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Local existence and uniqueness in the largest critical space for a surface growth model

Dirk Blömker and Marco Romito

Abstract. We show the existence and uniqueness of solutions (either local or global for small data) for an equation arising in different aspects of surface growth. Following the work of Koch and Tataru we consider spaces critical with respect to scaling and we prove our results in the largest possible critical space such that weak solutions are defined, which turns out to be a Besov space. Similarly to 3D-Navier Stokes, the uniqueness of global weak solutions remains unfortunately open, unless the initial conditions are sufficiently small.

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1. Introduction

The analysis of mathematical models for the study of surface growth has attracted a lot of attention in recent years, one can see for example the reviews in [1, 11] and numerous recent publications [7, 9, 19, 20, 23–25].

In this article we consider a model arising in the growth of amorphous surfaces which is described by the following partial differential equation,

$$\partial_t h + \Delta^2 h + \Delta|\nabla h|^2 = 0. \tag{1.1}$$

on the whole $\mathbf{R}^d$ or with periodic boundary conditions. The function $h(t,\cdot)$ models a height profile at time $t > 0$, so $d = 1$ and $d = 2$ are the physically relevant dimensions. In view of this and of Proposition 2.2 we will restrict

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the analysis to the case $d \leq 3$ throughout this paper (although most of the computation holds without restrictions on the dimension).

Equation (1.1), which is sometimes referred to as a conservative version of the Kuramoto-Sivashinsky equation, arises also in several other models for surface growth. The two-dimensional version was suggested in [23–25] as a phenomenological model for the growth of an amorphous surface ($Zr_{65}Al_{7,5}Cu_{27,5}$) and more recently as a model in surface erosion using ion-beam sputtering [7,19,20]. The one-dimensional equation appeared as a model for the boundaries of terraces in the epitaxy of Silicon [9].

For simplicity of presentation we consider the rescaled version (1.1) with a-dimensional length-scales. Furthermore, we have ignored lower order terms like the Kuramoto-Sivashinsky term $-|\nabla h|^2$ or a linear instability given by $+\Delta h$. These terms can easily be incorporated in the result.

In the physical literature equation (1.1) is usually subject to space–time white noise, which we also have neglected for simplicity of presentation. Indeed, using the standard method of looking at the difference between h and the stochastic convolution, the stochastic PDE can be transformed into a random PDE. If the stochastic convolution is sufficiently regular, then for each instance of chance the path-wise solvability for the stochastic PDE is completely analogous to the results presented here and one only needs to consider additional lower order terms. This will be done with more details later in Sect. 5.

A crucial open problem for equation (1.1) is the fact that the uniqueness of global solutions is not known. We remark that numerical experiments do not report any problems of blow up, see Hoppe and Nash [13,14], or the previously stated physics literature. Numerical experiments from Blömker, Gugg and Raible [4] furthermore indicate a fast convergence of spectral Galerkin methods for averaged surface roughness for the stochastic PDE.

The existence of global weak solutions in dimension $d = 1$ on bounded domains has been studied in [4] (see also the references therein), based on spectral Galerkin methods. The crucial estimates are energy-type inequalities which allow for uniform bounds on the L^2-norm. The method has been significantly extended by Blömker, Flandoli and Romito [2] in order to verify the existence of a solution that defines a Markov process. Winkler and Stein [27] used Rothe's method to verify the existence of a global weak solution, this result has been recently extended by Winkler [30] to the two-dimensional case, using energy type estimates for $\int e^h \, dx$.

The authors have showed in [5] the uniqueness of local solutions with initial values in the critical Hilbert space $H^{1/2}$ in the one dimensional case. Local uniqueness of continuous solutions in $W^{1,4}$ for the stochastic PDE in dimension $d = 1, 2$ can be found in [3]. The existence and uniqueness of a regularized problem with a cut-off in the nonlinearity in dimension $d = 2$ has been studied in Hoppe, Linz and Litvinov [12].

In this paper, following the technique introduced in the remarkable paper by Koch and Tataru [15] on the Navier–Stokes equations, we study existence and uniqueness of solutions with initial data in the largest possible critical space where weak solutions make sense. This space contains all previous spaces where analogous results were proved. Let us remark that the same method has been applied to other fourth order problems [16, 29].

Our main motivation to study (1.1) in the largest possible space actually comes from the stochastically perturbed equation, where, due to noise, solutions are not regular enough. For instance, in dimension $d = 2$, for space–time white noise, due to the nonlinearity neither the uniqueness nor the existence of global solutions are known yet.

Our *main results* first show that the largest critical space introduced by Koch and Tataru is in our setting and at least for dimension $d \leq 3$ equivalent to the homogeneous Besov space $\dot{B}^{0,\infty}_{\infty}(\mathbf{R}^d)$. This space is much simpler to work with, and this simplifies our result on existence and uniqueness in that space, which classically holds for small data or for short times (see Theorem 4.3). Moreover, all such solutions are smooth in the space variable (Theorem 6.1).

The paper is organized as follows. In Sect. 2 we discuss the space of initial conditions according to the ideas of [15] and we show that, in the physically relevant case of small dimension, it coincides with the homogeneous Besov space $\dot{B}^{0,\infty}_{\infty}(\mathbf{R}^d)$. Some admissible initial conditions and examples are discussed in Sect. 3. Based on Banach's fixed-point iteration scheme, Sect. 4 provides the existence and uniqueness results. Section 5 contains some details on the extension of such results to the stochastically forced case. We close the paper with Sect. 6, where we show smoothness of solutions.

2. Function spaces

Recall first the following result, an easy consequence of Poincaré's inequality, which ensures that all integrals in (2.1) are well defined.

Lemma 2.1. *If u is a distribution on $\mathbf{R}^d$ such that $\nabla u \in L^2_{\text{loc}}([0,\infty) \times \mathbf{R}^d)$, then $u \in L^2_{\text{loc}}([0,\infty) \times \mathbf{R}^d)$ and thus $u \in L^2_{\text{loc}}([0,\infty), H^1_{\text{loc}}(\mathbf{R}^d))$.*

A weak solution for (1.1) with initial condition $h_0 \in L^1_{\text{loc}}(\mathbf{R}^d)$ is any distribution h on $\mathbf{R}^d$ with locally square integrable gradient $\nabla h \in L^2_{\text{loc}}([0,\infty) \times \mathbf{R}^d)$ such that for every smooth and compactly supported function $\phi \in C^\infty_c([0,\infty) \times \mathbf{R}^d)$,

$$\int_0^\infty \int_{\mathbf{R}^d} h(t,x) \frac{\partial \phi}{\partial t}(t,x)\,dx\,dt - \int_0^\infty \int_{\mathbf{R}^d} h(t,x) \Delta^2 \phi(t,x)\,dx\,dt + \\ - \int_0^\infty \int_{\mathbf{R}^d} |\nabla h(t,x)|^2 \Delta \phi(t,x)\,dx\,dt = - \int_{\mathbf{R}^d} h_0(x) \phi(0,x)\,dx. \qquad (2.1)$$

Lemma 2.1 ensures that all terms in the formula above are well defined. Moreover, the solution is only defined up to constants. As the equation is translation invariant (in space) and invariant with respect to the scaling

$$h(t,x) \longrightarrow h(\lambda^4 t, \lambda x), \qquad (2.2)$$

we consider the scaling-aware invariant version of the L^2_{loc} space for the gradient ∇h, defined by the following norm.

$$\|h\|_{\mathcal{X}^0} := \left(\sup_{x\in\mathbf{R}^d, R>0} \left\{ \frac{1}{R^{d+2}} \int_0^{R^4} \int_{B_R(x)} |\nabla h|^2 \, dy \, dt \right\} \right)^{\frac{1}{2}} .$$

This definition follows the ideas of [15]. Our aim in this section is to prove a simpler representation of the bi-caloric extension in this space.

Define therefore the linear space $\mathcal{X}$ of functions such that the following quantity is finite,

$$\|k\|_{\mathcal{X}} = \sup_{t>0} \{ t^{\frac{1}{4}} \|\nabla k(t)\|_\infty \}.$$

A local in time version of this space can be defined for any $R > 0$ by

$$\|k\|_{\mathcal{X}_R} := \sup_{t\leq R^4} (t^{\frac{1}{4}} \|\nabla k(t)\|_\infty).$$

for functions $k : [0, R^4] \times \mathbf{R}^d \to \mathbf{R}$. Note that we always identify functions that differ only by a constant. This is motivated by the fact that the equation is mass-conservative, if the total mass $\int h \, dx$ is finite.

In order to track the corresponding spaces for initial values, let $A = \Delta^2$. Consider the Green's function $G : [0, \infty) \times \mathbf{R}^d \to \mathbf{R}$ associated to the operator A, where $G(t, x)$ has the Fourier transform (with respect to the space variable) $\widehat{G}(t, \xi) = \mathrm{e}^{-t|\xi|^4}$. By scaling we obtain

$$G(t, x) = t^{-d/4} g(xt^{-1/4}), \quad \text{where } g(x) = G(1, x).$$

The function g is in the Schwartz class since $\widehat{g}(\xi) = \mathrm{e}^{-|\xi|^4}$ (see [16] for decay properties in the x variable).

Define the semigroup e^{-tA} by the convolution $\mathrm{e}^{-tA} k = G(t, \cdot) \star k$, which is also denoted as the *bi-caloric* extension of k.

Denote by $\mathcal{B}$ the homogeneous Besov space $\dot{B}^{0,\infty}_{\infty}(\mathbf{R}^d)$, and note that an equivalent norm on $\mathcal{B}$ is given by

$$\|k\|_{\mathcal{B}} := \| \mathrm{e}^{-tA} k\|_{\mathcal{X}}$$

(see for instance [17] for a characterization of Besov spaces in terms of heat kernels. Here we use the bi-Laplace operator A, which changes the scaling of time). Define similarly the local version $\mathcal{B}_R$.

In contrast to the case of Navier–Stokes in dimension three [15], here for low dimension the problem is easier, since their space coincides with $\mathcal{B}$, as shown by the proposition stated below. This proposition is the only reason why we restrict to dimension $d \leq 3$, as we rely for simplicity on the simpler structure of $\mathcal{B}$.

Proposition 2.2. *Assume $d \leq 3$. Then $e^{tA}k \in \mathcal{X}^0$ if and only if $k \in \mathcal{B}$ and the norm in $\mathcal{B}$ is equivalent to the norm defined by the bi-caloric extension in $\mathcal{X}^0$.*

This means, that there are constants $c_1, c_2 > 0$ such that

$$c_1 \|k\|_{\mathcal{B}} \leq \|e^{tA}k\|_{\mathcal{X}^0} \leq c_2 \|k\|_{\mathcal{B}} . \tag{2.3}$$

Proof. We start by proving (2.3). The inequality on the right holds in any dimension $d \geq 1$ since it is straightforward to check that there is $c > 0$ such that $\|\cdot\|_{\mathcal{X}^0} \leq c\|\cdot\|_{\mathcal{X}}$. For the inequality on the left, we need to show that for $k \in \mathcal{B}^0, x \in \mathbf{R}^d, t > 0$,

$$|t^{\frac{1}{4}} \nabla(\mathrm{e}^{-tA} k)(x)| = t^{\frac{1}{4}} \left| \nabla \int_{\mathbf{R}^d} G(t, x-y) k(y)\, dy \right| \leq c\|k\|_{\mathcal{B}^0}.$$

By scaling and translations invariance, it is sufficient to show the statement for $t = 1$ and $x = 0$. Since

$$\mathrm{e}^{-A} = \int_0^1 \mathrm{e}^{-(1-s)A}\, \mathrm{e}^{-sA}\, ds,$$

it follows by the Cauchy–Schwartz inequality that

$$\begin{aligned}
|\nabla(\mathrm{e}^{-A} k)(0)| &\leq \sum_{n\in\mathbf{Z}^d} \left| \int_0^1 (1-s)^{-\frac{d}{4}} \int_{B_n} g(y(1-s)^{-\frac{1}{4}}) \nabla(\mathrm{e}^{-sA} k)(y)\, dy\, ds \right| \\
&\leq \sum_{n\in\mathbf{Z}^d} \left(\int_0^1 (1-s)^{-\frac{d}{2}} \int_{B_n} g(y(1-s)^{-\frac{1}{4}})^2\, dy\, ds \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^1 \int_{B_n} |\nabla(\mathrm{e}^{-sA} k)(y)|^2\, dy\, ds \right)^{\frac{1}{2}} \\
&\leq c\|k\|_{\mathcal{B}^0} \sum_{n\in\mathbf{Z}^d} \left(\int_0^1 s^{-\frac{d}{2}} \int_{B_n} g(ys^{-\frac{1}{4}})^2\, dy\, ds \right)^{\frac{1}{2}},
\end{aligned}$$

where B_n are the balls of centre $2d^{-1/2}n$ and radius 1 (so that their union covers $\mathbf{R}^d$). By a change of variables,

$$I_n := \int_0^1 s^{-\frac{d}{2}} \int_{B_n} g(ys^{-\frac{1}{4}})^2\, dy\, ds = \int_0^1 s^{-\frac{d}{4}} \int_{s^{-1/4}B_n} g(z)^2\, dz\, ds.$$

First, $|I_n| \leq C$ for all $n \in \mathbf{Z}^d$, as $d \leq 3$ and $g \in L^2(\mathbf{R}^d)$. Note that $d \leq 3$ is necessary, as for $0 \in B_n$ we have $\int_{s^{-1/4}B_n} g(z)^2\, dz \uparrow \|g\|_{L^2}^2$ for $s \downarrow 0$.

For the convergence of the series consider for $s \in (0,1)$ and $0 \notin B_n$ (i.e. $2|n| > \sqrt{d}$) that

$$\begin{aligned}
\int_{s^{-1/4}B_n} g(z)^2\, dz &\leq \int_{\mathbf{R}^d} |g(z)|\, dz \cdot \sup\{|g(z)| \ : \ |z| \in s^{-1/4}B_n\} \\
&\leq C \sup\{|g(z)| \ : \ |z| > 2|n|d^{-1/2} - 1\},
\end{aligned}$$

which can be bounded by a summable term, since g is in the Schwartz class. □

3. Examples

The Besov space $\mathcal{B}$ is well understood, it contains for example the space BMO of functions of bounded mean oscillations. In view of Theorem 4.3 we wish to discuss for which initial conditions it is possible to find R such that the initial condition is small in the $\mathcal{B}_R$ norm. To this aim define

$$\mathcal{Z} = \{k : \mathbf{R}^d \to \mathbf{R} : \|k\|_{\mathcal{B}_R} \to 0 \text{ as } R \downarrow 0\}.$$

We will later see in Theorem 4.3 that functions in $\mathcal{Z}$ correspond to initial conditions where it is possible to solve the Eq. (1.1) locally for a small time interval.

In this section we give a few general examples (see the lemma below, which also extends a result proved in dimension $d = 1$ in [5]) of subspaces of $\mathcal{Z}$. Then we show that $L^\infty(\mathbf{R}^d)$, although it is contained in $\mathcal{B}$, is not contained in $\mathcal{Z}$. Hence the method of proof presented here fails to provide local uniqueness of solutions for initial conditions in $L^\infty(\mathbf{R}^d)$ with large norm. In the last part of the section we discuss our main example, which in particular shows, at least in space dimension 4, non-uniqueness of the mild formulation.

Lemma 3.1. *The following statements hold,*

- *if $k : \mathbf{R}^d \to \mathbf{R}$ is bounded and uniformly continuous, then $k \in \mathcal{Z}$,*
- *if $k : \mathbf{R}^d \to \mathbf{R}$ has bounded gradient on $\mathbf{R}^d$, then $k \in \mathcal{Z}$,*
- *if $k \in \dot{H}^{d/2}(\mathbf{R}^d)$, then $k \in \mathcal{Z}$,where the homogeneous space is defined by (denoting by $\widehat{k}$ the Fourier transform of k)*

$$\dot{H}^{d/2}(\mathbf{R}^d) = \left\{k : \mathbf{R}^d \to \mathbf{R} : \|k\|_{\dot{H}^{d/2}} := \int_{\mathbf{R}^d} |\xi|^d |\widehat{k}(\xi)|^2 \, d\xi < \infty\right\}$$

Proof. For the *first result* fix $\epsilon > 0$. By uniform continuity there is $\delta > 0$ such that $|k(x) - k(y)| \le \epsilon$ for all $x, y \in \mathbf{R}^d$ with $|x - y| \le \delta$. Since the integral of ∇g is zero,

$$\begin{aligned} t^{\frac14} |\nabla(\mathrm{e}^{-tA} k)(x)| &\le \left| \int_{t^{\frac14}|z| \ge \delta} \nabla g(z)(k(x - zt^{\frac14}) - k(x))\, dz \right| \\ &\quad + \left| \int_{t^{\frac14}|z| \le \delta} \nabla g(z)(k(x - zt^{\frac14}) - k(x))\, dz \right| \\ &\le 2\|k\|_\infty \int_{t^{\frac14}|z| \ge \delta} |\nabla g(z)|\, dz + \epsilon \|\nabla g\|_{L^1(\mathbf{R}^d)}, \end{aligned}$$

hence $\limsup_{R\to 0} \|k\|_{\mathcal{B}_R} \le \epsilon \|\nabla g\|_{L^1}$ and as $\epsilon \downarrow 0$, the claim follows.

The *second claim* follows easily from the bound $t^{\frac14} |\nabla(\mathrm{e}^{-tA} k)(x)| \le t^{\frac14} \|g\|_{L^1} \|\nabla k\|_{L^\infty}$.

Let us now turn to the *third result.* If $t > 0$ and $x \in \mathbf{R}^d$, by using the properties of Fourier transform and convolution,

$$\begin{aligned} t^{\frac14} |\nabla(\mathrm{e}^{-tA} k)(x)| &= \frac{1}{t^{\frac{d}{4}}} \left| \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (\nabla g)((x - y)t^{-\frac14}) \widehat{k}(\xi)\, \mathrm{e}^{\mathrm{i}\xi\cdot y}\, dy\, d\xi \right| \\ &= \left| \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \nabla g(y)\, \mathrm{e}^{-\mathrm{i}\xi z t^{\frac14}}\, \widehat{k}(\xi)\, \mathrm{e}^{\mathrm{i}\xi\cdot x}\, dy\, d\xi \right| \\ &\le t^{\frac14} \int_{\mathbf{R}^d} |\xi|\, |\widehat{k}(\xi)|\, \mathrm{e}^{-|\xi|^4 t}\, d\xi. \end{aligned}$$

Given $a > 0$, split the integral in the last line of formula above in two pieces, corresponding to the domains of integration $\{|\xi| \le a\}$ and $\{|\xi| > a\}$ respectively. We can now easily bound the terms using Cauchy–Schwarz inequality and change of variables.

In conclusion we obtain

$$\|k\|_{\mathcal{B}_R} \le caR\|k\|_{\dot{H}^{d/2}} + c\left(\int_{|\xi|\ge a} |\xi|^d |\widehat{k}(\xi)|^2\, d\xi\right)^{\frac{1}{2}}.$$

So we see that $\limsup_{R\to 0} \|k\|_{\mathcal{B}_R}$ is bounded by a quantity which converges to 0 as $a \uparrow \infty$. □

Example 3.2. *There are functions in $L^\infty(\mathbf{R}^d)$ not belonging to $\mathcal{Z}$. Since the Green's function tensorizes, it is enough to find a counterexample in dimension $d = 1$. Define $k(x) = \mathbb{1}_{[-1,1]}(x)$, then it is easy to see that*

$$\|k\|_{\mathcal{B}_R} \ge |g(0) - g\left(\frac{-2}{R^{1/4}}\right)| \longrightarrow g(0) > 0.$$

where we have chosen $x = -1$ and $t = R$ and used $g(x) \to 0$ as $x \to \infty$.

On the other hand, the space $\mathcal{Z}$ contains unbounded functions, which are not in $\mathcal{B}$. Define for $\alpha > 0$, $k_\alpha(x) = |x|^\alpha$. Now it is easy to see that $\|t^{\frac{1}{4}}\partial_x(\mathrm{e}^{-tA}\, k_\alpha)\|_\infty = t^{\frac{\alpha}{4}}\|\partial_x(\mathrm{e}^{-A}\, k_\alpha)\|_\infty$. Hence $\|k_\alpha\|_{\mathcal{B}_R} \to 0$ for $R \to 0$ but $\|k_\alpha\|_{\mathcal{B}} = \infty$.

We turn to the main example of the section. Consider first the case $d = 1$, then $k(x) = \log|x|$ is (formally) a stationary solution for problem (1.1) (see [5]). Similarly, there are other examples such as $\log(\sin(x))$ or $\log(\cosh(x))$, for instance. Nevertheless, in dimension $d = 1$ these functions k are neither a weak nor a mild solution, as ∇k is not locally square summable. Here we show that $k \in \mathcal{B}$ but $k \notin \mathcal{Z}$. Indeed, consider first, by a change of variables

$$t^{\frac{1}{4}}\left|(\mathrm{e}^{-tA}\, k)_x(x)\right| = \left|\int_{\mathbf{R}} g'(z)\log|t^{-\frac{1}{4}}x - z|\, dz\right|,$$

where we used that the integral over g' is zero. Now $\tilde{x} = t^{-\frac{1}{4}}x$ yields

$$\sup_{x\in\mathbf{R}}\{t^{\frac{1}{4}}|(\partial_x\, \mathrm{e}^{-tA}\, k)(x)|\} = \sup_{x\in\mathbf{R}}\left\{\left|\int_{\mathbf{R}} g'(z)\log|x-z|\, dz\right|\right\} = \|\partial_x\, \mathrm{e}^{-A}\, k\|_\infty.$$

Thus $\|k\|_{\mathcal{B}}$ is finite, but $\|k\|_{\mathcal{B}_R}$ is independent of R and does not converge to 0.

This example becomes more interesting if one turns to higher dimension. Let $k(x) = \alpha_d \log(|x|)$ for $x \in \mathbf{R}^d$, where $\alpha_d \in \mathbf{R}$ is a suitable constant that will be specified later. As before $k \in \mathcal{B}$, but $k \notin \mathcal{Z}$.

The function k is not even formally a weak solution for $d = 2$ and all $\alpha_d \ne 0$. But as the weak derivative $\nabla k \in L^2_{\mathrm{loc}}$ in dimension $d \ge 3$, it is straightforward to check that both the weak formulation (2.1) and the mild formulation are satisfied with $\alpha_d = 2 - d$.

A special case is dimension $d = 4$, where k is a weak solution for any $\alpha_d \in \mathbf{R}$. But in that case the $\mathcal{B}$-norm of k can be made arbitrarily small with

$\alpha_d \to 0$, and thus by Theorems 4.3 and 6.1 there is also a smooth mild solution starting at k.

In conclusion this example shows the non-uniqueness and lack of smoothness of mild solutions, at least in dimension 4. Nevertheless there is no contradiction with Theorem 4.3, since $k \notin \mathcal{X}$ if considered as a function of time.

4. The fixed point argument

Define the map

$$\mathscr{V}(h,k)(t) = \int_0^t \Delta(\mathrm{e}^{-(t-s)A}\, \nabla h(s) \cdot \nabla k(s))\, ds$$

and set

$$\mathscr{F}(h)(t) = \mathrm{e}^{-tA}\, h_0 - \mathscr{V}(h,h)(t). \tag{4.1}$$

We will use the following concept of a mild solution, which is given as a solution of the variation of constants formula in (4.2).

Definition 4.1. *We say that $h \in \mathcal{X}$ solves (1.1) with initial condition $h_0 \in \mathcal{B}$, if for all $t > 0$*

$$h(t) = \mathrm{e}^{-tA}\, h_0 - \mathscr{V}(h,h)(t). \tag{4.2}$$

We call $h \in \mathcal{X}_R$ a local solution, if (4.2) holds only for $t \in [0, R^4]$.

The following Lemma is crucial for the proof of uniqueness and existence. It verifies that the nonlinear part is locally Lipschitz continuous.

Lemma 4.2. *The map $\mathscr{V}$ is bi-linear continuous from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{X}$ and from $\mathcal{X}_R \times \mathcal{X}_R$ to $\mathcal{X}_R$, for all $R > 0$.*

Proof. The bilinearity is obvious. For the boundedness let $x \in \mathbf{R}^d$ and $t > 0$, then

$$\begin{aligned}
|\nabla \mathscr{V}(h,k)(t,x)| &= \left| \int_0^t \int_{\mathbf{R}^d} \nabla\Delta G(t-s, x-y) \nabla h(s,y) \nabla k(s,y)\, dy\, ds \right| \\
&= \left| \int_0^t \frac{1}{(t-s)^{\frac{d+3}{4}}} \int_{\mathbf{R}^d} (\nabla\Delta g) \left(\frac{x-y}{(t-s)^{1/4}} \right) \nabla h(s,y) \nabla k(s,y)\, dy\, ds \right| \\
&\le \|h\|_{\mathcal{X}} \|k\|_{\mathcal{X}} \int_0^t \frac{1}{(t-s)^{\frac{d+3}{4}} \sqrt{s}} \int_{\mathbf{R}^d} \left| (\nabla\Delta g) \left(\frac{x-y}{(t-s)^{1/4}} \right) \right| dy\, ds \\
&\le \|h\|_{\mathcal{X}} \|k\|_{\mathcal{X}} \|g\|_{W^{3,1}(\mathbf{R}^d)} \int_0^t \frac{1}{(t-s)^{3/4} \sqrt{s}}\, ds \\
&\le t^{-1/4} B\left(\frac{1}{2}, \frac{1}{4} \right) \|h\|_{\mathcal{X}} \|k\|_{\mathcal{X}} \|g\|_{W^{3,1}(\mathbf{R}^d)} \\
&= c_4 t^{-1/4} \|h\|_{\mathcal{X}} \|k\|_{\mathcal{X}},
\end{aligned}$$

where B is the Beta function. The corresponding inequality for the local space $\mathcal{X}_R$ proceeds similarly. □

Using the previous Lemma, we can now state and prove our main result. The first part states global existence of unique solutions, while the second part is about local existence of solutions. Let us point out, that the theorem holds in any dimension, but for dimension $d \geq 4$, the space $\mathcal{B}$ might not be the largest possible one.

Theorem 4.3. *Fix $0 < \delta < 1/4c_4$, where c_4 is the constant defined in the proof of Lemma 4.2.*

If $\|h_0\|_{\mathcal{B}} \leq \delta$, then there exists a unique (global) solution in $\mathcal{X}$ of (1.1) with initial condition h_0.

Moreover, if $\|h_0\|_{\mathcal{B}_R} \leq \delta$, then there is a unique local solution in $\mathcal{X}_R$ of (1.1) on $[0, R^4]$ with initial condition h_0.

Finally, if h_0 is periodic and small in $\mathcal{B}_R$ for some $R > 0$ (or it is small in $\mathcal{B}$), then the solution is also periodic.

In particular, $\|h_0\|_{\mathcal{B}_R} \leq \delta$ is true for a suitable value of R for all $h_0 \in \mathcal{Z}$.

Proof. We prove the first statement by a fixed point iteration argument. Choose $K > 0$ such that

$$\frac{1}{2c_4}(1 - \sqrt{1 - 4c_4\delta}) \leq K < \frac{1}{2c_4}.$$

Define

$$H_0 = 0, \quad H_{n+1} = \mathscr{F}(H_n) = \mathrm{e}^{-tA}\, h_0 - \mathscr{V}(H_n, H_n), \tag{4.3}$$

then $\|H_1\|_{\mathcal{X}} \leq \delta$ and it is easy to check by induction (and by the choice of δ and K) that $\|H_n\|_{\mathcal{X}} \leq K$ for all n. Then

$$\|H_{n+1} - H_n\|_{\mathcal{X}} = \|\mathscr{V}(H_n, H_n) - \mathscr{V}(H_{n-1}, H_{n-1})\|_{\mathcal{X}} \leq 2c_4 K \|H_n - H_{n-1}\|_{\mathcal{X}}$$

and so $(H_n)_{n\in\mathbf{N}}$ is convergent in $\mathcal{X}$ to a fixed point of $\mathscr{F}$.

The same proof works for local spaces, since both constants c_1 and c_4 do not depend on R. Finally, if h_0 is periodic, the statement of periodicity follows by translation invariance and uniqueness. □

Remark 4.4. (Forward self-similar solutions) The theorem above allows to show the existence of self-similar solutions, namely solutions invariant under the scaling (2.2). Indeed, assume to have $h_0 \in \mathcal{B}$ (or in a local space) such that $h_0(\lambda x) = h_0(x)$ for all $\lambda > 0$, then it is easy to verify that H_1 is invariant under the scaling (2.2) and that $\mathscr{V}(h, h)$ is also invariant if so is h. In conclusion the whole sequence $(H_n)_{n\in\mathbf{N}}$ defined in (4.3) is invariant, as well as its limit.

Given a (forward) self-similar solution h, one can write $h(t, x) = \psi(x/t^{1/4})$, where $\psi(x) = h(1, x)$ solves the equation

$$\Delta^2\psi + \Delta|\nabla\psi|^2 - \frac{1}{4}x \cdot \nabla\psi = 0.$$

Due to the scaling property, the only admissible initial conditions are the 0-homogeneous functions, namely $h_0(\lambda x) = h_0(x)$ for all $\lambda > 0$. The simplest case corresponds to $d = 1$, where the only 0-homogeneous functions are those constant on $(-\infty, 0)$ and on $(0, \infty)$ (possibly with different values on the two half-lines). For any such function h_0, $\|h_0\|_{\mathcal{B}_R} = |h_0(1) - h_0(-1)|\, \|g\|_{L^\infty}$.

Backward self-similar solutions might provide examples of solutions with blow-up. Due to the scaling of the problem, the quantity blowing up is related to the derivative of the solution. We do not know if backward self-similar solutions exist (notice that backward self-similar solutions do not exist for the Navier–Stokes equations, see [21]).

5. The stochastic problem

In this section we give a short outline of the proof of local existence for the stochastic PDE, without many details on probability theory. For details we refer to [6,8,18]. Consider

$$\partial_t h + \Delta^2 h + \Delta|\nabla h|^2 = \partial_t W \ , \tag{5.1}$$

where $\partial_t W$ is the generalized derivative of a Hilbert-space value Wiener process. Define the corresponding Ornstein–Uhlenbeck process for $t > 0$ as the following Itô-integral

$$Z(t) = \int_0^t e^{-(t-s)A}\, dW. \tag{5.2}$$

Note that Z solves $\partial_t Z + \Delta^2 Z = \partial_t W$ with $Z(0) = 0$. The mild solution of (5.1) is analogous to Definition 4.1 given by a solution of

$$h(t) = \mathrm{e}^{-tA}\, h_0 - \mathscr{V}(h,h)(t) + Z(t)$$

Now the main problem in this setting is to determine the regularity of Z, which can be read in terms of the covariance of the driving noise. Once we know this, we can solve the equation using again Banach's fixed point argument.

This is a standard extension to Theorem 4.3 which can be carried on for instance in the case of bounded intervals (i.e. $d = 1$) with periodic boundary conditions and space–time white noise. In fact in this case the stochastic convolution Z and its derivative $\partial_x Z$ are continuous in both space and time (this can be verified using the methods in [8], see for example [22]) and almost surely $\|Z\|_{\mathcal{X}_R} \to 0$ for $R \to 0$. Hence the equation can be uniquely solved in $\mathcal{X}_R$, for a random $R > 0$.

An interesting question appears in the case of periodic boundary conditions and $d = 2$, since for space–time white noise the convolution Z just fails to be differentiable in space (Z will be differentiable if we consider slightly more regular noise). In this case one can consider the new unknown $v = h - Z$, solution to the following random PDE,

$$\partial_t v + \Delta^2 v + \Delta|\nabla v|^2 = -\Delta|\nabla Z|^2 - 2\Delta|\nabla v \cdot \nabla Z|^2, \quad v(0) = h_0. \tag{5.3}$$

The new variable v is expected to be more regular than h, since the above equation contains only some additional lower order terms that do not change the proofs, once Z is sufficiently regular. Nevertheless, using renormalisation techniques it is in some cases possible to define $\Delta|\nabla Z|^2$ for Gaussian fields that are not differentiable (this is the subject of a work in progress).

For stochastic PDEs on unbounded domains one can use the formulation of Walsh [28], although one has to consider that for space–time white noise both the stochastic convolution $Z(t,x)$ and its derivative $\nabla Z(t,x)$ (if it exists) are unbounded for $|x| \to \infty$.

6. Smoothness of solutions

Following the same methods of [10], we show that solutions in $\mathcal{X}$ (or $\mathcal{X}_R$) are smooth. Define for $m \geq 1$,

$$\|k\|_{\mathcal{X},m} := \sup_{t>0} \left\{ t^{\frac{m+1}{4}} \sum_{|\alpha|=m+1} \|D^\alpha k\|_\infty \right\}$$

and denote by $\|\cdot\|_{\mathcal{X}_R,m}$ the corresponding local version, where for $\alpha = (\alpha_1,\dots,\alpha_d)$ we used $D^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

Let $\mathcal{X}^m$ be the space

$$\mathcal{X}^m = \{k : \mathbf{R}^d \to \mathbf{R} : \|k\|_{\mathcal{X}^m} := \max_{0\leq j\leq m} \|k\|_{\mathcal{X},j} < \infty\},$$

and denote by $\mathcal{X}_R^m$ the corresponding local version. For simplicity of notations we understand that $\|\cdot\|_{\mathcal{X},0} = \|\cdot\|_{\mathcal{X}}$ and for $R = \infty$ that $\mathcal{X}_\infty^m = \mathcal{X}^m$. The main theorem of this section is the following result on smoothness in space. Smoothness in time then follows from the PDE by a standard bootstrapping argument.

Theorem 6.1. *Let h be a solution of (1.1) in $\mathcal{X}_R$, with $0 < R \leq \infty$. Then $h(t) \in C_b^\infty(\mathbf{R}^d)$ for all $t \in (0,R)$.*

Proof. If the initial condition is small enough in $\mathcal{B}_R$, the statement follows from Proposition 6.3 below. In the general case we notice that if $h \in \mathcal{X}_R$, then $\nabla h(t)$ is bounded for all $t \in (0,R)$, therefore $h(t) \in \mathcal{Z}$, by Lemma 3.1. The conclusion then follows again from Proposition 6.3. □

Remark 6.2. A more careful estimate of the constants appearing in the results that follow (for instance the constants K_m in Lemma 6.6) would show that the solutions are analytic in space. We point out that Koch and Lamm [16] have a shorter and more elegant proof of analyticity (in a different context). An advantage of the method used here is to provide the behaviour at $t = 0$ of solutions through the spaces $\mathcal{X}^m$ and $\mathcal{X}_R^m$.

In order to complete the proof of the above theorem, we need the following proposition, which also gives a better estimate of the solution near $t = 0$ if the initial condition is small enough.

Proposition 6.3. *There exists $\delta > 0$ such that if $\|h_0\|_{\mathcal{B}} < \delta$, then the solution to (1.1) granted by Theorem 4.3 is in $\mathcal{X}^m$ for all $m \geq 1$.*

If $R > 0$ and $\|h_0\|_{\mathcal{B}_R} < \delta$, then the solution to (1.1) granted by Theorem 4.3 is in $\mathcal{X}_R^m$ for all $m \geq 1$.

We start by giving a slight generalization of (2.3).

Lemma 6.4. *Let $0 < R \le \infty$ and $k \in \mathcal{B}_R$, then for every $m \ge 0$,*

$$\sup_{t\le R}\left\{ t^{\frac{m+1}{4}} \sum_{|\alpha|=m+1} \left\| D^\alpha(\mathrm{e}^{-tA}\,k)\right\|_\infty \right\} \le cm^d(m+1)^{\frac{m+1}{4}}\|\nabla g\|^m_{L^1(\mathbf{R}^d)}\|k\|_{\mathcal{B}_R}. \tag{6.1}$$

Proof. Since for $|\alpha| = m+1$,

$$D^\alpha(\mathrm{e}^{-tA}\,k) = \prod_{i=1}^d (\partial^{\alpha_i}_{x_i}\,\mathrm{e}^{-\frac{\alpha_i}{m+1}tA})k,$$

it is sufficient to show that the operator $\partial_{x_i}\,\mathrm{e}^{-tA}$ maps $L^\infty(\mathbf{R}^d)$ into itself with operator norm $\|\partial_{x_i}\,\mathrm{e}^{-tA}\|_{L^\infty\to L^\infty} \le t^{-1/4}\|\nabla g\|_{L^1(\mathbf{R}^d)}$. This is immediate since by a change of variables,

$$t^{\frac14}|\partial_{x_i}(\mathrm{e}^{-tA}\,k)(x)| = t^{\frac{d}{4}}\left|\int_{\mathbf{R}^d}(\partial_{x_i}g)\left(\frac{x-y}{t^{1/4}}\right)k(y)\,dy\right| \le \|\nabla g\|_{L^1(\mathbf{R}^d)}\|k\|_\infty.$$

Finally, $\#(\{\alpha : |\alpha| = m+1\}) = \binom{m+d}{d-1} \le cm^d$. □

Lemma 6.5. *There is $c_5 > 0$ such that for $m \ge 1, 0 < R \le \infty$ and $h, k \in \mathcal{X}^m_R$,*

$$\begin{aligned}\|\mathscr{V}(h,k)\|_{\mathcal{X}_{R,m}} \le{}& c_5 m^d(m+1)^{\frac{m+3}{2}}\|\nabla g\|^m_{L^1(\mathbf{R}^d)}\|g\|_{W^{3,1}(\mathbf{R}^d)}\|h\|_{\mathcal{X}_R}\|k\|_{\mathcal{X}_R}\\ &+c_5\|h\|_{\mathcal{X}_R}\|k\|_{\mathcal{X}_{R,m}} + c_5\|h\|_{\mathcal{X}_{R,m}}\|k\|_{\mathcal{X}_R}\\ &+c_5 m^d\sum_{j=1}^{m-1}\binom{m}{j}\|h\|_{\mathcal{X}_{R,j}}\|k\|_{\mathcal{X}_{R,m-j}}.\end{aligned} \tag{6.2}$$

Proof. Fix $m \ge 1, 0 < R \le \infty, t \le R$ and $h, k \in \mathcal{X}^m_R$. Consider a value $\epsilon \in (0,1)$ which will be specified later, and let $|\alpha| = m+1$. Since $|\alpha| \ge 1$, there is $i \le d$ such that $\alpha_i \ge 1$. So assume without loss of generality that $a_1 \ge 1$ and let $\alpha' = \alpha - (1, 0, \ldots, 0)$.

$$\begin{aligned}D^\alpha\mathscr{V}(h,k)(t) &= \int_0^t D^\alpha\Delta(\mathrm{e}^{-(t-s)A}(\nabla h(s)\nabla k(s)))\,ds\\ &= \int_0^{t(1-\epsilon)} D^\alpha\Delta(\mathrm{e}^{-(t-s)A}(\nabla h(s)\nabla k(s)))\,ds\\ &\quad+\int_{t(1-\epsilon)}^t D^\alpha\Delta(\mathrm{e}^{-(t-s)A}(\nabla h(s)\nabla k(s)))\,ds\\ &= ① + ②.\end{aligned}$$

For the term ① we use the factorization introduced in the previous lemma and we proceed as in the proof of Lemma 4.2,

$$\begin{aligned}|\,①\,| &\le (m+1)^{\frac{m+3}{4}}\|\nabla g\|^m_{L^1}\|\partial_{x_1}\Delta g\|_{L^1}\|h\|_{\mathcal{X}_R}\|k\|_{\mathcal{X}_R}\int_0^{t(1-\epsilon)} s^{-\frac12}(t-s)^{-\frac{m+3}{4}}\,ds\\ &\le 2\sqrt{1-\epsilon}\left(\frac{m+1}{\epsilon}\right)^{\frac{m+3}{4}} t^{-\frac{m+1}{4}}\|\nabla g\|^m_{L^1}\|\partial_{x_1}\Delta g\|_{L^1}\|h\|_{\mathcal{X}_R}\|k\|_{\mathcal{X}_R}.\end{aligned}$$

For the second term we use Leibniz formula,

$$②=\sum_{\beta\le\alpha'}\binom{\alpha'}{\beta}\int_{t(1-\epsilon)}^{t}\partial_{x_1}\Delta\,\mathrm{e}^{-(t-s)A}(D^{\beta}\nabla h)(D^{\alpha'-\beta}\nabla k)\,ds$$

and, as in the proof of Lemma 4.2,

$$\begin{aligned}②&\le\sum_{\beta\le\alpha'}\binom{\alpha'}{\beta}\|\partial_{x_1}\Delta g\|_{L^1}\|h\|_{\mathcal{X}_{R,|\beta|}}\|k\|_{\mathcal{X}_{R,m-|\beta|}}\int_{t(1-\epsilon)}^{t}\frac{1}{s^{\frac{m+2}{4}}(t-s)^{\frac34}}\,ds\\&\le\frac{4\epsilon^{\frac14}}{(1-\epsilon)^{\frac{m+2}{4}}}t^{-\frac{m+1}{4}}\|\partial_{x_1}\Delta g\|_{L^1}\sum_{\beta\le\alpha'}\binom{\alpha'}{\beta}\|h\|_{\mathcal{X}_{R,|\beta|}}\|k\|_{\mathcal{X}_{R,m-|\beta|}}\end{aligned}$$

If we set $\epsilon=\frac{1}{(m+d)^{4d}}$ the term $4\epsilon^{1/4}(1-\epsilon)^{-(m+2)/4}(m+d)^d$ is uniformly bounded in m (we recall that the number of multi-indices α such that $|\alpha|=m+1$ is bounded by $(m+d)^d$) and so by summing up over α the estimates for ① and ② together show the lemma. □

As in the proof of Theorem 4.3, define $H_0=0$ and

$$H_{n+1}(t)=\mathrm{e}^{-tA}\,h_0-\mathscr{V}(H_n,H_n)(t).$$

Lemma 6.6. *There is $\delta'>0$ such that if $0<R\le\infty$ and $\|h_0\|_{\mathcal{B}_R}<\delta$, then for every $m\ge0$ there is $K_m>0$ such that*

$$\|H_n\|_{\mathcal{X}_{R,m}}\le K_m.$$

Proof. If $\|h_0\|_{\mathcal{B}_R}$ is small enough, the proof of Theorem 4.3 shows that there is K_0 such that $\|H_n\|_{\mathcal{X}_R}\le K_0$. By possibly taking $\|h_0\|_{\mathcal{B}_R}$ smaller, we can assume that $\lambda=2c_5K_0<1$, where c_5 is given in Lemma 6.5. We prove the statement by induction: the case $m=0$ has been already proved. Set $a_m=c_5m^d(m+1)^{(m+3)/2}\|\nabla g\|_{L^1}^m\|g\|_{W^{3,1}}$ [this is the coefficient appearing in the first line of formula (6.2)] and $b_m=cm^d(m+1)^{(m+1)/4}\|\nabla g\|_{L^1}^m$ [this appears in formula (6.1)], then by Lemmas 6.4 and 6.5,

$$\begin{aligned}\|H_{n+1}\|_{\mathcal{X}_{R,m}}&\le\|H_1\|_{\mathcal{X}_{R,m}}+\|\mathscr{V}(H_n,H_n)\|_{\mathcal{X}_{R,m}}\\&\le b_mK_0+a_mK_0^2+2c_5K_0\|H_n\|_{\mathcal{X}_{R,m}}\\&\quad+c_5m^d\sum_{j=1}^{m-1}\tbinom{m}{j}\|H_n\|_{\mathcal{X}_{R,j}}\|H_n\|_{\mathcal{X}_{R,m-j}}\\&\le\left(b_mK_0+a_mK_0^2+c_5m^d\sum_{j=1}^{m-1}\tbinom{m}{j}K_jK_{m-j}\right)+\lambda\|H_n\|_{\mathcal{X}_{R,m}},\end{aligned}$$

so that by recurrence and again Lemma 6.4,

$$
\begin{aligned}
&\|H_{n+1}\|_{\mathcal{X}_R,m}\\
&\quad\le \left(b_mK_0+a_mK_0^2+c_5m^d\sum_{j=1}^{m-1}\tbinom{m}{j}K_jK_{m-j}\right)(1+\cdots+\lambda^{n-1})+\lambda^n\|H_1\|_{\mathcal{X}_R,m}\\
&\quad\le \frac{1}{1-\lambda}\left(b_mK_0+a_mK_0^2+c_5m^d\sum_{j=1}^{m-1}\tbinom{m}{j}K_jK_{m-j}\right)+b_mK_0,
\end{aligned}
$$

and the last line in the formula above provides K_m. □

Proof of Proposition 6.3. Theorem 4.3 ensures that if $\|h_0\|_{\mathcal{B}_R}$ is small enough, then there is $\lambda = 2c_4K_0 < 1$ (where the number K_0 is given by previous lemma) such that $\|H_{n+1} - H_n\|_{\mathcal{X}_R} \le c\lambda^n$. We prove by induction that there are numbers $C_m > 0$ and $\mu \in (0, 1)$ such that

$$\|H_{n+1} - H_n\|_{\mathcal{X}_R,m} \le C_m\mu^n, \quad m \ge 0,$$

if $\|h_0\|_{\mathcal{B}_R}$ is small enough. Let $\lambda = 2c_5K_0$ (where c_5 has been introduced in Lemma 6.5), assume $\lambda < 1$ and let $\lambda < \mu < 1$. We have already verified that the inductive claim is true for $m = 0$. Assume the claim is true for $0,\dots,m-1$, then by Lemma 6.5 and the inductive assumption,

$$
\begin{aligned}
&\|H_{n+1} - H_n\|_{\mathcal{X}_R,m}\\
&\quad\le \|\mathscr{V}(H_n, H_n - H_{n-1})\|_{\mathcal{X}_R,m} + \|\mathscr{V}(H_n - H_{n-1}, H_{n-1})\|_{\mathcal{X}_R,m}\\
&\quad\le [a_m(\|H_n\|_{\mathcal{X}_R}+\|H_{n-1}\|_{\mathcal{X}_R})+c_5(\|H_n\|_{\mathcal{X}_R,m}+\|H_{n-1}\|_{\mathcal{X}_R,m})]\,\|H_n - H_{n-1}\|_{\mathcal{X}_R}\\
&\qquad+c_5(\|H_n\|_{\mathcal{X}_R} + \|H_{n-1}\|_{\mathcal{X}_R})\|H_n - H_{n-1}\|_{\mathcal{X}_R,m}\\
&\qquad+c_5m^d\sum_{j=1}^{m-1}\tbinom{m}{j}(\|H_n\|_{\mathcal{X}_R,j} + \|H_{n-1}\|_{\mathcal{X}_R,j})\|H_n - H_{n-1}\|_{\mathcal{X}_R,m-j}\\
&\quad\le \lambda\|H_n - H_{n-1}\|_{\mathcal{X}_R,m} + \widetilde{K}_m\mu^{n-1},
\end{aligned}
$$

where we have set $a_m = c_5m^d(m+1)^{(m+3)/2}\|\nabla g\|_{L^1}^m\|g\|_{W^{3,1}}$ [the coefficient in the first line of (6.2)], $\widetilde{K}_m = 2C_0(a_mK_0 + c_5K_m) + 2c_5m^d\sum_{j=1}^{m-1}\binom{m}{j}K_jC_{m-j}$, and the constants K_j are given by the previous lemma. By recurrence (notice that $\mu > \lambda$), it is easy to see that for every n,

$$
\begin{aligned}
\|H_{n+1}-H_n\|_{\mathcal{X}_R,m} &\le \lambda^{n+1}\|H_1-H_0\|_{\mathcal{X}_R,m}+\widetilde{K}_m\left(\lambda^{n-1}+\lambda^{n-2}\mu+\cdots+\mu^{n-1}\right)\\
&\le \left(\lambda K_m + \frac{1}{\mu-\lambda}\widetilde{K}_m\right)\mu^n,
\end{aligned}
$$

which concludes the induction. In conclusion, the sequence $(H_n)_{n\in\mathbf{N}}$ converges in all spaces $\mathcal{X}_R^m$. □

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Dirk Blömker
Institut für Mathematik
Universität Augsburg
86135 Augsburg
Germany
e-mail: dirk.bloemker@math.uni-augsburg.de
http://www.math.uni-augsburg.de/ana/bloemker.html

Marco Romito
Dipartimento di Matematica, Università di Pisa
Largo Bruno
Pontecorvo 5
56127 Pisa, Italia
e-mail: romito@dm.unipi.it
http://www.dm.unipi.it/pages/romito

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