - \varphi(y_1, u_1, u_1) + \varphi(y_2, u_2, u_1) - \varphi(y_2, u_2, u_2) \\ &\qquad + j^0(u_1; u_2 - u_1) + j^0(u_2; u_1 - u_2) + \langle f_1 - f_2, u_1 - u_2\rangle. \end{aligned}$$

We now use hypotheses (2.2)(b), (2.4)(c) and (3.3)(b) to obtain

$$\begin{aligned} &m_A \, \|u_1 - u_2\|_X \\ &\quad \leq \alpha_\varphi \, \|u_1 - u_2\|_X + \beta_\varphi \, \|y_1 - y_2\|_Y + \alpha_j \, \|u_1 - u_2\|_X + \|f_1 - f_2\|_{X^*}. \end{aligned}$$

Thus, the smallness asumption (3.4) implies that

$$\|u_1 - u_2\|_X \le C\left(\|y_1 - y_2\|_Y + \|f_1 - f_2\|_{X^*}\right) \tag{3.11}$$

with

$$C = \frac{1}{m_A - \alpha_\varphi - \alpha_j}\ \max\{\beta_\varphi, 1\} > 0. \tag{3.12}$$

Inequallity (3.11) shows that $t \mapsto u_\eta(t)\colon \mathbb{R}_+ \to K$ is a continuous function, which concludes the existence part of the lemma. The uniqueness part is a direct consequence of the uniqueness of the element $u_\eta(t)$ which solves the variational-hemivariational inequality (3.7) for each $t \in \mathbb{R}_+$. □

In the next step we use Lemma 6 to define the operator $\Lambda_\eta\colon C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; K) \subset C(\mathbb{R}_+; X)$ by equality

$$\Lambda\eta = u_\eta \quad \text{for all } \ \eta \in C(\mathbb{R}_+; X). \tag{3.13}$$

We have the following fixed point result.

Lemma 7. *The operator Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; K)$.*

Proof. Let η_1, $\eta_2 \in C(\mathbb{R}_+; X)$ and let y_i be the functions defined by (3.6) for $\eta = \eta_i$, i.e., $y_i = y_{\eta_i}$ for $i = 1$, 2. We also denote by u_i the solution of the variational-hemivariational inequality (3.7) for $\eta = \eta_i$, i.e., $u_i = u_{\eta_i}$, $i = 1$, 2. Let $n \in \mathbb{N}$ and let $t \in [0, n]$. From definition (3.13), we have

$$\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_X = \|u_1(t) - u_2(t)\|_X. \tag{3.14}$$

Moreover, an argument similar to that used in the proof of (3.11) shows that

$$\|u_1(t) - u_2(t)\|_X \le C\,\|y_1(t) - y_2(t)\|_Y \tag{3.15}$$

with C given by (3.12). Next, we use (3.6) and the property (2.11) of the operator $\mathcal{S}$, to see that

$$\|y_1(t) - y_2(t)\|_Y \le s_n \int_0^t \|\eta_1(s) - \eta_2(s)\|_X\, ds.$$

Therefore, using this inequality in (3.15) yields

$$\|u_1(t) - u_2(t)\|_X \le C s_n \int_0^t \|\eta_1(s) - \eta_2(s)\|_X\, ds. \tag{3.16}$$

We now combine (3.14) and (3.16) to see that

$$\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_X \le C s_n \int_0^t \|\eta_1(s) - \eta_2(s)\|_X\, ds. \tag{3.17}$$

Finally, we use (3.17) and Theorem 2 with $k = 1$, $c_n = C s_n$ and $d_n = 0$ to see that the operator Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$. Since Λ has values in $C(\mathbb{R}_+; K)$, we deduce that $\eta^* \in C(\mathbb{R}_+; K)$, which concludes the proof. □

We now have all the ingredients needed to prove Theorem 5.

Proof. Existence. Let $\eta^* \in C(\mathbb{R}_+;K)$ be the fixed point of the operator Λ. It follows from (3.6) and (3.13) that, for all $t \in \mathbb{R}_+$, the following equalities hold

$$y_{\eta^*}(t) = (\mathcal{S}\eta^*)(t) \quad \text{and} \quad u_{\eta^*}(t) = \eta^*(t). \tag{3.18}$$

We now write the inequality (3.7) for $\eta = \eta^*$ and then use the equalities (3.18) to conclude that the function $\eta^* \in C(\mathbb{R}_+;K)$ is a solution to the variational-hemivariational inequality (3.1).

Uniqueness. The uniqueness part is a consequence of the uniqueness of the fixed point of the operator Λ. A direct proof can be obtained by using a Gronwall-type argument and is as follows. Assume that u_1, u_2 are two solutions of the variational inequality (3.1) with regularity $C(\mathbb{R}_+;K)$ and let $n \in \mathbb{N}$. Also, let $t \in [0,n]$. We use (3.1) to see that

$$\begin{aligned}
&\langle Au_1(t) - Au_2(t), u_1(t) - u_2(t)\rangle \\
&\quad \le \varphi((\mathcal{S}u_1)(t), u_1(t), u_2(t)) - \varphi((\mathcal{S}u_1)(t), u_1(t), u_1(t)) \\
&\qquad + \varphi((\mathcal{S}u_2)(t), u_2(t), u_1(t)) - \varphi((\mathcal{S}u_2(t)), u_2(t), u_2(t)) \\
&\qquad + j^0(u_1(t); u_2(t) - u_1(t)) + j^0(u_2(t); u_1(t) - u_2(t))
\end{aligned}$$

and then, by arguments similar to those used in the proof of (3.11), we deduce that

$$\|u_1(t) - u_2(t)\|_X \le C\,\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t))\|_Y.$$

We use this inequality and assumption (2.11) to find that

$$\|u_1(t) - u_2(t)\|_X \le C s_n \int_0^t \|u_1(s) - u_2(s)\|_X\, ds.$$

Next, it follows from the Gronwall inequality that $u_1(t) = u_2(t)$ for all $t \in [0,n]$. This implies that $u_1(t) = u_2(t)$ for all $t \in \mathbb{R}_+$ and concludes the proof of the uniqueness part. □

We end this section with a version of Theorem 5 in the study of first order evolutionary variational-hemivariational inequalities which is useful in various applications. For this, we assume that

$$\left\{\begin{array}{l} B\colon X \to Y \ \text{ is a Lipschitz continuous operator, i.e.,} \\ \quad \text{there exists } L_B > 0 \text{ such that} \\ \qquad \|Bu_1 - Bu_2\|_Y \le L_B\|u_1 - u_2\|_X \quad \text{for all } \ u_1,\, u_2 \in X. \end{array}\right. \tag{3.19}$$

$$u_0 \in X. \tag{3.20}$$

Then, the problem under consideration can be formulated as follows.

Problem 8. *Find a function $u \in C^1(\mathbb{R}_+;X)$ such that $u(0) = u_0$ and, for all $t \in \mathbb{R}_+$, the following inequality holds*

$$\begin{aligned}
&u'(t) \in K, \quad \langle Au'(t), v - u'(t)\rangle + \varphi(Bu(t), u'(t), v) - \varphi(Bu(t), u'(t), u'(t)) \\
&\qquad + j^0(u'(t); v - u'(t)) \ge \langle f, v - u'(t)\rangle \quad \text{for all } v \in K.
\end{aligned} \tag{3.21}$$

Theorem 9. *Let X be a reflexive Banach space, Y a normed space and assume that (2.1), (2.2), (2.4), (2.6), (3.3)–(3.5), (3.19) and (3.20) hold. Then, Problem 8 has a unique solution which satisfies $u \in C^1(\mathbb{R}_+;X)$.*

Proof. Let $w = u'$ and denote by $\mathcal{S}\colon C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;Y)$ the operator given by

$$(\mathcal{S}v)(t) = B\Big(\int_0^t w(s)\,ds + u_0\Big) \quad \text{for all } \ v \in C(\mathbb{R}_+;X),\ t \in \mathbb{R}_+. \tag{3.22}$$

Then, it is easy to see that the function u is a solution to Problem 8 with regularity $u \in C^1(\mathbb{R}_+;X)$ if and only if w is a solution to the history dependent variational-hemivariational inequality

$$\begin{aligned}&\langle Aw(t), v - w(t)\rangle + \varphi((Sw)(t), w(t), v) - \varphi((Sw)(t), w(t), w(t))\\ &\qquad + j^0(w(t); v - w(t)) \geq \langle f, v - w(t)\rangle \quad \text{for all } v \in K,\ t \in \mathbb{R}_+,\end{aligned} \tag{3.23}$$

with regularity $w \in C(\mathbb{R}_+;K)$. Next, the existence of a unique solution to (3.23) results from Theorem 5, since assumptions (3.19) and (3.20) imply that the operator (3.22) is a history-dependent operator. This concludes the proof of the theorem. □

4. A convergence result

In this section we study the continuous dependence of the solution to Problem 4 on the data. To this end, we assume in what follows that (2.1), (2.2), (2.4)–(2.6), (3.2)–(3.5) hold and denote by $u \in C(\mathbb{R}_+;K)$ the solution of (3.1) obtained in Theorem 1. For each $\rho > 0$, let φ_ρ, j_ρ and f_ρ be perturbed data corresponding to φ, j and f, which satisfy conditions (3.3), (2.4) and (3.5), respectively. We denote by α_{φ_ρ}, β_{φ_ρ} and α_{j_ρ} the constants involved in assumptions (3.3) and (2.4).

We consider the following perturbed version of Problem 4.

Problem 10. *Find a function $u_\rho \in C(\mathbb{R}_+;K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds*

$$\begin{aligned}&\langle Au_\rho(t), v - u_\rho(t)\rangle + \varphi_\rho((\mathcal{S}u_\rho)(t), u_\rho(t), v) - \varphi((\mathcal{S}u_\rho)(t), u_\rho(t), u_\rho(t))\\ &\qquad + j^0_\rho(u_\rho(t); v - u_\rho(t)) \geq \langle f, v - u_\rho\rangle \quad \textit{for all } \ v \in K.\end{aligned} \tag{4.1}$$

It follows from Theorem 5 that Problem 10 has a unique solution with regularity $u_\rho \in C(\mathbb{R}_+;K)$. Our interest lies in the behavior of the solution u_ρ, as ρ tends to zero. To this end, we consider the following additional hypotheses.

$$\begin{cases}\text{there exists } \ m_0 > 0 \text{ and } \beta_0 \ \text{ such that}\\ \text{(a)} \quad \alpha_{\varphi_\rho} + \alpha_{j_\rho} \leq m_0 < m_A \ \text{ for all } \ \rho > 0.\\ \text{(b)} \quad \beta_{\varphi_\rho} \leq \beta_0 \ \text{ for all } \rho > 0.\end{cases} \tag{4.2}$$

$$\begin{cases}\text{there exists a function } \ G\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } g \in \mathbb{R}_+ \text{ such that}\\ \text{(a)} \quad \varphi(y,\xi,\eta) - \varphi(y,\xi,\xi) - \varphi_\rho(y,\xi,\eta) + \varphi_\rho(y,\xi,\xi)\\ \qquad \leq G(\rho)(\|y\|_Y + \|\xi\|_X + g)\|\xi - \eta\|_X\\ \qquad\quad \text{for all } y \in Y,\ \xi,\ \eta \in K,\ \rho > 0.\\ \text{(b)} \quad \lim\limits_{\rho\to 0} G(\rho) = 0.\end{cases} \tag{4.3}$$

$$\begin{cases} \text{there exists a function } H\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } h \in \mathbb{R}_+ \text{ such that} \\ \text{(a)} \quad j^0(\xi;\eta) - j^0_\rho(\xi;\eta) \le H(\rho)(\|\xi\| + h)\|\xi - \eta\| \\ \qquad\qquad \text{for all } \xi,\, \eta \in X,\, \rho > 0. \\ \text{(b)} \quad \lim_{\rho\to 0} H(\rho) = 0. \end{cases} \tag{4.4}$$

$$f_\rho \to f \text{ in } C((\mathbb{R}_+;X), \quad \text{as } \rho \to 0. \tag{4.5}$$

The main result in this section is the following.

Theorem 11. *Assume (4.2)–(4.5). Then the solution u_ρ of Problem 10 converges to the solution u of Problem 4, i.e.,*

$$u_\rho \to u \quad \text{in} \quad C(\mathbb{R}_+;X), \quad \text{as } \rho \to 0. \tag{4.6}$$

Proof. Let $\rho > 0$, $n \in \mathbb{N}$ and $t \in [0,n]$. We take $v = u_\rho(t)$ in (3.1) and $v = u(t)$ in (4.1), and then we add the resulting inequalities to obtain

$$\begin{aligned} &\langle Au_\rho(t) - Au(t), u_\rho(t) - u(t)\rangle \\ &\quad \le \varphi((Su)(t), u(t), u_\rho(t)) - \varphi((Su)(t), u(t), u(t)) \\ &\qquad + \varphi_\rho((Su_\rho)(t), u_\rho(t), u(t)) - \varphi_\rho((Su_\rho)(t), u_\rho(t), u_\rho(t)) \\ &\qquad + j^0(u(t); u_\rho(t) - u(t)) + j^0_\rho(u_\rho(t); u - u_\rho(t)) \\ &\qquad + \langle f_\rho(t) - f(t), u_\rho(t) - u(t)\rangle. \end{aligned} \tag{4.7}$$

We estimate each term of the last inequality. First, it follows from assumption (2.2)(b) that

$$\langle Au_\rho(t) - Au(t), u_\rho(t) - u(t) \ge m_A\|u_\rho(t) - u(t)\|_X^2. \tag{4.8}$$

Next, we write

$$\begin{aligned} &\varphi((Su(t), u(t), u_\rho(t)) - \varphi((Su(t), u(t), u(t)) \\ &\qquad + \varphi_\rho((Su_\rho)(t), u_\rho(t), u(t)) - \varphi_\rho((Su_\rho)(t), u_\rho(t), u_\rho(t)) \\ &\quad = \varphi((Su)(t), u(t), u_\rho(t)) - \varphi((Su)(t), u(t), u(t)) \\ &\qquad - \varphi_\rho((Su)(t), u(t), u_\rho(t)) + \varphi_\rho((Su(t)), u(t), u(t)) \\ &\qquad + \varphi_\rho((Su)(t), u(t), u_\rho(t)) - \varphi_\rho((Su)(t), u(t), u(t)) \\ &\qquad + \varphi_\rho((Su_\rho)(t), u_\rho(t), u(t)) - \varphi_\rho((Su_\rho)(t), u_\rho(t), u_\rho(t)) \end{aligned}$$

and, using hypotheses (4.3)(a) and (3.3)(b), we infer that

$$\begin{aligned} &\varphi((Su(t), u(t), u_\rho(t)) - \varphi((Su(t), u(t), u(t)) \\ &\qquad + \varphi_\rho((Su_\rho)(t), u_\rho(t), u(t)) - \varphi_\rho((Su_\rho)(t), u_\rho(t), u_\rho(t)) \\ &\quad \le G(\rho)(\|(Su)(t)\|_Y + \|u(t)\|_X + g)\|u_\rho(t) - u(t)\|_X \\ &\qquad + \alpha_{\varphi_\rho}\|u_\rho(t) - u(t)\|_X^2 + \beta_{\varphi_\rho}\|(Su)(t) - (Su_\rho)(t)\|_Y\|u_\rho(t) - u(t)\|_X. \end{aligned} \tag{4.9}$$

In a similar way, we write

$$\begin{aligned} &j^0(u(t); u_\rho(t) - u(t)) + j^0_\rho(u_\rho(t); u(t) - u_\rho(t)) \\ &\quad = j^0(u(t); u_\rho(t) - u(t)) - j^0_\rho(u(t); u_\rho(t) - u(t)) \\ &\qquad + j^0_\rho(u(t); u_\rho(t) - u(t)) + j^0_\rho(u_\rho(t); u(t) - u_\rho(t)) \end{aligned}$$

and, using hypotheses (4.4)(a) and (2.4), we find that

$$j^0(u(t);u_\rho(t)-u(t))+j^0_\rho(u_\rho(t);u(t)-u_\rho(t)) \\ \le H(\rho)(\|u\|_X+h)\|u_\rho(t)-u(t)\|_X+\alpha_{j_\rho}\|u_\rho(t)-u(t)\|_X^2. \tag{4.10}$$

Finally, we note that

$$\langle f_\rho(t)-f(t),u_\rho(t)-u(t)\rangle \le \delta_n(\rho)\|u_\rho(t)-u(t)\|_X, \tag{4.11}$$

where

$$\delta_n(\rho)=\max_{t\in[0,n]}\|f_\rho(t)-f(t)\|_{X^*}. \tag{4.12}$$

Now, we combine inequalities (4.7)–(4.11) and use assumptions (4.2) and (3.2) to see that

$$\big(m_A-m_0)\|u_\rho(t)-u(t)\|_X \le G(\rho)(\|(Su)(t)\|_Y+\|u(t)\|_X+g) \\ +H(\rho)(\|u(t)\|_X+h)+\delta_n(\rho)+\beta_0 s_n\int_0^t\|u_\rho(t)-u(t)\|_X\,ds. \tag{4.13}$$

Let

$$F_n(u)=\max_{t\in[0,n]}\Big(\|(Su)(t)\|_Y+\|u(t)\|_X+g+h\Big).$$

Then, inequality (4.13) implies

$$\|u_\rho(t)-u(t)\|_X\le\frac{1}{(m_A-m_0)}\Big(\big(G(\rho)+H(\rho)\big)F_n(u)+\delta_n(\rho)\Big) \\ +\frac{\beta_0 s_n}{(m_A-m_0)}\int_0^t\|u_\rho(t)-u(t)\|_X\,ds$$

and, therefore, the Gronwall argument yields

$$\|u_\rho(t)-u(t)\|_X\le\frac{1}{m_A-m_0}\Big(\big(G(\rho)+H(\rho)\big)F_n(u)+\delta_n(\rho)\Big)e^{\frac{\beta_0 s_n t}{m_A-m_0}}.$$

Hence, we conclude that

$$\max_{t\in[0,n]}\|u_\rho(t)-u(t)\|_X \\ \le\frac{1}{m_A-m_0}\Big(\big(G(\rho)+H(\rho)\big)F_n(u)+\delta_n(\rho)\Big)e^{\frac{\beta_0 s_n n}{m_A-m_0}}. \tag{4.14}$$

Note that assumption (4.5), definitions (2.10) and (4.12) imply

$$\delta_n(\rho)\to 0,\qquad \text{as}\ \ \rho\to 0. \tag{4.15}$$

Therefore, using (4.3)(b), (4.4)(b) and (4.15), it follows from (4.14) that

$$\max_{t\in[0,n]}\|u_\rho(t)-u(t)\|_V\to 0,\qquad \text{as}\ \ \rho\to 0. \tag{4.16}$$

The convergence (4.6) is now a consequence of (4.16) and (2.10). This completes the proof. □

5. A model of viscoelastic frictionless contact

A large number of quasistatic contact problems with elastic, viscoelastic or viscoplastic materials lead to history-dependent variational-hemivariational inequalities of the form (3.1) in which the unknown is either the displacement or the velocity field. For a variety of such inequalities, the results in Sects. 3 and 4 can be applied. In this section we illustrate this point for a viscoelastic contact problem.

The physical setting is the following. A viscoelastic body occupies, in its reference configuration, a regular domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with boundary $\partial\Omega$. The boundary is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that the measure of Γ_1, denoted $m(\Gamma_1)$, is positive. The body is clamped on Γ_1 and so the displacement field vanishes there. Time-dependent surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and time-dependent volume forces of density $\boldsymbol{f}_0$ act in Ω. The body is in contact on Γ_3 with an obstacle, the so-called foundation. The contact is frictionless and is modeled with a nonmonotone normal compliance condition associated with a unilateral contact condition. The process is quasistatic and the time interval of interest is $\mathbb{R}_+$. Then, the mathematical model of the contact problem (that we state here and explain later in this section) is the following.

Problem 12. *Find a displacement field* $\boldsymbol{u}\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}\colon \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mu\Big(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - P_{M(\kappa(\zeta(t)))}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \quad \text{in } \Omega, \tag{5.1}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{5.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g,\ \ \sigma_\nu(t) + \xi_\nu(t) \le 0, \\ (\sigma_\nu(t) + \xi_\nu(t))(u_\nu(t) - g) = 0 \\ \xi_\nu(t) \in \partial j_\nu(u_\nu(t)) \end{array}\right\} \quad \text{on } \Gamma_3, \tag{5.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on } \Gamma_3, \tag{5.6}$$

for all $t \in \mathbb{R}_+$.

Here we use the notation $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ for the displacement vector, the stress tensor, and the linearized strain tensor, respectively, and $\boldsymbol{\nu} = (\nu_i)$ for the outward unit normal at $\partial\Omega$. Recall that $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$ where the index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable, denoted $\boldsymbol{x} = (x_i)$. Moreover, we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of a vector field $\boldsymbol{v}$ on $\partial\Omega$, i.e., $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. In addition, σ_ν and $\boldsymbol{\sigma}_\tau$ represent the normal and tangential components of the stress field $\boldsymbol{\sigma}$ and are defined by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, respectively.

In the study of Problem 12 we use standard notation for Lebesgue and Sobolev spaces. For all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$ we still denote by $\boldsymbol{v}$ the trace of $\boldsymbol{v}$ on $\partial\Omega$ and, recall, we use the notation v_ν and $\boldsymbol{v}_\tau$ for its normal and tangential traces. In addition, we introduce spaces V and $\mathcal{H}$ defined by

$$V = \{ \ \boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ a.e. on } \ \Gamma_1 \ \},$$
$$\mathcal{H} = \{ \ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega;\mathbb{S}^d) \mid \tau_{ij} = \tau_{ji}, \ 1 \le i,j \le d \ \}.$$

The space $\mathcal{H}$ is a real Hilbert space with the canonical inner product given by

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}(\boldsymbol{x})\,\tau_{ij}(\boldsymbol{x})\,dx,$$

and the associated norm $\|\cdot\|_{\mathcal{H}}$. Since $m(\Gamma_1) > 0$, it is well known that V is a real Hilbert space with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \boldsymbol{u},\boldsymbol{v} \in V \tag{5.7}$$

and the associated norm $\|\cdot\|_V$. Moreover, by the Sobolev trace theorem, we have

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le \|\gamma\|\,\|\boldsymbol{v}\|_V \quad \text{ for all } \ \boldsymbol{v} \in V. \tag{5.8}$$

Here and below $\|\gamma\|$ represents the norm of the trace operator $\gamma\colon V \to L^2(\Gamma_3;\mathbb{R}^d)$.

Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty = \{ \ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \ \ 1 \le i,j,k,l \le d \ \}.$$

We note that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \sum_{0 \le i,j,k,l \le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

In addtition, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_{\mathcal{H}} \le \ \|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{\tau}\|_{\mathcal{H}} \quad \text{ for all } \ \mathcal{E} \in \mathbf{Q}_\infty, \ \boldsymbol{\tau} \in \mathcal{H}. \tag{5.9}$$

We now present a short description of the equations and conditions in Problem 12 in which, for simplicity, we do not indicate explicitly the dependence of the variables on $\boldsymbol{x}$. We refer the reader to [12,21,29,33] for more details and mechanical interpretation.

First, Eq. (5.1) is the constitutive law for viscoelastic materials in which $\mathcal{A}$ represents the elasticity tensor, $\mathcal{B}$ is the relaxation tensor, μ is a given positive constant and $P_{M(\kappa(\cdot))}$ is the projection operator on the von Mises convex (2.12), defined with the bound $\kappa = \kappa(\zeta(t))$. Here $\zeta(t)$ is given by

$$\zeta(t) = \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\|_{\mathcal{H}}\,ds \tag{5.10}$$

and represents the total strain (or, alternatively, the accumulated strain) in the body over the time period $[0,t]$, for any $t \in \mathbb{R}_+$. Considering such a dependence represents the first trait of novelty of our contact model and is reasonable from the physical point of view. Indeed, it incorporates the changes in the constitutive law resulting from cycles of deformation, and, therefore, it could describe the hardening or the softening of the material. Moreover, considering such a dependence makes Problem 12 more interesting from a mathematical point of view. Note that the nonhomogeneous case could be considered, under appropriate assumptions on μ and κ. Neverthleless, for simplicity, we restrict here to the homogeneous case, i.e., we assume that the coeficient μ and the bound κ do not depend on the spatial variable $\boldsymbol{x}$.

It follows from above that the elasticity operator in the constitutive law (5.1) is nonlinear and is given by $\boldsymbol{\varepsilon} \mapsto \mathcal{A}\boldsymbol{\varepsilon}+\mu\,(\boldsymbol{\varepsilon}-P_{M(\cdot)}\boldsymbol{\varepsilon})$. where, for simplicity, we write $M(\cdot) = M(\kappa(\zeta(t))$. Such kind of operators have been intensively used in the literature, see for instance [29,33] and the references therein. Since $\boldsymbol{\varepsilon} = P_{M(\cdot)}\boldsymbol{\varepsilon}$ if and only if $\boldsymbol{\varepsilon} \in M(\cdot)$, from (5.1) we see that that the material has a linearly viscoelastic behavior if and only if the strain tensor $\boldsymbol{\varepsilon}$ belongs to $M(\cdot)$. The nonlinear elastic response of the material appears only for strain tensors $\boldsymbol{\varepsilon}$ which satisfy $\boldsymbol{\varepsilon} \notin M(\cdot)$.

Equation (5.2) is the equilibrium equation for the quasistatic contact process. On Γ_1, we have the clamped boundary condition (5.3) and, on Γ_2, the surface traction boundary condition (5.4). Relation (5.5) is the contact condition in which $g > 0$ and ∂j_ν denotes the Clarke subdifferential of a given function j_ν. It represents the second trait of novelty of our contact model. Indeed, condition (5.5) models the contact with a foundation which is made of a rigid body covered by a layer made of elastic material, say asperities. It shows that the penetration is restricted, since $u_\nu \le g$ where g represents the thickness of the elastic layer. Also, when there is penetration, as far as the normal displacement does not reach the bound g, the contact is described by a nonmonotone normal compliance condition since, in this case $-\sigma_\nu = \xi_\nu \in \partial j_\nu(u_\nu)$. Due to the nonmonotonicity of ∂j_ν, the condition allows to describe the hardening or the softening phenomena of the foundation. Various examples and mechanical interpretation associated with the nonmonotone normal compliance condition can be found in [21].

We now list the assumptions on the problem data. First, we assume that the elasticity tensor $\mathcal{A}$ and the relaxation tensor $\mathcal{B}$ satisfy the following conditions.

$$\left\{\begin{array}{l} \text{(a) } \mathcal{A} = (a_{ijkl})\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) } a_{ijkl} = a_{klij} = a_{jikl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d. \\ \text{(c) } \ \text{there exists } m_{\mathcal{A}} > 0 \ \text{ such that} \\ \qquad \mathcal{A}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge m_{\mathcal{A}}\|\boldsymbol{\tau}\|^2 \ \text{ forall } \boldsymbol{\tau} \in \mathbb{S}^d, \ \text{a.e. in } \Omega. \end{array}\right. \tag{5.11}$$

$$\mathcal{B} \in C(\mathbb{R}_+;\mathbf{Q}_\infty). \tag{5.12}$$

The bound κ and the potential function j_ν satisfy

$$\left\{\begin{array}{l} \kappa\colon \mathbb{R} \to \mathbb{R}_+ \text{ is a Lipschitz continuous function, i.e.,} \\ \qquad \text{there exists } \ L_\kappa > 0 \text{ such that} \\ \qquad\quad |\kappa(r_1) - \kappa(r_2)| \le L_\kappa |r_1 - r_2| \quad \text{for all } \ r_1, r_2 \in \mathbb{R}. \end{array}\right. \tag{5.13}$$

$$\begin{cases} j_\nu\colon \Gamma_3\times\mathbb{R}\to\mathbb{R} \text{ is such that} \\ \quad \text{(a) } j_\nu(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R} \text{ and there} \\ \qquad \text{exists } \bar{e}\in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot,\bar{e}(\cdot))\in L^1(\Gamma_3). \\ \quad \text{(b) } j_\nu(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x}\in\Gamma_3. \\ \quad \text{(c) } |\partial j_\nu(\boldsymbol{x},r)|\le \bar{c}_0+\bar{c}_1\,|r| \text{ for a.e. } \boldsymbol{x}\in\Gamma_3, \\ \qquad \text{for all } r\in\mathbb{R} \text{ with } \bar{c}_0,\,\bar{c}_1\ge 0. \\ \quad \text{(d) } j_\nu^0(\boldsymbol{x},r_1;r_2-r_1)+j_\nu^0(\boldsymbol{x},r_2;r_1-r_2)\le\bar{\alpha}_\nu\,|r_1-r_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x}\in\Gamma_3, \text{ all } r_1,\,r_2\in\mathbb{R} \text{ with } \bar{\alpha}_\nu\ge 0. \end{cases} \tag{5.14}$$

Finally, we assume that the densities of body forces and surface tractions have the regularity

$$\boldsymbol{f}_0\in C(\mathbb{R}_+;L^2(\Omega;\mathbb{R}^d)),\quad \boldsymbol{f}_2\in C(\mathbb{R}_+;L^2(\Gamma_2;\mathbb{R}^d)). \tag{5.15}$$

We now turn to the variational formulations of Problem 12. To this end, we introduce the set of admissible displacements U and the function $\boldsymbol{f}:\mathbb{R}_+\to V^*$ defined by

$$U=\{\,\boldsymbol{v}\in V\mid v_\nu\le g \ \text{ a.e. on } \Gamma_3\,\}, \tag{5.16}$$

$$\langle \boldsymbol{f}(t),\boldsymbol{v}\rangle_{V^*\times V}=(\boldsymbol{f}_0(t),\boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)}+(\boldsymbol{f}_2(t),\boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \text{for all } \boldsymbol{v}\in V,\ t\in\mathbb{R}_+. \tag{5.17}$$

Let $\boldsymbol{v}\in U$ and $t\in\mathbb{R}_+$. We perform integrations by parts, split the resulting surface integral on three integrals on Γ_1, Γ_2 and Γ_3, and then we use the boundary conditions (5.3), (5.4) and the equilibrium equation (5.2) to obtain

$$\begin{aligned}&\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx \\ &\quad=\int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,dx+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,d\Gamma \\ &\qquad+\int_{\Gamma_3}\sigma_\nu(t)(v_\nu-u_\nu(t))\,d\Gamma+\int_{\Gamma_3}\boldsymbol{\sigma}_\tau(t)\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau(t))\,d\Gamma.\end{aligned} \tag{5.18}$$

Next, we use (5.5) and the definition of the subdifferential to find

$$j_\nu^0(u_\nu(t);v_\nu-u_\nu(t))+\sigma_\nu(t)(v_\nu-u_\nu(t))\ge 0 \quad \text{a.e. on } \Gamma_3,$$

which implies that

$$\int_{\Gamma_3} j_\nu^0(u_\nu(t);v_\nu-u_\nu(t))\,d\Gamma+\int_{\Gamma_3}\sigma_\nu(t)(v_\nu-u_\nu(t))\,d\Gamma\ge 0. \tag{5.19}$$

In addition, (5.6) shows that

$$\int_{\Gamma_3}\boldsymbol{\sigma}_\tau(t)(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau(t))\,d\Gamma=0. \tag{5.20}$$

Then, combinining (5.17)–(5.20), we find that

$$\begin{aligned}&(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ &\quad+\int_{\Gamma_3} j_\nu^0(u_\nu(t);v_\nu-u_\nu(t))\,d\Gamma\ge\langle\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t)\rangle_{V^*\times V}.\end{aligned} \tag{5.21}$$

In addition, it follows from (5.5) that $\boldsymbol{u}(t) \in U$ for all $t \in \mathbf{R}_+$. Then, using the constitutive law (5.1) and inequality (5.21) we obtain the following variational formulation of Problem 8, in terms of displacement.

Problem 13. *Find a displacement field $\boldsymbol{u}\colon \mathbb{R}_+ \to U$ such that the inequality*

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ &\quad - \mu(P_{M(\kappa(\zeta(t)))}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ &\quad + \Big(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t))\, d\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle_{V^* \times V} \quad \textit{for all } \ \boldsymbol{v} \in U\end{aligned} \tag{5.22}$$

holds for all $t \in \mathbb{R}_+$, where $\zeta(t)$ is given by (5.10).

The analysis of Problem 13, including its unique solvability and some comments on the dependence of the solution with respect to the data, will be presented in the next section.

6. Analysis of the contact problem

We start with the following existence and uniqueness result for Problem 13.

Theorem 14. *Assume hypotheses (5.11)–(5.15) and the smallness condition*

$$\bar{\alpha}_\nu \, \|\gamma\|^2 \leq m_{\mathcal{A}}. \tag{6.1}$$

Then Problem 13 has a unique solution $\boldsymbol{u} \in C(\mathbb{R}_+; U)$.

Proof. We introduce the space $Y = \mathbb{R} \times \mathcal{H}$ endowed with the norm

$$\|y\|_Y = |r| + \|\boldsymbol{\theta}\|_{\mathcal{H}} \quad \text{for all } \ y = (r, \boldsymbol{\theta}) \in Y, \tag{6.2}$$

together with the operators $A\colon V \to V^*$, $\mathcal{S}\colon C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Y)$ and the functions $\varphi\colon Y \times V \times V \to \mathbb{R}$, $j\colon V \to \mathbb{R}$ defined by

$$\begin{aligned}\langle A\boldsymbol{u}, \boldsymbol{v} \rangle_{V^* \times V} &= (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + \mu(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ &\quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V,\end{aligned} \tag{6.3}$$

$$\begin{aligned}(\mathcal{S}u)(t) &= \Big(\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\|_{\mathcal{H}}\, ds, \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds \Big) \\ &\quad \text{for all } \boldsymbol{u} \in C(\mathbb{R}_+; V),\end{aligned} \tag{6.4}$$

$$\begin{aligned}\varphi(y, \boldsymbol{u}, \boldsymbol{v}) &= -\mu(P_{M(\kappa(r))}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\theta}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ &\quad \text{for all } y = (r, \boldsymbol{\theta}) \in Y,\ \boldsymbol{u},\ \boldsymbol{v} \in V,\end{aligned} \tag{6.5}$$

$$j(\boldsymbol{v}) = \int_{\Gamma_3} j_\nu(v_\nu)\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V. \tag{6.6}$$

Then, it is easy to see that (5.22) represents an inequality of the form (3.1) in which $X = V$, $K = U$ and $f \equiv \boldsymbol{f}$, where U and $\boldsymbol{f}$ are defined by (5.16) and (5.17), respectively. Our aim in what follows is to apply the abstract existence and uniqueness result provided by Theorem 5. To this end, we verify the assumptions of this theorem.

First, we note that condition (2.1) on the set K is clearly satisfied. Moreover, using (5.11), it is easy to see that the operator defined by (6.3) satisfies condition (2.2) with $\alpha_A = m_A = m_{\mathcal{A}} + \mu$ and $\beta_A = \gamma_A = 0$. Also, assumption (5.14) on the function j_ν imply that the function j given by (6.6) satisfy condition (2.4) with constants $c_0 = \bar{c}_0$, $c_1 = \bar{c}_1$ and $\alpha_j = \bar{\alpha}_\nu \, \|\gamma\|^2$. In addition, assumption (6.1) implies that condition (2.6) is also satisfied.

Next, an elementary calculation based on assumption (5.12) and inequality (5.9) shows that for all $n \in \mathbb{N}$, we have

$$\|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_Y \le c_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds$$
$$\text{for all } \ \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V), \quad \text{for all } t \in [0,n] \quad \text{ with } \ c_n > 0$$

which shows that the operator $\mathcal{S}$ as a history-dependent operator, i.e., it satisfies condition (3.2).

Note also that the function φ defined by (6.5) satisfies condition (3.3)(a). In order to verify condition (3.3)(b), let $y_i = (r_i, \boldsymbol{\theta}_i) \in Y$, $\boldsymbol{u}_i$, $\boldsymbol{v}_i \in V$ for $i = 1$, 2. Then, by definition (6.5) we deduce

$$\begin{aligned}
&\varphi(y_1, \boldsymbol{u}_1, \boldsymbol{v}_2) - \varphi(y_1, \boldsymbol{u}_1, \boldsymbol{v}_1) + \varphi(y_2, \boldsymbol{u}_2, \boldsymbol{v}_1) - \varphi(y_2, \boldsymbol{u}_2, \boldsymbol{v}_2)\\
&\quad = \mu(P_{M(\kappa(r_2))}\boldsymbol{\varepsilon}(\boldsymbol{u}_2) - P_{M(\kappa(r_1))}\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \boldsymbol{\varepsilon}(\boldsymbol{v}_2) - \boldsymbol{\varepsilon}(\boldsymbol{v}_1))_{\mathcal{H}}\\
&\qquad + (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2, \boldsymbol{\varepsilon}(\boldsymbol{v}_2) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1))_{\mathcal{H}}\\
&\quad \le \Big(\mu \|P_{M(\kappa(r_1))}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - P_{M(\kappa(r_2))}\boldsymbol{\varepsilon}(\boldsymbol{u}_2)\|_{\mathcal{H}} + \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathcal{H}}\Big)\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V
\end{aligned}$$

and, therefore

$$\begin{aligned}
&\varphi(y_1, \boldsymbol{u}_1, \boldsymbol{v}_2) - \varphi(y_1, \boldsymbol{u}_1, \boldsymbol{v}_1) + \varphi(y_2, \boldsymbol{u}_2, \boldsymbol{v}_1) + \varphi(y_2, \boldsymbol{u}_2, \boldsymbol{v}_1)\\
&\quad \le \Big(\mu \|P_{M(\kappa(r_1))}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - P_{M(\kappa(r_1))}\boldsymbol{\varepsilon}(\boldsymbol{u}_2)\|_{\mathcal{H}}\\
&\qquad + \mu \|P_{M(\kappa(r_1))}\boldsymbol{\varepsilon}(\boldsymbol{u}_2) - P_{M(\kappa(r_2))}\boldsymbol{\varepsilon}(\boldsymbol{u}_2)\|_{\mathcal{H}} + \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{\mathcal{H}}\Big)\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V
\end{aligned} \tag{6.7}$$

We now use the nonexpansivity of the projector operator to see that

$$\|P_{M(\kappa(r_1))}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - P_{M(\kappa(r_1))}\boldsymbol{\varepsilon}(\boldsymbol{u}_2)\|_{\mathcal{H}} \le \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V. \tag{6.8}$$

On the other hand, Proposition 3 combined with hypothesis (5.13) implies

$$\|P_{M(\kappa(r_1))}\boldsymbol{\varepsilon}(\boldsymbol{u}_2) - P_{M\kappa(r_2))}\boldsymbol{\varepsilon}(\boldsymbol{u}_2)\|_{\mathcal{H}} \le L_\kappa |r_1 - r_2|. \tag{6.9}$$

We now use estimates (6.7)–(6.9) and (6.2) to see that

$$\begin{aligned}
&\varphi(y_1, \boldsymbol{u}_1, \boldsymbol{v}_2) - \varphi(y_1, \boldsymbol{u}_1, \boldsymbol{v}_1) + \varphi(y_2, \boldsymbol{u}_2, \boldsymbol{v}_1) + \varphi(y_2, \boldsymbol{u}_2, \boldsymbol{v}_1)\\
&\quad \le \mu \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V + \max\{L_\kappa, 1\}\, \|y_1 - y_2\|_V \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V
\end{aligned}$$

which shows that φ satisfies condition (3.3)(b) with $\alpha_\varphi = \mu$ and $\beta_\varphi = \max\{L_\kappa, 1\}$. Moreover, recall that $\alpha_j = \bar{\alpha}_\nu \, \|\gamma\|^2$, $\alpha_A = m_A = m_{\mathcal{A}} + \mu$ and, therefore, condition (3.4) is also satisfied. Finally, regularity (3.5) follows from (5.17) and (5.15). Theorem 14 is now a direct consequence of Theorem 5. $\square$

We also note that Theorem 11 can be used to study the dependence of the weak solution of Problem 13 with respect to perturbations of the data and to prove its continuous dependence on the bound κ, the normal compliance function j, and densities of body forces and surface tractions, $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$, respectively. Here, we omit the details and we restict ourselves to present an example of functions φ and φ_ρ defined by (6.5) that satisfy condition (4.3).

In the following, let ρ be a positive parameter and κ, κ_ρ be two functions which satisfy hypothesis (5.13). Moreover, we assume that

$$\begin{cases} \text{there exists a function } \ \omega\colon \mathbb{R}_+ \to \mathbb{R}_+ \ \text{ and } \ \delta \in \mathbb{R}_+ \ \text{ such that} \\ \ \text{(a)} \quad |\kappa_\rho(r) - \kappa(r)| \le \omega(\rho)(|r| + \delta) \ \text{ for all } \ r \in \mathbb{R},\, \rho > 0. \\ \ \text{(b)} \quad \lim\limits_{\rho \to 0} \omega(\rho) = 0. \end{cases} \tag{6.10}$$

With functions κ and κ_ρ, we construct the Von Mises convexes $M(\kappa(\cdot))$ and $M(\kappa_\rho(\cdot))$, respectively, defined by (2.12). Then, we consider the corresponding functions φ and φ_ρ given by (6.5). We claim that these functions satisfy condition (4.3) with the spaces $X = V$ and $Y = \mathbb{R} \times \mathcal{H}$. Indeed, assume that $y = (r, \boldsymbol{\theta}) \in Y$ and $\boldsymbol{u}$, $\boldsymbol{v} \in V$ are given. Then, by using (6.5) and Proposition 3, we deduce

$$\begin{aligned} &\varphi(y, \boldsymbol{u}, \boldsymbol{v}) - \varphi(y, \boldsymbol{u}, \boldsymbol{u}) + \varphi_\rho(y, \boldsymbol{u}, \boldsymbol{v}) - \varphi_\rho(y, \boldsymbol{u}, \boldsymbol{u}) \\ &\quad = \mu(P_{M(\kappa(r))}\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_{M(\kappa_\rho(r))}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ &\quad \le \mu\, \|P_{M(\kappa(r))}\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_{M(\kappa_\rho(r))}\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{\mathcal{H}} \|\boldsymbol{u} - \boldsymbol{v}\|_V \\ &\quad \le \mu\, |\kappa(r) - \kappa_\rho(r)|\, \|\boldsymbol{u} - \boldsymbol{v}\|_V. \end{aligned}$$

We now use assumption (6.10) together with (6.2) to see that condition (4.3) is satisfied.

We also recall that examples of functions j_ν and $j_{\nu\rho}$ which satisfy condition (5.14) such that the corresponding functions j and j_ρ defined by (6.6) satisfy assumptions (3.4), (4.2) and (4.4) can be found in [31]. Hence, we conclude that the convergence result of Theorem 11 can be used in the study of the corresponding frictionless contact problem.

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Mircea Sofonea
Laboratoire de Mathématiques et Physique
Université de Perpignan Via Domitia
52 Avenue Paul Alduy
66860
Perpignan
France
e-mail: sofonea@univ-perp.fr

Stanisław Migórski
Faculty of Mathematics and Computer Science
Institute of Computer Science
Jagiellonian University
ul. Stanisława Łojasiewicza 6
30348
Kraków
Poland
e-mail: migorski@ii.uj.edu.pl

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