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Passing from bulk to bulk-surface evolution in the Allen–Cahn equation

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Abstract. In this paper we formulate a boundary layer approximation for an Allen–Cahn-type equation involving a small parameter ε. Here, ε is related to the thickness of the boundary layer and we are interested in the limit $\varepsilon \to 0$ in order to derive nontrivial boundary conditions. The evolution of the system is written as an energy balance formulation of the L^2-gradient flow with the corresponding Allen–Cahn energy functional. By transforming the boundary layer to a fixed domain we show the convergence of the solutions to a solution of a limit system. This is done by using concepts related to Γ- and Mosco convergence. By considering different scalings in the boundary layer we obtain different boundary conditions.

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1. Introduction

In the recent years there has been a growing interest in the coupling of bulk and surface processes. One important example is the theory of spinodal decomposition of binary mixtures where dynamic boundary conditions are used to model the effective short-range interaction between the two mixture components and the wall (i.e., the boundary), see e.g. [20,29] and the references therein. Moreover, we refer to [7,8,15,17,19,26,30,37] for an (incomplete) list of articles related to the mathematical analysis of dynamic boundary conditions for various evolutionary systems including the heat equation, the iso- and non-isothermal Allen–Cahn equation, the Cahn–Hilliard equation and the Caginalp system. In addition, we point out to the book [39] for the connection to Feller semigroups and Markov processes.

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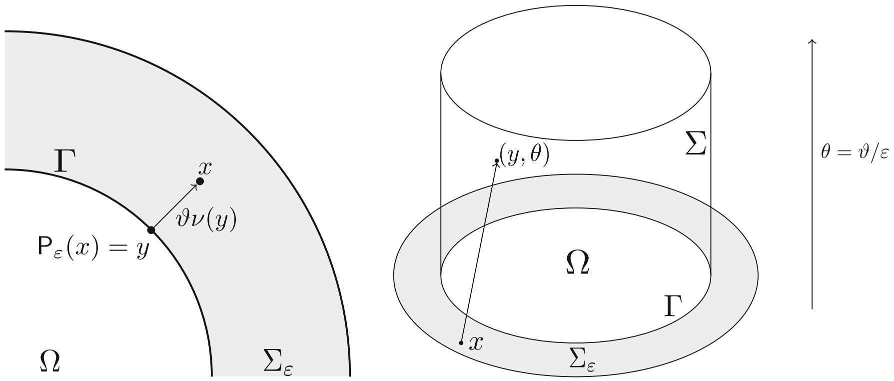

Figure 1. Transformation of the boundary layer

In this paper we discuss the question whether such dynamic boundary conditions can be obtained as a limit of a family of bulk systems in the case of the Allen–Cahn equation. More precisely, for a domain Ω with C^2-boundary Γ we introduce a boundary layer of thickness $\varepsilon > 0$, denoted by Σ_ε, that shrinks to Γ as ε tends to 0 (see Fig. 1). In the domains Ω and Σ_ε we consider the following system of (bulk) Allen–Cahn-type equations

$$\begin{aligned} \tau_{\mathrm{b}}\partial_t u_\varepsilon - A_{\mathrm{b}}\Delta u_\varepsilon + W_{\mathrm{b}}'(u_\varepsilon) = 0 \quad \text{in } \Omega, \\ \tau_\varepsilon\partial_t u_\varepsilon - A_\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W_{\mathrm{s}}'(u_\varepsilon) = 0 \quad \text{in } \Sigma_\varepsilon, \end{aligned}$$

subject to natural continuity and transmission conditions (see (1)) at the common boundary between the fixed bulk domain Ω and the thin boundary layer Σ_ε. Here, W_{b} and W_{s} are given, in general nonconvex, bulk and surface potentials.

In order to derive nontrivial boundary conditions when ε goes to 0 we assume that the relaxation time τ_ε and the diffusion coefficient A_ε depend on ε in the boundary layer Σ_ε. This amounts to different length and time scales in the bulk and in the boundary layer. We then show that the solutions of this system converge (up to subsequences) to a solution of a limit system which describes the coupling of bulk and surface evolution. The specific form of the derived limit system depends on the scalings of the coefficients τ_ε and A_ε. In particular, we will derive a hierarchy of dynamic and static boundary conditions depending on the scalings.

This approach is quite common in the derivation of lower-dimensional models in static elasticity, see e.g. [9,16]. Furthermore, we refer to [35] for the derivation of models for conductive thin sheets using asymptotic expansion and to [10] for the (non-rigorous) derivation of boundary conditions for the heat equation.

Here, however, we give a rigorous convergence proof which is based on an *energy balance formulation* of the underlying gradient flow structure of the

Allen–Cahn equation. More precisely, by defining the Allen–Cahn energy functionals $\mathcal{E}_\varepsilon$ the bulk equations can be written as L^2-gradient flow in form of a *force balance* between the dissipative forces and the potential restoring forces given by the derivative of $\mathcal{E}_\varepsilon$. This force balance formulation is equivalent to a scalar *energy balance* equation written in terms of the energy functionals and quadratic dissipation potentials $\mathcal{R}_\varepsilon$, which in this case are given by the squares of the L^2-norm (see also [1,24])

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) + \int_0^t \left[\mathcal{R}_\varepsilon(\dot{u}_\varepsilon) + \mathcal{R}_\varepsilon^*(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\right] \mathrm{d}s = \mathcal{E}_\varepsilon(u_\varepsilon(0)),$$

where $\mathcal{R}_\varepsilon^*$ denotes the dual dissipation potential, i.e., the Legendre transform of $\mathcal{R}_\varepsilon$.

The energy balance formulation opens the door for the application of notions of variational convergence such as Mosco and Γ-convergence [3,6,11]. Here we follow the ideas in [33] (see also [5,21,22]) where a method to prove the convergence of gradient flows for Γ-converging energy functionals was presented and applied to derive the limiting dynamics of vortices for the heat flow of the Ginzburg–Landau energy. However, we emphasize that the convergence of the gradient flow cannot follow from the Γ-convergence of the energy functionals only and extra conditions are required for the interplay of the convergence of the energy and the dissipation potentials. These extra conditions amount to the construction of mutual recovery curves for the energy and dissipation potentials.

Additionally, for λ-convex energy functionals the evolution of the system can be equivalently described by an *evolution variational inequality*

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) + \langle \mathcal{G}_\varepsilon \dot{u}_\varepsilon(t), u_\varepsilon(t) - \widetilde{u}\rangle \le \mathcal{E}_\varepsilon(\widetilde{u}) - \Lambda_\varepsilon(u_\varepsilon(t) - \widetilde{u}) \quad \forall\, \widetilde{u},$$

where $\mathcal{G}_\varepsilon$ denotes the linear and self-adjoint operator associated with $\mathcal{R}_\varepsilon$ and Λ_ε corresponds to the λ-convexity of $\mathcal{E}_\varepsilon$.

We show that we can pass to the limit in the energy balance and the evolution variational inequality, respectively, in order to obtain corresponding limit formulations, written in terms of limit functionals $\mathcal{E}_0$ and $\mathcal{R}_0$, which describe the coupling of bulk and surface evolution.

Let us remark here that the purpose of this paper is twofold: first, we want to identify the relevant scalings in the boundary layer system for deriving nontrivial boundary conditions. In particular, this identification can be used to obtain more information about the structure of the limit systems (see, e.g. [4]). As a longterm goal we shall apply this approach to the related problem of deriving interface conditions in reaction-diffusion systems. Nonstandard interface and transmission conditions in semiconductor heterostructures and biological systems are of great importance (see [14,18,36]). Especially in organic photovoltaics interfaces are the fundamental building block, see [28, Sect. 8].

Second, the paper contributes to the theory of application of Γ-convergence methods to evolutionary problems, especially to gradient flows. We refer to [2,23,25] for the application of the principles of Γ-convergence to rate-independent evolution, Hamiltonian systems and Wasserstein gradient flows, respectively.

The paper is organized as follows: in Sect. 2 we introduce the underlying geometry of the boundary layer approximation and present the system of Allen–Cahn-type equations along with technical details such as growth conditions, etc. The bulk system will then be cast into the gradient flow framework, in particular in the energy balance formulation. Furthermore, we introduce a change of coordinates in order to transform the system to a fixed domain (see Sect. 2.3). In this change of coordinates we characterize a point in the boundary layer by its projection and distance onto, resp., to Γ. Therefore we can decompose directions in Σ_ε into tangential and normal parts relative to Γ. The normal direction is then rescaled in order to obtain a fixed domain.

In Sect. 3 we present the main result of the paper, i.e., the limit passage in the energy balance. This is based on the results in [33] which for the convenience of the reader will be reformulated here. Applied to our specific problem the construction of the mutual recovery curves is akin to the construction of the recovery sequences for the energy functionals in the sense of Γ-convergence.

In the final Sect. 4 the derived limit models will be discussed. In particular, depending on the scaling of the relaxation time and the diffusion coefficient in the boundary layer we obtain the usual Dirichlet- and Neumann boundary conditions as well as dynamic boundary conditions and boundary conditions that are to our knowledge not addressed in the literature so far, e.g., coupling of the bulk equation to an elliptic equation for the trace on Γ (see (17))

Notably, we also obtain the system recently considered in [37] where it was studied regarding existence and uniqueness of global solutions, as well as asymptotic behavior and the existence of a global attractor. The system consists of the following bulk equation and dynamic boundary condition for the bulk and surface order parameters u and $U = u|_\Gamma$

$$\begin{aligned} \tau_\mathrm{b}\partial_t u - A_\mathrm{b}\Delta u + W_\mathrm{b}'(u) = 0 \quad &\text{in } \Omega,\\ \tau_\mathrm{s}\partial_t U - A_\mathrm{s}\Delta_\Gamma U + A_\mathrm{b}\tfrac{\partial u}{\partial \nu} + W_\mathrm{s}'(U) = 0 \quad &\text{on } \Gamma, \end{aligned}$$

where Δ_Γ denotes the Laplace–Beltrami operator on Γ.

2. Setting of the model

2.1. Definitions and notations

We consider an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with a C^2-boundary denoted by $\Gamma := \partial\Omega$. For a sufficiently small parameter $\varepsilon > 0$ we introduce the domain Ω_ε defined by

$$\Omega_\varepsilon := \big\{x \in \mathbb{R}^d \,:\, \mathrm{dist}(x, \Omega) < \varepsilon\big\},$$

where $\mathrm{dist}(x, \Omega) := \inf_{y\in\Omega} |x - y|$ denotes the distance to Ω. We call the set $\Sigma_\varepsilon := \Omega_\varepsilon \setminus \overline{\Omega}$ the boundary layer (or ε-neighborhood) of Ω. Obviously, we have the convergence $\Omega_\varepsilon \to \Omega$ for $\varepsilon \to 0$ with respect to the Hausdorff distance.

Let $T > 0$ be a finite time horizon. In the domain Ω_ε we consider the following system of Allen–Cahn-type equations:

$$\begin{aligned} \tau_{\mathrm{b}} \partial_t u_\varepsilon - A_{\mathrm{b}} \Delta u_\varepsilon + W_{\mathrm{b}}'(u_\varepsilon) &= 0 \quad \text{in } [0,T] \times \Omega, \\ \tau_\varepsilon \partial_t u_\varepsilon - A_\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W_{\mathrm{s}}'(u_\varepsilon) &= 0 \quad \text{in } [0,T] \times \Sigma_\varepsilon, \end{aligned} \tag{AC$_\varepsilon$}$$

where $\tau_{\mathrm{b}}, \tau_\varepsilon > 0$ denote the relaxation times, $A_{\mathrm{b}}, A_\varepsilon$ the diffusion coefficients, and W_{b}', W_{s}' are the derivatives of potentials $W_{\mathrm{b}}, W_{\mathrm{s}} \in \mathrm{C}^1(\mathbb{R})$ in the bulk and in the boundary layer, respectively. The equations above are subjected to the following natural boundary and transmission conditions at $\partial\Omega_\varepsilon$ and at the interface Γ

$$\begin{aligned} A_\varepsilon \tfrac{\partial u_\varepsilon}{\partial \nu} &= 0 && \text{on } [0,T] \times \partial\Omega_\varepsilon, \\ A_{\mathrm{b}} \tfrac{\partial u_\varepsilon}{\partial \nu} &= A_\varepsilon \tfrac{\partial u_\varepsilon}{\partial \nu} && \text{on } [0,T] \times \Gamma, \\ [\![u_\varepsilon]\!] &= 0 && \text{on } [0,T] \times \Gamma, \end{aligned} \tag{1}$$

where ν denotes the outer unit normal on Γ and $\partial\Omega_\varepsilon$ and $[\![\cdot]\!]$ denotes the jump across the interface Γ. Finally, the system is completed by imposing the initial condition $u_\varepsilon(0) = u_\varepsilon^0$ in Ω_ε.

We assume that in the boundary layer Σ_ε the coefficients satisfy the scalings

$$\tau_\varepsilon = \varepsilon^{-\alpha_{\mathsf{R}}} \tau_{\mathrm{s}} \quad \text{and} \quad A_\varepsilon = \varepsilon^{-\beta_{\mathsf{E}}} A_{\mathrm{s}}$$

for given $\tau_{\mathrm{s}}, A_{\mathrm{s}} > 0$ and $\alpha_{\mathsf{R}} \in \mathbb{R}$, $\beta_{\mathsf{E}} \in]-1, \infty[$.

The nonlinearities W_{b} and W_{s} are at least of quadratic growth and satisfy the growth conditions

$$|W_{\mathrm{b/s}}'(u)| \leq C(1+|u|^p) \quad \text{with } p \in [1, q[\quad \text{and} \quad q = \begin{cases} \infty & d = 2, \\ \frac{d+2}{d-2} & d \geq 3. \end{cases} \tag{W_{Grow}}$$

These are the same growth conditions imposed in [37] for the bulk potential W_{b}, while we have a stronger growth condition for the boundary potential since we are in the full d-dimensional domain Σ_ε in contrast to the $(d-1)$-dimensional boundary Γ in [37].

A prominent example for the (nonconvex) potentials W_{b} and W_{s} is the double well potential $u \mapsto \frac{1}{4}(1-u^2)^2$, which obviously satisfies the above growth conditions for $d = 2, 3$.

We show that solutions of the system above converge in a certain sense to a solution of a limit system which consists of the bulk equation in Ω in (AC$_\varepsilon$) coupled to an equation posed on the boundary Γ. As we will see, the form of the latter equation will heavily depend on the choices for the scaling exponents α_{R} and β_{E}.

To put the above system in an abstract framework we introduce the function spaces $V_\varepsilon := \mathrm{H}^1(\Omega_\varepsilon)$ and $H_\varepsilon := \mathrm{L}^2(\Omega_\varepsilon)$. Then, the weak formulation of the system (AC$_\varepsilon$) reads: Find $u_\varepsilon \in \mathrm{H}^1(0,T; H_\varepsilon) \cap \mathrm{L}^2(0,T; V_\varepsilon)$ with $u_\varepsilon(0) = u_\varepsilon^0$ such that for all $\varphi \in V_\varepsilon$ and almost all $t \in [0,T]$ we have

$$0 = \int_{\Omega_\varepsilon} \Big[\mathbb{G}_\varepsilon(x) \partial_t u_\varepsilon \varphi + \mathbb{A}_\varepsilon(x) \nabla u_\varepsilon \cdot \nabla \varphi + \mathbb{W}_\varepsilon'(x, u_\varepsilon) \varphi \Big] \, \mathrm{d}x, \tag{w-AC$_\varepsilon$}$$

where we use the notation

$$\mathbb{G}_\varepsilon(x) = \begin{cases} \tau_\mathrm{b} & \text{in } \Omega, \\ \tau_\varepsilon & \text{in } \Sigma_\varepsilon, \end{cases} \qquad \mathbb{A}_\varepsilon(x) = \begin{cases} A_\mathrm{b} & \text{in } \Omega, \\ A_\varepsilon & \text{in } \Sigma_\varepsilon, \end{cases} \qquad \mathbb{W}_\varepsilon(x,\cdot) = \begin{cases} W_\mathrm{b}(\cdot) & \text{in } \Omega, \\ \frac{1}{\varepsilon} W_\mathrm{s}(\cdot) & \text{in } \Sigma_\varepsilon. \end{cases}$$

The existence of solutions of (AC_ε), resp. ($\mathsf{w}\text{-}\mathsf{AC}_\varepsilon$), follows from standard arguments, see e.g. [31,37].

Theorem 2.1. (Existence of solutions) *For fixed $\varepsilon > 0$ let $u_\varepsilon^0 \in V_\varepsilon$ be given. Moreover, assume that the growth condition (W_{Grow}) holds. Then, there exists a solution $u_\varepsilon \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_\varepsilon)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega_\varepsilon))$ of the system (AC_ε).*

2.2. Different formulations of gradient flows

It is well know that Eq. (AC_ε) is the L^2-gradient flow of the Allen–Cahn functional $\mathcal{E}_\varepsilon : V_\varepsilon \to \mathbb{R}$ defined by

$$\mathcal{E}_\varepsilon(u) = \int_{\Omega_\varepsilon} \left[\frac{\mathbb{A}_\varepsilon(x)}{2} |\nabla u|^2 + \mathbb{W}_\varepsilon(x,u) \right] \mathrm{d}x.$$

More precisely, by defining the symmetric and positive metric tensor $\mathcal{G}_\varepsilon : H_\varepsilon \to H_\varepsilon^*$ via $\langle \mathcal{G}_\varepsilon \dot{u}, \dot{v}\rangle = \int_{\Omega_\varepsilon} \mathbb{G}_\varepsilon(x) \dot{u}\dot{v}\,\mathrm{d}x$ the equation in ($\mathsf{w}\text{-}\mathsf{AC}_\varepsilon$) can then be written in the form

$$\mathcal{G}_\varepsilon \dot{u}_\varepsilon(t) = -\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t)), \qquad (\mathsf{fb}_\varepsilon)$$

with $\mathrm{D}\mathcal{E}_\varepsilon(u)$ denoting the Gâteaux derivative of $\mathcal{E}_\varepsilon$ which is well-defined due to (W_{Grow}). Note that we (notationally) distinguish between H_ε and H_ε^* since the former is the space of velocities $\dot{u}$, while the latter is the space of forces $\xi = \mathrm{D}\mathcal{E}_\varepsilon(u)$. Thus, $\mathcal{G}_\varepsilon$ maps velocities to forces. The equation above can be seen as a *force balance* formulation of the gradient flow, where $\mathcal{G}_\varepsilon \dot{u}_\varepsilon$ and $\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)$ are the dissipative and potential restoring forces, respectively. Defining the inverse operator $\mathcal{K}_\varepsilon = \mathcal{G}_\varepsilon^{-1}{:}H_\varepsilon^* \to H_\varepsilon$, mapping forces to velocities, we can write the force balance (fb_ε) as *rate equation* in H_ε

$$\dot{u}_\varepsilon(t) = -\mathcal{K}_\varepsilon \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t)) =: -\nabla_{\mathcal{G}_\varepsilon}\mathcal{E}_\varepsilon(u_\varepsilon(t)), \qquad (\mathsf{re}_\varepsilon)$$

where $\nabla_{\mathcal{G}_\varepsilon}\mathcal{E}$ denotes the gradient of $\mathcal{E}_\varepsilon$ with respect to the metric tensor $\mathcal{G}_\varepsilon$. Note that we have $\langle \xi, \mathcal{K}_\varepsilon \eta\rangle = \int_{\Omega_\varepsilon} \mathbb{G}_\varepsilon(x)^{-1}\xi\eta\,\mathrm{d}x$. The operator $\mathcal{G}_\varepsilon$ defines the quadratic dissipation potential $\mathcal{R}_\varepsilon(\dot{u}) = \frac{1}{2}\langle \mathcal{G}_\varepsilon \dot{u}, \dot{u}\rangle$ whose Legendre transform is given by $\mathcal{K}_\varepsilon$, i.e., we have $\mathcal{R}_\varepsilon^*(\xi) = \frac{1}{2}\langle \xi, \mathcal{K}_\varepsilon \xi\rangle$, where ξ denotes the *"dual variable"* (also called chemical potential or thermodynamically conjugated driving force, see [24]). Furthermore, by using the chain rule we have that

$$\begin{aligned} \mathcal{E}_\varepsilon(u_\varepsilon(0)) - \mathcal{E}_\varepsilon(u_\varepsilon(t)) &= \int_0^t \langle \mathcal{G}_\varepsilon \dot{u}_\varepsilon, \dot{u}_\varepsilon\rangle \,\mathrm{d}s \\ &= \int_0^t \langle \mathrm{D}\mathcal{E}(u_\varepsilon), \mathcal{K}_\varepsilon \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)\rangle \,\mathrm{d}s \\ &= \int_0^t \Big[\mathcal{R}_\varepsilon(\dot{u}_\varepsilon) + \mathcal{R}_\varepsilon^*(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)) \Big] \,\mathrm{d}s. \end{aligned}$$

Hence, we see that the force balance (fb_ε) and the rate equation (re_ε) are equivalent to the *energy balance*

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) + \int_0^t \Big[\mathcal{R}_\varepsilon(\dot u_\varepsilon) + \mathcal{R}_\varepsilon^*(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\Big]\,\mathrm{d}s = \mathcal{E}_\varepsilon(u_\varepsilon(0)). \qquad (\mathsf{eb}_\varepsilon)$$

This formulation (which is based on a principle by De Giorgi and whose solutions are also called curves of maximal slope see [1, Sect. 1.3]) is indeed equivalent due to the Legendre Fenchel equivalences for convex potentials, i.e.,

$$\mu = \mathrm{D}\mathcal{R}_\varepsilon(v) \iff v = \mathrm{D}\mathcal{R}_\varepsilon^*(\mu) \iff \mathcal{R}_\varepsilon(v) + \mathcal{R}_\varepsilon^*(\mu) = \langle v, \mu\rangle.$$

We also used the chain rule $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(u) = \langle \mathrm{D}\mathcal{E}_\varepsilon(u), \dot u\rangle$.

In fact, in (eb_ε) we only need the lower estimate "$\leq$", the reverse estimate follows from the definition of the Legendre transform. The advantage of (eb_ε) is that it is a scalar equation in $\mathbb{R}$ in contrast to Eqs. (fb_ε) and (re_ε) in H_ε^* and H_ε, respectively.

Let us remark here that $2\mathcal{R}_\varepsilon(\dot u)$ and $2\mathcal{R}_\varepsilon^*(-\mathrm{D}\mathcal{E}_\varepsilon(u))$ are nothing but the squares of the so called metric derivative of u and the metric slope of $\mathcal{E}_\varepsilon$ calculated with respect to the metric induced by $\mathcal{G}_\varepsilon$, see [1].

If the potentials W_b and W_s are λ_b-convex, resp., λ_s-convex, ($s \mapsto W_{\mathrm{b/s}}(s) - \frac{\lambda_{\mathrm{b/s}}}{2}|s|^2$ is convex) the energy functional satisfies the convexity estimate

$$\mathcal{E}_\varepsilon(\widetilde u) \geq \mathcal{E}_\varepsilon(u) + \langle \mathrm{D}\mathcal{E}_\varepsilon(u), \widetilde u - u\rangle + \Lambda_\varepsilon(\widetilde u - u) \quad \forall\, \widetilde u \in V_\varepsilon,$$

where $\Lambda_\varepsilon(w) = \int_\Omega \frac{\lambda_\mathrm{b}}{2}|w|^2\,\mathrm{d}x + \int_{\Sigma_\varepsilon} \frac{\lambda_\mathrm{s}}{2\varepsilon}|w|^2\,\mathrm{d}x$. Note, that λ_b and λ_s do not have to be positive and therefore W_b and W_s are in general nonconvex. The double well potential $u \mapsto \frac{1}{4}(1-u^2)^2$ is λ-convex with $\lambda = -1$. Moreover, every $W \in \mathrm{C}^{1,1}(\mathbb{R})$ is λ-convex.

Using the force balance formulation (fb_ε) we arrive at the equivalent formulation as *evolution variational inequality* (see [1,12])

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) + \langle \mathcal{G}_\varepsilon \dot u_\varepsilon(t), u_\varepsilon(t) - \widetilde u\rangle \leq \mathcal{E}_\varepsilon(\widetilde u) - \Lambda_\varepsilon(u_\varepsilon(t) - \widetilde u) \quad \forall\, \widetilde u \in V_\varepsilon. \quad (\mathsf{evi}_\varepsilon)$$

Note that this formulation is written only in terms of the functional $\mathcal{E}_\varepsilon$ and the operator $\mathcal{G}_\varepsilon$, and is therefore derivative free.

We study the behavior of the solutions u_ε when $\varepsilon \to 0$. In this case the boundary layer Σ_ε shrinks to Γ and we show that the "limit" of the sequence $u_\varepsilon|_{\Sigma_\varepsilon}$ describes the evolution on Γ.

2.3. Transformation of the problem

In order to provide a notion of convergence of the solutions u_ε we transform the variable domain Ω_ε to a fixed domain.

For this, note that due to the smoothness of the boundary Γ and for sufficiently small ε a point $x \in \Sigma_\varepsilon$ can be characterized in the following way: there exist unique $y \in \Gamma$ and $\vartheta \in]0, \varepsilon[$ such that $x = y + \vartheta\nu(y)$ (see, e.g. [40, Chap. 2]), where ν denotes the unit outer normal on Γ (see Fig. 1). Hence, we introduce the change of coordinates in Σ_ε

$$X_\varepsilon(y,\theta) := y + \varepsilon\theta\nu(y), \quad (y,\theta) \in \Gamma\times]0,1[,$$
$$Y_\varepsilon(x) := \big(y_\varepsilon(x), \theta_\varepsilon(x)\big) := \big(\mathsf{P}_\varepsilon(x), \mathsf{d}_\varepsilon(x)/\varepsilon\big), \quad x \in \Omega_\varepsilon,$$

where P_ε and d_ε denote the projection from Σ_ε on Γ and the distance to Γ, respectively.

With this change of coordinates we define $\Sigma := \Gamma\times]0,1[$ and for a function $u : \Sigma_\varepsilon \to \mathbb{R}$ we set $U = u \circ X_\varepsilon : \Sigma \to \mathbb{R}$. Since the boundary Γ is of class C^2 we have that the outer unit normal satisfies $\nu \in \mathrm{C}^1(\Gamma;\mathbb{R}^d)$. Therefore, if $u \in \mathrm{H}^1(\Sigma_\varepsilon)$ we have $U \in \mathrm{H}^1(\Sigma)$. More precisely, it holds

$$\begin{pmatrix} \nabla_\Gamma U \\ \partial_\theta U \end{pmatrix} = \begin{pmatrix} \mathbb{P}(y) - \varepsilon\theta\,\mathbb{S}(y) \\ \varepsilon\,\nu(y)^\top \end{pmatrix} \nabla u, \quad \text{and} \quad \nabla u = \left(\mathbb{Q}_\varepsilon(x) \Big| \frac{1}{\varepsilon}\nu(\mathsf{P}_\varepsilon(x)) \right) \begin{pmatrix} \nabla_\Gamma U \\ \partial_\theta U \end{pmatrix},$$

where $\nabla_\Gamma U \in \mathcal{T}(\Gamma)$ denotes the tangential gradient of U on Γ, $\mathbb{P}(y)$ is the projection onto the tangential space at $y \in \Gamma$, $\mathbb{S} = -\partial\nu/\partial y$ is the so-called shape operator (see, e.g. [13]) and $\mathbb{Q}_\varepsilon$ is such that $\mathbb{Q}_\varepsilon(\mathbb{P} - \varepsilon\theta\mathbb{S}) = \mathbb{P}$.

The tangential gradient $\nabla_\Gamma U$ on Γ can be characterized in the following way (see [13,34]): For $V : \Gamma \to \mathbb{R}$ denote by $\widetilde{V}$ a smooth extension of V to $\mathbb{R}^d$, then $\nabla_\Gamma V(y) = \mathbb{P}(y)[\nabla\widetilde{V}]$. It is easy to check that this definition is well-defined and independent of the extension $\widetilde{V}$, moreover, we have that $\mathbb{P} = I - \nu \otimes \nu$. Similarly, the divergence on Γ for vector fields V can be defined as

$$\mathrm{div}_\Gamma \mathsf{V} = \mathrm{div}\widetilde{\mathsf{V}} - \nabla(\widetilde{\mathsf{V}} \cdot \nu)\nu,$$

where $\widetilde{\mathsf{V}}$ is again a smooth extension of V. In this framework the Laplace-Beltrami operator Δ_Γ has the simple form $\Delta_\Gamma U = \mathrm{div}_\Gamma(\nabla_\Gamma U)$. For a vector field $\mathsf{V} \in \mathrm{L}^2(\Gamma;\mathcal{T}(\Gamma))$ such that $\mathrm{div}_\Gamma \mathsf{V} \in \mathrm{L}^2(\Gamma)$ and $U \in \mathrm{H}^1(\Gamma)$ we have Green's formula

$$-\int_\Gamma \nabla_\Gamma U \cdot \mathsf{V}\,\mathrm{d}\Gamma = \int_\Gamma U \mathrm{div}_\Gamma \mathsf{V}\,\mathrm{d}\Gamma.$$

In contrast to Σ_ε we leave the bulk domain Ω untransformed. Hence, we introduce the spaces for the bulk variable $u : \Omega \to \mathbb{R}$ and the surface variable $U : \Sigma \to \mathbb{R}$

$$\mathcal{V} := \big\{ (u,U) \in \mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Sigma) \,:\, u|_\Gamma = U|_{\{\theta=0\}} \big\}, \quad \mathcal{H} := \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Sigma).$$

The measure on Σ is given by $\mathrm{d}\mu = \mathrm{d}\Gamma \otimes \mathrm{d}\lambda^1$, i.e., the product of the surface measure on Γ and the one-dimensional Lebesgue measure on $]0,1[$. The space $\mathrm{H}^1(\Sigma)$ is defined in the usual way, i.e., the closure of $\mathrm{C}^1(\Sigma)$ with respect to the norm $\|\cdot\|_{\mathrm{H}^1(\Sigma)}$, where

$$\|U\|^2_{\mathrm{H}^1(\Sigma)} = \int_\Sigma \Big[|U|^2 + |\nabla_\Gamma U|^2 + |\partial_\theta U|^2 \Big]\,\mathrm{d}\mu.$$

Now, substituting the above transformations in $\mathcal{E}_\varepsilon$ we arrive at the transformed energy functional $\mathsf{E}_\varepsilon : \mathcal{V} \to [0,\infty[$, for $\boldsymbol{u} = (u,U)$ defined by

$$\begin{aligned} \mathsf{E}_\varepsilon(\boldsymbol{u}) = &\int_\Omega \left[\frac{A_\mathrm{b}}{2}|\nabla u|^2 + W_\mathrm{b}(u) \right] \mathrm{d}x \\ &+ \int_\Sigma \left[\frac{A_\varepsilon}{2}\left(\nabla_\Gamma U \cdot \mathbb{B}_\varepsilon(y,\theta)\nabla_\Gamma U + \frac{1}{\varepsilon^2}\,|\partial_\theta U|^2 \right) + W_\varepsilon(U) \right] \mathbb{J}_\varepsilon(y,\theta)\,\mathrm{d}\mu, \end{aligned}$$

where $\mathbb{B}_\varepsilon = \mathbb{Q}_\varepsilon^\top \mathbb{Q}_\varepsilon$ and $\mathbb{J}_\varepsilon$ describes the change of volume due to the transformation. Additionally, the transformed dissipation potential $\mathsf{R}_\varepsilon : \mathcal{H} \to [0, \infty[$ reads

$$\mathsf{R}_\varepsilon(\dot{\boldsymbol{u}}) = \int_\Omega \frac{\tau_\mathrm{b}}{2} |\dot{u}|^2 \,\mathrm{d}x + \int_\Sigma \frac{\tau_\varepsilon}{2} |\dot{U}|^2 \mathbb{J}_\varepsilon(y,\theta) \,\mathrm{d}\mu.$$

We denote by $\mathsf{G}_\varepsilon : \mathcal{H} \to \mathcal{H}^*$ the associated operator, i.e., $\mathsf{R}_\varepsilon(\dot{\boldsymbol{u}}) = \frac{1}{2}\langle \mathsf{G}_\varepsilon \dot{\boldsymbol{u}}, \dot{\boldsymbol{u}} \rangle$. The inverse operator $\mathsf{K}_\varepsilon = \mathsf{G}_\varepsilon^{-1} : \mathcal{H}^* \to \mathcal{H}$ gives the dual dissipation potential R_ε^*, more precisely, for a dual variable $\boldsymbol{\xi} = (\xi, \Xi)$ it reads

$$\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon(t)) + \int_0^t \Big[\mathsf{R}_\varepsilon(\dot{\boldsymbol{u}}_\varepsilon) + \mathsf{R}_\varepsilon^*(-\mathrm{D}\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon)) \Big] \,\mathrm{d}s = \mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon(0)). \qquad (\mathsf{EB}_\varepsilon)$$

Moreover, in the λ-convex case the evolution of the transformed system is equivalently described by the following evolution variational inequality which corresponds to (evi_ε)

$$\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon(t)) + \langle \mathsf{G}_\varepsilon \dot{\boldsymbol{u}}_\varepsilon(t), \boldsymbol{u}_\varepsilon(t) - \widetilde{\boldsymbol{u}} \rangle \le \mathsf{E}_\varepsilon(\widetilde{\boldsymbol{u}}) - \boldsymbol{\Lambda}_\varepsilon(\boldsymbol{u}_\varepsilon(t) - \widetilde{\boldsymbol{u}}), \qquad (\mathsf{EVI}_\varepsilon)$$

where $\boldsymbol{\Lambda}_\varepsilon(\boldsymbol{u}) = \int_\Omega \frac{\lambda_\mathrm{b}}{2} |u|^2 \,\mathrm{d}x + \int_\Sigma \frac{\lambda_\mathrm{s}}{2} |U|^2 \frac{\mathbb{J}_\varepsilon}{\varepsilon} \,\mathrm{d}\mu$.

We will use both formulations, (EB_ε) and (EVI_ε), for the convergence analysis. Note that (EB_ε) contains the derivative of the energy functional E_ε while (EVI_ε) does not. Conversely, (EVI_ε) contains the derivative of the dissipation potential R_ε while (EB_ε) is free of it.

The following lemma is concerned with the convergences of the geometrical quantities $\mathbb{B}_\varepsilon$ and $\mathbb{J}_\varepsilon$.

Lemma 2.2. *It holds that $\mathbb{B}_\varepsilon \to \mathbb{I}$ uniformly in Σ, with $\mathbb{I}$ denoting the identity in the tangent bundle of Γ, and $\mathbb{J}_\varepsilon/\varepsilon \to 1$ uniformly in Σ.*

The easiest (although not most elegant) way to see that the convergence is indeed as stated, is to switch to local coordinates and calculate $\mathbb{B}_\varepsilon$ and $\mathbb{J}_\varepsilon$ explicitly in terms of the covariant and contravariant basis vectors (see [9] for a related problem in the theory of elastic shells).

3. Convergence of the system

Our result is formulated abstractly in terms of Mosco convergence of E_ε towards a limit E_0 and of R_ε towards R_0. For functionals F_n, defined on a Banach space $\mathcal{Q}$, the definition of Mosco convergence is as follows:

$$\mathsf{F}_n \xrightarrow{\mathrm{M}} \mathsf{F} \;\Leftrightarrow\; \begin{cases} \text{(i) Liminf estimate for weakly converging sequences:} \\ \quad \boldsymbol{q}_n \rightharpoonup \boldsymbol{q} \Longrightarrow \mathsf{F}(\boldsymbol{q}_n) \le \liminf\limits_{n\to\infty} \mathsf{F}_n(\boldsymbol{q}), \\ \text{(ii) Existence of strongly converging recovery sequences:} \\ \quad \forall\, \widehat{\boldsymbol{q}} \in \mathcal{Q}\; \exists\, (\widehat{\boldsymbol{q}}_n)_n :\; \widehat{\boldsymbol{q}}_n \to \widehat{\boldsymbol{q}} \text{ and } \mathsf{F}(\widehat{\boldsymbol{q}}) \ge \limsup\limits_{n\to\infty} \mathsf{F}_n(\widehat{\boldsymbol{q}}_n). \end{cases}$$

Hence, Mosco convergence is nothing but Γ-convergence in the weak *and* in the strong topology.

Since it is essential to choose the right topology for computing the Γ- or Mosco limits, the first step in our convergence proof is to derive a priori estimates for the solutions $(u_\varepsilon, U_\varepsilon)$. This is addressed in the following lemma.

Lemma 3.1. (A priori estimate) *Let* $\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon(0)) \le C < \infty$ *and define* $\Omega_T = \Omega \times]0,T[$ *and* $\Sigma_T = \Sigma \times]0,T[$*. Then, there exist constants* $C_1, C_2, C_3, C_4 > 0$*, independent of* ε*, such that*

$$\begin{aligned} &\big\|\dot{u}_\varepsilon\big\|^2_{\mathrm{L}^2(\Omega_T)} + \varepsilon^{1-\alpha_\mathsf{R}}\big\|\dot{U}_\varepsilon\big\|^2_{\mathrm{L}^2(\Sigma_T)} \le C_1, \\ &\big\|\mathrm{D}_u\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon)\big\|^2_{\mathrm{L}^2(\Omega_T)} + \frac{1}{\varepsilon^{1-\alpha_\mathsf{R}}}\big\|\mathrm{D}_U\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon)\big\|^2_{\mathrm{L}^2(\Sigma_T)} \le C_2, \end{aligned} \tag{2a}$$

and

$$\begin{aligned} &\big\|\nabla u_\varepsilon(t)\big\|^2_{\mathrm{L}^2(\Omega)} + \big\|u_\varepsilon(t)\big\|^2_{\mathrm{L}^2(\Omega)} + \big\|U_\varepsilon(t)\big\|^2_{\mathrm{L}^2(\Sigma)} \le C_3, \\ &\varepsilon^{1-\beta_\mathsf{E}}\big\|\nabla_\Gamma U_\varepsilon(t)\big\|^2_{\mathrm{L}^2(\Sigma)} + \frac{1}{\varepsilon^{1+\beta_\mathsf{E}}}\big\|\partial_\theta U_\varepsilon(t)\big\|^2_{\mathrm{L}^2(\Sigma)} \le C_4, \end{aligned} \tag{2b}$$

for all $t \in [0,T]$*.*

Proof. The estimates in (2) are a direct consequence of the energy balance (EB_ε). We remind that the relaxation time and the diffusion coefficient are given by $\tau_\varepsilon = \tau_\mathrm{s}\varepsilon^{-\alpha_\mathsf{R}}$, $A_\varepsilon = A_\mathrm{s}\varepsilon^{-\beta_\mathsf{E}}$. The energy functional satisfies the estimate

$$\begin{aligned} \mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) \ge C(&\|\nabla u_\varepsilon\|^2_{\mathrm{L}^2(\Omega)} + \|u_\varepsilon\|^2_{\mathrm{L}^2(\Omega)} + \|U_\varepsilon\|^2_{\mathrm{L}^2(\Omega)} \\ &+\varepsilon^{1-\beta_\mathsf{E}}\|\nabla_\Gamma U_\varepsilon\|^2_{\mathrm{L}^2(\Sigma)} + \varepsilon^{-(\beta_\mathsf{E}+1)}\|\partial_\theta U_\varepsilon\|^2_{\mathrm{L}^2(\Sigma)}) - c, \end{aligned}$$

where we have used the quadratic growth of the nonlinearities W_b and W_s as well as Lemma 2.2. The dissipation potential satisfies

$$\begin{aligned} \mathsf{R}_\varepsilon(\dot{\boldsymbol{u}}_\varepsilon) &\ge C(\|\dot{u}_\varepsilon\|^2_{\mathrm{L}^2(\Omega)} + \varepsilon^{1-\alpha_\mathsf{R}}\|\dot{U}_\varepsilon\|^2_{\mathrm{L}^2(\Sigma)}), \\ \mathsf{R}^*_\varepsilon(\boldsymbol{\xi}_\varepsilon) &\ge C(\|\mathrm{D}_u\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon)\|^2_{\mathrm{L}^2(\Omega)} + \varepsilon^{\alpha_\mathsf{R}-1}\|\mathrm{D}_U\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon)\|^2_{\mathrm{L}^2(\Sigma)}). \end{aligned}$$

By assumption the lefthand-side in the energy balance (EB_ε) is bounded, thus we arrive at (2). □

The a priori estimates show that the critical scaling for the relaxation time $\tau_\varepsilon = \varepsilon^{-\alpha_\mathsf{R}}\tau_\mathrm{s}$ is $\alpha_\mathsf{R} = 1$. For $\alpha_\mathsf{R} < 1$ we expect the time derivatives in Σ to blow up while the thermodynamically conjugated driving forces tend to 0 in the limit. This means that we have a much faster timescale in the boundary layer, such that in the limit the system is always in equilibrium on the boundary. Conversely, $\alpha_\mathsf{R} > 1$ amounts to a slower timescale in the boundary layer with no evolution. In contrast to these degenerate cases $\alpha_\mathsf{R} = 1$ results in a nontrivial dynamic boundary condition as in [37].

In addition, we find the characteristic values $\beta_\mathsf{E} \in \{-1, +1\}$ for the scalings of the diffusion coefficient $A_\varepsilon = \varepsilon^{-\beta_\mathsf{E}}A_\mathrm{s}$ in the boundary layer. For $\beta_\mathsf{E} > 1$ all derivatives have to vanish such that U is constant (in every connected component of Σ). However, it is not fixed and may evolve in time, we refer to this as the fast diffusion case. Conversely, for $\beta_\mathsf{E} < 1$ we expect the tangential derivatives to blow up in the boundary layer (no diffusion case). For $\beta_\mathsf{E} = 1$ we expect genuine surface diffusion.

The crucial point is that in all of the cases above the derivative with respect to θ has to vanish. Hence, in the limit the surface variable U is given only by its trace on Γ which allows for the reduction to surface evolution, see Sect. 4 for the final discussion.

Lemma 3.1 shows that we can extract a (not relabeled) subsequence $\boldsymbol{u}_\varepsilon = (u_\varepsilon, U_\varepsilon)$ such that for the bulk variable u_ε we have the convergence

$$\begin{aligned} u_\varepsilon &\overset{*}{\rightharpoonup} u \ \text{ in } \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)),\\ \dot{u}_\varepsilon &\rightharpoonup \dot{u} \ \text{ in } \mathrm{L}^2(\Omega_T). \end{aligned} \tag{3}$$

Moreover, the second estimate in (2a) shows (by eventually extracting another subsequence) that we have the convergence $\mathsf{DE}_\mathrm{b}(u_\varepsilon) \rightharpoonup \xi$ in $\mathrm{L}^2(\Omega_T)$, where $\mathsf{E}_\mathrm{b}(u)$ denotes the bulk energy part. However, due to (3) we can argue that $u_\varepsilon \to u$ in $\mathrm{L}^q(\Omega_T)$ with $1 \le q < \infty$ for $d = 2$ and $1 \le q < 2d/(d-2)$ for $d \ge 3$. In particular, considering an almost everywhere converging subsequence and using the growth condition (W_Grow) the Dominated Convergence theorem yields $\xi = \mathsf{DE}_\mathrm{b}(u)$, hence

$$\mathsf{DE}_\mathrm{b}(u_\varepsilon) \rightharpoonup \mathsf{DE}_\mathrm{b}(u) \quad \text{in } \mathrm{L}^2(\Omega_T). \tag{4}$$

Moreover, we have the following convergences for U_ε

$$\begin{aligned} U_\varepsilon &\overset{*}{\rightharpoonup} U \quad \text{in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)),\\ \partial_\theta U_\varepsilon &\to 0 \quad \text{in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Sigma)), \end{aligned} \tag{5}$$

where the last convergence follows from $\beta_\mathsf{E} > -1$ and $\varepsilon^{-(1+\beta_\mathsf{E})}\|\partial_\theta U_\varepsilon(t)\|^2_{\mathrm{L}^2(\Sigma)}$ being bounded. Depending on the value of β_E we find a subsequence such that the tangential gradients of U_ε satisfy

$$\left.\begin{aligned} \nabla_\Gamma U_\varepsilon &\overset{*}{\rightharpoonup} \nabla_\Gamma U && \text{for } \beta_\mathsf{E} = 1\\ \nabla_\Gamma U_\varepsilon &\to 0 && \text{for } \beta_\mathsf{E} > 1 \end{aligned}\right\} \text{ in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Sigma)). \tag{6}$$

Furthermore, we can assume that

$$\text{for } \alpha_\mathsf{R} = 1: \quad \dot{U}_\varepsilon \rightharpoonup \dot{U} \quad \text{and} \quad \mathsf{DE}_{\mathrm{s},\varepsilon}(U_\varepsilon) \rightharpoonup \Xi, \tag{7}$$

where we denoted by $\mathsf{E}_{\mathrm{s},\varepsilon}$ the surface energy part such that $\mathsf{E}_\varepsilon(u,U) = \mathsf{E}_\mathrm{b}(u) + \mathsf{E}_{\mathrm{s},\varepsilon}(U)$. The limit $\Xi \in \mathrm{L}^2(\Sigma_T)$ is to be determined. For the remaining cases $\alpha_\mathsf{R} < 1$ and $\alpha_\mathsf{R} > 1$ we have

$$\left.\begin{aligned} \mathsf{DE}_{\mathrm{s},\varepsilon}(U_\varepsilon) &\to 0 && \text{for } \alpha_\mathsf{R} < 1\\ \dot{U}_\varepsilon &\to 0 && \text{for } \alpha_\mathsf{R} > 1 \end{aligned}\right\} \text{ in } \mathrm{L}^2(\Sigma_T). \tag{8}$$

Obviously, the energy functionals E_ε blow up if the derivative with respect to θ does not vanish (for $\beta_\mathsf{E} > 1$ the same holds for the tangential derivatives). Thus, we expect the limit problems to be defined on the subspace of functions that are constant in normal direction (and tangential direction for $\beta_\mathsf{E} > 1$).

Let us consider the case $\beta_\mathsf{E} \ge 1$ first: We define the reduced spaces $\mathcal{V}_\mathrm{tang}$, $\mathcal{V}_\mathrm{const}$ and their closures in $\mathcal{H}$ via

$$\mathcal{V}_\mathrm{tang} := \{(u,U) \in \mathcal{V} : \partial_\theta U = 0 \text{ a.e. in } \Sigma\}, \quad \mathcal{H}_\mathrm{tang} := \overline{\mathcal{V}_\mathrm{tang}}^{\mathcal{H}},$$

$$\mathcal{V}_\mathrm{const} := \{(u,U) \in \mathcal{V} : U = \text{const a.e. in } \Sigma\}, \quad \mathcal{H}_\mathrm{const} := \overline{\mathcal{V}_\mathrm{const}}^{\mathcal{H}}.$$

In the following theorem we prove the Mosco convergence of the energy functionals E_ε for $\beta_\mathsf{E} \ge 1$ in $\mathcal{V}$.

Theorem 3.2. (Mosco convergence for $\beta_{\mathsf{E}} \geq 1$) *For* $\beta_{\mathsf{E}} = 1$ *the energy functionals* E_ε *converge in the sense of Mosco to the limit functional* $\mathsf{E}_{\mathrm{tang}} : \mathcal{V} \to \mathbb{R}_\infty$ *given by*

$$\mathsf{E}_{\mathrm{tang}}(\boldsymbol{u}) = \begin{cases} \mathsf{E}_{\mathrm{b}}(u) + \int_\Sigma \left[\frac{A_{\mathrm{s}}}{2}|\nabla_\Gamma U|^2 + W_{\mathrm{s}}(U)\right] \mathrm{d}\mu & \text{if } \boldsymbol{u} \in \mathcal{V}_{\mathrm{tang}}, \\ +\infty & \text{otherwise.} \end{cases}$$

For $\beta_{\mathsf{E}} > 1$ *the Mosco limit of* E_ε*, denoted* $\mathsf{E}_{\mathrm{const}}$*, is given by*

$$\mathsf{E}_{\mathrm{const}}(\boldsymbol{u}) = \begin{cases} \mathsf{E}_{\mathrm{b}}(u) + \int_\Sigma W_{\mathrm{s}}(U)\,\mathrm{d}\mu & \text{if } \boldsymbol{u} \in \mathcal{V}_{\mathrm{const}}, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Here we only consider the case $\beta_{\mathsf{E}} = 1$. The result for the other case follows analogously.

Liminf estimate for weak convergence. For all sequences $\boldsymbol{u}_\varepsilon = (u_\varepsilon, U_\varepsilon) \rightharpoonup \boldsymbol{u} = (u, U)$ in $\mathcal{V}$ we have to show $\mathsf{E}_{\mathrm{tang}}(\boldsymbol{u}) \leq \liminf_{\varepsilon\to 0} \mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon)$. Assuming that $\liminf_{\varepsilon\to 0} \mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) < \infty$ due to the weak lower semicontinuity of the norm on $\mathcal{V}$ we necessarily have that $\boldsymbol{u} \in \mathcal{V}_{\mathrm{tang}}$.

The compact embedding $\mathcal{V} \subset\subset \mathrm{L}^q(\Omega)\times\mathrm{L}^q(\Sigma)$, where $q \in [1, \infty[$ for $d = 2$ and $q < 2d/(d-2)$ otherwise, yields the strong convergence $(u_\varepsilon, U_\varepsilon) \to (u, U)$ in $\mathrm{L}^q(\Omega)\times\mathrm{L}^q(\Sigma)$. Thus, using the growth conditions for W_{b} and W_{s} we conclude that

$$\int_\Omega W_{\mathrm{b}}(u_\varepsilon)\,\mathrm{d}x \to \int_\Omega W_{\mathrm{b}}(u)\,\mathrm{d}x \quad \text{and} \quad \int_\Sigma W_{\mathrm{s}}(U_\varepsilon)\,\mathrm{d}\mu \to \int_\Sigma W_{\mathrm{s}}(U)\,\mathrm{d}\mu.$$

As before, we denote the bulk and surface energy parts of E_ε by E_{b} and $\mathsf{E}_{\mathrm{s},\varepsilon}$, such that $\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) = \mathsf{E}_{\mathrm{b}}(u_\varepsilon) + \mathsf{E}_{\mathrm{s},\varepsilon}(U_\varepsilon)$. It holds that

$$\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) \geq \mathsf{E}_{\mathrm{b}}(u_\varepsilon) + \int_\Sigma \left[\frac{A_{\mathrm{s}}}{2}\nabla_\Gamma U_\varepsilon \cdot \mathbb{B}_\varepsilon(y,\theta)\nabla_\Gamma U_\varepsilon + W_{\mathrm{s}}(U_\varepsilon)\right] \frac{\mathbb{J}_\varepsilon(y,\theta)}{\varepsilon}\,\mathrm{d}\mu.$$

Hence, by the uniform convergence of $\mathbb{B}_\varepsilon$ and $\mathbb{J}_\varepsilon/\varepsilon$ we obtain the lim inf estimate.

Limsup estimate for strongly converging recovery sequences. The construction of recovery sequences $\widehat{\boldsymbol{u}}_\varepsilon$ such that $\widehat{\boldsymbol{u}}_\varepsilon \to \boldsymbol{u}$ and $\mathsf{E}_\varepsilon(\widehat{\boldsymbol{u}}_\varepsilon) \to \mathsf{E}_{\mathrm{tang}}(\boldsymbol{u})$ is straightforward: For $\boldsymbol{u} \notin \mathcal{V}_{\mathrm{tang}}$ the result is trivial since $\mathsf{E}_{\mathrm{tang}}(\boldsymbol{u}) = \infty$ and we may take $\widehat{\boldsymbol{u}}_\varepsilon = \boldsymbol{u}$ and argue as in the first step.

For $\boldsymbol{u} \in \mathcal{V}_{\mathrm{tang}}$ we can choose the constant sequence $\widehat{\boldsymbol{u}}_\varepsilon = \boldsymbol{u}$ since the derivative with respect to θ does not appear in E_ε and we can conclude

$$\mathsf{E}_\varepsilon(\boldsymbol{u}) = \mathsf{E}_{\mathrm{b}}(u) + \int_\Sigma \left[\frac{A_{\mathrm{s}}}{2}\nabla_\Gamma U \cdot \mathbb{B}_\varepsilon(y,\theta)\nabla_\Gamma U + W_{\mathrm{s}}(U)\right] \frac{\mathbb{J}_\varepsilon(y,\theta)}{\varepsilon}\,\mathrm{d}\mu \to \mathsf{E}_{\mathrm{tang}}(\boldsymbol{u}),$$

where we used Lemma 2.2 again. □

The remaining case $\beta_{\mathsf{E}} \in]-1, 1[$ is more complicated since we lose the uniform coercivity of the energy functionals on $\mathcal{V}$. Hence, we have to work in the coarser topology of the bigger space $\mathcal{W}$ defined by

$$\mathcal{W} := \left\{(u, U) \in \mathrm{H}^1(\Omega)\times\mathrm{L}^2(\Sigma) : \partial_\theta U \in \mathrm{L}^2(\Sigma),\ u|_\Gamma = U|_{\{\theta=0\}}\right\}.$$

Let us point out here that the existence of the derivative with respect to θ in $\mathrm{L}^2(\Sigma)$ suffices for the well-definedness of the trace on Γ since for arbitrary $U \in \mathrm{C}^\infty(\Sigma)$ it holds that

$$\|U|_{\{\theta=0\}}\|_{\mathrm{L}^2(\Gamma)} \leq C\big(\|U\|_{\mathrm{L}^2(\Sigma)} + \|\partial_\theta U\|_{\mathrm{L}^2(\Sigma)}\big).$$

As before we introduce a reduced space of functions which are constant in normal direction

$$\mathcal{W}_{\text{nodiff}} := \{(u, U) \in \mathcal{W} \,:\, \partial_\theta U = 0 \text{ a.e. in } \Sigma\}.$$

Since the convergence of the surface variable U_ε is in general only weak in $\mathrm{L}^2(\Sigma)$ and the nonlinearity W_s is allowed to be nonconvex we have to replace W_s in the limit by its convex envelope, denoted W_s^{**} in the following (see, e.g. [6,11]).

Theorem 3.3. (Mosco convergence for $-1 < \beta_\mathsf{E} < 1$) *The energy functionals* E_ε Γ*-converge on* $\mathcal{W}$ *to the limit functional* $\mathsf{E}_{\text{nodiff}} : \mathcal{W} \to \mathbb{R}_\infty$ *given by*

$$\mathsf{E}_{\text{nodiff}}(\boldsymbol{u}) = \begin{cases} \mathsf{E}_\mathrm{b}(u) + \int_\Sigma W_\mathrm{s}^{**}(U)\,\mathrm{d}\mu & \text{if } \boldsymbol{u} \in \mathcal{W}_{\text{nodiff}}, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Liminf estimate for weak convergence. Let $\boldsymbol{u}_\varepsilon = (u_\varepsilon, U_\varepsilon) \rightharpoonup \boldsymbol{u} = (u, U)$ in $\mathcal{W}$. By arguing as in Theorem 3.2 we can assume that $\boldsymbol{u} \in \mathcal{W}_{\text{nodiff}}$ and $\sup_{0<\varepsilon<\varepsilon_0} \mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) < \infty$. We have the estimate

$$\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) \geq \mathsf{E}_\mathrm{b}(u_\varepsilon) + \int_\Sigma W_\mathrm{s}^{**}(U_\varepsilon)\frac{\mathbb{J}_\varepsilon(y,\theta)}{\varepsilon}\,\mathrm{d}\mu.$$

Applying $\liminf_{\varepsilon\to 0}$ to both sides of the estimate and using the uniform convergence of $\mathbb{J}_\varepsilon/\varepsilon$ and the weak lower semicontinuity of $U \mapsto \int_\Sigma W^{**}(U)\,\mathrm{d}\mu$ on $\mathrm{L}^2(\Sigma)$ we conclude that $\liminf_{\varepsilon\to 0} \mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) \geq \mathsf{E}_{\text{nodiff}}(\boldsymbol{u})$.

Limsup estimate for recovery sequences. Let $\boldsymbol{u} \in \mathcal{W}_{\text{nodiff}}$ be such that $\mathsf{E}_{\text{nodiff}}(\boldsymbol{u}) < \infty$. By the density of $\mathcal{V}_{\text{tang}}$ in $\mathcal{W}_{\text{nodiff}}$ we can find a sequence $(\widehat{\boldsymbol{u}}_\varepsilon)_{\varepsilon>0} \subset \mathcal{V}_{\text{tang}}$ such that $\widehat{\boldsymbol{u}}_\varepsilon \to \boldsymbol{u}$ (strongly) in $\mathcal{W}$ and $\varepsilon^\sigma\|\nabla_\Gamma \widehat{U}_\varepsilon\|^2_{\mathrm{L}^2(\Sigma)} \to 0$, where $\sigma = 1-\beta_\mathsf{E} \in]0,2[$. Since $\widehat{\boldsymbol{u}}_\varepsilon = (\widehat{u}_\varepsilon, \widehat{U}_\varepsilon)$ converges strongly in $\mathcal{W}$ we can extract a (not relabeled) sequence such that $\widehat{U}_\varepsilon(y,\theta) \to U(y,\theta)$ a.e. in Σ. Using Fatou's lemma we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathsf{E}_\varepsilon(\widehat{\boldsymbol{u}}_\varepsilon) &\leq \limsup_{\varepsilon\to 0}\left\{\mathsf{E}_\mathrm{b}(\widehat{u}_\varepsilon) + \int_\Sigma \left[C\varepsilon^\sigma|\nabla_\Gamma \widehat{U}_\varepsilon|^2 + W_\mathrm{s}(U_\varepsilon)\right]\frac{\mathbb{J}_\varepsilon(y,\theta)}{\varepsilon}\,\mathrm{d}\mu\right\} \\ &\leq \mathsf{E}_\mathrm{b}(u) + \int_\Sigma W_\mathrm{s}(U)\,\mathrm{d}\mu.\end{aligned}$$

The left-hand side, also known as Γ-limes superior (or upper Γ-limit), is weakly lower semicontinuous on $\mathcal{W}$ (see [6,11]). Hence, by taking the lower semicontinuous envelope on both sides we arrive at $\limsup_{\varepsilon\to 0} \mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) \leq \mathsf{E}_{\text{nodiff}}(\boldsymbol{u})$. □

Let us emphasize here that for W_s satisfying the growth condition (W_{Grow}) the directional derivative $\langle \mathsf{DE}_{\text{nodiff}}(\boldsymbol{u}), \boldsymbol{w}\rangle$ is in general not well-defined for $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{W}_{\text{nodiff}}$ since we do not have the embedding $\mathcal{W}_{\text{nodiff}} \subset$

$L^q(\Omega) \times L^q(\Sigma)$ for $1 \le q < \infty$ for $d = 2$ and $1 \le q < 2d/(d-2)$ for $d \ge 3$. Thus, we restrict ourselves to the case of a quadratic potential, such that $W_\mathrm{s}(U) = \frac{\omega_\mathrm{s}}{2}|U|^2$ with $\omega_\mathrm{s} > 0$. In this much simpler case the (strongly converging) recovery sequences are given by $\widehat{\boldsymbol{u}}_\varepsilon$ in the proof of Theorem 3.3. Hence, E_ε Mosco converges to $\mathsf{E}_\mathrm{nodiff}$ in $\mathcal{W}$.

The limits for the dissipation potential R_ε and the dual dissipation potentials R_ε^* for the cases $\alpha_\mathsf{R} = 1$, $\alpha_\mathsf{R} > 1$ and $\alpha_\mathsf{R} < 1$ are easily computed. Note that for the last two cases the uniform coercivity of R_ε^* and R_ε on $\mathcal{H}^*$ and $\mathcal{H}$, respectively, is lost.

For the nondegenerate case $\alpha_\mathsf{R} = 1$ we have the convergence

$$\mathsf{R}_\varepsilon \xrightarrow{\mathrm{M}} \mathsf{R}_\mathrm{dyn} \quad \text{with} \quad \mathsf{R}_\mathrm{dyn}(\dot{\boldsymbol{u}}) = \int_\Omega \frac{\tau_\mathrm{b}}{2}|\dot{u}|^2\,\mathrm{d}x + \int_\Sigma \frac{\tau_\mathrm{s}}{2}|\dot{U}|^2\,\mathrm{d}\mu$$

while for the other two cases (the slow and the fast evolution cases, see discussion in Sect. 3) it holds

$$\alpha_\mathsf{R} > 1: \ \mathsf{R}_\varepsilon(\dot{\boldsymbol{u}}) \to \mathsf{R}_\mathrm{slow}(\dot{\boldsymbol{u}}) \quad \text{with} \quad \mathsf{R}_\mathrm{slow}(\dot{u}, \dot{U}) = \begin{cases} \displaystyle\int_\Omega \frac{\tau_\mathrm{b}}{2}|\dot{u}|^2\,\mathrm{d}x & \text{if } \dot{U} = 0, \\ \infty & \text{else,} \end{cases}$$

$$\alpha_\mathsf{R} < 1: \ \mathsf{R}_\varepsilon(\dot{\boldsymbol{u}}) \to \mathsf{R}_\mathrm{fast}(\dot{\boldsymbol{u}}) \quad \text{with} \quad \mathsf{R}_\mathrm{fast}(\dot{u}, \dot{U}) = \int_\Omega \frac{\tau_\mathrm{b}}{2}|\dot{u}|^2\,\mathrm{d}x.$$

The Legendre transforms are easily computed as

$$\mathsf{R}^*_\mathrm{slow}(\xi, \Xi) = \int_\Omega \frac{\tau_\mathrm{b}^{-1}}{2}|\xi|^2\,\mathrm{d}x \quad \text{and} \quad \mathsf{R}^*_\mathrm{fast}(\xi, \Xi) = \begin{cases} \displaystyle\int_\Omega \frac{\tau_\mathrm{b}^{-1}}{2}|\xi|^2\,\mathrm{d}x & \text{if } \Xi = 0, \\ \infty & \text{else.} \end{cases}$$

We see that the limits for R_ε correspond to the observations made in Sect. 3. For $\alpha_\mathsf{R} > 1$ we obtain the static condition $\dot{U} = 0$, i.e., fixed (boundary) evolution. While for $\alpha_\mathsf{R} < 1$ the condition $\Xi = 0$ for the thermodynamically conjugated driving force means that the (boundary-)system is in equilibrium.

3.1. Passing to the limit in the energy balance ($\mathsf{EB}_{\boldsymbol{\varepsilon}}$)

In this subsection we focus on the energy balance formulation (EB_ε) and show that the limit $\boldsymbol{u} = (u, U)$ in (3)–(7) is a solution of the limit system $(\mathsf{E}_0, \mathsf{R}_0)$ with $\mathsf{E}_0 = \mathsf{E}_\mathrm{tang}, \mathsf{E}_\mathrm{const}, \mathsf{E}_\mathrm{nodiff}$ and $\mathsf{R}_0 = \mathsf{R}_\mathrm{slow}, \mathsf{R}_\mathrm{dyn}$. In particular, we do not treat the case $\mathsf{R}_0 = \mathsf{R}_\mathrm{fast}$ since in this limit case the chain rule is not available. Hence, the abstract framework discussed in Sect. 2.2 does not apply and we are not able to characterize the limit $\boldsymbol{u}$ as a solution of a corresponding force balance formulation, i.e., a system of partial differential equations. However, we show in the following subsection that for λ-convex energies the EVI-formulation can be used instead.

In particular, we show in this subsection that

$$\liminf_{\varepsilon \to 0} \left\{ \mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon(t)) + \int_0^t \left[\mathsf{R}_\varepsilon(\dot{\boldsymbol{u}}_\varepsilon) + \mathsf{R}_\varepsilon^*(-\mathrm{D}\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon))\right]\mathrm{d}s \right\}$$
$$\ge \mathsf{E}_0(\boldsymbol{u}(t)) + \int_0^t \left[\mathsf{R}_0(\dot{\boldsymbol{u}}) + \mathsf{R}_0^*(-\mathrm{D}\mathsf{E}_0(\boldsymbol{u}))\right]\mathrm{d}s.$$

Here and subsequently we use the the notation $\mathcal{V}_0 = \mathcal{V}_{\mathrm{tang}}, \mathcal{V}_{\mathrm{const}}$ and $\mathcal{W}_{\mathrm{nodiff}}$ when we refer to the domains of the corresponding limit energy functionals $\mathsf{E}_0 = \mathsf{E}_{\mathrm{tang}}$, etc. Note that the situation for $\mathsf{E}_{\mathrm{tang}}$ and $\mathsf{E}_{\mathrm{const}}$ is quite different from that for $\mathsf{E}_{\mathrm{nodiff}}$ due to the different underlying space, which fails to be compact in $\mathrm{L}^2(\Omega)$.

Remark 3.4. In order to pass to the limit we use the pointwise (in time) weak convergence of the solutions in the space $\mathcal{V}$ (resp., $\mathcal{W}$), i.e., $\boldsymbol{u}_\varepsilon(t) \rightharpoonup \boldsymbol{u}(t)$ in $\mathcal{V}$ (resp., $\mathcal{W}$). Indeed, let $\mathcal{V}_{\mathrm{weak}}$ denote the space $\mathcal{V}$ endowed with the weak topology then the continuous embedding $\mathrm{L}^\infty(0,T;\mathcal{V}) \cap \mathrm{H}^1(0,T;\mathcal{H}) \subset \mathrm{C}([0,T];\mathcal{V}_{\mathrm{weak}})$ (see, e.g. [31, Sect. 8.3]) implies that the weak* convergence $\boldsymbol{u}_\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $\mathrm{L}^\infty(0,T;\mathcal{V})\cap\mathrm{H}^1(0,T;\mathcal{H})$ implies $\boldsymbol{u}_\varepsilon(t) \rightharpoonup \boldsymbol{u}(t)$ in $\mathcal{V}$ (the same holds for $\mathcal{V}$ replaced by $\mathcal{W}$). This can be seen by means of a simple contradiction argument.

Following the ideas in [33] we define for a given curve $u_\varepsilon\colon [0,T] \to \mathcal{V}$ with $u_\varepsilon(t) \rightharpoonup u(t)$ in $\mathcal{V}$ (resp., in $\mathcal{W}$) the *energy excess* $\mathsf{D} : [0,T] \to [0,\infty]$ by

$$\mathsf{D}_\varepsilon(t) = \mathsf{E}_\varepsilon(u_\varepsilon(t)) - \mathsf{E}_0(u(t)), \quad \mathsf{D}(t) = \limsup_{\varepsilon\to 0} \mathsf{D}_\varepsilon(t) \geq 0.$$

We call u_ε *well-prepared initially* if $\mathsf{D}(0) = 0$.

The additional conditions for the convergence of the gradient flow given in [33] can be directly translated in our case to

1. (*Lower Bound*) There exists $f \in \mathrm{L}^1(0,T)$ such that for every $t \in [0,T]$
$$\liminf_{\varepsilon\to 0} \int_0^t \mathsf{R}_\varepsilon(\dot{\boldsymbol{u}}_\varepsilon)\,\mathrm{d}s \geq \int_0^t \big[\mathsf{R}_0(\dot{\boldsymbol{u}}) - f(s)\mathsf{D}(s)\big]\,\mathrm{d}s. \tag{9}$$
2. (*Construction*) There exists a locally bounded function g on $[0,T]$ such that for any $t_0 \in]0,T[$ and any smooth curve $\widehat{\boldsymbol{u}} :]t_0{-}\delta, t_0{+}\delta[\to \mathcal{V}_0$ satisfying $\widehat{\boldsymbol{u}}(t_0) = \boldsymbol{u}(t_0)$ there exists a $\widehat{\boldsymbol{u}}_\varepsilon :]t_0{-}\delta, t_0{+}\delta[\to \mathcal{V}$ such that $\widehat{\boldsymbol{u}}_\varepsilon(t_0) = \boldsymbol{u}_\varepsilon(t_0)$ and
$$\limsup_{\varepsilon\to 0} \mathsf{R}_\varepsilon\big(\dot{\widehat{\boldsymbol{u}}}_\varepsilon(t_0)\big) \leq \mathsf{R}_0\big(\dot{\widehat{\boldsymbol{u}}}(t_0)\big) + g(t_0)\mathsf{D}(t_0), \tag{10a}$$
$$\liminf_{\varepsilon\to 0} -\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}_\varepsilon(\widehat{\boldsymbol{u}}_\varepsilon)|_{t=t_0} \geq -\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}_0(\widehat{\boldsymbol{u}})|_{t=t_0} - g(t_0)\mathsf{D}(t_0). \tag{10b}$$

The energy excess D should be interpreted as a small perturbation. It is shown in [33] that $\mathsf{D} \equiv 0$ holds using Gronwall 's lemma. However, in the proof of the convergence result in Theorem 3.5 we show that one can actually take $f = g = 0$. While the first condition in (9) asks for a liminf estimate for the (integrated) dissipation potential R_ε the second condition in (10) can be interpreted as a liminf estimate for the dual dissipation potential along the derivative of the energy functionals. Indeed, adding (10a) to (10b) we arrive at the following

$$\begin{aligned}\liminf_{\varepsilon\to 0} \mathsf{R}_\varepsilon^*(-\mathrm{D}\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon)) &\geq \liminf_{\varepsilon\to 0} \Big[-\big\langle \mathrm{D}\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon), \dot{\widehat{\boldsymbol{u}}}_\varepsilon\big\rangle - \mathsf{R}_\varepsilon\big(\dot{\widehat{\boldsymbol{u}}}_\varepsilon\big)\Big] \\ &\geq -\big\langle \mathrm{D}\mathsf{E}_0(\boldsymbol{u}), \dot{\widehat{\boldsymbol{u}}}\big\rangle - \mathsf{R}_0\big(\dot{\widehat{\boldsymbol{u}}}\big) - 2g\mathsf{D}.\end{aligned}$$

Taking the supremum over all $\dot{\widehat{\boldsymbol{u}}}$ yields the limit dual dissipation potential $\mathsf{R}_{\rm dyn}(-\mathrm{D}\mathsf{E}_0(\boldsymbol{u}))$ at the lefthand side.

Let us point out that the limit system considered in [33] is finite dimensional. Therefore, we have to adapt the results for our purpose. In particular, we have to show that the Gâteaux derivative of the limit energy functional is well-defined in $\mathcal{H}$.

The main result for $\mathsf{E}_0 = \mathsf{E}_{\rm tang}, \mathsf{E}_{\rm const}$ and $\mathsf{E}_{\rm nodiff}$ and $\mathsf{R}_0 = \mathsf{R}_{\rm dyn}$ reads as follows:

Theorem 3.5. (Convergence of gradient flow for $\beta_{\mathsf{E}} > -1$ and $\alpha_{\mathsf{R}} = 1$) *Let $\boldsymbol{u}_\varepsilon$ be a family of solutions of the energy balance (EB_ε) converging as in (3)–(7) to a limit $\boldsymbol{u}$. If $\mathsf{D}(0) = 0$, i.e., $\boldsymbol{u}_\varepsilon$ is well prepared initially, then $\mathsf{D} \equiv 0$ on $[0, T]$ and $\boldsymbol{u}$ is the solution of the gradient flow for E_0 and $\mathsf{R}_{\rm dyn}$, i.e., it holds that*

$$\mathsf{E}_0(\boldsymbol{u}(t)) + \int_0^t \Big[\mathsf{R}_{\rm dyn}(\dot{\boldsymbol{u}}) + \mathsf{R}^*_{\rm dyn}\big(-\mathrm{D}\mathsf{E}_0(\boldsymbol{u})\big)\Big]\,\mathrm{d}s \le \mathsf{E}_0(\boldsymbol{u}(0)). \tag{11}$$

Proof. The weak convergence $\mathrm{D}\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) \rightharpoonup \boldsymbol{\xi} = (\mathrm{D}\mathsf{E}_{\rm b}(u), \Xi)$ in $\mathrm{L}^2(0, T; \mathcal{H}^*)$ implies that $\mathrm{D}\mathsf{E}_0(\boldsymbol{u}) \in \mathrm{L}^2(0, T; \mathcal{H}_0^*)$, where $\mathcal{H}_0 = \overline{\mathcal{V}_0}^{\mathcal{H}}$. Indeed, multiplying with a fixed $\widehat{\boldsymbol{u}} \in \mathrm{L}^2(0, T; \mathcal{V}_0 \cap \mathcal{V})$ leads to the convergence

$$\int_0^T \langle \mathrm{D}\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon), \widehat{\boldsymbol{u}}\rangle\,\mathrm{d}t \to \int_0^T \langle \mathrm{D}\mathsf{E}_0(\boldsymbol{u}), \widehat{\boldsymbol{u}}\rangle\,\mathrm{d}t = \int_0^T \langle \boldsymbol{\xi}, \widehat{\boldsymbol{u}}\rangle\,\mathrm{d}t.$$

Here we used the continuity properties of the associated Nemytskii operators $u \mapsto W_{\rm b}'(u)$ and $U \mapsto W_{\rm s}'(U)$, respectively (see [31]). The density of $\mathcal{V}_0 \cap \mathcal{V}$ in $\mathcal{H}_0$ yields now $\mathrm{D}\mathsf{E}_0(\boldsymbol{u}) \in \mathrm{L}^2(0, T; \mathcal{H}_0^*)$.

We see that $\widehat{\boldsymbol{u}} \in \mathrm{L}^2(0, T; \mathcal{V}_0 \cap \mathcal{V})$ satisfies the conditions (10a) and (10b): We easily check that $\int_0^t \mathsf{R}_\varepsilon(\widehat{\boldsymbol{u}})\,\mathrm{d}s \to \int_0^t \mathsf{R}_{\rm dyn}(\widehat{\boldsymbol{u}})\,\mathrm{d}s$ holds and conclude that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_0^t \mathsf{R}^*_\varepsilon(-\mathrm{D}\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon))\,\mathrm{d}s &\ge \liminf_{\varepsilon\to 0} \int_0^t \big[-\big\langle \mathrm{D}\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon), \widehat{\boldsymbol{u}}\big\rangle - \mathsf{R}_\varepsilon(\widehat{\boldsymbol{u}})\big]\,\mathrm{d}s \\ &= \int_0^t \big[-\big\langle \mathrm{D}\mathsf{E}_0(\boldsymbol{u}), \widehat{\boldsymbol{u}}\big\rangle - \mathsf{R}_{\rm dyn}(\widehat{\boldsymbol{u}})\big]\,\mathrm{d}s.\end{aligned}$$

Taking the supremum over all $\widehat{\boldsymbol{u}} \in \mathrm{L}^2(0, T; \mathcal{H}_0)$ we arrive at the liminf estimate for the dual dissipation along $\mathrm{D}\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon)$.

The Mosco convergence of the energy functionals and Remark 3.4 lead together with the liminf estimate for R_ε to the lower energy estimate

$$\mathsf{E}_0(\boldsymbol{u}(t)) + \int_0^t \mathsf{R}_{\rm dyn}(\dot{\boldsymbol{u}}) + \mathsf{R}^*_{\rm dyn}(-\mathrm{D}\mathsf{E}_0(\boldsymbol{u}))\,\mathrm{d}s \le \mathsf{E}_0(\boldsymbol{u}(0)),$$

which is actually an equality due to the chain rule for $t \mapsto \mathsf{E}_0(\boldsymbol{u}(t))$ and the characterization of the Legendre transform. □

The derivation of the corresponding energy balance for $\mathsf{R}_0 = \mathsf{R}_{\rm slow}$ is remarkably easier and follows by the same arguments as in the proof of Theorem 3.5 with $\widehat{\boldsymbol{u}} = 0$.

Theorem 3.6. (Convergence of gradient flow for $\beta_{\mathsf{E}} > -1$ and $\alpha_{\mathsf{R}} > 1$) *Let $\boldsymbol{u}_\varepsilon$ be a family of solutions of the energy balance (EB$_\varepsilon$) converging as in (3)–(7) to a limit $\boldsymbol{u}$. If $\mathsf{D}(0) = 0$ then $\mathsf{D} \equiv 0$ on $[0,T]$ and $\boldsymbol{u}$ is the solution of the gradient flow for E_0 and R_{slow}, i.e., it holds that*

$$\mathsf{E}_{\text{b}}(u(t)) + \int_0^t \mathsf{R}_{\text{b}}(\dot{u}) + \mathsf{R}_{\text{b}}^*(-\mathrm{D}\mathsf{E}_{\text{b}}(u))\,\mathrm{d}s = \mathsf{E}_{\text{b}}(u(0)),$$

where E_{b} and R_{b} denote the bulk part of the limit energy and dissipation potential, such that $\mathsf{E}_0(\boldsymbol{u}) = \mathsf{E}_{\text{b},0}(u) + \mathsf{E}_{\text{s},0}(U)$ and $\mathsf{R}_{\text{slow}}(\dot{\boldsymbol{u}}) = \mathsf{R}_{\text{b}}(\dot{u})$.

Remark 3.7. The well preparedness of the initial conditions $\boldsymbol{u}_\varepsilon(0)$ can be translated into asking that $\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon(0)) \to \mathsf{E}_0(\boldsymbol{u}(0))$, i.e., the initial energies converge.

3.2. Passing to the limit in the variational inequality (EVI$_\varepsilon$)

In order to derive limit systems for the case $\mathsf{R}_0 = \mathsf{R}_{\text{fast}}$ we turn to the evolution variational inequality (EVI$_\varepsilon$) which is an equivalent formulation in case of λ-convex energy functionals. It reads (integrated over time)

$$\int_0^T [\mathsf{E}_\varepsilon(\boldsymbol{u}_\varepsilon) + \langle \mathsf{G}_\varepsilon \dot{\boldsymbol{u}}_\varepsilon, \boldsymbol{u}_\varepsilon - \widetilde{\boldsymbol{u}} \rangle]\,\mathrm{d}t \le \int_0^T [\mathsf{E}_\varepsilon(\widetilde{\boldsymbol{u}}) - \boldsymbol{\Lambda}_\varepsilon(\boldsymbol{u}_\varepsilon - \widetilde{\boldsymbol{u}})]\,\mathrm{d}t \tag{12}$$

for all $\widetilde{\boldsymbol{u}} \in \mathrm{L}^2(0,T;\mathcal{V})$. Note that we consider here the time-integrated version of (EVI$_\varepsilon$). This is due to the fact that we have no estimates for the time derivative of the surface variable U. Hence, we cannot argue with pointwise in time convergence of the solution.

However, working with the integrated inequality bears problems since the Γ-convergence of the time-integrated functionals is in general not trivial. We refer to [32,38] for the following result.

Proposition 3.8. *Let F_ε denote a sequence of weakly lower semicontinuous functionals on a reflexive and separable Banach space $\mathcal{X}$ satisfying the liminf estimate for the weak convergence in $\mathcal{X}$. Moreover, let $w_\varepsilon \rightharpoonup w$ (weakly-* if $p = \infty$) in $\mathrm{L}^p(0,T;\mathcal{X})$. Then, it holds that*

$$\int_0^T \mathsf{F}_0(w(t))\,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^T \mathsf{F}_\varepsilon(w_\varepsilon(t))\,\mathrm{d}t.$$

The main result for the case $\mathsf{R}_0 = \mathsf{R}_{\text{fast}}$ reads as follows

Theorem 3.9. (Convergence of gradient flow for $\beta_{\mathsf{E}} > -1$ and $\alpha_{\mathsf{R}} < 1$) *Let $\boldsymbol{u}_\varepsilon$ be a family of solutions of the evolution variational inequality (12) converging as in (3)–(7) to the limit $\boldsymbol{u}$. Then, $\boldsymbol{u}$ is the solution of the following evolution variational inequality for E_0 and R_{fast}*

$$\int_0^T \mathsf{E}_0(\boldsymbol{u})\,\mathrm{d}t + \int_0^T \int_\Omega \tau_{\text{b}} \dot{u}(u - \widetilde{u})\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T [\mathsf{E}_0(\widetilde{\boldsymbol{u}}) - \boldsymbol{\Lambda}(\boldsymbol{u} - \widetilde{\boldsymbol{u}})]\,\mathrm{d}t \tag{13}$$

for all $\widetilde{\boldsymbol{u}} \in \mathrm{L}^2(0,T;\mathcal{V}_0)$, where $\boldsymbol{\Lambda}(\boldsymbol{u}) = \int_\Omega \frac{\lambda_{\text{b}}}{2}|u|^2\,\mathrm{d}x + \int_\Sigma \frac{\lambda_{\text{s}}}{2}|U|^2\,\mathrm{d}\mu$.

Proof. Let $\widetilde{\boldsymbol{u}} = (\widetilde{u}, \widetilde{U}) \in \mathrm{L}^2(0,T;\mathcal{V}_0 \cap \mathcal{V})$. It is easy to check that $\int_0^T \mathsf{E}_\varepsilon(\widetilde{\boldsymbol{u}})\,\mathrm{d}t \to \int_0^T \mathsf{E}_0(\widetilde{\boldsymbol{u}})\,\mathrm{d}t$. Moreover, from the estimates in Lemma 3.1 we infer that $\dot{u}_\varepsilon \rightharpoonup \dot{u}$ in $\mathrm{L}^2([0,T]\times\Omega)$ and $\varepsilon^{1-\alpha_{\mathsf{R}}}\dot{U}_\varepsilon \to 0$ in $\mathrm{L}^2([0,T]\times\Sigma)$. Hence, we have that

$$\int_0^T \langle \mathsf{G}_\varepsilon \dot{\boldsymbol{u}}_\varepsilon, \boldsymbol{u}_\varepsilon - \widetilde{\boldsymbol{u}} \rangle \, \mathrm{d}t = \int_0^T \int_\Omega \tau_\mathrm{b} \dot{u}_\varepsilon (u_\varepsilon - \widetilde{u}) \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Sigma \tau_\mathrm{s} \varepsilon^{1-\alpha_\mathsf{R}} \dot{U}_\varepsilon (U_\varepsilon - \widetilde{U}) \frac{\mathbb{J}_\varepsilon}{\varepsilon} \, \mathrm{d}\mu \, \mathrm{d}t \to \int_0^T \int_\Omega \tau_\mathrm{b} \dot{u}(u - \widetilde{u}) \, \mathrm{d}x \, \mathrm{d}t.$$

Thus, applying liminf to (12) and using Proposition 3.8 we obtain (13). □

4. Discussion of the limit models

In this section we show that the limit models obtained in Sect. 3 can be reduced to a real bulk–surface evolutionary system in $\overline{\Omega}$. The main observation is that for a pair (u, U) in $\mathcal{V}_\mathrm{tang}, \mathcal{V}_\mathrm{const}$ or $\mathcal{W}_\mathrm{nodiff}$ we can characterize U by a function defined only on the boundary $\Gamma = \partial\Omega$. More precisely, these spaces are isomorph to the spaces V_tang, V_const and W_nodiff given by (Table 1)

$$\begin{aligned} V_\mathrm{tang} &:= \big\{ (u,U) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Gamma) \, : \, u|_\Gamma = U \big\}, \\ V_\mathrm{const} &:= \big\{ (u,U) \in \mathrm{H}^1(\Omega) \times \mathbb{R}^{N_\Gamma} \, : \, u|_{\Gamma_i} = U^i, \ i = 1, \ldots, N_\Gamma \big\}, \\ W_\mathrm{nodiff} &:= \big\{ (u,U) \in \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Gamma) \, : \, u|_\Gamma = U \big\} \end{aligned}$$

where $N_\Gamma \in \mathbb{N}$ is the number of connected components $\Gamma_i \subset \Gamma$. We denote by H_tang, H_const and H_nodiff the closures of the spaces above with respect to the L^2-norm, such that

$$H_\mathrm{tang} = H_\mathrm{nodiff} = \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma) \quad \text{and} \quad H_\mathrm{const} = \mathrm{L}^2(\Omega) \times \mathbb{R}^{N_\Gamma}.$$

With these characterizations the energy functionals E_tang and $\mathsf{E}_\mathrm{nodiff}$ can be reduced by integration over the variable $\theta \in]0,1[$ while for $\mathsf{E}_\mathrm{const}$ we integrate over y as well. The reduced energy functionals, denoted $\mathcal{E}_\mathrm{tang}, \mathcal{E}_\mathrm{const}$ and $\mathcal{E}_\mathrm{nodiff}$ are then given by

$$\begin{aligned} \mathcal{E}_\mathrm{tang}(u,U) &:= \mathcal{E}_\mathrm{b}(u) + \int_\Gamma \left[\frac{A_\mathrm{s}}{2} |\nabla_\Gamma U|^2 + W_\mathrm{s}(U) \right] \mathrm{d}\Gamma, \\ \mathcal{E}_\mathrm{const}(u,U) &:= \mathcal{E}_\mathrm{b}(u) + |\Gamma| \sum_{i=1}^{N_\Gamma} W_\mathrm{s}(U^i), \end{aligned}$$

Table 1. Boundary conditions of the limit system for the different parameter regimes of α_R and β_E with all constants set to 1

	$\alpha_\mathsf{R} < 1$	$\alpha_\mathsf{R} = 1$	$\alpha_\mathsf{R} > 1$
$\lvert\beta_\mathsf{E}\rvert < 1$	$\frac{\partial u}{\partial \nu} + U = 0$	$\dot{U} + \frac{\partial u}{\partial \nu} + U = 0$	$u\lvert_\Gamma = u_\mathrm{D} \in \mathrm{H}^{1/2}(\partial\Omega)$
$\beta_\mathsf{E} = 1$	$\frac{\partial u}{\partial \nu} + W_\mathrm{s}'(U) = \Delta_\Gamma U$	$\dot{U} + \frac{\partial u}{\partial \nu} + W_\mathrm{s}'(U) = \Delta_\Gamma U$	$u\lvert_\Gamma = u_\mathrm{D} \in \mathrm{H}^1(\partial\Omega)$
$\beta_\mathsf{E} > 1$	$\left[\frac{\partial u}{\partial \nu}\right]_i + W_\mathrm{s}'(U^i) = 0$	$\dot{U}^i + \left[\frac{\partial u}{\partial \nu}\right]_i + W_\mathrm{s}(U^i) = 0$	$u\lvert_\Gamma = \text{const on } \Gamma$

$$\mathcal{E}_{\rm nodiff}(u,U) := \mathcal{E}_{\rm b}(u) + \frac{\omega_{\rm s}}{2}\|U\|^2_{\mathrm{L}^2(\Gamma)},$$

where in each case $\mathcal{E}_{\rm b}(u) = \int_\Omega [\frac{A_{\rm b}}{2}|\nabla u|^2 + W_{\rm b}(u)]\,\mathrm{d}x$ denotes the bulk energy.

Starting with the case $\alpha_{\mathsf{R}} = 1$ we see that the limit energy balance in (11) can be written in terms of $\mathcal{E}_0 \in \{\mathcal{E}_{\rm tang}, \mathcal{E}_{\rm const}, \mathcal{E}_{\rm nodiff}\}$ and the dissipation potential $\mathcal{R}_{\rm dyn}$. Here, in slight abuse of notation, $\mathcal{R}_{\rm dyn}$ is for each of the energy functionals $\mathcal{E}_{\rm tang}$, $\mathcal{E}_{\rm const}$ and $\mathcal{E}_{\rm nodiff}$ defined on the spaces $H_{\rm tang}$, $H_{\rm const}$ and $H_{\rm nodiff}$ and obtained as before via integration with respect to the variable θ or (y,θ), respectively. Thus, the reduced energy balance reads

$$\mathcal{E}_0(u(t),U(t)) + \int_0^t \mathcal{R}_{\rm dyn}(\dot u,\dot U) + \mathcal{R}^*_{\rm dyn}(-\mathrm{D}\mathcal{E}_0(u,U))\,\mathrm{d}s = \mathcal{E}_0(u(0),U(0)).$$

To highlight the structure of the limit systems we now write down the corresponding force balance equation written in terms of the reduced energy and dissipation functional. It consists of two equations for the bulk and the surface variable u and $U = u|_\Gamma$, respectively. Using the chain rule and the Fenchel equivalences we obtain

$$\begin{pmatrix} \tau_{\rm b}\dot u + \mathrm{D}_u\mathcal{E}_0(u,U) \\ \tau_{\rm s}\dot U + \mathrm{D}_U\mathcal{E}_0(u,U)\end{pmatrix} = 0.$$

For each of the energy functionals the first equation is formally equivalent to the well-known Allen–Cahn equation in $[0,T]\times\Omega$

$$\tau_{\rm b}\partial_t u - A_{\rm b}\Delta u + W'_{\rm b}(u) = 0. \qquad (\mathrm{AC}_{\rm bulk})$$

This equation is coupled to the boundary evolution of $u|_\Gamma = U$, which for the energy functional $\mathcal{E}_{\rm tang}$ (limit case for $\beta_{\mathsf{E}} = 1$) is described by

$$\tau_{\rm s}\partial_t U - A_{\rm s}\Delta_\Gamma U + A_{\rm b}\tfrac{\partial u}{\partial\nu} + W'_{\rm s}(U) = 0. \qquad (14)$$

Hence, we obtain the surface Allen–Cahn equation with a contribution given by the conormal derivative of the bulk variable u. The system ($\mathrm{AC}_{\rm bulk}$) and (14) was studied in [37].

In the limit case for $\beta_{\mathsf{E}} > 1$ the limit energy functional is given by $\mathcal{E}_{\rm const}$ and we obtain a simpler boundary condition, which consists of a system of ordinary differential equations for each of the connected components Γ_i of the boundary Γ, namely

$$\tau_{\rm s}\partial_t U^i + A_{\rm b}\big[\tfrac{\partial u}{\partial\nu}\big]_i + W'_{\rm s}(U^i) = 0, \qquad (15)$$

where $[g]_i := \frac{1}{|\Gamma_i|}\int_{\Gamma_i} g\,\mathrm{d}\Gamma$ denotes the mean value of $g:\Gamma_i\to\mathbb{R}$ over $\Gamma_i\subset\Gamma$.

Finally, for $\mathcal{E}_0 = \mathcal{E}_{\rm nodiff}$ $(-1<\beta_{\mathsf{E}}<1)$ the boundary condition reads

$$\tau_{\rm s}\partial_t U + A_{\rm b}\tfrac{\partial u}{\partial\nu} + \omega_{\rm s}U = 0. \qquad (16)$$

This boundary condition can be found as a special case in [27].

In the case $\alpha_{\mathsf{R}} > 1$ ($\mathsf{R}_0 = \mathsf{R}_{\rm slow}$) we obtain the bulk Allen–Cahn equation ($\mathrm{AC}_{\rm bulk}$) and have no evolution on the boundary, i.e. $\dot U = 0$. Which means that the boundary values are fixed by the initial conditions. Since we assumed in the convergence analysis that the initial energies converge, the initial values $(u(0),U(0))$ have to lie in the spaces $V_{\rm tang}$, $V_{\rm const}$ and $V_{\rm nodiff}$ for $\beta_{\mathsf{E}} = 1$,

$\beta_{\mathsf{E}} > 1$ and $-1 < \beta_{\mathsf{E}} < 1$, respectively. In particular, in the first case we have $u|_\Gamma = u|_\Gamma(0) \in \mathrm{H}^1(\Gamma)$, while in the second case the boundary values are constant (on each connected component) and in the last case we have $u|_\Gamma = u(0)|_\Gamma \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$.

At last we discuss the fast evolution case $\alpha_{\mathsf{R}} < 1$ ($\mathsf{R}_0 = \mathsf{R}_{\text{fast}}$). Choosing $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - h\boldsymbol{w}$, $h > 0$ in the limit evolution variational inequality (13) and letting $h \to 0$ we obtain the system

$$\begin{pmatrix} \tau_{\mathrm{b}}\dot{u} + \mathrm{D}_u\mathcal{E}_0(u,U) \\ \mathrm{D}_U\mathcal{E}_0(u,U) \end{pmatrix} = 0.$$

Hence, for $\beta_{\mathsf{E}} = 1$ the limit energy functional is given by $\mathcal{E}_{\text{tang}}$ and we deduce that the bulk equation (AC_{bulk}) is in this case coupled to the nonlinear elliptic surface equation

$$-A_{\mathrm{s}}\Delta_\Gamma U + A_{\mathrm{b}}\frac{\partial u}{\partial \nu} + W_{\mathrm{s}}'(U) = 0. \tag{17}$$

While for $\mathcal{E}_0 = \mathcal{E}_{\text{const}}$ ($\beta_{\mathsf{E}} > 1$) we have the following nonlinear equation for each connected component of the boundary Γ

$$A_{\mathrm{b}}\left[\frac{\partial u}{\partial \nu}\right]_i + W_{\mathrm{s}}'(U^i) = 0. \tag{18}$$

In the last case $-1 < \beta_{\mathsf{E}} < 1$ and therefore $\mathcal{E}_0 = \mathcal{E}_{\text{nodiff}}$ we obtain the usual Robin boundary condition

$$A_{\mathrm{b}}\frac{\partial u}{\partial \nu} + \omega_{\mathrm{s}}U = 0. \tag{19}$$

Figure 2 shows details of numerical simulations of equation (AC_{bulk}) in a (polygonal approximation of a) circular domain using continuous piecewise

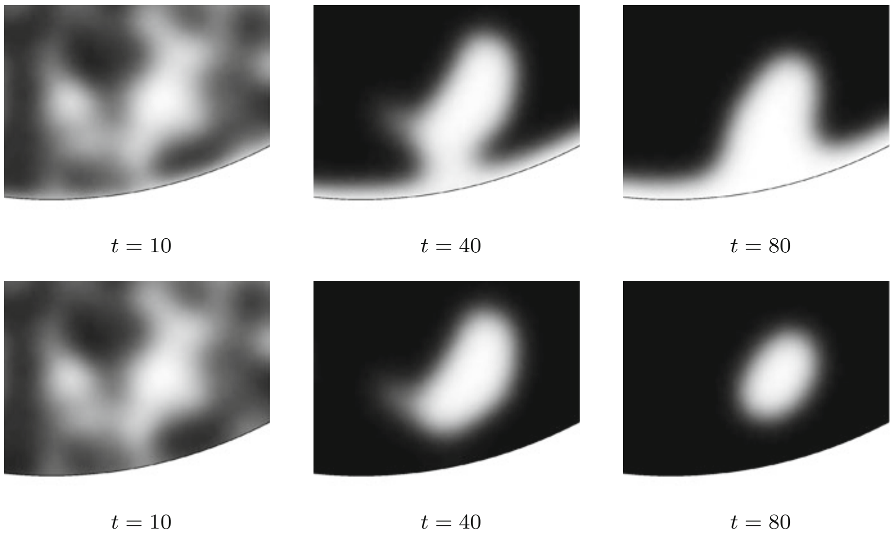

FIGURE 2. Detail of two solutions of Eq. (AC_{bulk}) near boundary with dynamic boundary condition (14) (*top*) and Neumann boundary condition (*bottom*) for subsequent times

affine finite elements in the bulk and on the surface (see, e.g. [14]). In particular, the behavior of two solutions of (AC_{bulk}) near the boundary Γ is depicted for subsequent times in case of the dynamical boundary condition (14) and the Neumann boundary $\frac{\partial u}{\partial \nu} = 0$. Here, both solutions are starting from the same initial condition. The potential in the bulk is given by the double-well potential $W_{\mathrm{b}}(u) = \frac{k_{\mathrm{b}}}{4}(1-u^2)^2$ with $k_{\mathrm{b}} > 0$ while for the dynamic boundary condition we additionally have the quadratic potential $W_{\mathrm{s}}(U) = \frac{k_{\mathrm{s}}}{2}(1-U)^2$ on Γ with $k_{\mathrm{s}} > 0$. The dynamic boundary condition models a strong interaction between the wall and the mixture components described by the order parameter u (resp., U). In particular, due to the potential W_{s} and the surface diffusion we have an accumulation of the phase $U = 1$ at the boundary Γ, which can be clearly seen in the pictures.

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