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On the Brezis–Lieb lemma without pointwise convergence

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Abstract. Brezis–Lieb lemma is an improvement of Fatou Lemma that evaluates the gap between the integral of a functional sequence and the integral of its pointwise limit. The paper proves some analogs of Brezis–Lieb lemma without assumption of convergence almost everywhere. While weak convergence alone brings no conclusive estimates, a lower bound for the gap is found in L^p, $p \geq 3$, under condition of weak convergence and weak convergence in terms of the duality mapping. We prove that the restriction on p is necessary and prove few related inequalities in connection to weak convergence.

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1. Introduction

Brezis–Lieb lemma ([1]) is a refinement of Fatou lemma that plays an important role in analysis of partial differential equations. Let Ω, μ be a measure space. The lemma says that if (u_k) is a bounded sequence in $L^p(\Omega, \mu)$, $p \in (1, \infty)$, and $u_k \to u$ a.e., then

$$\int_\Omega |u_k|^p d\mu - \int_\Omega |u|^p d\mu - \int_\Omega |u_k - u|^p d\mu \to 0. \tag{1.1}$$

In concrete applications convergence a.e. might be hard to verify, while the weak convergence condition rarely presents a difficulty, since $L^p(\Omega, \mu)$ with $p \in (1, \infty)$ is reflexive and any bounded sequence there has a weakly convergent subsequence. Thus it is natural to ask what possible analogs of (1.1) may exist for sequences in L^p that do not necessarily converge everywhere. This situation arises in applications to quasilinear elliptic PDE when u_k are vector-valued functions of the form $\nabla w_k \in L^p$ and one cannot rely on compactness of local Sobolev imbeddings that yield a.e. convergence of w_k but not of their gradients. An immediate analog is given by weak semicontinuity of the norm, namely

$$u_k \rightharpoonup u \implies \int_\Omega |u_k|^p d\mu \geq \int_\Omega |u|^p d\mu + o(1),$$

but this inequality is quite crude as it does not account for the norm of the remainder $u_k - u$.

On the other hand, there are some cases where Brezis–Lieb lemma holds under assumption of weak convergence alone. One is when Ω is a countable set equipped with the counting measure, because in this case pointwise convergence follows from weak convergence. Another is the case $p = 2$, when the conclusion of Brezis–Lieb lemma holds even if convergence a.e. is not assumed. This follows from an elementary relation in the general Hilbert space:

$$u_k \rightharpoonup u \implies \|u_k\|^2 = \|u_k - u\|^2 + \|u\| - 2(u_k - u, u) = \|u_k - u\|^2 + \|u\| + o(1). \tag{1.2}$$

Since in both examples the norm satisfies the Opial condition [4], it would be tempting to conjecture that the condition of a.e. convergence may be dropped whenever the Opial condition holds, or, in case of a strictly convex Banach space X with single-valued duality map, whenever the following sharp sufficient condition, which implies Opial condition (see [4]), holds: $u_k \rightharpoonup 0$ in $X \implies u_k^* \rightharpoonup 0$. This prompted the authors of a forthcoming paper [5] to prove the following analog of Brezis–Lieb Lemma with a.e. convergence replaced by weak convergence of a dual sequence. However, as we show in Corollary 3.5 below, the condition $p \geq 3$ (that has nothing to do with Opial's condition or dual mapping) cannot be relaxed. The condition $|u_k - u|^{p-2}(u_k - u) \rightharpoonup 0$ below is not arbitrary, but is an assumption of weak convergence of the duality mapping, which can be equivalently expressed as $(u_k - u)^* \rightharpoonup 0$.

Theorem 1.1. ([5]) *Let (Ω, μ) be a measure space and let $p \in [3, \infty)$. Assume that $u_k \rightharpoonup u$ in $L^p(\Omega, \mu)$ and $|u_k - u|^{p-2}(u_k - u) \rightharpoonup 0$ in $L^{p'}(\Omega, \mu)$, $p' = \frac{p}{p-1}$. Then*

$$\int_\Omega |u_k|^p d\mu \geq \int_\Omega |u|^p d\mu + \int_\Omega |u_k - u|^p d\mu + o(1). \tag{1.3}$$

The proof of the theorem follows immediately from the following elementary inequality, verified in [5],

$$|1 + t|^p \geq 1 + |t|^p + p|t|^{p-2}t + pt, \quad |t| \leq 1, \tag{1.4}$$

which in turn implies $|u_k|^p \geq |u_k - u|^p + |u|^p + p|u|^{p-2}u(u_k - u) + p|u_k - u|^{p-2}(u_k - u)u$, with the integrals of the last two terms vanishing by assumption. Remarkably, (1.4) is false for all $p \in (1, 3)$, but this does not imply that (1.3) is false for these p, moreover, as we mentioned above, it is true in the case of ℓ^p, although as we show in this note, it is false for $L^p([0, 1])$. The inequality in (1.3) can be strict. Indeed, one can easily calculate by binomial expansion for $p = 4$ that if $u_k \rightharpoonup u$ and $(u_k - u)^3 \rightharpoonup 0$ in $L^{4/3}$, then

$$\int_\Omega |u_k|^4 d\mu = \int_\Omega |u|^4 d\mu + \int_\Omega |u_k - u|^4 d\mu + 6 \int u^2 (u_k - u)^2 d\mu + o(1).$$

There have been some modifications of Brezis–Lieb lemma, in literature, namely [2,3], but we could not find any related results without the assumption of the a.e. convergence. In this note we prove a generalization of (1.3) to the case of vector-valued functions and $p \geq 3$, and show in Corollary 3.5 that the inequality (1.3) is false for all $p \in (1,3)$. Other results in this note are: a different weak convergence condition that yields (1.3) for all $p \geq 2$ (Theorem 4.1), a version of Theorem 1.1 for vector-valued functions (Theorem 2.1), and the analysis, in Sect. 3, of weak limits for sequences of the form $\varphi \circ v_k$ with different functions φ.

2. Theorem 1.1 for vector-valued functions

Theorem 2.1. *Let (Ω, μ) be a measure space and let $p \in [3, \infty)$ and $m \in \mathbb{N}$. Assume that $u_k \rightharpoonup u$ in $L^p(\Omega, \mu; \mathbb{R}^m)$ and $|u_k - u|^{p-2}(u_k - u) \rightharpoonup 0$ in $L^{p'}(\Omega, \mu; \mathbb{R}^m)$, $p' = \frac{p}{p-1}$. Then*

$$\int_\Omega |u_k|^p d\mu \geq \int_\Omega |u|^p d\mu + \int_\Omega |u_k - u|^p d\mu + o(1). \tag{2.1}$$

Proof. Once we prove the inequality

$$F(t,\theta) := |1 + t^2 + 2t\theta|^{p/2} - 1 - |t|^p - p|t|^{p-2}t\theta - pt\theta \geq 0, \quad |t| \leq 1, \ |\theta| \leq 1, \tag{2.2}$$

the assertion of the theorem will follow similarly to that of Theorem 1.1.

Note that for each $t \in [-1, 1]$, the function $\theta \mapsto F(t, \theta)$ is convex on $[-1, 1]$. An elementary computation shows that, for any $t \in [-1, 1]$, $\frac{\partial F(t,\theta)}{\partial \theta} \neq 0$, and thus $F(t, \theta) \geq \min\{F(t, -1), F(t, 1)\}$. Since $F(t, -1) = F(-t, 1)$ it suffices to show that $F(t, 1) \geq 0$ for all $t \in [-1, 1]$. This inequality, however, is nothing but (1.4). □

Writing the statement of Theorem 2.1 in terms of gradients of functions, and noting that $|\nabla u_k - \nabla u|^{p-2}(\nabla u_k - \nabla u) \rightharpoonup 0$ in $L^{p'}(\Omega; \mathbb{R}^N)$ can be rewritten in terms of the p-Laplacian, as $-\Delta_p(u_k - u) \rightharpoonup 0$ in the sense of distributions (the relevant norm bound is already given as the L^p bound for the gradient in the first condition), we have

Corollary 2.2. *Let $\Omega \in \mathbb{R}^N$, $N \in \mathbb{N}$, be an open set and let $p \in [3, \infty)$. Assume that $\nabla u_k \rightharpoonup \nabla u$ in $L^p(\Omega; \mathbb{R}^N)$ and $-\Delta_p(u_k - u) \rightharpoonup 0$ in the sense of distributions. Then*

$$\int_\Omega |\nabla u_k|^p dx \geq \int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla u_k - \nabla u|^p dx + o(1).$$

3. Weak convergence of compositions.

Let $p \in (1, \infty)$. It is possible to construct a sequence $v_k \rightharpoonup 0$ in $L^p([0, 1])$ such that $|v_k|^{q-1}v_k$ has a nonzero weak limit in $L^{p/q}([0, 1])$ for any $q \in (1, p]$. We consider here a more general case, comparing weak limits of sequences of the form $\varphi(v_k)$ with different odd continuous functions φ.

We focus here only on the measure space $[0, 1]$ equipped with the Lebesgue measure, but the argument can be easily adapted to domains in $\mathbb{R}^N$. Let $T_j v(x) = v(jx)$ for $x \in [0, 1/j]$, $j \in \mathbb{N}$, extended periodically to the rest of the interval $[0,1]$. Note that operators T_j are isometries on $L^p([0,1])$. Oscillatory sequences $T_j v$ always converge weakly to a constant function as indicated in the following statement.

Lemma 3.1. *If $v \in L^p([0,1])$, $p \in (1,\infty)$, then $T_j v \rightharpoonup \int_{[0,1]} v\,dx$ in $L^p([0,1])$.*

Proof. Since $\|T_j v\|_p = \|v\|_p$, it suffices to verify that $\int T_j v\psi \ dx \to \int_{[0,1]} v\,dx \int_{[0,1]} \psi\,dx$ for all step functions ψ, since they form a dense subspace of $L^{p'}([0,1])$. This, however, easily follows from a particular case $\psi = 1$, which in turn can be handled by applying periodicity and rescaling of the integration variable. □

Lemma 3.2. *Let $1 < q \le p < \infty$. If φ is a continuous real-valued function on $\mathbb{R}$ such that for some $C > 0$, $|\varphi(t)| \le C(1 + |t|^q)$, and $v \in L^p([0,1])$, then $\varphi(T_j v) = T_j\varphi(v) \rightharpoonup \int_{[0,1]} \varphi(v(s))ds$ in $L^{p/q}([0,1])$.*

Proof. Let v be first a step function with values t_j on intervals of length m_j, $j = 1,\dots,M$. By Lemma 3.1, $\varphi(T_k v) \rightharpoonup \sum_j \varphi(t_j)m_j = 0$ in L^r for any $r \in [1,\infty)$. The assertion of the lemma follows then from the density of step functions in $L^{p/q}$. □

Theorem 3.3. *Let φ_i, $i = 1,\dots,M$, be continuous functions $\mathbb{R} \to \mathbb{R}$, odd for each $i \ne M$, and assume that for some $q \ge 1$, $C > 0$, $|\varphi_i(t)| \le C(1 + |t|^q)$, $i = 1,\dots,M$. If for every sequence $v_k \in L^\infty([0,1])$, such that $\varphi_i(v_k) \rightharpoonup 0$ in $L^1([0,1])$, $i = 1,\dots M-1$, one also has $\varphi_M(v_k) \rightharpoonup 0$ in $L^1([0,1])$, then the functions $\{\varphi_i\}_{i=1,\dots M}$ are linearly dependent.*

Proof. Let $\psi \ge 1$ be a Lipschitz continuous function on $[-a,a] \subset \mathbb{R}$, $a > 0$, and let v be a solution of the equation

$$v'(t) = \frac{\gamma}{\psi(v(t))}, \quad v(0) = -a,$$

with the value of $\gamma = \gamma(\psi) > 0$ set to satisfy $v(1) = a$. Such γ always exists, since $v'(t) \le \gamma$ and thus $v(1) \le -a + \gamma$, and on the other hand, $v(1) \ge -a + \frac{\gamma}{\psi(-a)+L(v(1)+a)}$, where L is the Lipschitz constant of ψ, and thus $v(1)$ is a continuous function of $\gamma \in (0,\infty)$ with the range $(-a,+\infty)$.

By Lemma 3.2,

$$\varphi_i(T_k v) \rightharpoonup \int_{[0,1]} \varphi_i(v(s))\,ds = \gamma^{-1}\int_{[-a,a]} \varphi_i(t)\psi(t)\,dt, \tag{3.1}$$

with the weak convergence in $L^p([0,1])$ for any $p \ge 1$.

Consider now a closure Y in $L^2([-a,a])$ of the span of all positive bounded continuous functions ψ on $[-a,a]$, such that $(\varphi_i,\psi)_{L^2([-a,a])} = 0$, $i = 1,\dots,M-1$. Note that Ycontains all positive even functions and thus is nontrivial. Furthermore, Y is the orthogonal complement of $\{\varphi_i\}_{i=1,\dots,M-1}$ in L^2: indeed, any function can be approximated by a bounded function in this orthogonal complement, and adding a large constant to the latter makes it a positive function orthogonal to $\{\varphi_i\}_{i=1,\dots,M-1}$. By assumption, it follows from (3.1) that $\varphi_M \perp Y$, and consequently it belongs to the span of $\varphi_1,\dots,\varphi_{M-1}$

as functions on $[-a,a]$. Since the value of $a>0$ is arbitrary, on may conclude (assuming without loss of generality that $\varphi_1,\dots,\varphi_{M-1}$ are linearly independent, so that the coefficients in expansion of φ_M as a linear combination of $\varphi_1,\dots,\varphi_{M-1}$ are unique), the functions $\varphi_1,\dots,\varphi_M$ are linearly dependent also as functions on $\mathbb{R}$. □

Corollary 3.4. *Let φ_i, $i=1,\dots,M$, be continuous linearly independent nonzero functions $\mathbb{R}\to\mathbb{R}$, odd for each $i\neq M$, and assume that for some $q\geq 1$, $C>0$, $|\varphi_i(t)|\leq C(1+|t|^q)$, $i=1,\dots,M$. There exists a sequence $v_k\in L^\infty([0,1])$, such that $\varphi_i(v_k)\rightharpoonup 0$ in $L^1([0,1])$, $i=1,\dots M-1$, while there is $\alpha\neq 0$ such that $\varphi_M(v_k)\rightharpoonup\alpha$. If the functions φ_i, $i=1,\dots,M$, are piecewise-C^1 and linearly independent on any interval, and φ_M changes sign, the sequence v_k can be chosen so that $\alpha<0$.*

Proof. The first assertion of the corollary is immediate from Theorem 3.3. Assume now, in view of Lemma 3.2, that for every $v\in L^\infty([0,1])$, such that $\varphi_i(T_kv)\rightharpoonup\int_{[0,1]}\varphi_i(v(s))ds=0$, $i=1,\dots,M-1$, we have $\alpha=\int_{[0,1]}\varphi_M(v(s))ds\geq 0$. We have therefore that

$$\inf_{\int_{[0,1]}\varphi_i(v(s))ds=0,\ i=1,\dots,M-1}\int_{[0,1]}\varphi_M(v(s))ds=0. \tag{3.2}$$

It is easy to show that there exists a non-zero bounded function v_0 such that $\int_{[0,1]}\varphi_M(v_0(s))ds=0$. Indeed, let $a,b\in\mathbb{R}$ be such that $\varphi_M(a)<0<\varphi_M(\mathrm{b})$. By continuity of φ_M there exist an $\epsilon>0$ such that for any functions v and w such that $\|v-a\|_\infty<\epsilon$ and $\|w-b\|_\infty<\epsilon$, one has $\varphi_M(v)<0$ and $\varphi_M(w)>0$. Fix any such $v,w\in C^1$ whose derivatives are linearly independent. Then the function $\theta\mapsto\int_{[0,1]}\varphi_M(\theta v+(1-\theta)w)$, $0\leq\theta\leq 1$, will change sign and thus it will vanish at some $\theta_0\in(0,1)$ by the intermediate value theorem. The function $v_0=\theta_0v+(1-\theta_0)w$ will not be a constant by the assumption of linear independence.

Then v_0 is a point of minimum in 3.2, and by the Lagrange multiplier rule, there exist real numbers $\lambda_1,\dots,\lambda_{M-1}$ such that for any t in the range of v_0 where the functions φ_i are differentiable,

$$\varphi_M'(t)=\lambda_1\varphi_1'(t)+\cdots+\lambda_{M-1}\varphi_{M-1}'(t).$$

Since functions $\{\varphi_i\}_{i=1,\dots,M}$ are linearly independent on any interval and are piecewise differentiable, we have a contradiction. □

Corollary 3.5. *Let $\Omega=[0,1]$, equipped with the Lebesgue measure. Then for any $p\in[1,3)$, $p\neq 2$, there exists a sequence $v_k\in L^\infty([0,1])$ such that $v_k\rightharpoonup 0$ in L^p, $|v_k|^{p-2}v_k\rightharpoonup 0$ in $L^{p'}([0,1])$, but the relation (1.3) with $u_k=1+v_k$ does not hold.*

Proof. Let $F_p(t)=|1+t|^p-1-|t|^p$. Given $1\leq p<3$, the function F_p changes sign. Apply Corollary 3.4 with $M=3$, $\varphi_1(t)=t$ and $\varphi_2(t)=|t|^{p-2}t$ and $\varphi_3(t)=F_p(t)$. □

Remark 3.6. *Note that this counterexample cannot be extended to all measure spaces, since, as we have noted,* (1.1) *holds in* ℓ^p *under the assumption of weak convergence alone.*

4. A version of Brezis–Lieb lemma

In the previous section we observed, roughly speaking, that weak limits of $\varphi_i(u_k)$ for linearly independent functions φ_i have independent values, and that the inequality $\int_{[0,1]} \varphi_M(v_k) \geq o(1)$ holds for all sequences satisfying $\varphi_i(v_k) \rightharpoonup 0$, $i = 1, \ldots, M$, only if $\varphi_M(t) - \sum_{i=1}^{M-1} \lambda_i \varphi_i(t) \geq 0$ for some real $\lambda_1, \ldots, \lambda_M$. Therefore one may as well use the condition $\Phi(v_k) \rightharpoonup 0$ with $\Phi(t) = \sum_{i=1}^{M-1} \lambda_i \varphi_i(t)$. In particular, the function $F_p(t) = |1 + t|^p - 1 - |t|^p$, $p \geq 2$, dominates the following function: $\Phi(t) = pt$ for $|t| \leq 1$, $\Phi(t) = p|t|^{p-2}t$ for $|t| > 1$.

Theorem 4.1. *Let Let* (Ω, μ) *be a measure space and let* $p \geq 2$*. Assume that* $u_k \in L^p(\Omega, \mu)$, $u \in L^p(\Omega, \mu)$ *and* $\Psi(u, u_k - u) \rightharpoonup 0$ *in* $L^1(\Omega, \mu)$*, where*

$$\Psi(s, t) = \begin{cases} |s|^{p-1}t, & |t| \leq |s|, \\ |s||t|^{p-2}t, & |t| \geq |s| \end{cases}.$$

Then

$$\int_\Omega |u_k|^p d\mu \geq \int_\Omega |u|^p d\mu + \int_\Omega |u_k - u|^p d\mu + o(1). \tag{4.1}$$

Proof. This follows from the inequality $F_p(\lambda) \geq \Phi(\lambda)$, from which, with $\lambda = \frac{u_k(x) - u(x)}{u(x)}$, whenever $u(x) \neq 0$, immediately follows

$$|u_k|^p - |u|^p - |u - u_k|^p \geq \Psi(u, u_k - u).$$

□

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