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On a singular elliptic problem involving critical growth in $\mathbb{R}^N$

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Abstract. In this paper, we prove a suitable Trudinger–Moser inequality with a singular weight in $\mathbb{R}^N$ and as an application of this result, using the mountain-pass theorem we establish sufficient conditions for the existence of nontrivial solutions to quasilinear elliptic partial differential equations of the form

$$-\Delta_N u + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^a} \quad \text{in } \mathbb{R}^N, \quad N \geq 2,$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential, $a \in [0, N)$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ behaves like $\exp(\alpha|u|^{N/(N-1)})$ when $|u| \to \infty$.

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1. Introduction

In this paper we deal with the existence of nontrivial solutions for the problem

$$-\Delta_N u + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^a} \quad \text{in } \mathbb{R}^N, \quad N \geq 2, \tag{1.1}$$

where $\Delta_N u \equiv \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian and $a \in [0, N)$. Motivated by a Trudinger–Moser inequality with a singular weight (see Lemma 2.3 and [22,28]) we consider here the maximal growth on the nonlinear term $f(x,u)$ which allows to treat Eq. (1.1) variationally in a subspace of $W^{1,N}(\mathbb{R}^N)$, the Sobolev space of functions in $L^N(\mathbb{R}^N)$ such that its weak derivatives are also in $L^N(\mathbb{R}^N)$ with the norm

$$\|u\|_{1,N} \doteq \left(\int_{\mathbb{R}^N} (|\nabla u|^N + |u|^N)\, \mathrm{d}x \right)^{1/N}.$$

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It is assumed that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(x,0) \equiv 0$ and f behaves like $\exp\left(\alpha|u|^{N/(N-1)}\right)$ when $|u| \to \infty$. More precisely, we assume the following growth condition on the nonlinearity $f(x,u)$:

(f_1) There exist constants $\alpha_0, b_1, b_2 > 0$ such that for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$,

$$|f(x,u)| \leq b_1|u|^{N-1} + b_2\Phi\left(\alpha_0|u|^{N/(N-1)}\right),$$

where

$$\Phi(t) = \exp(t) - \sum_{k=0}^{N-2} \frac{t^j}{j!}.$$

The main features of this class of problems, considered in this paper, are the presence of the singularity $|x|^{-a}$, it is defined in the whole $\mathbb{R}^N$, involves critical growth and the nonlinear operator N-Laplacian. In spite of a possible failure of the Palais–Smale compactness condition, in this article we apply minimax methods, more precisely, the mountain-pass theorem, to obtain the existence of weak solutions of (1.1) in the subspace $E \subset W^{1,N}(\mathbb{R}^N)$ given by

$$E = \left\{ u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N \, \mathrm{d}x < \infty \right\}.$$

We say that $u \in E$ is a weak solution of problem (1.1) if for all $v \in E$ we have

$$\int_{\mathbb{R}^N} (|\nabla u|^{N-2}\nabla u \nabla v + V(x)|u|^{N-2}uv) \, \mathrm{d}x - \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^a} v \, \mathrm{d}x = 0.$$

Under the condition on the potential

$(V_1) V$ is a continuous function such that

$$V(x) \geq V_0 > 0, \quad \text{for all } x \in \mathbb{R}^N.$$

We can see that E is a reflexive Banach space when endowed with the norm

$$\|u\| \doteq \left(\int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N) \, \mathrm{d}x \right)^{1/N}$$

and for all $N \leq q < \infty$,

$$E \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \tag{1.2}$$

with continuous embedding (see [8,26]). Moreover,

$$\lambda_1(N) \doteq \inf_{u \in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N) \, \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^N/|x|^a \, \mathrm{d}x} > 0. \tag{1.3}$$

We also assume suitable conditions on the potential V in order to get the compactness of the following embedding

$$E \hookrightarrow L^p(\mathbb{R}^N) \quad \text{for all } p \geq N. \tag{1.4}$$

Notice that each one of the two conditions below are sufficient to get (1.4) (this fact can be found in [8,20,24,26]):

(V_2) $V(x) \to \infty$ as $|x| \to \infty$; or more generally, for every $M > 0$,

$$\mu\left(\{x \in \mathbb{R}^N : V(x) \leq M\}\right) < \infty,$$

with μ denoting the Lebesgue measure in $\mathbb{R}^N$;

(V_3) $\int_{\mathbb{R}^N} V(x)^{-1}\,\mathrm{d}x < \infty$.

We assume the following conditions on the nonlinearity $f(x,u)$:

(f_2) There exists $\theta > N$ such that for all $x \in \mathbb{R}^N$ and $u \geq 0$,

$$0 \leq \theta F(x,u) \equiv \mu \int_0^u f(x,t)\,\mathrm{d}t \leq u f(x,u).$$

(f_3) There exist constants $R_0, M_0 > 0$ such that, for all $x \in \mathbb{R}^N$ and $u \geq R_0$,

$$0 < F(x,u) \leq M_0 f(x,u).$$

(f_4) $\limsup_{u\to 0^+} \dfrac{NF(x,u)}{|u|^N} < \lambda_1(N)$ uniformly in $x \in \mathbb{R}^N$.

(f_5) $\lim_{u\to+\infty} uf(x,u)\exp(-\alpha_0|u|^{N/(N-1)}) \geq \beta_0 > 0$ uniformly on compact subsets of $\mathbb{R}^N$.

The main result of this paper is the following

Theorem 1.1. *Suppose* (V_1) *and* (V_2) *(or* (V_3)*) and* $(f_1)-(f_5)$ *are satisfied. Then, the problem* (1.1) *has a nontrivial solution* $u \in E$.

Remark 1.1. We point out that Theorem 1.1 is closely related with some results in [10] where the the existence and multiplicity of weak solutions for the following class of problems $-\Delta u + V(x)u = \frac{g(u)}{|x|^a} + h(x)$ *in* $\mathbb{R}^2$, where $a \in [0,2), V(x)$ is a continuous positive potential bounded away from zero and which can be "large" at the infinity, the nonlinearity $g(s)$ behaves like $e^{\alpha s^2}$ when $|s| \to +\infty$ and $h \in (H^1(\mathbb{R}^2))^*$ is a small perturbation. See also [17] for others related results.

Remark 1.2. Notice that, in the semilinear case the hypotheses of Theorem 1.1 are for example satisfied by the nonlinearity

$$f(x,u) = 3u^2 + 2u\exp(u^2) - 2u.$$

The condition (f_4) is essential to guarantee our main results. For example, if $f(x,u) = \lambda_1 u + \exp(u^2) - 1 + 2u^2\exp(u^2)$, (1.1) does not have positive solution.

Remark 1.3. It is well known that problems involving the $p-$Laplacian operator appear in many contexts. Some of these problems come from different areas of applied mathematics and physics. For example, they may be found in the study of non-Newtonian fluids, non-linear elasticity, and reaction-diffusions. For discussions about problems modelled by these boundary value problems, see for example [11].

Remark 1.4. The results in this paper were in part motivated by several recent papers on elliptic problems involving critical growth in the Trudinger–Moser case, see [1,2,4,6,9,12,13,15,16,27] and references therein. Here we complement some the results mentioned above by establishing sufficient conditions for the existence of nontrivial solutions for singular case with $a \in [0,N)$. For problem (1.1), when $a \equiv 0$, the existence of nontrivial solutions has been studied on bounded domains by [9] in the semilinear case and by [1,12,23] for the quasilinear equations. For problems in unbounded domains see [6,13].

This paper is organized as follows.

Section 2 we show a singular Trudinger–Moser inequality in $\mathbb{R}^N$ and contains some technical results that allows to give a variational approach of our work.

Section 3 contains the variational framework and we check the geometric conditions of the associated functional.

In Sect. 4 we get a more precise information about the minimax level obtained by the mountain-pass theorem.

Section 5 we prove our main results.

Throughout this paper we will choose subsequences of given sequences and relabeling them as the original sequence and we shall do without explicitly stating it.

$C, C_1, C_2, \ldots$ design positive generic constants and B_r denotes a ball centered at the origin and radius r.

2. A singular Trudinger–Moser inequality in $\mathbb{R}^N$

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. The well known Trudinger–Moser inequality (see [22,28]) asserts that

$$\exp(\alpha|u|^{N/(N-1)}) \in L^1(\Omega), \quad \forall u \in W_0^{1,N}(\Omega),\ \forall \alpha > 0.$$

Moreover, that there exists a constant $C = C(N)$, which depends only on N, such that

$$\sup_{\|\nabla u\|_N \leq 1} \int_\Omega \exp(\alpha|u|^{N/(N-1)})\, \mathrm{d}x \leq C(N)|\Omega|, \quad \text{if } \alpha \leq \alpha_N,$$

where $|\Omega| = \int_\Omega \mathrm{d}x, \alpha_N = N w_{N-1}^{1/(N-1)}$ and w_{N-1} is the $(N-1)-$dimensional measure of the $(N-1)-$sphere.

In the case of unbounded domains, we have the following version of the Trudinger–Moser inequality (see [13, Lemma 1], [6,23]).

Lemma 2.1. *If $N \geq 2, \alpha > 0$ and $u \in W^{1,N}\left(\mathbb{R}^N\right)$ then*

$$\int_{\mathbb{R}^N} \Phi\left(\alpha|u|^{N/(N-1)}\right)\, \mathrm{d}x < \infty. \tag{2.1}$$

Moreover, if $\|\nabla u\|_N^N \leq 1$, $\|u\|_N \leq M < \infty$ and $\alpha < \alpha_N$ then there exists a constant $C = C(N, M, \alpha)$, which depends only on N, M and α, such that

$$\int_{\mathbb{R}^N} \Phi\left(\alpha|u|^{N/(N-1)}\right)\, \mathrm{d}x \leq C(N, M, \alpha). \tag{2.2}$$

In a recent paper Adimurthi–Sandeep (see [3]) extended the Trudinger–Moser inequality with a singular weight, more precisely:

Lemma 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$, containing the origin and $u \in W_0^{1,N}(\Omega)$. Then for every $\alpha > 0$ and $a \in [0, N)$*

$$\int_\Omega \frac{\exp\left(\alpha|u|^{N/(N-1)}\right)}{|x|^a}\, \mathrm{d}x < \infty.$$

Moreover,

$$\sup_{\|\nabla u\|_N \leq 1} \int_\Omega \frac{\exp\left(\alpha|u|^{N/(N-1)}\right)}{|x|^a}\,\mathrm{d}x < \infty,$$

if and only if $\alpha/\alpha_N + a/N \leq 1$.

Next, inspired this last inequality and in [13, Lemma 1], we establish a version the Trudinger-Moser inequality with singular weight in $\mathbb{R}^N$. The proof of this result follows the same ideas in [17] and we omit it.

Lemma 2.3. *If* $\alpha > 0, a \in [0, N)$ *and* $u \in W^{1,N}(\mathbb{R}^N)$ *then*

$$\int_{\mathbb{R}^N} \frac{\Phi\left(\alpha|u|^{N/(N-1)}\right)}{|x|^a}\,\mathrm{d}x < \infty. \tag{2.3}$$

Moreover, if $\|\nabla u\|_N \leq 1, \|u\|_N \leq M < \infty$ *and* $\alpha/\alpha_N + a/N < 1$. *Then there exist a constant* $C = C(N, M, \alpha)$, *which depends only on* M, *and* α, *such that*

$$\int_{\mathbb{R}^N} \frac{\Phi\left(\alpha|u|^{N/(N-1)}\right)}{|x|^a}\,\mathrm{d}x \leq C(N, M, \alpha). \tag{2.4}$$

The next results are essential to establish the mountain-pass geometry of the associated functional.

Lemma 2.4. *If* $u \in E$ *and* $\|u\| \leq M$ *with* $\beta M^{N/(N-1)}/\alpha_N + a/N < 1$ *then*

$$\int_{\mathbb{R}^N} \frac{\Phi\left(\beta|u|^{N/(N-1)}\right)|u|}{|x|^a}\,\mathrm{d}x \leq C(N, M, \beta)\|u\|_s$$

for some $s > N$.

Proof. Using the Hölder inequality, we have

$$\begin{aligned}\int_{|x|\leq\sigma} \frac{\Phi\left(\beta|u|^{N/(N-1)}\right)|u|}{|x|^a}\,\mathrm{d}x &\leq \int_{|x|\leq\sigma} \frac{\exp\left(\beta|u|^{N/(N-1)}\right)|u|}{|x|^a}\,\mathrm{d}x \\ &\leq \left(\int_{|x|\leq\sigma} \frac{\exp\left(\beta r|u|^{N/(N-1)}\right)}{|x|^{ar}}\,\mathrm{d}x\right)^{1/r} \\ &\quad \times \left(\int_{|x|\leq\sigma} |u|^s\,\mathrm{d}x\right)^{1/s}\end{aligned}$$

where $r > 1$ is sufficiently close 1 such that $\beta r M^{N/(N-1)}/\alpha_N + a/N < 1, s = r/(r-1) > N$ and σ is a number to be determined later. Now, using the Lemma 2.3, we obtain

$$\int_{|x|\leq\sigma} \frac{\Phi\left(\beta|u|^{N/(N-1)}\right)|u|}{|x|^a}\,\mathrm{d}x \leq C(\beta, N)\|u\|_s. \tag{2.5}$$

Now, choosing $\sigma \geq 1$, we have

$$\int_{|x|\geq\sigma} \frac{\Phi\left(\beta|u|^{N/(N-1)}\right)|u|}{|x|^a}\,\mathrm{d}x \leq \int_{|x|\geq\sigma} \Phi\left(\beta|u|^{N/(N-1)}\right)|u|\,\mathrm{d}x,$$

hence, by Lemma 2.2 in [14] there exist $C_2(N, M, \beta) > 0$ such that

$$\int_{|x|\geq\sigma} \frac{\Phi\left(\beta|u|^{N/(N-1)}\right)|u|}{|x|^a}\,\mathrm{d}x \leq C_2(N, M, \beta)\|u\|_s. \tag{2.6}$$

From (2.5) and (2.6), we obtain the Lemma. □

Lemma 2.5. *For $\beta/\alpha_N + a/N < 1, \|u\| \leq M$ with M sufficiently small and $q > N$, we have*

$$\int_{\mathbb{R}^N} \frac{\Phi\left(\beta|u|^{N/(N-1)}\right)|u|^q}{|x|^a}\,\mathrm{d}x \leq C(\beta, N)\|u\|^q. \tag{2.7}$$

Proof. The proof is analogous to the preceding proof. □

3. The variational framework

We are interested in obtaining nonnegative weak solutions, it is convenient to define

$$f(x, u) = 0, \quad \forall (x, u) \in \mathbb{R}^N \times (-\infty, 0].$$

From (f_1),we obtain for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$,

$$|F(x, u)| \leq b_3\Phi\left(\alpha_0|u|^{N/(N-1)}\right) \tag{3.1}$$

for some constant $b_3 > 0$. Thus, by Lemma 2.3, we have $F(x, u)/|x|^a \in L^1(\mathbb{R}^N)$ for all $u \in W^{1,N}(\mathbb{R}^N)$. Therefore, the functional $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{N}\|u\|^N - \int_{\mathbb{R}^N} \frac{F(x, u)}{|x|^a}\,\mathrm{d}x$$

is well defined. Using the next proposition and standard arguments (see [5, Theorem A.VI] and [25, Appendix B]), we see that I is a C^1 functional on E with

$$I'(u)v = \int_{\mathbb{R}^N} (|\nabla u|^{N-2}\nabla u\nabla v + V(x)|u|^{N-2}uv)\,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{f(x, u)v}{|x|^a}\,\mathrm{d}x,$$

for $v \in E$. Consequently, critical points of the functional I are precisely the weak solutions of problem (1.1).

Proposition 3.1. *Suppose that (u_n) be a sequence in $W^{1,N}(\mathbb{R}^N)$ strongly convergent. Then there exist a subsequence (u_{n_k}) of (u_n) and $v \in W^{1,N}(\mathbb{R}^N)$ such that $|u_{n_k}(x)| \leq v(x)$ almost everywhere in $\mathbb{R}^N$.*

Proof. See proposition 1 in [14]. □

Here, like in [26], we are going to use a mountain-pass theorem without a compactness condition such like the one of the Palais–Smale type. This version of mountain-pass theorem is a consequence of Ekeland's variational principle (see [7,18,21]). In the next two lemmas we check that the functional I satisfies the geometric conditions of the mountain-pass theorem.

Lemma 3.2. *Suppose that* $(V_1), (f_1) - (f_5)$ *hold. Then there exists* $\rho > 0$ *such that*

$$I(u) > 0 \quad if \ \|u\| = \rho.$$

Proof. From (f_4), there exist $\tau, \delta > 0$ in such a way that $|u| \leq \delta$ implies

$$F(x,u) \leq \frac{(\lambda_1(N) - \tau)}{N} |u|^N, \tag{3.2}$$

for all $x \in \mathbb{R}^N$. On the other hand, using (f_1) for each $q > N$, there exists a constant $C = C(q, \delta)$ such that

$$F(x,u) \leq C\Phi\left(\alpha_0 |u|^{N/(N-1)}\right) |u|^q, \tag{3.3}$$

for $|u| \geq \delta$ and $x \in \mathbb{R}^N$. From (3.2) and (3.3) we obtain

$$F(x,u) \leq \frac{(\lambda_1(N) - \tau)}{N} |u|^N + C\Phi\left(\alpha_0 |u|^{N/(N-1)}\right) |u|^q,$$

for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Now, using Lemma 2.5, (1.3) and the continuous embedding $E \hookrightarrow L^N(\mathbb{R}^N)$, we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{N}\|u\|^N - \frac{(\lambda_1(N) - \tau)}{N} \int_{\mathbb{R}^N} \frac{|u|^N}{|x|^a}\, \mathrm{d}x - C\|u\|^q \\ &\geq \frac{1}{N}\left(1 - \frac{(\lambda_1(N) - \tau)}{\lambda_1(N)}\right) \|u\|^N - C\|u\|^q. \end{aligned}$$

Hence

$$I(u) \geq \|u\| \left[\frac{1}{N}\left(1 - \frac{(\lambda_1(N) - \tau)}{\lambda_1(N)}\right) \|u\|^{N-1} - C\|u\|^{q-1}\right]. \tag{3.4}$$

Since $\tau > 0$ and $q > N$, we may choose $\rho > 0$ such that

$$\frac{1}{N}\left(1 - \frac{(\lambda_1(N) - \tau)}{\lambda_1(N)}\right) \rho^{N-1} - C\rho^{q-1} > 0.$$

□

Lemma 3.3. *There exists* $e \in E$ *with* $\|e\| > \rho$ *such that*

$$I(e) < \inf_{\|u\| = \rho} I(u).$$

Proof. Let $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}, u \geq 0$, with compact support $K = supp(u)$. By (f_2) and (f_3) there exist $c > 0$ and $d > 0$ such that

$$F(x,s) \geq cs^\theta - d, \quad \forall\ (x,s) \in K \times [0, +\infty). \tag{3.5}$$

Thus, since $\theta > N$, for $t > 0$ we have

$$I(tu) \leq \frac{t^N}{N}\|u\|^N - ct^\theta \int_{\mathbb{R}^N} \frac{u^\theta}{|x|^a}\, \mathrm{d}x + d \int_K \frac{\mathrm{d}x}{|x|^a},$$

which implies that $I(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t sufficiently large, the proof of the lemma follows. □

4. The minimax level

In order to get a more precise information about the minimax level obtained by the mountain-pass theorem, let us consider the following sequence of scaled and truncated Green's functions also considered by Moser (see [22]).

$$\widetilde{M}_n(x,r)=\frac{1}{w_{N-1}^{1/N}}\begin{cases}(\log n)^{(N-1)/N} & \text{if } |x|\le r/n\\ \dfrac{\log\left(\frac{r}{|x|}\right)}{(\log n)^{1/N}} & \text{if } r/n\le |x|\le r\\ 0 & \text{if } |x|\ge r.\end{cases}$$

Notice that $\widetilde{M}_n(\cdot,r)\in W^{1,N}(\mathbb{R}^N)$, the support of $\widetilde{M}_n(x,r)$ is the ball B_r,

$$\int_{\mathbb{R}^N}|\nabla\widetilde{M}_n(x,r)|^N\,\mathrm{d}x=1\quad\text{and}\quad\int_{\mathbb{R}^N}|\widetilde{M}_n(x,r)|^N\,\mathrm{d}x=O(1/\log n)\tag{4.1}$$

as $n\to\infty$. Moreover, considering $M_n(x,r)=\widetilde{M}_n(x,r)/\|\widetilde{M}_n\|$, we can write

$$M_n^{N/(N-1)}(x,r)=w_{N-1}^{-1/(N-1)}\log n+d_n,\quad\forall|x|\le r/n,\tag{4.2}$$

where

$$d_n=w_{N-1}^{-1/(N-1)}\log n\left(\|\widetilde{M}_n\|^{-1/(N-1)}-1\right).$$

Using (4.1), we conclude that $\|\widetilde{M}_n\|\to 1$ as $n\to\infty$. Consequently,

$$\frac{d_n}{\log n}\to 0\quad\text{as}\quad n\to\infty.\tag{4.3}$$

Lemma 4.1. *Let* $\zeta_n=\|\widetilde{M}_n\|$, *then*

$$\lim_{n\to+\infty}\zeta_n\log n\int_0^{\zeta_n^{-1}}\exp[(N-a)\log n(\xi^{N/(N-1)}-\zeta_n\xi)]\,\mathrm{d}\xi=\frac{N}{N-a}.$$

Proof. Talking $t=\zeta_n\zeta$, we get

$$\int_0^{\zeta_n^{-1}}\exp[(N-a)\log n(\xi^{N/(N-1)}-\zeta_n\xi)]\,\mathrm{d}\xi$$
$$=\zeta_n^{-1}\int_0^1\exp[(N-a)\log n[(t\zeta_n^{-1})^{N/(N-1)}-t]]\,\mathrm{d}t.$$

Now, let $g:[0,1]\to\mathbb{R}$ the function defined by $g(t)=(N-a)\log n[(t\zeta_n^{-1})^{N/(N-1)}-t]$. Then

$$g'(0)=-(N-a)\log n$$

and

$$g'(1)=(N-a)\log n\left(\frac{N}{N-1}(\zeta_n^{-1})^{N/(N-1)}-1\right).$$

Let $\epsilon>0$ sufficiently small, then

$$g(t)=-[(N-a)\log n]t+o(t),\quad t\in[0,\epsilon],$$

and

$$g(t) = (N-a)\log n\left(\frac{N}{N-1}(\zeta_n^{-1})^{N/(N-1)} - 1\right)(t-1) + o(t), \quad t \in [1-\epsilon, 1].$$

Therefore,

$$\lim_{n\to\infty} \zeta_n \log n\left(\zeta_n^{-1}\int_0^{\epsilon} \exp[(N-a)\log n[(t\zeta_n^{-1})^{N/(N-1)} - t]]\,\mathrm{d}t\right) = \frac{1}{N-a} \tag{4.4}$$

and

$$\lim_{n\to\infty} \zeta_n \log n\left(\zeta_n^{-1}\int_{1-\epsilon}^{1} \exp[(N-a)\log n[(t\zeta_n^{-1})^{N/(N-1)} - t]]\,\mathrm{d}t\right) = \frac{N-1}{N-a}. \tag{4.5}$$

On the other hand, we have

$$\lim_{n\to\infty} \zeta_n \log n\left(\zeta_n^{-1}\int_{\epsilon}^{1-\epsilon} \exp[(N-a)\log n[(t\zeta_n^{-1})^{N/(N-1)} - t]]\,\mathrm{d}t\right) = 0. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we obtain the result. □

Lemma 4.2. *Suppose that* (V_1) *and* (f_1)–(f_5) *hold. Then there exists* $n \in \mathbb{N}$ *such that*

$$\max_{t\geq 0}\left\{\frac{t^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, tM_n)}{|x|^a}\,\mathrm{d}x\right\} < \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. Fix $r > 0$ such that

$$\beta_0 > \frac{N-a}{Nr^{N-a}}\left(\frac{N-a}{\alpha_0}\right)^{N-1}, \tag{4.7}$$

where β_0 has been given in the assumption (f_5). Suppose, by contradiction, that for all n we have

$$\max_{t\geq 0}\left\{\frac{t^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, tM_n)}{|x|^a}\,\mathrm{d}x\right\} \geq \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

where $M_n(x) = M_n(x, r)$. In view of (3.5), for each n there exists $t_n > 0$ such that

$$\frac{t_n^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, t_n M_n)}{|x|^a}\,\mathrm{d}x = \max_{t\geq 0}\left\{\frac{t^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, tM_n)}{|x|^a}\,\mathrm{d}x\right\}.$$

Thus,

$$\frac{t_n^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, t_n M_n)}{|x|^a}\,\mathrm{d}x \geq \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

and using the fact that $F(x, u) \geq 0$, we obtain

$$t_n^N \geq \left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{4.8}$$

Since at $t = t_n$ we have

$$\frac{d}{dt}\left(\frac{t^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, tM_n)}{|x|^a}\,\mathrm{d}x\right) = 0,$$

it follows that

$$t_n^N = \int_{\mathbb{R}^N} \frac{t_n M_n f\,(x, t_n M_n)}{|x|^a}\,\mathrm{d}x = \int_{|x|\le r} \frac{t_n M_n f\,(x, t_n M_n)}{|x|^a}\,\mathrm{d}x. \tag{4.9}$$

Using hypothesis (f_5), given $\tau > 0$ there exists $R_\tau > 0$ such that for all $u \ge R_\tau$ and $|x| \le r$, we get

$$uf(x, u) \ge (\beta_0 - \tau)\exp(\alpha_0 |u|^{N/(N-1)}). \tag{4.10}$$

From (4.9) and (4.10), for large n, we obtain

$$\begin{aligned} t_n^N &\ge (\beta_0 - \tau)\int_{|x|\le r/n} \frac{\exp(\alpha_0 |t_n M_n|^{N/(N-1)})}{|x|^a}\,\mathrm{d}x \\ &= (\beta_0 - \epsilon)\frac{w_{N-1}}{N-a}\left(\frac{r}{n}\right)^{N-a} \exp(\alpha_0 t_n^{N/(N-1)} w_{N-1}^{-1/(N-1)} \log n + \alpha_0 t_n^{N/(N-1)} d_n). \end{aligned}$$

Thus, setting

$$H_n \doteq \frac{\alpha_0 N \log n}{\alpha_N} {t_n}^{N/(N-1)} + \alpha_0 {t_n}^{N/(N-1)} d_n - N \log t_n - (N-a)\log n$$

we have

$$1 \ge (\beta_0 - \tau)\frac{w_{N-1}}{N-a} r^{N-a} \exp H_n.$$

Consequently, the sequence (t_n) is bounded. Otherwise, up to subsequences, we would have

$$\lim_{n\to\infty} H_n = +\infty,$$

which leads to a contradiction. Moreover, by (4.3), (4.8) and

$$\begin{aligned} t_n^N &\ge (\beta_0 - \tau)\frac{w_{N-1}}{N-a} r^{N-a} \\ &\quad\times \exp\left\{\left[\frac{\alpha_0 {t_n}^{N/(N-1)}}{\alpha_N} - 1\right](N-a)\log n + \alpha_0 {t_n}^{N/(N-1)} d_n\right\}, \end{aligned}$$

it follows that

$$t_n^N \to \left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \quad as\ n \to \infty. \tag{4.11}$$

In order to estimate (4.9) more precisely, we consider the sets

$$A_n = \{x \in B_r : t_n M_n \ge R_\tau\} \quad \text{and} \quad B_n = B_r \backslash A_n.$$

From (4.9) and (4.10) we achieve

$$\begin{aligned} t_n^N &\ge (\beta_0 - \tau)\int_{|x|\le r} \frac{\exp(\alpha_0 |t_n M_n|^{N/(N-1)})}{|x|^a}\,\mathrm{d}x + \int_{B_n} \frac{t_n M_n f\,(x, t_n M_n)}{|x|^a}\,\mathrm{d}x \\ &\quad -(\beta_0 - \tau)\int_{B_n} \frac{\exp(\alpha_0 |t_n M_n|^{N/(N-1)})}{|x|^a}\,\mathrm{d}x. \end{aligned} \tag{4.12}$$

Notice that $M_n(x) \to 0$ and the characteristic functions $\chi_{B_n} \to 1$ for almost every x such that $|x| \le r$. Therefore, the Lebesgue's dominated convergence theorem implies

$$\int_{B_n} \frac{t_n M_n f(x, t_n M_n)}{|x|^a} \, \mathrm{d}x \to 0$$

and

$$\int_{B_n} \frac{\exp(\alpha_0 |t_n M_n|^{N/(N-1)})}{|x|^a} \, \mathrm{d}x \to \frac{w_{N-1}}{N-a} r^{N-a}.$$

Moreover, using that

$$t_n^N \ge \left(\frac{N-a}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1},$$

we have

$$\begin{aligned}
&\int_{|x|\le r} \frac{\exp(\alpha_0 |t_n M_n|^{N/(N-1)})}{|x|^a} \, \mathrm{d}x \\
&\quad \ge \int_{|x|\le r} \frac{\exp(\frac{N-a}{N} \alpha_N |M_n|^{N/(N-1)})}{|x|^a} \, \mathrm{d}x \\
&\quad = \int_{|x|\le r/n} \frac{\exp(\frac{N-a}{N} \alpha_N |M_n|^{N/(N-1)})}{|x|^a} \, \mathrm{d}x \\
&\qquad + \int_{r/n \le |x| \le r} \frac{\exp(\frac{N-a}{N} \alpha_N |M_n|^{N/(N-1)})}{|x|^a} \, \mathrm{d}x.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{|x|\le \frac{r}{n}} \frac{\exp(\frac{N-a}{N} \alpha_N |M_n|^{N/(N-1)})}{|x|^a} \, \mathrm{d}x \\
&\quad = \int_{|x|\le r/n} \frac{\exp[\frac{N-a}{N} \alpha_N \omega_{N-1}^{-1/(N-1)} \log n + d_n \frac{N-a}{N} \alpha_N]}{|x|^a} \, \mathrm{d}x \\
&\quad = \frac{\omega_{N-1}}{N-a} \frac{r^{N-a}}{n^{N-a}} n^{[(N-a) + \frac{N-a}{N} \alpha_N \frac{d_n}{\log n}]} \to \frac{\omega_{N-1}}{N-a} r^{N-a},
\end{aligned}$$

where we have used (4.3). Now, using the change of variable

$$\xi = \frac{\log \frac{r}{s}}{\zeta_n \log n} \text{ with } \zeta_n = \|\widetilde{M}_n\|,$$

by straightforward computation, we have

$$\begin{aligned}
&\int_{\frac{r}{n} \le |x| \le r} \frac{\exp(\frac{N-a}{N} \alpha_N |M_n|^{N/(N-1)})}{|x|^a} \, \mathrm{d}x \\
&\quad = w_{N-1} r^{N-a} \zeta_n \log n \int_0^{\zeta_n^{-1}} \exp[(N-a) \log n (\xi^{N/(N-1)} - \zeta_n \xi)] \, \mathrm{d}\xi
\end{aligned}$$

which converges to $\frac{w_{N-1}}{N-a}r^{N-a}N$ as $n \to \infty$, by Lemma 4.1. Finally, taking $n \to \infty$ in (4.12) and using (4.11) we obtain

$$\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \geq (\beta_0 - \tau)\frac{w_{N-1}}{N-a}r^{N-a}N,$$

which implies that

$$\beta_0 \leq \frac{N-a}{Nr^{N-a}}\left(\frac{N-a}{\alpha_0}\right)^{N-1},$$

contrary to (4.7), and the proof is complete. □

5. Proof of Theorem 1.1

In order to prove that a Palais–Smale sequence converges to a nontrivial solution of problem (1.1) we need to establish the following preliminary lemma.

Lemma 5.1. *Let $(v_n) \subset E$ be a sequence such that $v_n \rightharpoonup u_0$ in E and $\frac{f(x,v_n)}{|x|^a} \to \frac{f(x,u_0)}{|x|^a}$ in $L^1(B_R)$ for any $R > 0$, then*

$$\frac{F(x,v_n)}{|x|^a} \to \frac{F(x,u_0)}{|x|^a} \text{ in } L^1(\mathbb{R}^N).$$

Proof. For any $R > 0$ we have

$$\frac{f(x,v_n)}{|x|^a} \to \frac{f(x,u_0)}{|x|^a} \text{ in } L^1(B_R).$$

Thus, there exists $g \in L^1(B_R)$ such that $\frac{|f(x,v_n)|}{|x|^a} \leq g$ almost everywhere in B_R. (f_3) implies that $F(x,v_n) \leq M_1 + M_0 f(x,v_n)$ almost everywhere in B_R, where $M_1 = \sup_{(x,v_n)\in B_R\times[0,R_0]} F(x,v_n)$. Thus, by generalized Lebesgue dominated convergence theorem (cf. [19])

$$\frac{F(x,v_n)}{|x|^a} \to \frac{F(x,u_0)}{|x|^a} \text{ in } L^1(B_R).$$

Setting $R \geq 1$ we have

$$\int_{|x|>R} \frac{F(x,v_n)}{|x|^a}\,\mathrm{d}x \leq \int_{|x|>R} F(x,v_n)\,\mathrm{d}x. \tag{5.1}$$

Using (f_1) and (f_2), we have

$$\int_{|x|>R} F(x,v_n)\,\mathrm{d}x \leq C_1 \int_{|x|>R} |v_n|^N\,\mathrm{d}x + C_2 \int_{|x|>R} |v_n|\Phi(\alpha_0|v_n|^{N/(N-1))})\,\mathrm{d}x \tag{5.2}$$

and

$$\int_{|x|>R} |v_n|\Phi(\alpha_0|v_n|^{N/(N-1))})\,\mathrm{d}x = \sum_{j=N-1}^{\infty} \frac{\alpha_0^j}{j!}\int_{|x|>R} |v_n||v_n|^{Nj/(N-1)}\,\mathrm{d}x$$

$$= \sum_{j=N-1}^{\infty} \frac{\alpha_0^j}{j!}\int_{|x|>R} |v_n^*||v_n^*|^{Nj/(N-1)}\,\mathrm{d}x,$$

where v_n^* is the Schwarz symmetrization of v_n. Notice that the estimate

$$\int_{|x|>R} \frac{1}{|x|^{1+Nj/(N-1)}}\,\mathrm{d}x = w_{N-1}\int_R^{\infty} \frac{t^{N-1}}{t^{1+Nj/(N-1)}}\,\mathrm{d}t$$

$$= \left(\frac{w_{N-1}}{\frac{Nj}{N-1} - N + 1}\right) R^{N-1-Nj/(N-1)} \leq \frac{w_{N-1}}{R},$$

for all $j \geq N-1$, together with the Radial Lemma leads to

$$\sum_{j=N-1}^{\infty} \frac{\alpha_0^j}{j!}\int_{|x|>R} |v_n^*||v_n^*|^{Nj/(N-1)}\,\mathrm{d}x$$

$$\leq C\left(\frac{N}{w_{N-1}}\right)^{1/N} \sum_{j=N-1}^{\infty} \frac{\alpha_0^j}{j!}\left(\frac{N}{w_{N-1}}\right)^{j/(N-1)} C^{Nj/(N-1)}$$

$$\int_{|x|>R} |x|^{-1-Nj/(N-1)}\,\mathrm{d}x$$

$$\leq \frac{C(N)}{R}.$$

Thus, given $\delta > 0$ there exists $R > 0$ such that

$$\int_{|x|>R} |u_0|^N\,\mathrm{d}x < \delta \quad \text{and} \quad \int_{|x|>R} |v_n|\Phi(\alpha_0|v_n|^{N/(N-1)})\,\mathrm{d}x < \delta,$$

which together with (5.2) imply that

$$\int_{|x|>R} F(x,v_n)\,\mathrm{d}x \leq C\delta \quad \text{and} \quad \int_{|x|>R} F(x,u_0)\,\mathrm{d}x \leq C\delta.$$

Using the estimate

$$\left|\int_{\mathbb{R}^N} \frac{F(x,v_n)}{|x|^a}\,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{F(x,u_0)}{|x|^a}\,\mathrm{d}x\right|$$

$$\leq \left|\int_{B_R} \frac{F(x,v_n)}{|x|^a}\,\mathrm{d}x - \int_{B_R} \frac{F(x,u_0)}{|x|^a}\,\mathrm{d}x\right|$$

$$+ \int_{|x|>R} F(x,v_n)\,\mathrm{d}x + \int_{|x|>R} F(x,u_0)\,\mathrm{d}x$$

we get

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} \frac{F(x,v_n)}{|x|^a}\,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{F(x,u_0)}{|x|^a}\,\mathrm{d}x\right| \leq C\delta,$$

and since δ is arbitrary, the claim is proved. □

In view of Lemmas 3.2 and 3.3 we can apply the mountain-pass theorem without a compactness condition such like the one of the Palais-Smale type. (see [7,18,21]) to obtain a sequence $(u_n) \subset E$ such that $I(u_n) \to c > 0$ and $I'(u_n) \to 0$, that is,

$$\frac{1}{N}\|u_n\|^N - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|x|^a}\,\mathrm{d}x \to c \quad \text{as } n \to \infty, \tag{5.3}$$

$$|I'(u_n)v| \le \tau_n\|v\|, \quad \text{for all } v \in E, \tag{5.4}$$

where $\tau_n \to 0$ as $n \to \infty$. Furthermore, by Lemma 4.2

$$c < \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

From now on, we shall be working in order to prove that (u_n) converges to a weak nontrivial solution u of problem (1.1). From (5.3), (5.4) and (f_2), we get

$$\begin{aligned} C + \tau_n\|u_n\| &\ge \left(\frac{\theta}{N} - 1\right)\|u_n\|^N - \int_{\mathbb{R}^N} \frac{(\theta F(x,u_n) - f(x,u_n)u_n)}{|x|^a}\,\mathrm{d}x \\ &\ge \left(\frac{\theta}{N} - 1\right)\|u_n\|^N, \end{aligned}$$

which implies that

$$\|u_n\| \le C, \quad \int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{|x|^a}\,\mathrm{d}x \le C \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|x|^a}\,\mathrm{d}x \le C. \tag{5.5}$$

Therefore, up to subsequences, we have $u_n \rightharpoonup u$ weakly in $E, u_n \to u$ in $L^q(\mathbb{R}^N), \forall q \ge N$ and $u_n(x) \to u(x)$ almost everywhere in$\mathbb{R}^N$. Moreover, using (5.5) and arguing as in Lemma 2.1 of [9] and Lemma 4 of [12] or Lemma 3.2 of [23], we get

$$\begin{cases} \dfrac{f(x,u_n)}{|x|^a} \to \dfrac{f(x,u)}{|x|^a} \quad \text{in} \quad L^1(B_R), \\ |\nabla u_n|^{N-2}\nabla u_n \to |\nabla u|^{N-2}\nabla u \text{ weakly in } \left(L^{N/(N-1)}(B_R)\right)^N, \end{cases} \tag{5.6}$$

for all $R > 0$. Therefore by (5.4) passing to the limit, we have

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{N-2}\nabla u\nabla\varphi + V(x)|u|^{N-2}u\varphi\right)\mathrm{d}x - \int_{\mathbb{R}^N} \frac{f(x,u)\varphi}{|x|^a}\,\mathrm{d}x = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in E then u is a weak solution of (1.1). Let us show that u is nontrivial. Assume, by contradiction, that $u \equiv 0$. By Lemma 5.1 we have

$$\int_{\mathbb{R}^N} \frac{F(x,u_n)}{|x|^a}\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty.$$

This together with (5.3) imply

$$\|u_n\|^N \to Nc \quad \text{as } n \to +\infty, \tag{5.7}$$

and hence given $\epsilon > 0$, we have $\|u_n\|^N \le Nc + \epsilon$, for large n. Using that

$$c < \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

and choosing $q > 1$ sufficiently close to 1 and ϵ sufficiently small, we obtain

$$q\alpha_0 \|u_n\|^{N/(N-1)}/\alpha_N + qa/N < 1.$$

Hence, by Lemma 2.3, we have

$$\int_{\mathbb{R}^N} \left(\frac{\Phi(\alpha |u_n|^{N/(N-1)})}{|x|^a} \right)^q \mathrm{d}x \le C, \quad \text{for large } n,$$

which, in combination with Hölder inequality and (f_1), implies that

$$\int_{\mathbb{R}^N} \frac{f(x, u_n)}{|x|^a} u_n \,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty.$$

Therefore, from (5.4) with $v = u_n$, we achieve $\|u_n\| \to 0$ as $n \to +\infty$, which contradicts (5.7) since $c > 0$. Thus, u is nontrivial and the proof of our main result is complete.

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References

[1] Adimurthi, A.: Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the N-Laplacian. Ann. Sc. Norm. Sup. Pisa **17**, 393–413 (1990)

[2] Adimurthi, A., Yadava, S.L.: Critical exponent problem in $\mathbb{R}^2$ with Neumann boundary condition. Commun. Partial Differ. Equ. **15**, 461–501 (1990)

[3] Adimurthi, A., Sandeep, K.: A singular Moser–Trudinger embedding and its applications. Nonlinear Differ. Equ. Appl. **13**, 585–603 (2007)

[4] Alves, C.O., do Ó, J.M., Miyagaki, O.H.: On nonlinear perturbations of a periodic elliptic problem in $\mathbb{R}^2$ involving critical growth. Nonlinear Anal. **56**, 781–791 (2004)

[5] Berestycki, H., Lions, P.L.: Nonlinear scalar field equations, I. Existence of ground state. Arch. Rat. Mech. Anal. **82**, 313–346 (1983)

[6] Cao, D.M.: Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$. Commun. Partial Diff. Equ. **17**, 407–435 (1992)

[7] Chabrowski, J.: Variational methods for potential operator equations. With applications to nonlinear elliptic equations. In: de Gruyter Studies in Mathematics, vol. 24. Walter de Gruyter & Co, Berlin (1997)

[8] Costa, D.G.: On a class of elliptic systems in $\mathbb{R}^N$. Electron. J. Differ. Equ. (07), 1–14 (1994)

[9] de Figueiredo, D.G., Miyagaki, O.H., Ruf, B.: Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range. Calc. Var. **3**, 139–153 (1995)

[10] de Souza, M., do Ó, J.M.: On singular Trudinger–Moser type inequality for unbounded domains in dimension two and its applications (preprint)

[11] Díaz, J.I.: Nonlinear partial differential equations and free boundaries. vol. I. Elliptic equations. In: Research Notes in Mathematics, vol. 106, 323 pp. Pitman, Boston (1985)

[12] do Ó, J.M.: Semilinear Dirichlet problems for the $N-$Laplacian in $\mathbb{R}^N$ with nonlinearities in critical growth range. Differ. Integral Equ. **9**(5), 967–979 (1996)

[13] do Ó, J.M.: N-Laplacian equations in $\mathbb{R}^N$ with critical growth. Abstr. Appl. Anal **2**(3-4), 301–315 (1997)

[14] do Ó, J.M., Medeiros, E., Severo, U.B.: On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$. J. Differ. Equ. **246**, 1363–1386 (2009)

[15] do Ó, J.M., Souto, M.A.S.: On a class of nonlinear Schrödinger equations in $\mathbb{R}^2$ involving critical growth. J. Differ. Equ. **174**, 289–311 (2001)

[16] do Ó, J.M., Ruf, B.: On a Schrödinger equation with periodic potential and critical growth in $\mathbb{R}^2$. NoDEA Nonlinear Differ. Equ. Appl. **13**, 167–192 (2006)

[17] Giacomoni, J., Sreenadh, K.: A multiplicity result to a nonhomogeneous elliptic equation in whole space $\mathbb{R}^2$. Adv. Math. Sci. Appl. **15**, 467–488 (2005)

[18] Ghoussoub, N.: Duality and Perturbation Methods in Critical Point Theory. Cambridge University Press, Cambridge (1993)

[19] Kavian, O.: Introduction la théorie des points critiques. Springer-Verlag France, Paris (1993)

[20] Kondratev, V., Shubin, M.: Discreteness of spectrum for the Schrödinger operators on manifolds of bounded geometry. Oper. Theory Adv. Appl. **110**, 185–226 (1999)

[21] Mawhim, J., Willem, M.: Critical Point Theory and Hamiltonian Systems. Springer, Belin (1989)

[22] Moser, J.: A sharp form of an inequality by N. Trudinger. Ind. Univ. Math. J. **20**, 1077–1092 (1971)

[23] Panda, R.: On semilinear Neumann problem with critical growth for the $n-$Laplacian. Nonlinear Anal. T.M.A. **26**, 1347–1366 (1996)

[24] Omana, W., Willem, M.: Homoclinic orbits for a class of Hamiltonian systems. Differ. Integral Equ. **5**, 1115–1120 (1992)

[25] Rabinowitz, P.H.: Minimax methods in critical point theory with applications to differential equations. CBMS Regional Conf. Ser. in Math., vol. 65 (1986)

[26] Rabinowitz, P.H.: On a class of nonlinear Schrödinger equations. Z. Angew Math. Phys (Zamp) **43**, 270–291 (1992)

[27] Silva, E., Soares, S.: Liouville–Gelfand type problems for the N-Laplacian on bounded domains of $\mathbb{R}^N$. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **28**(4), 1–30 (1999)

[28] Trudinger, N.S.: On the imbedding into Orlicz spaces and some applications. J. Math. Mech. **17**, 473–484 (1967)

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