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Nonlinear Differential Equations and Applications NoDEA

On the lower semicontinuity and approximation of L^∞-functionals

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Abstract. In this paper we show that if the supremal functional

$$F(V,B) = \operatorname*{ess\,sup}_{x\in B} f(x, DV(x))$$

is sequentially weak* lower semicontinuous on $W^{1,\infty}(B,\mathbb{R}^d)$ for every open set $B \subseteq \Omega$ (where Ω is a fixed open set of $\mathbb{R}^N$), then $f(x,\cdot)$ is rank-one level convex for a.e $x \in \Omega$. Next, we provide an example of a weak Morrey quasiconvex function which is not strong Morrey quasiconvex. Finally we discuss the L^p-approximation of a supremal functional F via Γ-convergence when f is a non-negative and coercive Carathéodory function.

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1. Introduction

In recent years, a new class of functionals is being considered with growing interest in the mathematical literature: these functionals are represented by the so called *supremal form*

$$F(V) = \operatorname*{ess\,sup}_{x\in \Omega} f(x, DV(x)) \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$ and $V \in W^{1,\infty}(\Omega,\mathbb{R}^d)$. Following an established convention, we will refer to a functional of the type (1.1) as a *supremal functional* or L^∞*-functional* (see [3,8]). Several papers in this context were motivated by the problem to finding the best Lipschitz extension in Ω of a function defined on the boundary $\partial\Omega$ ([5,7,20]). Actually, the L^∞-variational problems arise naturally in several practical contexts: for example, in models describing dielectric breakdown in a composite material (see [17]) or polycrystal plasticity (see [10]), and in the optimal transportation problem (see [13]). In order to apply the direct method of the calculus of variations the main issue is the lower semicontinuity of F. Semicontinuity properties for supremal functionals have recently been studied by many authors; we refer for instance to Barron-Jensen [6], Barron-Jensen-Wang [8], Prinari [22,23] and to the recent papers by Ansini-Prinari [2] and Ribeiro-Zappale [24]. In the context of supremal functionals, in [8] Barron, Jensen and Wang introduce the notion of the weak Morrey quasiconvexity as the natural extension of the notion of Morrey quasiconvexity (see [14] and references therein).

Definition 1.1. *A Borel function $f : \mathbb{M}^{d\times N} \mapsto \mathbb{R}$ is* **weak Morrey quasiconvex** *if for all $\Sigma \in \mathbb{M}^{d\times N}$*

$$f(\Sigma) \leq \operatorname*{ess\,sup}_{x\in Q} f(\Sigma + D\varphi(x)) \qquad \forall \varphi \in W_0^{1,\infty}(Q;\mathbb{R}^d) \tag{1.2}$$

where $Q = [0,1]^N$ stands for the standard unit cube and $W_0^{1,\infty}(Q;\mathbb{R}^d)$ stands for the weak closure of $C_c^{\infty}Q;\mathbb{R}^d)$ in $W^{1,\infty}(Q;\mathbb{R}^d)$.*

They show that this property is a necessary condition for the lower semicontinuity of a supremal functional with respect to the weak* topology of $W^{1,\infty}$. Moreover, in the scalar case, i.e. $d = 1$ or $N = 1$, this condition is also sufficient since it coincides with the notion of level convexity (for a proof, see Theorem 5.5 (ii) and (iii) in [24]). We recall that $f : \mathbb{R}^k \to \mathbb{R}$ is level convex if for every $t \in \mathbb{R}$ the level set $\left\{\xi \in \mathbb{R}^k : f(\xi) \leq t\right\}$ is convex. In the vectorial case, Barron, Jensen and Wang do not characterize the weak* lower semicontinuity of a supremal functional (1.1) by the weak Morrey quasiconvexity and introduce another class of functions: the strong Morrey quasiconvex functions.

Definition 1.2. *A function $f : \mathbb{M}^{d\times N} \to \mathbb{R}$ is* **strong Morrey quasiconvex** *if $\forall \varepsilon > 0$, $\forall \Sigma \in \mathbb{M}^{d\times N}$ and $\forall K > 0$ $\exists\, \delta = \delta(\varepsilon, K, \Sigma) > 0$ such that $\forall \varphi \in W^{1,\infty}(Q,\mathbb{R}^d)$ satisfying*

$$\|D\varphi\|_{L^\infty(Q)} \leq K, \quad \max_{\partial Q} |\varphi(x)| \leq \delta,$$

it holds:

$$f(\Sigma) \leq \operatorname*{ess\,sup}_{x\in Q} f(\Sigma + D\varphi(x)) + \epsilon.$$

It easily follows that any strong Morrey quasiconvex function is also weak Morrey quasiconvex. In their paper, Barron, Jensen and Wang show that, under suitable assumptions for f, the strong Morrey quasiconvexity is necessary and sufficient for the lower semicontinuity of supremal functionals defined on $W^{1,\infty}(\Omega,\mathbb{R}^d)$ (see Theorem 2.7 in [8]). They also raise the important question if the notions of weak and strong Morrey quasiconvexity are equivalent in the vectorial case. They speculate that this is not the case. In this paper, we show that these two classes of functions are in general different. With this aim, under a continuity assumption on $f(\cdot, \Sigma)$, in Theorem 2.4 we show that if the supremal functional

$$F(V, A) = \operatorname*{ess\,sup}_{x\in A} f(x, DV(x)) \tag{1.3}$$

is sequentially weakly* lower semicontinuous on $W^{1,\infty}(A,\mathbb{R}^d)$ for every open set $A \subset \Omega$, then for every $x \in \Omega$ the function $f(x,\cdot)$ is weak Morrey quasiconvex and **rank-one level convex**, i.e.

$$f(x, t\Sigma^1 + (1-t)\Sigma^2) \leq \max\{f(x,\Sigma^1),\ f(x,\Sigma^2)\}$$

for every $\Sigma^1, \Sigma^2 \in \mathbb{M}^{d\times N}$ such that rank $(\Sigma^1 - \Sigma^2) \leq 1$. In particular every strong Morrey quasiconvex function is rank-one level convex. This result is applied in Example 2.7 in order to find a weak Morrey quasiconvex function which cannot be strong Morrey quasiconvex: more precisely we exhibit a weak Morrey quasiconvex function f which is not rank-one level convex. A fortiori, such a function f cannot be strong Morrey quasiconvex.

Summarizing, by Theorem 2.4 and by [1, Proposition 5.2] we have that

$$\begin{aligned} &f \text{ strong Morrey quasiconvex} \\ &\quad \Longrightarrow f \text{ weak Morrey quasiconvex and rank-one quasiconvex,} \end{aligned}$$

$$\begin{aligned} &f \text{ upper semicontinuous and weak Morrey quasiconvex} \\ &\quad \Longrightarrow f \text{ rank-one quasiconvex} \end{aligned}$$

while in general

$$f \text{ weak Morrey quasiconvex} \not\Longrightarrow f \text{ strong Morrey quasiconvex}$$

$$\begin{aligned} &f \text{ lower semicontinuous and weak Morrey quasiconvex} \\ &\quad \not\Longrightarrow f \text{ rank-one quasiconvex} \end{aligned}$$

(see Example 2.7). In the integral case the famous counterexample of Sverak [25] shows that the rank-one convexity does not imply quasiconvexity at least when $N \geq 3$, $d \geq 2$. In the supremal case the question remains open whether the rank-one level convexity implies any type of Morrey quasiconvexity.

The last part of this paper is devoted to study the Γ-convergence, as $p \to \infty$, of the family of the integral functionals $F_p : C(\bar{\Omega};\mathbb{R}^d) \to [0,+\infty]$ given by

$$F_p(V) := \begin{cases} \left(\int_\Omega f^p(x, DV(x))dx\right)^{1/p} & \text{if } V \in W^{1,p}(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

where $f : \Omega \times \mathbb{M}^{d\times N} \to [0,+\infty)$ is a Carathéodory function satisfying the growth condition: there exist $\alpha, \beta > 0$ such that

$$\alpha|\Sigma| \leq f(x,\Sigma) \leq \beta(|\Sigma| + 1) \tag{1.5}$$

for a.e. $x \in \Omega$ and for every $\Sigma \in \mathbb{M}^{d\times N}$. This problem has been studied by Garroni-Nesi-Ponsiglione in the special case $f(x,\Sigma) = \lambda(x)|\Sigma|$ (see [17]), by Champion-De Pascale-Prinari when f satisfies a generalized Jensen inequality for gradient Young measures (see [12]) and by Bocea-Nesi and Ansini-Prinari when $V \in L^\infty(\Omega;\mathbb{M}^{d\times N})$ is constrained to satisfy a more general rank-constant differential constraint (see [1,10]). In this paper we give a complete representation of the Γ-limit by showing that, as $p \to \infty$, the family $(F_p)_p$ Γ-converges, with respect to the uniform convergence, to the supremal functional F

$$F(V) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, DV(x)) & \text{if } V \in W^{1,\infty}(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise} \end{cases} \tag{1.6}$$

where the function f_∞ is given by the formula

$$f_\infty(x,\Sigma) = \sup_{p\geq 1}(Qf^p(x,\Sigma))^{1/p} \quad \text{for a.e. } x \in \Omega, \quad \forall\, \Sigma \in \mathbb{M}^{d\times N}.$$

Here, for every $p \geq 1$ and for a.e. $x \in \Omega$, the function $Qf^p(x,\cdot)$ is the quasiconvex envelope of the function $f^p(x,\cdot)$, i.e.

$$Qf^p(x,\Sigma) := \inf\left\{\int_Q f^p(x,\Sigma + D\varphi(y))dy : \ \varphi \in C_0^\infty(Q;\mathbb{R}^d)\right\}. \tag{1.7}$$

(see Theorem 3.2). Notice that if $d = 1$ and $f : \mathbb{R}^N \to [0,+\infty)$ then

$$f_\infty(\xi) = \sup_{p\geq 1}(f^p)^{**}(\xi)^{1/p} \quad \forall\, \xi \in \mathbb{R}^N$$

since, in the scalar case, the function Qf^p coincides with the convex envelope $(f^p)^{**}$ of the function f^p. In particular if $f : \mathbb{R}^N \to [0,+\infty)$ is a continuous function satisfying a linear growth condition, then f_∞ coincides with the greatest lower semicontinuous and level convex function which is less or equal to f (see [23, Corollary3.11]).

In [1] a function $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ which satisfies condition

$$f(\Sigma) = \sup_{p\geq 1}(Qf^p(\Sigma))^{1/p}\ \forall\, \Sigma \in \mathbb{M}^{d\times N}$$

has been referred as **curl-∞ quasiconvex**. The class of curl-∞ quasiconvex functions is quite large since it contains the coercive functions which are continuous and quasiconvex (see [1, Proposition 3.6(1)]) or lower semicontinuous and level convex (see [2, Proposition 2.9]) or continuous and polylevel convex (see [1, Proposition 5.7]). When $N = 1$ or $d = 1$ the curl-∞ quasiconvexity reduces to level convexity in the class of the functions $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ which are upper semicontinuous (see [1, Proposition 5.2(3)]). In the general case, since the Γ-limit F given by (1.6) is weakly* lower semicontinuous, we obtain that every continuous curl-∞ quasiconvex function satisfying the growth condition (1.5) is strong Morrey quasiconvex. It remains an open question whether every strong Morrey quasiconvex function with a suitable growth condition is curl-∞ quasiconvex.

1.1. Notation

Let Ω be a bounded open subset of $\mathbb{R}^N$. We denote by $\mathcal{O}(\Omega)$ the family of open subsets of Ω. We write $\mathcal{L}^N(E)$ for the Lebesgue measure of $E \subset \mathbb{R}^N$. Let $\Sigma \in \mathbb{M}^{d\times N}$, where $\mathbb{M}^{d\times N}$ stands for the space of $d \times N$ real matrices, with a slight abuse of notation, we denote $|\Sigma| = \sum_{i=1}^d |\Sigma_i|$, where Σ_i is the i^{th} row of Σ and $|\Sigma_i|$ its Euclidean norm. We use ξ_i also to denote the i^{th} component of a vector ξ. The notation $\Sigma \cdot a$ stands for the matrix $\Sigma \in \mathbb{M}^{d\times N}$ that acts on the vector $a \in \mathbb{R}^N$ while $x \cdot y$ denotes the scalar product between two vectors.

2. Necessary conditions for the lower semicontinuity

In this section first of all we recall the main results shown by Barron-Jensen-Wang in [8]. They prove that the strong Morrey quasiconvexity provides a necessary and sufficient condition for the weak* lower semicontinuity of a supremal functional under a continuity assumption on $f(\cdot,\Sigma)$. Note that the proof of the following theorem relies on a intermediate result (see Lemma 2.8 in [8]) which shows that the weak Morrey quasiconvexity is a necessary condition for the weak* lower semicontinuity.

Theorem 2.1 ([8], Theorems 2.6–2.7**).** *Let $f : \Omega \times \mathbb{M}^{d\times N} \to \mathbb{R}$ be a Borel function. Assume that there exists a function $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ continuous in its first variable, non-decreasing in its second variable, $\omega(0,t) = 0$ for every $t > 0$ and such that*

$$|f(x_1,\Sigma) - f(x_2,\Sigma)| \leq \omega(|x_1 - x_2|, |\Sigma|) \tag{2.1}$$

for every $x_1, x_2 \in \mathbb{R}^N$, $\Sigma \in \mathbb{M}^{d\times N}$. Let

$$F(V,B) := \operatorname*{ess\,sup}_{x\in B} f(x, DV(x)), \qquad V \in W^{1,\infty}(\Omega;\mathbb{R}^d), \quad B \in \mathcal{O}(\Omega). \tag{2.2}$$

Then the following facts are equivalent:

(i) *For any $x \in \Omega$ $f(x,\cdot)$ is strong Morrey quasiconvex and lower semicontinuous;*

(ii) *The functional $F(\cdot,B)$ is weakly* lower semicontinuous in $W^{1,\infty}(B,\mathbb{R}^d)$ for every $B \in \mathcal{O}(\Omega)$.*

Remark 2.2.
1. The class of strong Morrey quasi-convex functions contain the (Morrey) quasi-convex functions with appropriate growth conditions (see [8, Proposition 2.4]).
2. Thanks to [8, Theorem 3.3], the level convexity is sufficient to provide the weak* lower semicontinuity of the functional $F(\cdot,B)$ given by (2.2). Therefore, thanks to the necessary condition stated in Theorem 2.1, it follows that any level convex function $f = f(\xi)$ is strong Morrey quasi-convex. In the scalar case $d = 1$ or $N = 1$, the class of strong Morrey quasi-convex functions coincides with the class of the level convex functions (for a proof see [8, Theorem 3.3] and [24, Theorem 5.5]).

In [8] the authors introduce a last notion of convexity: the rank-one convexity.

Definition 2.3. *A measurable function $f : \mathbb{R}^N \to \mathbb{R}$ is* **rank-one level convex** *if for every $t \in [0,1]$ it holds*

$$f(t\Sigma^1 + (1-t)\Sigma^2) \leq \max\{f(\Sigma^1),\, f(\Sigma^2)\} \tag{2.3}$$

for every $\Sigma^1, \Sigma^2 \in \mathbb{M}^{d\times N}$ such that rank $(\Sigma^1 - \Sigma^2) \leq 1$.

Any upper semicontinuous and weak Morrey quasi-convex function is rank-one level convex (see [1, Proposition 5.2]) but in general the only weak Morrey quasiconvexity is not sufficient to provide the rank-one quasiconvexity: in fact in [24, Remark 5.2], the authors exhibit an example of a lower semicontinuous and weak Morrey quasi-convex function which is not rank-one level convex. In the following theorem we derive the rank-one level convexity as a necessary condition for the weak* lower semicontinuity of a supremal functional. A version of this theorem has been given in [2, Theorem 6.2], in order to characterize the weak* lower semicontinuity of functionals of the form

$$F(V) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, V(x))$$

where $V \in L^\infty(\Omega; \mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$ and $\mathcal{A}$ is a constant-rank partial differential operator. Note that in our setting we do not require the continuity of the supremand function f with respect to second variable.

Theorem 2.4 (Necessary condition). *Let $f : \Omega\times\mathbb{M}^{d\times N} \to \mathbb{R}$ be a Borel function. Assume that there exists a function $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ continuous in its first variable, non-decreasing in its second variable, $\omega(0, t) = 0$ for every $t > 0$ and satisfying (2.1). Assume that the functional $F(\cdot, B)$ given by (2.2) is weakly* lower semicontinuous in $W^{1,\infty}(B, \mathbb{R}^d)$ for every $B \in \mathcal{O}(\Omega)$. Then*

(i) $\forall x \in \mathbb{R}^N$

$$\Sigma \mapsto f(x, \Sigma)$$

is lower semicontinuous in $\mathbb{R}^N$;

(ii) $\forall x_0 \in \Omega$ *and for every N-cube $Y \subset \mathbb{R}^N$*

$$f\left(x_0, \fint_Y DV(x)\, dx\right) \le \operatorname*{ess\,sup}_{x\in Y} f(x_0, DV(x)) \tag{2.4}$$

for every $V \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{M}^{d\times N})$ whose gradient is Y-periodic. In particular $f(x_0, \cdot)$ is weak Morrey quasiconvex;

(iii) $\forall x_0 \in \Omega$ *$f(x_0, \cdot)$ is rank-one level convex. In particular, if $d = 1$ or $N = 1$, $f(x_0, \cdot)$ is level convex.*

Proof. (i) It follows by [8, Lemma 2.8(1)].

(ii) Let $V \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{M}^{d\times N})$ whose gradient DV is Y-periodic. We define $V_n(x) = \frac{1}{n}V(n(x - x_0))$. Then $V_n \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{M}^{d\times N})$ and DV_n is $(1/n)Y$-periodic. Since $(V_n)_n$ converges weakly* to the function $U(x) = (\fint_Y DV(y)\, dy)\cdot x$ in $W^{1,\infty}(B; \mathbb{M}^{d\times N})$ for every bounded open set $B \subset \mathbb{R}^N$, by the lower semicontinuity of the functional F we have that for every $0 < \rho < d(x_0, \partial\Omega)$

$$\operatorname*{ess\,sup}_{x\in B_\rho(x_0)} f\left(x, \fint_Y DV(y)\, dy\right) \le \liminf_{n\to\infty} \operatorname*{ess\,sup}_{x\in B_\rho(x_0)} f(x, DV_n(x)). \tag{2.5}$$

By (2.1) we have that

$$\begin{aligned} f(x, DV_n(x)) &= f(x, DV_n(x)) - f(x_0, DV_n(x)) + f(x_0, DV_n(x)) \\ &\le \omega(|x - x_0|, |DV_n(x)|) + f(x_0, DV_n(x)) \end{aligned}$$

hence,

$$f(x, DV_n(x)) \leq \omega(|x - x_0|, C) + \operatorname*{ess\,sup}_{x \in B_\rho(x_0)} f(x_0, DV_n(x)), \tag{2.6}$$

Since DV_n is $(1/n)$ Y-periodic, for n big enough, we have that

$$\operatorname*{ess\,sup}_{x \in B_\rho(x_0)} f(x_0, DV_n(x)) = \operatorname*{ess\,sup}_{x \in Y} f(x_0, DV(x)).$$

Therefore, by (2.6) it follows that

$$\operatorname*{ess\,sup}_{x \in B_\rho(x_0)} f(x, DV_n(x)) \leq \operatorname*{ess\,sup}_{x \in B_\rho(x_0)} \omega(|x - x_0|, c) + \operatorname*{ess\,sup}_{x \in Y} f(x_0, DV(x)).$$

Gathering the last inequality with (2.5) we get that for every $0 < \rho < d(x_0, \partial\Omega)$

$$\operatorname*{ess\,sup}_{x \in B_\rho(x_0)} f\left(x, \fint_Y DV(y)\, dy\right) \leq \operatorname*{ess\,sup}_{x \in B_\rho(x_0)} \omega(|x - x_0|, c) + \operatorname*{ess\,sup}_{x \in Y} f(x_0, DV(x)).$$

Since f is continuous in its first variable, by passing to the limit as $\rho \to 0^+$ it follows that

$$f\left(x_0, \fint_Y DV(y)\, dy\right) \leq \operatorname*{ess\,sup}_{x \in Y} f(x_0, DV(x)).$$

Finally if $W \in W_0^{1,\infty}(Y; \mathbb{M}^{d\times N})$ and $\tilde{W}$ is its Y-periodic extension, then the function $V(x) = \Sigma \cdot x + \tilde{W} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{M}^{d\times N})$ has Y-periodic gradient and $\fint_Y DV(x)\, dx = \Sigma$; hence,

$$\begin{aligned} f(x_0, \Sigma) = f\Big(x_0, \fint_Y DV(x)\, dx\Big) &\leq \operatorname*{ess\,sup}_{x \in Y} f(x_0, DV(x)) \\ &= \operatorname*{ess\,sup}_{x \in Y} f(x_0, \Sigma + DW(x)) \end{aligned}$$

for every fixed $x_0 \in \Omega$, $\Sigma \in \mathbb{M}^{d\times N}$ and $W \in W_0^{1,\infty}(Y; \mathbb{M}^{d\times N})$. In particular, when $Y = Q$ we obtain that $f(x_0, \cdot)$ is weak Morrey quasiconvex.

(iii) Let $\Sigma^1, \Sigma^2 \in \mathbb{M}^{d\times N}$ such that rank $(\Sigma^1 - \Sigma^2) \leq 1$. Let $a \in \mathbb{R}^d, w \in \mathbb{R}^N$ such that $\Sigma^1 - \Sigma^2 = a \otimes w = (a_i w_j)$ and consider the function $V \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^d)$ defined by

$$V(x) := \begin{cases} \Sigma^1 \cdot x + (x \cdot w)a - (1 - t)ja, & \text{if } x \in A_1 \\ \Sigma^1 \cdot x + (1 + j)ta, & \text{if } x \in A_2 \end{cases}$$

where

$$A_1 = \left\{x \in \mathbb{R}^N : \, j \leq x \cdot w < j + t, \; j \in \mathbb{Z}\right\},$$

$$A_2 = \left\{x \in \mathbb{R}^N : \, j + t \leq x \cdot w < j + 1, \; j \in \mathbb{Z}\right\}$$

for fixed $t \in (0, 1)$. Then the gradient

$$DV(x) = \begin{cases} \Sigma^1, & \text{if } x \in A_1 \\ \Sigma^2, & \text{if } x \in A_2 \end{cases}$$

is Y-periodic, where Y is any N-cube with one face orthogonal to w and side lenght 1. Hence, by (2.4)

$$f\left(t\Sigma_2+(1-t)\Sigma_1\right)=f\left(\fint_Y DV(y)\right)\le \operatorname*{ess\,sup}_{x\in Y} f(DV(x))$$
$$=\max\left\{f(\Sigma^1),\ f(\Sigma^2)\right\}.$$

□

Remark 2.5. In order to show part (iii) of Theorem 2.4 it is possible to proceed also in the following way (see [21]): for a fixed unit vector $a\in\mathbb{R}^N$ and for a fixed vector $w\in\mathbb{R}^d$, let $W\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^d)$ be the function given by

$$W(x):=w\int_0^{x\cdot a}(2\chi(s)-1)\,ds$$

where χ is the characteristic function of $(0,\frac{1}{2})$ in $(0,1)$ and extended periodically to $\mathbb{R}$. Then

$$DW(x)=(2\chi(x\cdot a)-1)a\otimes w$$

and, if $Y\subset\mathbb{R}^N$ is a cube with one axis along the direction of a, then, for fixed $\Sigma^1\in\mathbb{M}^{d\times N}$, we have that the function

$$V(x):=\Sigma_1\cdot x+W(x)\in W^{1,\infty}_{\mathrm{loc}}\left(\mathbb{R}^N;\mathbb{R}^d\right)$$

has Y-periodic gradient satisfying $\fint_Y DV(x)\,dx=\Sigma_1$. Therefore, by (2.4) it follows that

$$f(x_0,\Sigma^1)\le f(x_0,\Sigma^1+a\otimes w)\vee f(x_0,\Sigma^1-a\otimes w)$$

for every $x_0\in\Omega$. In particular, for every $\Sigma^1,\Sigma^2\in\mathbb{M}^{d\times N}$ such $\Sigma^1-\Sigma^2=a\otimes w$ we obtain

$$f\left(x_0,\frac{1}{2}\Sigma^1+\frac{1}{2}\Sigma^2\right)\le f(x_0,\Sigma^1)\vee f(x_0,\Sigma^2)\tag{2.7}$$

for every $x_0\in\Omega$. By using the fact that $f(x_0,\cdot)$ is lower semicontinuous, we can easily conclude that $f(x_0,\cdot)$ is rank-one level convex.

Corollary 2.6. *Let $f:\mathbb{M}^{d\times N}\to\mathbb{R}$ be a lower semicontinuous and strong Morrey quasiconvex function. Then f is rank-one level convex.*

Proof. Thanks to Theorem 2.1 the functional $F(V,\Omega)=\operatorname{ess\,sup}_{x\in\Omega}f(DV(x))$ is weakly* lower semicontinuous on $W^{1,p}(\Omega,\mathbb{R}^d)$ for every bounded open set $\Omega\subset\mathbb{R}^N$. Therefore, by applying Theorem 2.4, we can conclude that f is rank-one level convex. □

Thanks to the result given in Theorem 2.4 we are in position to show that in general the weak Morrey quasiconvexity is not equivalent to the strong Morrey quasiconvexity.

Example 2.7. Let $\Sigma^1, \Sigma^2 \in \mathbb{M}^{d\times N}$ be such that rank $(\Sigma^1 - \Sigma^2) = 1$. We consider the lower semicontinuous function

$$f(\Sigma) = 1 - \chi_S(\Sigma)$$

where $S = \{\Sigma^1, \Sigma^2\}$, and χ_S is the characteristic function of S. Then

(i) The function f is not rank-one level convex; therefore, thanks to Corollary 2.6, f is not strong Morrey quasiconvex.

(ii) As shown in [24], we have that f is weak Morrey quasiconvex. To this end it is enough to consider the case where $Z \notin S$. Then, the property (1.2) follows from the fact that there is no $\varphi \in W_0^{1,\infty}(Q; \mathbb{M}^{d\times N})$ such that $D\varphi(x) \in \{\Sigma^1 - Z; \Sigma^2 - Z\}$ a.e. in Q.

3. An approximation result

In this section we study the Γ-convergence, as $p \to \infty$, of the family of integral functionals $F_p : C(\bar{\Omega}; \mathbb{R}^d) \to [0, +\infty]$ given by

$$F_p(V) := \begin{cases} \left(\int_\Omega f^p(x, DV(x))dx \right)^{1/p} & \text{if } V \in W^{1,p}(\Omega; \mathbb{R}^d) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

For a comprehensive study of Γ-convergence we refer to the book of Dal Maso [15]. Here we recall only the sequential characterization of the Γ-limit when X is a metric space.

Proposition 3.1 (**[15]** Proposition 8.1)**.** *Let X be a metric space and let $\varphi_n : X \to \mathbb{R} \cup \{\pm\infty\}$ for every $n \in \mathbb{N}$. Then (φ_n) Γ-converges to φ with respect to the strong topology of X (and we write $\Gamma(X)\text{-}\lim_{n\to\infty} \varphi_n = \varphi$) if and only if*

(i) *For every $x \in X$ and for every sequence (x_n) converging to x, it is*

$$\varphi(x) \le \liminf_{n\to\infty} \varphi_n(x_n);$$

(ii) *For every $x \in X$ there exists a sequence (x_n) (recovering sequence) converging to $x \in X$ such that*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x_n).$$

The main result of this section is the following theorem.

Theorem 3.2. *Let Ω be a bounded open set with Lipschitz continuous boundary. Let $f : \Omega \times \mathbb{M}^{d\times N} \to \mathbb{R}$ be a Carathéodory function satisfying the following growth condition: there exist $\alpha, \beta > 0$ such that*

$$\alpha|\Sigma| \le f(x, \Sigma) \le \beta(|\Sigma| + 1) \qquad \text{for a.e. } x \in \Omega \text{ and for every } \Sigma \in \mathbb{M}^{d\times N}. \tag{3.2}$$

For every $p \ge 1$ let $F_p : C(\bar{\Omega}; \mathbb{R}^d) \to [0, +\infty]$ be the functional given by (3.1) and for every $(x, \Sigma) \in \Omega \times \mathbb{M}^{d\times N}$ let

$$(Qf^p)(x, \Sigma) := \inf \left\{ \int_Q f^p(x, \Sigma + DV(y))\, dy \;:\; V \in W_0^{1,\infty}(Q; \mathbb{R}^d) \right\}.$$

Then, as $p \to \infty$, $(F_p)_{p\geq 1}$ Γ*-converges with respect to the uniform convergence to the functional* $\bar{F} : C(\bar{\Omega}; \mathbb{R}^d) \to [0, +\infty]$ *given by*

$$\bar{F}(V) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, DV(x)) & \text{if } V \in W^{1,\infty}(\Omega; \mathbb{R}^d), \\ +\infty & \text{otherwise} \end{cases} \tag{3.3}$$

where

$$f_\infty(x, \Sigma) = \lim_{p\to\infty} (Qf^p(x, \Sigma))^{1/p}. \tag{3.4}$$

Remark 3.3. Note that

(1) Thanks to the growth condition (3.2), we have that Qf^p is a Carathéodory function which is quasiconvex with respect to the second variable (see [14, Proposition 9.5]);

(2) Applying Hölder's inequality, it is easy to show that the family $(\left(Qf^p\right)^{1/p})_{p\geq 1}$ is not decreasing. Therefore it follows that for every $x \in \Omega$ and for every $\Sigma \in \mathbb{M}^{d\times N}$ there exists the pointwise limit

$$\lim_{p\to\infty} \left(Qf^p\right)^{1/p}(x, \Sigma) = \sup_{p\geq 1} \left(Qf^p\right)^{1/p}(x, \Sigma).$$

(3) The Γ-lim inf inequality

$$F_\infty \leq \Gamma\text{-}\liminf_{p\to\infty} F_p$$

can be obtained also as application of Theorem 4.2 in [1] to the particular case $\mathcal{A} = curl$. However this theorem does not give the existence of a recovering sequence in the space $C(\bar{\Omega}, \mathbb{R}^d)$.

Proof (of Theorem 3.2). The proof will be achieved in several steps.

Step 1. For every $p \geq 1$ let $\bar{F}_p : C(\bar{\Omega}; \mathbb{R}^d) \to [0, +\infty]$ be the lower semicontinuous envelope of the functional F_p with respect to the uniform convergence. Since the family $(F_p)_{p\geq 1}$ is non decreasing, by [15, Proposition 5.4], we have that

$$\Gamma(L^\infty)\text{-}\lim_{p\to\infty} F_p(V) = \lim_{p\to\infty} \bar{F}_p(V) = \sup_{p\geq 1} \bar{F}_p(V). \tag{3.5}$$

Now we show that for every $p > N$ the lower semicontinuous envelope $\bar{F}_p$ coincides with the functional $\phi_p : C(\bar{\Omega}; \mathbb{R}^d) \to [0, +\infty]$ given by

$$\phi_p(V) := \begin{cases} \left(\int_\Omega Qf^p(x, DV(x))dx\right)^{1/p} & \text{if } V \in W^{1,p}(\Omega, \mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

With this aim, let $G_p, \bar{G}_p : W^{1,p}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ be the functionals given by

$$G_p(V) := \left(\int_\Omega f^p(x, DV(x))dx\right)^{1/p}$$

and

$$\bar{G}_p(V) := \left(\int_\Omega Qf^p(x, DV(x))dx\right)^{1/p}.$$

By [4, Statement III(7)] it follows that for every p the functional $\bar{G}_p$ is the sequentially lower semicontinuous envelope of the functional G_p on $W^{1,p}(\Omega,\mathbb{R}^d)$ with respect to the weak convergence of $W^{1,p}(\Omega,\mathbb{R}^d)$. In order to show that $\phi_p \le \bar{F}_p$ we notice that for every $p>1$ the functional ϕ_p is lower semicontinuous on $C(\bar{\Omega};\mathbb{R}^d)$ with respect to the uniform convergence. In fact, if $(V_n)\subseteq C(\bar{\Omega},\mathbb{R}^d)$ is such that $V_n \to V$ with respect to the uniform convergence and $\liminf_{n\to\infty}\phi_p(V_n)<+\infty$ then, thanks to the coercivity assumption (3.2), we have that there exists $n_0\in\mathbb{N}$ such that the sequence $(V_n)_{n\ge n_0}$ is bounded in $W^{1,p}(\Omega,\mathbb{R}^d)$. Therefore, up to a subsequence, $(V_n)_{n\ge n_0}$ weakly converges to V in $W^{1,p}(\Omega,\mathbb{R}^d)$. In particular

$$\phi_p(V)=\bar{G}_p(V)\le \liminf_{n\to\infty}\bar{G}_p(V_n)=\liminf_{n\to\infty}\phi_p(V_n).$$

Since $\phi_p\le F_p$ on $C(\bar{\Omega},\mathbb{R}^d)$ and ϕ_p is lower semicontinuous with respect to the uniform convergence, we obtain that

$$\phi_p(V)\le \bar{F}_p(V) \quad \forall\, V\in C(\bar{\Omega},\mathbb{R}^d). \tag{3.6}$$

On the other hand, for every $p>N$ the functional $\bar{F}_p$ is sequentially lower semicontinuous on $W^{1,p}(\Omega,\mathbb{R}^d)$ with respect to the weak convergence of $W^{1,p}(\Omega,\mathbb{R}^d)$. In fact, if $(V_n)\subseteq W^{1,p}(\Omega,\mathbb{R}^d)$ is such that $V_n\to V$ with respect to weak convergence of $W^{1,p}(\Omega,\mathbb{R}^d)$ then, thanks to Rellich-Kondrachov Theorem, we have that $V_n\in C(\bar{\Omega},\mathbb{R}^d)$ and $V_n\to V$ with respect to the uniform convergence. In particular it follows that $\bar{F}_p(V)\le\liminf_{n\to\infty}\bar{F}_p(V_n)$. Since $\bar{F}_p\le F_p=G_p$ on $W^{1,p}(\Omega,\mathbb{R}^d)$ we get that for every $p>N$

$$\bar{F}_p(V)\le \bar{G}_p(V)=\phi_p(V) \quad \forall\, V\in W^{1,p}\left(\Omega,\mathbb{R}^d\right). \tag{3.7}$$

Inequalities (3.6) and (3.7) imply that for every $p>N$

$$\bar{F}_p(V)=\phi_p(V)=\left(\int_\Omega Qf^p(x,DV(x))dx\right)^{1/p} \quad \forall\, V\in W^{1,p}\left(\Omega,\mathbb{R}^d\right).$$

If we show that$\bar{F}_p(V)<+\infty$ if and only if $V\in W^{1,p}(\Omega,\mathbb{R}^d)$ then we can conclude that $\bar{F}_p=\phi_p$ on $C(\bar{\Omega},\mathbb{R}^d)$ for every $p>N$. In fact if $V\in C(\bar{\Omega},\mathbb{R}^d)$ is such that $\bar{F}_p(V)<+\infty$ then there exists a sequence $(V_n)\subseteq C(\bar{\Omega},\mathbb{R}^d)$ such that $V_n\to V$ with respect to the uniform convergence and $\lim_{n\to\infty}F_p(V_n)=\bar{F}_p(V)<+\infty$. Then there exists $n_0\in\mathbb{N}$ such that $V_n\in W^{1,p}(\Omega,\mathbb{R}^d)$ for every $n\ge n_0$. Thanks to the coercivity assumption (3.2), we have that the sequence $(V_n)_{n\ge n_0}$ is bounded in $W^{1,p}(\Omega,\mathbb{R}^d)$ and, up to a subsequence, weakly converges to V in $W^{1,p}(\Omega,\mathbb{R}^d)$ when $p>1$. In particular $V\in W^{1,p}(\Omega,\mathbb{R}^d)$. The viceversa is trivial.

Step 2. Since $((Qf^p)^{1/p})$ is a not decreasing family converging to f_∞, for every $p>1$ we have that

$$\bar{F}_p(V)=\left(\int_\Omega Qf^p(x,DV(x))dx\right)^{1/p}\le \mathcal{L}^N(\Omega)^{\frac{1}{p}}\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x,DV(x))$$

for every $V\in W^{1,\infty}(\Omega,\mathbb{R}^d)$. In particular,

$$\sup_{p\ge 1}\bar{F}_p(V)\le \lim_{p\to\infty}\mathcal{L}^N(\Omega)^{\frac{1}{p}}\tilde{F}(V)=\tilde{F}(V). \tag{3.8}$$

By (3.5) and (3.8) we get that

$$\Gamma(L^\infty)\text{-}\lim_{p\to\infty} F_p(V) \le \tilde{F}(V)$$

for every $V \in C(\bar{\Omega}, \mathbb{R}^d)$.

We now prove the converse inequality; i.e.,

$$\Gamma(L^\infty)\text{-}\lim_{p\to\infty} F_p(V) \ge \tilde{F}(V)$$

for every $V \in C(\bar{\Omega}, \mathbb{R}^d)$.

Without loss of generality, we consider the case when $V \in C(\bar{\Omega}, \mathbb{R}^d)$ is such that $M = \sup_{p>1} \bar{F}_p(V) < +\infty$. This implies that $V \in W^{1,p}(\Omega, \mathbb{R}^d)$ for every $1 \le p < +\infty$ and thanks to the coercivity assumption (3.2), we have that for for every $1 \le p < +\infty$

$$\alpha ||V||_{W^{1,p}(\Omega,\mathbb{R}^d)} \le M.$$

In particular it follows that $V \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ and

$$\tilde{F}(V) = \operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, DV(x)) \le \beta ||V||_{W^{1,\infty}(\Omega,\mathbb{R}^d)} < +\infty.$$

Therefore, for every fixed $\varepsilon > 0$, there exists a measurable set $B_\varepsilon \subset \Omega$ such that $\mathcal{L}^N(B_\varepsilon) > 0$ and

$$\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, DV(x)) \le f_\infty(x, DV(x)) + \varepsilon$$

for every $x \in B_\varepsilon$. This implies

$$\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, DV(x))\mathcal{L}^N(B_\varepsilon) \le \int_{B_\varepsilon} f_\infty(x, DV(x))dx + \varepsilon\mathcal{L}^N(B_\varepsilon).$$

By Remark 3.3(2), Beppo Levi's Theorem, and Hölder's inequality we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, DV(x))\mathcal{L}^N(B_\varepsilon) \\
&\quad \le \lim_{p\to\infty} \int_{B_\varepsilon} (Qf^p)^{\frac{1}{p}}(x, DV(x))dx + \varepsilon\mathcal{L}^N(B_\varepsilon) \\
&\quad \le \lim_{p\to\infty} \left(\int_{B_\varepsilon} Qf^p(x, DV(x))dx\right)^{\frac{1}{p}} \mathcal{L}^N(B_\varepsilon)^{1-\frac{1}{p}} + \varepsilon\mathcal{L}^N(B_\varepsilon).
\end{aligned}$$

It follows that

$$\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, DV(x)) \le \lim_{p\to\infty} \bar{F}_p(V)\mathcal{L}^N(B_\varepsilon)^{-\frac{1}{p}} + \varepsilon = \sup_{p\ge 1} \bar{F}_p(V) + \varepsilon. \quad (3.9)$$

By (3.5), (3.9), and the arbitrariness of ε we conclude that

$$\Gamma(L^\infty)\text{-}\lim_{p\to\infty} F_p(V) \ge \operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, DV(x)) = \tilde{F}(V).$$

□

Corollary 3.4. *Let Ω be a bounded open set with Lipschitz continuous boundary. Let $f : \Omega \times \mathbb{M}^{d\times N} \to \mathbb{R}$ be a Carathéodory function satisfying the growth* *condition (3.2). Assume that for a.e. $x \in \Omega$ $f(x,\cdot)$ is curl-∞ quasiconvex, i.e. for a.e. $x \in \Omega$*

$$f(x,\Sigma) = \sup_{p\geq 1}(Qf^p(x,\Sigma))^{1/p} \ \forall\, \Sigma \in \mathbb{M}^{d\times N}. \tag{3.10}$$

For every $p \geq 1$ let $F_p : C(\bar{\Omega};\mathbb{R}^d) \to [0,+\infty]$ be the functional given by (3.1). Then, as $p \to \infty$, the family $(F_p)_{p\geq 1}$ Γ-converges with respect to the uniform convergence to the functional $F : C(\bar{\Omega};\mathbb{R}^d) \to [0,+\infty]$ given by

$$F(V) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f(x, DV(x)) & \text{if } V \in W^{1,\infty}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.11}$$

Now we show that, if f does not depend explicitly on x, then the curl-∞ quasiconvexity is also a necessary condition for the L^p-approximation of a supremal functional.

Theorem 3.5. *Let $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ be a continuous function satisfying the following growth condition: there exist $\alpha, \beta > 0$ such that*

$$\alpha|\Sigma| \leq f(\Sigma) \leq \beta(|\Sigma|+1) \ \forall \Sigma \in \mathbb{M}^{d\times N}. \tag{3.12}$$

Let Ω be an open set with Lipschitz continuous boundary and let $F_p, F : C(\Omega;\mathbb{R}^d) \to [0,+\infty]$ be given by

$$F_p(V) := \begin{cases} \left(\int_\Omega f^p(DV(x))dx\right)^{1/p} & \text{if } V \in W^{1,p}(\Omega;\mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.13}$$

and

$$F(V) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f(DV(x)) & \text{if } V \in W^{1,\infty}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.14}$$

respectively. Then the following statement are equivalent:

(i) *f is curl-∞ quasiconvex function;*
(ii) *F_p Γ-converges to F, as $p \to \infty$, with respect to the uniform convergence.*

Proof. (i) $\Longrightarrow$ (ii): it follows by Theorem 3.2.

(ii) $\Longrightarrow$ (i): by Theorem 3.2 we have that

$$\Gamma(L^\infty)\text{-}\lim_{p\to\infty} F_p(V) = \bar{F}(V)$$

for every $V \in W^{1,\infty}(\Omega;\mathbb{R}^d)$, with $\bar{F}$ given by (3.3). Therefore, by assumption we have that

$$\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(DV(x)) = F(V) = \operatorname*{ess\,sup}_{x\in\Omega} f(DV(x))$$

for every $V \in W^{1,\infty}(\Omega;\mathbb{R}^d)$. In particular, for fixed $\Sigma \in \mathbb{M}^{d\times N}$, choosen $V(x) := \Sigma \cdot x$ where $x \in \Omega$, we get

$$f_\infty(\Sigma) = f(\Sigma)$$

for every $\Sigma \in \mathbb{M}^{d\times N}$. □

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