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Nonlinear Differential Equations and Applications NoDEA

Multiplicity results for elliptic fractional equations with subcritical term

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Abstract. In the present paper, by using variational methods, we study the existence of multiple nontrivial weak solutions for parametric nonlocal equations, driven by the fractional Laplace operator $(-\Delta)^s$, in which the nonlinear term has a sublinear growth at infinity. More precisely, a critical point result for differentiable functionals is exploited, in order to prove the existence of an open interval of positive eigenvalues for which the treated problem admits at least two nontrivial weak solutions in a suitable fractional Sobolev space.

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Contents

1. Introduction

Nonlocal fractional equations appear in many fields and a lot of interest has been devoted to this kind of problems and to their concrete applications; see, for instance the seminal papers [5–7] and [1,2,4,8,9,11,16,20,25], as well as, the references therein.

In this note, motivated by this large interest in the current literature, exploiting variational methods, we study the existence of multiple weak solutions for the following fractional problem

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ ($n > 2s$ and $s \in]0,1[$) with Lipschitz boundary $\partial\Omega$, λ is real parameter, $(-\Delta)^s$ denotes the fractional Laplace operator which, up to normalization factor, may be defined as

$$(-\Delta)^s u(x) := -\int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \quad (\forall\, x \in \mathbb{R}^n) \tag{1.2}$$

while f is a subcritical term.

Precisely, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that is *superlinear at zero*, that is,

$$\lim_{t \to 0} \frac{f(t)}{t} = 0, \tag{1.3}$$

and *sublinear at infinity*, that is,

$$\lim_{|t| \to +\infty} \frac{f(t)}{t} = 0. \tag{1.4}$$

Further, we require that

$$\sup_{t \in \mathbb{R}} F(t) > 0, \tag{1.5}$$

where

$$F(t) := \int_0^t f(z) dz,$$

for any $t \in \mathbb{R}$.

Note that conditions (1.3) and (1.4) are standard assumptions to be satisfied in presence of subcritical terms. Now, let us denote

$$X_0 := \{v \in H^s(\mathbb{R}^n) : v = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}, \tag{1.6}$$

our ambient framework endowed by the norm

$$\|v\|_{X_0} := \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2s}}\, dxdy \right)^{1/2},$$

in which the functional space $H^s(\mathbb{R}^n)$ denotes the fractional Sobolev space of the functions $v \in L^2(\mathbb{R}^n)$ such that

$$\text{the map } (x, y) \mapsto \frac{v(x) - v(y)}{|x-y|^{\frac{n+2s}{2}}} \text{ is in } L^2(\mathbb{R}^n \times \mathbb{R}^n, dxdy).$$

This analytical context is inspired by (but not equivalent to) the fractional Sobolev spaces, in order to correctly encode the Dirichlet boundary datum in the variational formulation.

Indeed, the nonlocal analysis that we perform here in order to use our abstract approach is quite general and successfully exploited for other goals in several recent contributions; see [13–15,19,21–24] and [10] for an elementary introduction to this topic and for a list of related references.

By a *weak solution* of (1.1) we mean a function $u \in X_0$ such that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dxdy \\ \displaystyle\quad = \lambda \int_\Omega f(u(x))\varphi(x)dx, \quad \forall\, \varphi \in X_0. \end{cases} \tag{1.7}$$

As direct computations prove, Problem (1.7) represents the Euler–Lagrange equation of the C^1-functional $\mathcal{J}_\lambda : X_0 \to \mathbb{R}$ defined as

$$\begin{aligned} \mathcal{J}_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dxdy \\ -\lambda\int_\Omega F(u(x))dx, \end{aligned} \tag{1.8}$$

for every $u \in X_0$.

The main multiplicity result of the present paper can be stated as follows:

Theorem 1. *Let $s \in]0,1[$, $n > 2s$, Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function verifying conditions from (1.3) to (1.5). Then, there exists an open interval $\Lambda \subset]0,+\infty[$ and a real positive number κ such that for every $\lambda \in \Lambda$ problem (1.1) has at least two distinct, nontrivial weak solutions in X_0 whose X_0-norms are less than κ.*

The above theorem will be proved by using variational techniques, in particular performing a direct consequence of some general results given by Ricceri in [17,18], which assures the existence of multiple critical points for a functional, under suitable regularity assumptions on it (see Theorem 2 below).

We note that, in Theorem 1, the superlinearity of the function f at the origin [that is, hypothesis (1.3)] is an indispensable fact. Indeed, for example, if

$$f(t) := (\arctan t)^2, \quad (\forall\, t \in \mathbb{R})$$

an easy computation shows that (1.1) possesses only the trivial solution, whenever

$$|\lambda| < \frac{1}{c_2^2\pi},$$

where $c_2 > 0$ is the best Sobolev constant of the continuous embedding $X_0 \hookrightarrow L^2(\mathbb{R}^n)$. For detailed comments and for a more general non-existence result, see Proposition 7 in Subsect. 3.2.

Furthermore, we will give additional information as far as the localization of the interval Λ is concerned. More precisely, by using the notations adopted along the paper, we show that

$$\Lambda \subset \left[\frac{\lambda_{1,s}}{c_f}, \frac{2}{(1-\sigma_0)^2}\frac{(1-\sigma_0^n)\kappa_1\kappa_2}{\left(F(t_0)\sigma_0^n - (1-\sigma_0^n)\max\limits_{|t|\leq|t_0|}|F(t)|\right)}\frac{t_0^2}{\tau^2}\right],$$

see Subsect. 3.2 for a detailed proof.

Finally, we point out that the results contained here can be proved, in a similar way, for the nonlocal fractional operator defined by

$$\mathcal{L}_K u(x) := -\int_{\mathbb{R}^n}(u(x+y)+u(x-y)-2u(x))K(y)dy, \quad x \in \mathbb{R}^n,$$

where the kernel function $K : \mathbb{R}^n\setminus\{0\} \to]0,+\infty[$ satisfies the following hypothesis: *there exists* $\gamma \in]0,1]$ *such that*

$$\gamma|x|^{-(n+2s)} \leq K(x) \leq \gamma^{-1}|x|^{-(n+2s)},$$

for any $x \in \mathbb{R}^n\setminus\{0\}$.

Clearly, the paradigm of the above K is given by the singular kernel $K(x) := |x|^{-(n+2s)}$; in this case $\mathcal{L}_K$ reduces to the fractional Laplace operator given in (1.2).

The present paper is organized as follows. In Sect. 2 we give some notations and we recall some properties of the functional space we work in. We also give some tools which will be useful along the manuscript. In Sect. 3 we study problem (1.1) and we prove Theorem 1.

2. Some preliminaries

This section is devoted to the notations used along the paper. We also give some preliminary results which will be useful in the sequel.

2.1. Notations and definitions

In this subsection we briefly recall some properties of the functional space X_0, firstly introduced in [21], and we give some notations. The reader familiar with this topic may skip this section and go directly to the next one.

The space X_0 is defined as in (1.6), where $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(g(x)-g(y))^2}{|x-y|^{n+2s}}\,dxdy\Big)^{1/2}. \tag{2.1}$$

For further details on the fractional Sobolev spaces we refer to [10] and to the references therein.

Of course, the space X_0 is non-empty, since $C_0^2(\Omega) \subseteq X_0$ by [21, Lemma 5.1] and it depends on the set Ω. Moreover, by [22, Lemma 6] and the fact that any function $v \in X_0$ is such that $v = 0$ a.e. in $\mathbb{R}^n\setminus\Omega$, in the sequel we can take

$$X_0 \ni v \mapsto \|v\|_{X_0} = \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2s}} \, dxdy \right)^{1/2} \tag{2.2}$$

as norm on X_0.

Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (for this see [22, Lemma 7]), with scalar product

$$\langle u, v \rangle_{X_0} := \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)}{|x-y|^{n+2s}} \, dxdy. \tag{2.3}$$

Finally, we recall that in [22, Lemma 8] and in [24, Lemma 9] the authors proved that the embedding $j : X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2^*]$, while it is compact whenever $\nu \in [1, 2^*[$. In the sequel for any $\nu \in [1, 2^*[$, we will denote by c_ν the positive constant such that

$$\|v\|_{L^\nu(\mathbb{R}^n)} \leq c_\nu \|v\|_{X_0}, \tag{2.4}$$

for any $v \in X_0$.

2.2. Some useful tools

The main tool used along this paper in order to prove our multiplicity result stated in Theorem 1 is given by a direct consequence of some general theorems due to Ricceri [17, 18] that we recall here below.

Theorem 2. *Let $(E, \|\cdot\|)$ be a separable and reflexive real Banach space, and let $\Phi, \Psi : E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $z_0 \in E$ such that $\Phi(z_0) = \Psi(z_0) = 0$ and $\inf_{z \in E} \Phi(z) \geq 0$ and that there exist $z_1 \in E$, $\varrho > 0$ such that*

(i) $\varrho < \Phi(z_1)$;

(ii) $\sup_{\Phi(z)<\varrho} \Psi(z) < \varrho \dfrac{\Psi(z_1)}{\Phi(z_1)}$.

Further, put

$$\bar{a} := \frac{\zeta \varrho}{\varrho \dfrac{\Psi(z_1)}{\Phi(z_1)} - \sup\limits_{\Phi(z)<\varrho} \Psi(z)},$$

with $\zeta > 1$, assume that the functional

$$J_\lambda(z) := \Phi(z) - \lambda \Psi(z), \quad (\forall\, z \in E)$$

is sequentially weakly lower semicontinuous, satisfies the (PS) *condition, and*

(iii) $\lim_{\|z\| \to +\infty} J_\lambda(z) = +\infty$,

for every $\lambda \in [0, \bar{a}]$.

Then there is an open interval $\Lambda \subset [0, \bar{a}]$ and a number $\kappa > 0$ such that for each $\lambda \in \Lambda$, the equation $J'_\lambda(z) = 0$ admits at least three solutions in E having norm less than κ.

Some details and related topics on the above result can be found in the recent monograph [12].

For the sake of completeness, we also recall that the C^1-functional $J_\lambda : E \to \mathbb{R}$ satisfies the Palais-Smale condition at level $\mu \in \mathbb{R}$ when

$(\mathrm{PS})_\mu$ *Every sequence* $\{z_j\}_{j\in\mathbb{N}} \subset E$ *such that*

$$J_\lambda(z_j) \to \mu, \qquad \text{and} \qquad \|J'_\lambda(z_j)\|_{E^*} \to 0,$$

as $j \to \infty$*, possesses a convergent subsequence in* E*.*

Here E^* denotes the topological dual of E. Finally, we say that J_λ satisfies the Palais-Smale condition (in short (PS)) if $(\mathrm{PS})_\mu$ holds for every $\mu \in \mathbb{R}$.

Now, fix an element $x_0 \in \Omega$ and choose $\tau > 0$ in such a way that

$$\bar{B}(x_0,\tau) := \{x \in \mathbb{R}^n : |x - x_0| \leq \tau\} \subseteq \Omega, \tag{2.5}$$

where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

At this point, let $\sigma \in]0,1[$, $t_0 \in \mathbb{R}$ and define $u_\sigma^{t_0} \in X_0$ as follows

$$u_\sigma^{t_0}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus B(x_0,\tau) \\ \dfrac{t_0}{(1-\sigma)\tau}(\tau - |x - x_0|) & \text{if } x \in B(x_0,\tau)\setminus B(x_0,\sigma\tau) \\ t_0 & \text{if } x \in B(x_0,\sigma\tau), \end{cases}$$

where $B(x_0,r)$ denotes the n-dimensional open ball with center $x_0 \in \Omega$ and radius $r > 0$.

This function will be useful in the sequel in the proof of our theorem as well as the next preparatory Lemma. Let $n > 2s$ and set

$$\nu^{(n)} := (2\pi)^n \left(1 + \frac{1}{\lambda_1}\right), \tag{2.6}$$

where

$$\lambda_1 := \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}. \tag{2.7}$$

The following result holds.

Lemma 3. *Let* $\sigma, s \in]0,1[$ *and* τ *defined as before,* S_{n-2} *be the Lebesgue measure of the unit sphere in* $\mathbb{R}^{n-1}$ *and*

$$\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z} dz, \quad (\forall\, t > 0)$$

be the usual Gamma function. If $t_0 \in \mathbb{R}$*, then* $u_\sigma^{t_0} \in X_0$ *and one has*

$$\|u_\sigma^{t_0}\|_{X_0} \leq \frac{|t_0|}{(1-\sigma)} \sqrt{\frac{\pi^{\frac{n}{2}} \tau^{n-2}(1-\sigma^n)}{\Gamma\left(1 + \frac{n}{2}\right)} \kappa_1 \kappa_2}, \tag{2.8}$$

where

$$\kappa_1 := \begin{cases} 2\nu^{(1)} & \text{if } n = 1 \\ \left(\pi + \dfrac{4}{1+2s}\right)\nu^{(2)} & \text{if } n = 2 \\ S_{n-2}\left(\dfrac{\pi}{2} + \dfrac{2}{1+2s}\right)\nu^{(n)} & \text{if } n \geq 3 \end{cases} \quad \text{and} \quad \kappa_2 := \left(\frac{1}{2(1-s)} + \frac{2}{s}\right).$$

Proof. Computing the standard seminorm of the function $u_\sigma^{t_0}$ in $H^1(\mathbb{R}^n)$, we easily have

$$\begin{aligned}[u_\sigma^{t_0}]^2_{H^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\nabla u_\sigma^{t_0}(x)|^2\,dx = \int_{B(x_0,\tau)\setminus B(x_0,\sigma\tau)} \frac{t_0^2}{(1-\sigma)^2\tau^2}\,dx \\ &= \frac{t_0^2}{(1-\sigma)^2\tau^2}\,[\mathrm{meas}(B(x_0,\tau)) - \mathrm{meas}(B(x_0,\sigma\tau))] \\ &= \frac{t_0^2}{(1-\sigma)^2}\frac{\pi^{\frac{n}{2}}\tau^{n-2}(1-\sigma^n)}{\Gamma\left(1+\frac{n}{2}\right)}, \end{aligned} \tag{2.9}$$

where "meas$(B(x_0,\tau))$" and "meas$(B(x_0,\sigma\tau))$" denote respectively the Lebesgue measure of $B(x_0,\tau)$ and $B(x_0,\sigma\tau)$.

Since $u_\sigma^{t_0} \in H_0^1(\Omega)$, by [10, Proposition 2.2] it follows that $u_\sigma^{t_0} \in H^{s,2}(\Omega)$. Moreover, the boundary $\partial\Omega$ is Lipschitz, $\bar{B}(x_0,\tau) \subset \Omega$, and $u_\sigma^{t_0} = 0$ in $\Omega\setminus\bar{B}(x_0,\tau)$. By [10, Lemma 5.1] one has that $u_\sigma^{t_0} \in H^s(\mathbb{R}^n)$.

Hence, since $s \in]0,1[$, Proposition 3.4 of [10], yields

$$\begin{aligned}\|u_\sigma^{t_0}\|^2_{X_0} &\le 2\left(\int_{\mathbb{R}^n} \frac{1-\cos x_1}{|x|^{n+2s}}\,dx\right)\int_{\mathbb{R}^n} |\xi|^{2s}|\mathfrak{F}u_\sigma^{t_0}(\xi)|^2\,d\xi \\ &\le 2\left(\int_{\mathbb{R}^n} \frac{1-\cos x_1}{|x|^{n+2s}}\,dx\right)\int_{\mathbb{R}^n} (1+|\xi|^2)|\mathfrak{F}u_\sigma^{t_0}(\xi)|^2\,d\xi, \end{aligned} \tag{2.10}$$

where

$$\mathfrak{F}u_\sigma^{t_0}(\xi) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}u_\sigma^{t_0}(x)dx,$$

stands for the classical Fourier transform of $u_\sigma^{t_0}$.

Now, by standard arguments on the Fourier transform and Poincaré inequality, one has

$$\int_{\mathbb{R}^n} (1+|\xi|^2)|\mathfrak{F}u_\sigma^{t_0}(\xi)|^2\,d\xi \le \nu^{(n)}[u_\sigma^{t_0}]^2_{H^1(\mathbb{R}^n)}. \tag{2.11}$$

Indeed

$$u_\sigma^{t_0} \in L^2(\mathbb{R}^n) \quad \text{if and only if} \quad \mathfrak{F}u_\sigma^{t_0} \in L^2(\mathbb{R}^n)$$

and

$$\|u_\sigma^{t_0}\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^{-n}\|\mathfrak{F}u_\sigma^{t_0}\|^2_{L^2(\mathbb{R}^n)}. \tag{2.12}$$

Further

$$|\nabla u_\sigma^{t_0}| \in L^2(\mathbb{R}^n) \quad \text{if and only if} \quad |\xi|\mathfrak{F}u_\sigma^{t_0} \in L^2(\mathbb{R}^n)$$

and

$$\|\nabla u_\sigma^{t_0}\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^{-n}\||\xi|\mathfrak{F}u_\sigma^{t_0}\|^2_{L^2(\mathbb{R}^n)}. \tag{2.13}$$

Relations (2.12) and (2.13) give

$$\int_{\mathbb{R}^n} (1+|\xi|^2)|\mathfrak{F}u_\sigma^{t_0}(\xi)|^2\,d\xi = (2\pi)^n\left(\|u_\sigma^{t_0}\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u_\sigma^{t_0}\|^2_{L^2(\mathbb{R}^n)}\right). \tag{2.14}$$

Hence, by (2.6) and (2.7) one has that inequality (2.11) is a direct consequence of (2.14) taking into account that $u_\sigma^{t_0} \in W_0^{1,2}(\Omega)$.

Then, by (2.10) and (2.11), il follows that

$$\|u_\sigma^{t_0}\|_{X_0}^2 \leq 2\nu^{(n)} \left(\int_{\mathbb{R}^n} \frac{1-\cos x_1}{|x|^{n+2s}}\, dx \right) [u_\sigma^{t_0}]_{H^1(\mathbb{R}^n)}^2.$$

Thus, for $n \geq 3$, we have

$$\begin{aligned} \|u_\sigma^{t_0}\|_{X_0}^2 &\leq 2\nu^{(n)} \left(\int_{\mathbb{R}^n} \frac{1-\cos x_1}{|x|^{n+2s}}\, dx \right) [u_\sigma^{t_0}]_{H^1(\mathbb{R}^n)}^2 \\ &= 2\nu^{(n)} \left(\int_{\mathbb{R}^{n-1}} \frac{1}{(1+|x|^2)^{\frac{n+2s}{2}}}\, dx \right) \\ &\quad \times \left(\int_{\mathbb{R}} \frac{1-\cos t}{|t|^{1+2s}}\, dt \right) [u_\sigma^{t_0}]_{H^1(\mathbb{R}^n)}^2. \end{aligned}$$

The conclusion follows by (2.9) taking into account that, by direct computations it follows that, one has

$$\int_{\mathbb{R}^{n-1}} \frac{1}{(1+|x|^2)^{\frac{n+2s}{2}}}\, dx < S_{n-2} \left(\frac{\pi}{4} + \frac{1}{1+2s} \right),$$

and

$$\int_{\mathbb{R}} \frac{1-\cos t}{|t|^{1+2s}}\, dt < \frac{1}{2(1-s)} + \frac{2}{s}.$$

Indeed

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|x|^2)^{\frac{n+2s}{2}}}\, dx &= S_{n-2} \int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n+2s}{2}}}\, d\rho \\ &= S_{n-2} \left[\int_0^1 \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n+2s}{2}}}\, d\rho + \int_1^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n+2s}{2}}}\, d\rho \right] \\ &< S_{n-2} \left[\int_0^1 \frac{(1+\rho^2)^{\frac{n-2}{2}}}{(1+\rho^2)^{\frac{n+2s}{2}}}\, d\rho + \int_1^{+\infty} \frac{\rho^{n-2}}{(\rho^2)^{\frac{n+2s}{2}}}\, d\rho \right] \\ &< S_{n-2} \left[\int_0^1 \frac{d\rho}{1+\rho^2} + \int_1^{+\infty} \frac{d\rho}{\rho^{2(1+s)}} \right] \\ &= S_{n-2} \left(\frac{\pi}{4} + \frac{1}{1+2s} \right), \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}} \frac{1-\cos t}{|t|^{1+2s}}\, dt &= 2 \int_0^{+\infty} \frac{1-\cos t}{t^{1+2s}}\, dt \\ &= 2 \left[\int_0^1 \frac{1-\cos t}{t^{1+2s}}\, dt + \int_1^{+\infty} \frac{1-\cos t}{t^{1+2s}}\, dt \right] \end{aligned}$$

$$\leq \left[\frac{1}{2}\int_0^1 \frac{t^2}{t^{1+2s}}\,dt + 2\int_1^{+\infty}\frac{1}{t^{(1+2s)}}\,dt\right]$$
$$= \frac{1}{2(1-s)} + \frac{2}{s}.$$

On the other hand, we also have

$$\int_{\mathbb{R}^n} \frac{1-\cos x_1}{|x|^{n+2s}}\,dx \leq \begin{cases} \dfrac{1}{2(1-s)} + \dfrac{2}{s} & \text{if } n = 1 \\ \left(\dfrac{\pi}{2} + \dfrac{2}{1+2s}\right)\left(\dfrac{1}{2(1-s)} + \dfrac{2}{s}\right) & \text{if } n = 2. \end{cases}$$

Clearly $u_\sigma^{t_0} : \mathbb{R}^n \to \mathbb{R}$ is a continuous, $u_\sigma^{t_0} \in L^2(\Omega)$ and by the above computations it follows that $u_\sigma^{t_0} \in X$. Finally $u_\sigma^{t_0} = 0$ in $\mathbb{R}^n \backslash \Omega$. Thus $u_\sigma^{t_0} \in X_0$ and (2.8) holds. □

3. Proof of the main result

The validity of the following Lemmas will be crucial in order to prove the main result.

Lemma 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function verifying condition* (1.4)*. Then, for every $\lambda \in \mathbb{R}$, the functional $\mathcal{J}_\lambda$ is weakly lower semicontinuous on X_0.*

Proof. The application

$$u \mapsto \int_\Omega F(u(x))dx$$

is continuous in the weak topology of X_0.

Indeed, if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in X_0 such that $u_j \rightharpoonup u$ weakly in X_0, then, by (2.4) and [3, Theorem IV.9], up to a subsequence, u_j converges to u strongly in $L^\nu(\Omega)$ and a.e. in Ω as $j \to +\infty$, and it is dominated by some function $h_\nu \in L^\nu(\Omega)$, that is,

$$|u_j(x)| \leq h_\nu(x) \quad \text{a.e. } x \in \Omega \quad \text{for any } j \in \mathbb{N} \tag{3.1}$$

for any $\nu \in [1, 2^*[$.

Due to (1.4), there exists $c > 0$ such that

$$|f(t)| \leq c(1+|t|), \quad (\forall t \in \mathbb{R}). \tag{3.2}$$

Then, by the continuity of F and (3.2) it follows that

$$F(u_j(x)) \to F(u(x)) \ \text{ a.e. } x \in \Omega$$

as $j \to \infty$ and

$$|F(u_j(x))| \leq c\left(|u_j(x)| + \frac{1}{2}u_j(x)^2\right) \leq c\left(h_1(x) + \frac{1}{2}h_2(x)^2\right) \in L^1(\Omega)$$

a.e. $x \in \Omega$ and for any $j \in \mathbb{N}$.

Hence, by applying the Lebesgue Dominated Convergence Theorem in $L^1(\Omega)$, we have that

$$\int_\Omega F(u_j(x))\,dx \to \int_\Omega F(u(x))\,dx$$

as $j \to \infty$, that is the map

$$u \mapsto \int_\Omega F(u(x))dx$$

is continuous from X_0 with the weak topology to $\mathbb{R}$. Thus, the functional Ψ is continuous from X_0 with the weak topology to $\mathbb{R}$.

On the other hand, the map

$$u \mapsto \|u\|^2_{X_0}$$

is lower semicontinuous in the weak topology of X_0, so that the functional Φ is lower semicontinuous in the weak topology of X_0. □

Lemma 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function verifying condition* (1.4). *Then, for every $\lambda \in \mathbb{R}$, the functional $\mathcal{J}_\lambda$ is coercive and satisfies the compactness* (PS) *condition.*

Proof. Let us fix $\lambda \in \mathbb{R}$. By (1.4), there exists a positive δ_λ such that

$$|f(t)| \leq \frac{|t|}{2c_2^2(1+|\lambda|)},$$

for every $|t| \geq \delta_\lambda$.

Hence, we get

$$|F(t)| \leq \frac{t^2}{2c_2^2(1+|\lambda|)} + \max_{|t|\leq\delta_\lambda} |f(t)||t|,$$

for every $t \in \mathbb{R}$.

Thus

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geq \Phi(u) - |\lambda| \left|\int_\Omega F(u(x))dx\right| \\ &\geq \frac{1}{2}\|u\|^2_{X_0} - |\lambda| \int_\Omega |F(u(x))|dx \\ &\geq \frac{1}{2(1+|\lambda|)}\|u\|^2_{X_0} - c_1|\lambda| \max_{|t|\leq\delta_\lambda} |f(t)|\|u\|_{X_0},\end{aligned}$$

for every $u \in X_0$.

Then the functional $\mathcal{J}_\lambda$ is bounded from below and $\mathcal{J}_\lambda(u) \to +\infty$ whenever $\|u\|_{X_0} \to +\infty$. Hence $\mathcal{J}_\lambda$ is coercive.

Now, let us prove that $\mathcal{J}_\lambda$ satisfies the condition $(\mathrm{PS})_\mu$ for $\mu \in \mathbb{R}$. For our goal, let $\{u_j\}_{j\in\mathbb{N}} \subset X$ be a Palais-Smale sequence, that is,

$$\mathcal{J}_\lambda(u_j) \to \mu, \qquad \text{and} \qquad \|\mathcal{J}_\lambda'(u_j)\|_{X_0^*} \to 0,$$

as $j \to +\infty$.

Taking into account the coercivity of $\mathcal{J}_\lambda$, the sequence $\{u_j\}_{j\in\mathbb{N}}$ is necessarily bounded in X_0. Since X_0 is reflexive, we may extract a subsequence, which for simplicity we call again $\{u_j\}_{j\in\mathbb{N}}$, such that $u_j \rightharpoonup u$ in X_0. This means that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(u_j(x)-u_j(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,dxdy \to \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,dxdy, \tag{3.3}$$

for any $\varphi\in X_0$, as $j\to\infty$.

We will prove that $\{u_j\}_{j\in\mathbb{N}}$ strongly converges to $u\in X_0$. Exploiting the derivative $\mathcal{J}_\lambda(u_j)(u_j-u)$, we obtain

$$\langle\Phi'(u_j),u_j-u\rangle = \langle\mathcal{J}'_\lambda(u_j),u_j-u\rangle + \lambda\int_\Omega f(u_j(x))(u_j-u)(x)dx, \tag{3.4}$$

where

$$\langle\Phi'(u_j),u_j-u\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_j(x)-u_j(y))^2}{|x-y|^{n+2s}}dxdy - \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(u_j(x)-u_j(y)\big)\big(u(x)-u(y)\big)}{|x-y|^{n+2s}}\,dxdy$$

Since $\|\mathcal{J}'_\lambda(u_n)\|_{X_0^*}\to 0$ and the sequence $\{u_j-u\}_{j\in\mathbb{N}}$ is bounded in X, taking into account that $|\langle\mathcal{J}'_\lambda(u_j),u_j-u\rangle|\le\|\mathcal{J}'_\lambda(u_j)\|_{X_0^*}\|u_j-u\|_{X_0}$, one has

$$\langle\mathcal{J}'_\lambda(u_j),u_j-u\rangle\to 0. \tag{3.5}$$

Further, by the growth condition (3.2), one has

$$\begin{aligned}
&\int_\Omega |f(u_j(x))||u_j(x)-u(x)|dx\\
&\le c\int_\Omega |u_j(x)-u(x)|dx + c\int_\Omega |u_j(x)||u_j(x)-u(x)|dx\\
&\le c((\mathrm{meas}(\Omega))^{1/2}+\|u_j\|_{L^2(\Omega)})\|u_j-u\|_{L^2(\Omega)}.
\end{aligned}$$

Now, the embedding $X_0\hookrightarrow L^2(\Omega)$ is compact, hence $u_j\to u$ strongly in $L^2(\Omega)$. So we obtain

$$\int_\Omega |f(u_j(x))||u_j(x)-u(x)|dx\to 0, \tag{3.6}$$

as $j\to\infty$.

By (3.4) relations (3.5) and (3.6) yield

$$\langle\Phi'(u_j),u_j-u\rangle\to 0, \tag{3.7}$$

as $j\to\infty$.

Hence by (3.7) we can write

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_j(x)-u_j(y))^2}{|x-y|^{n+2s}}dxdy - \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(u_j(x)-u_j(y)\big)\big(u(x)-u(y)\big)}{|x-y|^{n+2s}}\,dxdy\to 0,$$

as $j\to\infty$.

Thus, by (3.8) and (3.3) it follows that

$$\lim_{j\to\infty}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u_j(x)-u_j(y))^2}{|x-y|^{n+2s}}\,dxdy = \int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\big(u(x)-u(y)\big)^2}{|x-y|^{n+2s}}\,dxdy.$$

In conclusion, thanks to [3, Proposition III.30], $u_j \to u$ in X_0. The proof is complete. □

Lemma 6. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function verifying condition* (1.3) *and* (1.4). *Then, the following property holds*

$$\lim_{\varrho\to 0^+}\frac{\displaystyle\sup_{u\in\Phi^{-1}(]-\infty,\varrho[)}\int_\Omega F(u(x))\,dx}{\varrho}=0.$$

Proof. Due to (1.3), for an arbitrary small $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that

$$|f(t)|<\frac{\varepsilon}{4c_2^2}|t|,$$

for every $|t|<\delta_\varepsilon$.

On the other hand, combining (3.2) with the above inequality, one has

$$|F(t)|\le \varepsilon\frac{t^2}{4c_2^2}+c(1+\delta_\varepsilon)\delta^{1-q}|t|^q, \tag{3.8}$$

for a fixed $q\in]2,2^*[$ and for all $t\in\mathbb{R}$.

Now, fix $\varrho>0$ and define the sets

$$S_\varrho:=\Phi^{-1}(]-\infty,\varrho[),\quad \text{and}\quad S_{2\varrho}:=\Phi^{-1}(]-\infty,2\varrho[).$$

Of course $S_\varrho\subseteq S_{2\varrho}$. Moreover, by using (3.8), we have

$$\int_\Omega F(u(x))\,dx\le \frac{\varepsilon}{2}\varrho+c(1+\delta_\varepsilon)\delta_\varepsilon^{1-q}c_q^q2^{q/2}\varrho^{q/2} \equiv \frac{\varepsilon}{2}\varrho+\widehat{c}(\varepsilon)\varrho^{q/2},$$

where we set

$$\widehat{c}(\varepsilon):=c(1+\delta_\varepsilon)\delta_\varepsilon^{1-q}c_q^q2^{q/2}.$$

Thus, there exists $\varrho(\varepsilon)>0$ such that, for every $0<\varrho<\varrho(\varepsilon)$, we have

$$0\le\frac{\displaystyle\sup_{u\in S_\varrho}\int_\Omega F(u(x))\,dx}{\varrho}\le\frac{\displaystyle\sup_{u\in S_{2\varrho}}\int_\Omega F(u(x))\,dx}{\varrho} \le\frac{\varepsilon}{2}+\widehat{c}(\varepsilon)\varrho^{\frac{q-2}{2}}<\varepsilon,$$

which completes the proof. □

3.1. Proof of Theorem 1

The idea of the proof consists in applying Theorem 2 to the functional $\mathcal{J}_\lambda$.

To this purpose, we write the functional $\mathcal{J}_\lambda$ as follows:

$$\mathcal{J}_\lambda(u) = \Phi(u) - \lambda\Psi(u), \quad (\forall\, u \in X_0)$$

where

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\, dxdy,$$

while

$$\Psi(u) := \int_\Omega F(u(x))\, dx,$$

for every $u \in X_0$.

First of all, note that X_0 is a separable Hilbert space (see [22, Lemma 7]) and the functionals Φ and Ψ are continuously Gâteaux differentiable. Moreover, the functional Φ is coercive in X_0, $\inf\limits_{u\in X_0} \Phi(u) = 0$ and, by choosing $z_0 \equiv 0$, of course

$$\Phi(z_0) = \Psi(z_0) = 0.$$

Now, by hypothesis (1.5), there exists $t_0 \in \mathbb{R}$ such that $F(t_0) > 0$. Further, let $\sigma_0 \in]0, 1[$ be such that

$$F(t_0)\sigma_0^n - (1-\sigma_0^n) \max_{|t|\le|t_0|} |F(t)| > 0.$$

Hence, set

$$u_{\sigma_0}^{t_0}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus B(x_0,\tau) \\ \dfrac{t_0}{(1-\sigma_0)\tau}(\tau - |x-x_0|) & \text{if } x \in B(x_0,\tau)\setminus B(x_0,\sigma_0\tau) \\ t_0 & \text{if } x \in B(x_0,\sigma_0\tau), \end{cases}$$

where τ is as in (2.5).

One has

$$\Psi(u_{\sigma_0}^{t_0}) \ge \left[F(t_0)\sigma_0^n - (1-\sigma_0^n)\max_{|t|\le|t_0|}|F(t)|\right]\omega_n\tau^n,$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Indeed, since

$$\|u_{\sigma_0}^{t_0}\|_\infty := \max_{x\in\bar{\Omega}} |u_{\sigma_0}^{t_0}(x)| \le |t_0|,$$

it follows that

$$\int_{B(x_0,\tau)\setminus B(x_0,\sigma_0\tau)} F(u_{\sigma_0}^{t_0}(x))\, dx \ge -(1-\sigma_0^n)\max_{|t|\le|t_0|}|F(t)|\tau^n\omega_n. \tag{3.9}$$

Consequently, relation (3.9) yields

$$\begin{aligned}\Psi(u^{t_0}_{\sigma_0}) &:= \int_\Omega F(u^{t_0}_{\sigma_0}(x))\,dx\\ &= \int_{B(x_0,\sigma_0\tau)} F(u^{t_0}_{\sigma_0}(x))\,dx + \int_{B(x_0,\tau)\setminus B(x_0,\sigma_0\tau)} F(u^{t_0}_{\sigma_0}(x))\,dx\\ &\geq F(t_0)\sigma_0^n\tau^n\omega_n + \int_{B(x_0,\tau)\setminus B(x_0,\sigma_0\tau)} F(u^{t_0}_{\sigma_0}(x))\,dx\\ &\geq \left[F(t_0)\sigma_0^n - (1-\sigma_0^n)\max_{|t|\leq|t_0|}|F(t)|\right]\omega_n\tau^n.\end{aligned}$$

Further, by Lemma 3 we have

$$\Phi(u^{t_0}_{\sigma_0}) \leq \beta, \tag{3.10}$$

where

$$\beta := \frac{t_0^2}{2(1-\sigma_0)^2}\frac{\pi^{\frac{n}{2}}\tau^{n-2}(1-\sigma_0^n)}{\Gamma\left(1+\dfrac{n}{2}\right)}\kappa_1\kappa_2.$$

Due to Lemma 6, we may take $\varrho_0 > 0$ such that the function $u^{t_0}_{\sigma_0} \in X_0$ verifies the following conditions:

$$u^{t_0}_{\sigma_0} \in \mathbb{R}\setminus\bar{S}_{\varrho_0}, \tag{3.11}$$

and

$$\frac{\sup\limits_{u\in S_{\varrho_0}}\Psi(u)}{\varrho_0} < \frac{\left[F(t_0)\sigma_0^n - (1-\sigma_0^n)\max\limits_{|t|\leq|t_0|}|F(t)|\right]\omega_n\tau^n}{\beta}. \tag{3.12}$$

Relation (3.11) clearly means

$$\varrho_0 < \Phi(u^{t_0}_{\sigma_0}).$$

Moreover, since (3.10) holds, it follows that

$$\frac{\left[F(t_0)\sigma_0^n - (1-\sigma_0^n)\max\limits_{|t|\leq|t_0|}|F(t)|\right]\omega_n\tau^n}{\beta} \leq \frac{\Psi(u^{t_0}_{\sigma_0})}{\Phi(u^{t_0}_{\sigma_0})}. \tag{3.13}$$

Hence, inequality (3.12) together to (3.13) give

$$\sup_{\Phi(u)<\varrho_0}\Psi(u) < \varrho_0\frac{\Psi(u^{t_0}_{\sigma_0})}{\Phi(u^{t_0}_{\sigma_0})}. \tag{3.14}$$

By choosing $z_1 \equiv u^{t_0}_{\sigma}$, hypotheses (i) and (ii) of Theorem 2 are verified.

Set

$$\bar{a} := \frac{1+\varrho_0}{\dfrac{\Psi(u^{t_0}_{\sigma_0})}{\Phi(u^{t_0}_{\sigma_0})} - \dfrac{\sup\limits_{\Phi(u)<\varrho_0}\Psi(u)}{\varrho_0}}.$$

Note that, in such a case $\zeta := 1+\varrho_0$ and, by (3.14), $\bar{a}$ is strictly positive.

Bearing in mind Lemma 4 the functional $\mathcal{J}_\lambda$ is sequentially lower continuous in the weak topology on X_0. Further, Lemma 5 guarantees the coercivity property.

Hence, all the assumptions of Theorem 2 are fulfilled. Thus there is an open interval $\Lambda \subseteq [0, \bar{a}]$ and a number $\kappa > 0$ such that for each $\lambda \in \Lambda$, the equation $J'_\lambda(u) = 0$ admits at least three solutions in X_0 having norm less than κ.

Since one of them may be the trivial one (note that by (1.1) one has $f(0) = 0$), we still have at least two distinct, nontrivial weak solutions of the problem (1.1).

3.2. Final comments

In spite of the fact that hypotheses of Theorem 1 are verified, the conclusions do not hold in general for every parameter $\lambda > 0$, as we mentioned in the first section. For instance, let f be a Lipschitz continuous function of constant L and take

$$0 \leq \lambda < \frac{1}{c_2^2 L}. \tag{3.15}$$

Further, let us assume that there exists a weak solution $u_0 \in X_0 \setminus \{0\}$ of the problem (1.1), that is,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u_0(x) - u_0(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}\, dxdy \\ \displaystyle\quad = \lambda \int_\Omega f(u_0(x))\varphi(x)dx, \quad \forall\, \varphi \in X_0. \end{cases} \tag{3.16}$$

In particular, testing (3.16) with $\varphi := u_0$, we have

$$\|u_0\|_{X_0}^2 = \lambda \int_\Omega f(u_0(x))u_0(x)\, dx. \tag{3.17}$$

On the other hand, bearing in mind that f is Lipschitz continuous and $f(0) = 0$, by using (3.15), it follows that

$$\begin{aligned} \int_\Omega f(u_0(x))u_0(x)\, dx &\leq \int_\Omega |f(u_0(x))||u_0(x)|\, dx \\ &= \int_\Omega |f(u_0(x)) - f(0)||u_0(x)|\, dx \\ &\leq L\|u_0\|_{L^2(\Omega)}^2 \leq c_2^2 L \|u_0\|_{X_0}^2. \end{aligned} \tag{3.18}$$

By (3.17) and (3.18) we get

$$\|u_0\|_{X_0}^2 \leq \lambda c_2^2 L \|u_0\|_{X_0}^2 < \|u_0\|_{X_0}^2,$$

which is a contradiction.

More generally, according to hypotheses (1.3)–(1.5) and defining the positive number

$$c_f := \max_{s \neq 0} \frac{|f(s)|}{|s|}, \tag{3.19}$$

a non-existence result for our problem can be proved whenever the parameter λ is small enough.

Namely, arguing substantially as before, and by [23, Proposition 9] we have

Proposition 7. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function which satisfies (1.3)–(1.5) and denote by $\lambda_{1,s}$ be the first positive eigenvalue of the linear problem*

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega. \end{cases} \tag{3.20}$$

Then for every parameter

$$0 \le \lambda < \frac{\lambda_{1,s}}{c_f},$$

where c_f is the constant defined in (3.19), problem (1.1) admits only the trivial solution.

Finally, a natural question arises about the interval Λ obtained in Theorem 1: *can we estimate it*?

To give a positive answer to this question, preserving the above notations, one has

$$\frac{1+\varrho_0}{\dfrac{\alpha}{\beta} - \dfrac{\sup\limits_{\Phi(u)<\varrho_0}\Psi(u)}{\varrho_0}} < \frac{2}{\dfrac{\alpha}{\beta} - \dfrac{\sup\limits_{\Phi(u)<\varrho_0}\Psi(u)}{\varrho_0}},$$

for every $\varrho_0 \in]0,1[$ and where we set

$$\alpha := \left(F(t_0)\sigma_0^n - (1-\sigma_0^n)\max_{|t|\le|t_0|}|F(t)|\right)\omega_n\tau^n.$$

Further, based on Lemma 6, one can take $\varrho_0 \in]0,1[$ and such that

$$\frac{1}{\dfrac{\alpha}{\beta} - \dfrac{\sup\limits_{\Phi(u)<\varrho_0}\Psi(u)}{\varrho_0}} < \frac{2\beta}{\alpha}.$$

Thus

$$\begin{aligned}\bar{a} := \frac{1+\varrho_0}{\dfrac{\Psi(u_{\sigma_0}^{t_0})}{\Phi(u_{\sigma_0}^{t_0})} - \dfrac{\sup\limits_{\Phi(u)<\varrho_0}\Psi(u)}{\varrho_0}} &\le \frac{1+\varrho_0}{\dfrac{\alpha}{\beta} - \dfrac{\sup\limits_{\Phi(u)<\varrho_0}\Psi(u)}{\varrho_0}} \\ &< 4\frac{\beta}{\alpha}.\end{aligned}$$

In conclusion, thanks to the above computations and bearing in mind Proposition 7, we have the following concrete localization

$$\Lambda \subset \left[\frac{\lambda_{1,s}}{c_f}, \bar{a}\right] \subset \left[\frac{\lambda_{1,s}}{c_f}, \frac{2}{(1-\sigma_0)^2}\frac{(1-\sigma_0^n)\kappa_1\kappa_2}{\left(F(t_0)\sigma_0^n - (1-\sigma_0^n)\max\limits_{|t|\le|t_0|}|F(t)|\right)}\frac{t_0^2}{\tau^2}\right],$$

as desired.

Remark 8. We point out that, encoding the exact normalization constant

$$C(n, s) := \left(\int_{\mathbb{R}^n} \frac{1 - \cos x_1}{|x|^{n+2s}}\, dx \right)^{-1},$$

defining the fractional Laplacian operator

$$(-\Delta)^s u(x) = -\frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \quad (\forall\, x \in \mathbb{R}^n)$$

our approach ensures the existence of an open interval

$$\Lambda \subset \left[\frac{\lambda_{1,s}}{c_f}, \frac{(1-\sigma_0^n)\nu^{(n)}}{(1-\sigma_0)^2 \left(F(t_0)\sigma_0^n - (1-\sigma_0^n) \max\limits_{|t| \le |t_0|} |F(t)| \right)} \frac{t_0^2}{\tau^2} \right],$$

and a real positive number κ such that for every $\lambda \in \Lambda$ problem (1.1) has at least two distinct, nontrivial weak solutions in X_0 whose X_0-norms are less than κ.

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