Nonlinear differ. equ. appl. 15 (2008), 535–556

1021-9722/040535-22
published online 26 November 2008
DOI 10.1007/s00030-008-7029-9

Nonlinear Differential Equations and Applications NoDEA

Robust Exponential Attractors for Singularly Perturbed Phase-Field Equations with Dynamic Boundary Conditions

Ciprian G. Gal, Maurizio Grasselli and Alain Miranville

Abstract. We consider a singularly perturbed phase-field model of Caginalp type which is thermally isolated and whose order parameter ϕ is subject to a dynamic boundary condition. More precisely, we indicate by ε a (small) coefficient multiplying $\partial_t u$ in the heat equation, u being the temperature, and we construct a family of exponential attractors which is robust as ε goes to 0. This is physically meaningful since the limiting problem is the viscous Cahn–Hilliard equation for the sole ϕ with a dynamic boundary condition. The upper semicontinuity of the global attractor is also analyzed. The paper extends and revisits some results previously obtained by A. Miranville *et al.*

Mathematics Subject Classification (2000). Primary 35B41, 35K55, 37L30; Secondary 80A22.

Keywords. Phase-field equations, non-isothermal Allen–Cahn systems, dynamic boundary conditions, global attractors, exponential attractors.

1. Introduction

A well-known mathematical model which describes phase transitions in presence of temperature variations, but in absence of mechanical stresses, is the phase-field system (see [7], cf. also [5, 25])

$$\delta\partial_t\phi - \Delta\phi + f_1(\phi) - \lambda u = 0\,, \tag{1.1}$$

$$\varepsilon\partial_t u + \lambda\partial_t\phi - \Delta u = 0\,, \tag{1.2}$$

in $\Omega \times (0, +\infty)$, Ω being a bounded domain in $\mathbb{R}^3$ with smooth boundary Γ. The variable $\phi(x, t)$ represents the order parameter (or phase-field), while $u(x, t)$ stands for the (relative) temperature. The given parameters δ and ε are positive, while the constant λ represents the latent heat. Moreover, the function f_1 is the derivative of a double-well like potential which accounts for the presence of different phases. Of course, Δ is the spatial Laplace operator.

There is a consistent literature on the mathematical analysis of system (1.1)–(1.2) when ϕ is subject to homogeneous Neumann (or Dirichlet) boundary conditions. In particular, the associated dissipative dynamical system has been analyzed in details as well as the convergence to steady states (see, e.g., [1–4, 10, 12, 17, 20–23, 34, 35, 39]). It is also worth observing that system (1.1)–(1.2) can be viewed as a singular perturbation of the celebrated Cahn–Hilliard equation that accounts for phase separation dynamics (see, e.g., [28, 29] and references therein). In fact, if we formally set $\varepsilon = 0$ in equation (1.2), then we can easily deduce the (viscous) Cahn–Hilliard equation

$$\lambda^2 \partial_t \phi - \Delta\big(\delta \partial_t \phi - \Delta\phi + f_1(\phi)\big) = 0\,, \tag{1.3}$$

in $\Omega \times (0, +\infty)$, which reduces to the classical Cahn–Hilliard equation when $\delta = 0$ (see [6], cf. also [33, 36]). More recently, system (1.1)–(1.2) has been endowed with a dynamic boundary condition for ϕ which accounts for possible interactions of the material with the walls (see [19], cf. also [8, 9, 11, 18, 26, 30, 32, 38] and references therein). This condition reads

$$\partial_t \phi = \alpha \Delta_\Gamma \phi - \partial_{\mathbf{n}} \phi - \beta\phi - f_2(\phi)\,, \tag{1.4}$$

on $\Gamma \times (0, +\infty)$. Here α and β are given positive constants, Δ_Γ denotes the Laplace–Beltrami operator on the surface Γ, $\partial_{\mathbf{n}}$ stands for the outward normal derivative, and f_2 is a given function satisfying suitable assumptions. The corresponding problem, with u subject to homogeneous Neumann conditions, has been interpreted and studied in [19] as a dissipative dynamical system, proving the existence of families of exponential attractors $\{\mathcal{M}_\varepsilon\}$ and global attractors $\{\mathcal{A}_\varepsilon\}$. The authors have also shown the uniformity of the former with respect to ε as well as the upper semicontinuity of the latter as ε goes to 0. Here we want to complete their analysis by proving the robustness of $\{\mathcal{M}_\varepsilon\}$, i.e., we obtain the explicit control of the Hausdorff distance between $\mathcal{M}_\varepsilon$ and $\mathcal{M}_0$ by a constant times some power of ε. This result says that the nontransient dynamics of the phase-field system (1.1)–(1.2) with the dynamic boundary condition (1.4) is close to the one of equation (1.3) subject to (1.4) (for similar results see, e.g., [26, 27] and references therein). In addition, we will also give a slight generalization of the upper semicontinuity result. We recall that the upper (and lower) semicontinuity of the global attractor for a phase-field system like (1.1)–(1.2) endowed with standard boundary conditions has been already analyzed in [13–16]. However, only in [27] the existence of a robust family of exponential attractors is established, provided that u and ϕ satisfy homogeneous Dirichlet (or Neumann) boundary conditions. We will follow a similar strategy.

Summing up, we are concerned with the study of the following boundary value problem:

$$\begin{cases} \delta \partial_t \phi = \Delta\phi - f_1(\phi) + \lambda u + g_1\,, & \text{in } \Omega \times (0, +\infty)\,, \\ \partial_t \phi = \alpha \Delta_\Gamma \phi - \partial_{\mathbf{n}} \phi - \beta\phi - f_2(\phi) + g_2\,, & \text{on } \Gamma \times (0, +\infty)\,, \end{cases} \tag{1.5}$$

$$\begin{cases} \varepsilon \partial_t u - \Delta u = -\lambda \partial_t \phi\,, & \text{in } \Omega \times (0, +\infty)\,, \\ \partial_{\mathbf{n}} u = 0\,, & \text{on } \Gamma \times (0, +\infty)\,, \end{cases} \tag{1.6}$$

endowed with the initial conditions

$$\phi_{|t=0} = \phi_0 \,, \quad u_{|t=0} = u_0 \,. \tag{1.7}$$

Here g_1 and g_2 are given external forces. The limiting equations of the above problem are formally obtained by taking $\varepsilon = 0$. This yields

$$\begin{cases} \delta\partial_t\overline{\phi}_0 = \Delta\overline{\phi}_0 - f_1\left(\overline{\phi}_0\right) + \lambda\overline{u}_0 + g_1 \,, & \text{in} \quad \Omega \times (0,+\infty)\,, \\ \partial_t\overline{\phi}_0 = \alpha\Delta_\Gamma\overline{\phi}_0 - \partial_{\mathbf{n}}\overline{\phi}_0 - \beta\overline{\phi}_0 - f_2\left(\overline{\phi}_0\right) + g_2 \,, & \text{on} \quad \Gamma \times (0,+\infty)\,, \end{cases} \tag{1.8}$$

$$\begin{cases} \lambda\partial_t\overline{\phi}_0 = \Delta\overline{u}_0 \,, & \text{in} \quad \Omega \times (0,+\infty)\,, \\ \partial_{\mathbf{n}}\overline{u}_0 = 0 \,, & \text{in} \quad \Gamma \times (0,+\infty)\,, \end{cases} \tag{1.9}$$

with initial condition

$$\overline{\phi}_{0|t=0} = \phi_0 \,. \tag{1.10}$$

Note that (1.3) can replace the first of (1.8) and (1.9). Moreover, $\overline{\phi}_0$ also fulfills the boundary condition

$$\partial_{\mathbf{n}}\big(\delta\partial_t\overline{\phi}_0 - \Delta\overline{\phi}_0 + f_1(\overline{\phi}_0) - g_1\big) = 0\,, \tag{1.11}$$

on $\Gamma \times (0,+\infty)$. Therefore, the spatial average of $\overline{\phi}_0$ is conserved.

The paper is organized as follows. In Section 2 we recall some existence results and several useful estimates proved in [19] (see also [18]) and state the main result, i.e., the existence of a robust family of exponential attractors. Section 3 is devoted to the most crucial step, namely, estimates on the difference between the solutions to problem (1.5)–(1.7) and problem (1.8)–(1.10). This result allows us to prove in Section 4 the existence of a family of exponential attractors that is robust with respect to ε. Finally, in Section 5, we slightly extend the result of [19] on the upper semicontinuity of the global attractors $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$, taking a larger phase-space.

2. Preliminaries and main results

Following [19,26], it is convenient to introduce an additional variable $\psi := \phi_{|\Gamma}$ and to interpret the dynamic boundary condition (1.5) as an evolution equation on the boundary Γ. Hence, for any $\varepsilon \in (0,1]$, problem (1.5)–(1.7) becomes

Problem $\mathcal{P}_\varepsilon$. *Find (ϕ,ψ,u) such that*

$$\begin{cases} \delta\partial_t\phi = \Delta\phi - f_1(\phi) + \lambda u + g_1 \,, & \textit{in} \quad \Omega \times (0,+\infty)\,, \\ \partial_t\psi = \alpha\Delta_\Gamma\psi - \partial_{\mathbf{n}}\phi - \beta\psi - f_2(\psi) + g_2 \,, & \textit{on} \quad \Gamma \times (0,+\infty)\,, \\ \psi = \phi_{|\Gamma}\,, \end{cases} \tag{2.1}$$

$$\begin{cases} \varepsilon\partial_t u - \Delta u = -\lambda\partial_t\phi \,, & \textit{in} \quad \Omega \times (0,+\infty)\,, \\ \partial_{\mathbf{n}} u = 0 \,, & \textit{on} \quad \Gamma \times (0,+\infty)\,, \end{cases} \tag{2.2}$$

with the initial conditions

$$\phi_{|t=0} = \phi_0 \,, \quad \psi_{|t=0} = \psi_0 \,, \quad u_{|t=0} = u_0 \,. \tag{2.3}$$

Observe that, due to the boundary conditions (2.2), the enthalpy is conserved, namely,

$$I_\varepsilon := \varepsilon\big\langle u(t)\big\rangle + \lambda\big\langle \phi(t)\big\rangle = \varepsilon \left\langle u_0\right\rangle + \lambda \left\langle \phi_0\right\rangle , \tag{2.4}$$

for any $t \geq 0$, where $\langle v\rangle$ denotes the spatial average of a function v on Ω.

Let us introduce the Hilbert spaces

$$\mathbb{V}_s := H^s(\Omega) \times H^s(\Gamma) \times H^s(\Omega),$$

for any $s \in \mathbb{N}$. The spaces $H^s(\Omega)$ and $H^s(\Gamma)$ are endowed with the norms induced by their standard inner products and are denoted by $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{H^s(\Gamma)}$, if $s > 0$, while, if $s = 0$, by $\|\cdot\|_2$ and $\|\cdot\|_{2,\Gamma}$, respectively. In particular, $\langle\cdot,\cdot\rangle_2$ denotes the standard scalar product in $L^2(\Omega)$. Then (see [19,26]) we introduce the function space

$$\mathbb{D}_\varepsilon^M := \Big\{(\phi,\psi,u) \in \mathbb{V}_2 : \psi = \phi_{|\Gamma},\ \ (\partial_{\mathbf{n}} u)_{|\Gamma} = 0,\ \ |I_\varepsilon| \leq M\Big\}, \tag{2.5}$$

for any given $\varepsilon \in (0,1]$ and any fixed $M \geq 0$. This space is a complete metric space endowed with the metric induced by the $\mathbb{V}_2$-norm.

We assume that

$$g_1 \in L^2(\Omega), \quad g_2 \in L^2(\Gamma), \tag{2.6}$$

while, concerning the nonlinear functions $f_i : \mathbb{R} \to \mathbb{R}$, we assume that they belong to $C'(\mathbb{R})$ and satisfy the conditions

$$\lim_{|y|\to+\infty} \inf f_i'(y) > 0, \tag{2.7}$$

$$f_i(y)\, y \geq \nu_i y^2 - \nu_i', \tag{2.8}$$

for some positive ν_1 and some nonnegative ν_2, ν_1', ν_2'. In addition, we assume the local Lipschitz continuity of f_i', $i = 1, 2$.

On account of [19, Lemma 2.1] (see also [27, Lemma 1.3]), the following a priori estimate can be proven.

Theorem 1. *Let assumptions* (2.6)–(2.8) *be satisfied. Then, every sufficiently smooth solution* $(\phi(t), \psi(t), u(t))$ *to* $\mathcal{P}_\varepsilon$ *satisfies the following estimate:*

$$\begin{aligned}\big\|\big(\phi(t), \psi(t), u(t)\big)\big\|_{\mathbb{V}_2}^2 &+ \varepsilon^2 \left\|\partial_t u(t)\right\|_2^2 + \left\|\partial_t \phi(t)\right\|_2^2 + \left\|\partial_t \psi(t)\right\|_{2,\Gamma}^2 \\ &+ \int_t^{t+1} \Big(\left\|\partial_t \phi(s)\right\|_{H^1(\Omega)}^2 + \left\|\partial_t \psi(s)\right\|_{H^1(\Gamma)}^2\Big)\, ds \\ &\leq Q_1\Big(\|(\phi_0,\psi_0,u_0)\|_{\mathbb{V}_2}^2\Big)\, e^{-\rho t} + Q_1\Big(\|g_1\|_2^2 + \|g_1\|_{2,\Gamma}^2\Big),\end{aligned} \tag{2.9}$$

where $\rho > 0$ *and the positive and monotone increasing function* Q_1 *are independent of* ε.

Existence and uniqueness for $\mathcal{P}_\varepsilon$ have also been proved in [19] (see also [18] for a slightly more general result). Clearly, Theorem 1 entails that the corresponding semiflow has a bounded absorbing set in $\mathbb{V}_2$.

Theorem 2. *Let assumptions* (2.6)–(2.8) *be satisfied. Then, for every* $(\phi_0, \psi_0, u_0) \in \mathbb{D}_\varepsilon^M$*, problem* $\mathcal{P}_\varepsilon$ *has a unique solution* $(\phi(t), \psi(t), u(t)) \in C([0, +\infty), \mathbb{D}_\varepsilon^M)$ *which satisfies estimate* (2.9)*. Consequently,* $\mathcal{P}_\varepsilon$ *defines a semiflow* $S_t^\varepsilon : \mathbb{D}_\varepsilon^M \to \mathbb{D}_\varepsilon^M$ *defined by*

$$S_t^\varepsilon (\phi_0, \psi_0, u_0) := \big(\phi(t), \psi(t), u(t)\big) , \quad \forall\, t \geq 0 . \tag{2.10}$$

Let us now consider the limiting problem $\mathcal{P}_0$ which can be formulated as

Problem $\mathcal{P}_0$**.** *Find* $(\overline{\phi}_0, \overline{\psi}_0, \overline{u}_0)$ *such that*

$$\begin{cases} \delta \partial_t \overline{\phi}_0 = \Delta \overline{\phi}_0 - f_1 \left(\overline{\phi}_0\right) + \lambda \overline{u}_0 + g_1 , \quad \textit{in} \quad \Omega \times (0, +\infty) , \\ \partial_t \overline{\psi}_0 = \alpha \Delta_\Gamma \overline{\psi}_0 - \partial_{\mathbf{n}} \overline{\phi}_0 - \beta \overline{\psi}_0 - f_2 \left(\overline{\psi}_0\right) + g_2 , \quad \textit{on} \quad \Gamma \times (0, +\infty) , \\ \overline{\psi}_0 = \overline{\phi}_{0|\Gamma} , \end{cases} \tag{2.11}$$

$$\begin{cases} \lambda \partial_t \overline{\phi}_0 = \Delta \overline{u}_0 , \quad \textit{in} \quad \Omega \times (0, +\infty) , \\ \partial_{\mathbf{n}} \overline{u}_0 = 0 , \quad \textit{on} \quad \Gamma \times (0, +\infty) , \end{cases} \tag{2.12}$$

with the initial conditions

$$\overline{\phi}_{0|t=0} = \phi_0 , \quad \overline{\psi}_{0|t=0} = \psi_0 . \tag{2.13}$$

Observe that, from the first equations of (2.11) and (2.12), it follows that

$$\begin{cases} -\delta \Delta \overline{u}_0 + \lambda^2 \overline{u}_0 = -\lambda \big(\Delta \overline{\phi}_0 - f_1(\overline{\phi}_0) + g_1\big) , \\ \partial_{\mathbf{n}} \overline{u}_0 = 0 . \end{cases} \tag{2.14}$$

Therefore $\overline{u}_0\,(t)$ is uniquely defined by (2.14), provided that $\overline{\phi}_0\,(t)$ is known. Then, by standard elliptic estimates, it follows that there exists a nonlinear operator (see [27])

$$\mathcal{L} \in C^1 \Big(H^2\,(\Omega) \, , \big\{ v \in H^2\,(\Omega) : \partial_{\mathbf{n}} v = 0 \big\} \Big) , \tag{2.15}$$

such that, for any $t \geq 0$ and any $\overline{\phi}_0\,(t) \in H^2(\Omega)$,

$$\overline{u}_0(t) = \mathcal{L}\big(\overline{\phi}_0(t)\big) . \tag{2.16}$$

Consequently, the solution to $\mathcal{P}_0$ exists only for initial data (ϕ_0, ψ_0, u_0) belonging to the infinite dimensional submanifold $\mathbb{L}^M$ of the phase space $\mathbb{V}_2$ defined by

$$\mathbb{L}^M := \big\{ \, (\phi_0, \psi_0, u_0) \in \mathbb{V}_2 : \psi_0 = \phi_{0|\Gamma} , \ u_0 = \mathcal{L}\,(\phi_0) , \\ \partial_{\mathbf{n}} u_0 = 0, \ |\lambda \, \langle \phi_0 \rangle| \leq M \big\} . \tag{2.17}$$

The following theorem is a direct consequence of the results in [19, 27].

Theorem 3. *Let assumptions* (2.6)–(2.8) *be satisfied. Then, for every* $(\phi_0, \psi_0, u_0) \in \mathbb{L}^M$*, problem* $\mathcal{P}_0$ *has a unique solution* $(\overline{\phi}_0(t), \overline{\psi}_0(t), \overline{u}_0(t)) \in C([0, +\infty); \mathbb{L}^M)$ *which* *satisfies the following estimate:*

$$\begin{aligned}&\left\|\left(\overline{\phi}_0\left(t\right),\overline{\psi}_0\left(t\right),\overline{u}_0\left(t\right)\right)\right\|_{\mathbb{V}_2}^2+\left\|\partial_t\overline{u}_0\left(t\right)\right\|_2^2+\left\|\partial_t\overline{\phi}_0\left(t\right)\right\|_2^2+\left\|\partial_t\overline{\psi}_0\left(t\right)\right\|_{2,\Gamma}^2\\&\quad+\int_t^{t+1}\Big(\left\|\left(\partial_t\overline{\phi}_0\left(s\right),\partial_t\overline{\psi}_0\left(s\right),\partial_t\overline{u}_0\left(s\right)\right)\right\|_{\mathbb{V}_1}^2+\left\|\partial_t^2\overline{u}_0\left(s\right)\right\|_{(H^1(\Omega))^*}^2\Big)ds\\&\le Q_2\big(\left\|(\phi_0,\psi_0,u_0)\right\|_{\mathbb{V}_2}^2\big)e^{-\rho t}+Q_2\big(\left\|g_1\right\|_2^2+\left\|g_1\right\|_{2,\Gamma}^2\big),\end{aligned}\tag{2.18}$$

for some $\rho>0$ and some positive and monotone increasing function Q_2. Consequently, $\mathcal{P}_0$ defines a semiflow S_t^0 on the manifold $\mathbb{L}^M$ by setting

$$S_t^0:\mathbb{L}^M\to\mathbb{L}^M,\quad S_t^0(\phi_0,\psi_0,u_0):=\big(\overline{\phi}_0(t),\overline{\psi}_0(t),\overline{u}_0(t)\big).\tag{2.19}$$

Let us now introduce the projection $\mathbb{P}:\mathbb{L}^M\to\widehat{\mathbb{L}}^M$, by setting $\mathbb{P}\left(\phi_0,\psi_0,u_0\right)=(\phi_0,\psi_0)$, where

$$\widehat{\mathbb{L}}^M:=\big\{\left(\phi_0,\psi_0\right)\in H^2\left(\Omega\right)\times H^2\left(\Gamma\right):\psi_0=\phi_{0|\Gamma},\ \left|\lambda\left\langle\phi_0\right\rangle\right|\le M\big\}.\tag{2.20}$$

Then, we define a semiflow $\widehat{S}_t^0$ on the complete metric space $\widehat{\mathbb{L}}^M$ by setting

$$\widehat{S}_t^0:\widehat{\mathbb{L}}^M\to\widehat{\mathbb{L}}^M,\quad \widehat{S}_t^0(\phi_0,\psi_0):=\big(\overline{\phi}_0(t),\overline{\psi}_0(t)\big),\tag{2.21}$$

where $(\overline{\phi}_0(t),\overline{\psi}_0(t))$ is the unique solution to (1.3)–(1.4) and (1.11) with $\overline{\psi}_0=\overline{\phi}_{0|\Gamma}$. This is nothing but the dynamical system associated with the viscous Cahn–Hilliard equation subject to no-flux and dynamic boundary conditions. It is clear that $\mathbb{P}S_t^0=\widehat{S}_t^0$. On the other hand, S_t^0 can be obtained from $\widehat{S}_t^0$ by a lifting of $\widehat{\mathbb{L}}^M$ to $\mathbb{L}^M$ defined through (2.14). It is known that $(\widehat{S}_t^0,\widehat{\mathbb{L}}^M)$ is a dissipative dynamical system and possesses an exponential attractor $\widehat{\mathcal{M}}_0^M$ which is contained in a bounded subset of $H^3(\Omega)\times H^3(\Gamma)$ (see [27]). Correspondingly, we set

$$\mathcal{M}_0^M:=\big\{\left(\phi,\psi,u\right)\in\mathbb{L}^M:\left(\phi,\psi\right)\in\widehat{\mathcal{M}}_0^M,\ u=\mathcal{L}\left(\phi\right)\big\}.\tag{2.22}$$

We are now ready to state the main result of this paper.

Theorem 4. *Let assumptions* (2.6)–(2.8) *be satisfied. Then, for every fixed $M\ge 0$, there exists a family of compact sets $\mathcal{M}_\varepsilon^M\subset\mathbb{D}_\varepsilon^M$, where $\varepsilon\in[0,1]$ and $\mathbb{D}_0^M:=\mathbb{L}^M$, with the following properties:*

(i) *The sets $\mathcal{M}_\varepsilon^M$ are semi-invariant with respect to the semiflows S_t^ε associated with problem $\mathcal{P}_\varepsilon$, that is,*

$$S_t^\varepsilon\left(\mathcal{M}_\varepsilon^M\right)\subseteq\mathcal{M}_\varepsilon^M,\quad\forall\, t\ge 0.\tag{2.23}$$

(ii) *The fractal dimension of the sets $\mathcal{M}_\varepsilon^M$ is finite and uniformly bounded with respect to ε, i.e.,*

$$\dim_F\left(\mathcal{M}_\varepsilon^M,\mathbb{D}_\varepsilon^M\right)\le C_M<+\infty,\tag{2.24}$$

where C_M is independent of ε.

(iii) *Each $\mathcal{M}_\varepsilon^M$ attracts exponentially any bounded subset of $\mathbb{D}_\varepsilon^M$, that is, there exist a positive constant ρ and a monotonic nonnegative function Q, depending on M but independent of ε, such that, for every bounded subset B of $\mathbb{D}_\varepsilon^M$, we have*

$$dist_{\mathbb{D}_\varepsilon^M}\left(S_t^\varepsilon B, \mathcal{M}_\varepsilon^M\right) \leq Q\left(\|B\|_{\mathbb{D}_\varepsilon^M}\right) e^{-\rho t}\,, \tag{2.25}$$

where $dist_{\mathbb{D}_\varepsilon^M}(X,Y) := \sup_{x\in X}\inf_{y\in Y}\|x-y\|_{\mathbb{V}_2}$ is the Hausdorff semidistance.

(iv) *There exist positive constants K and $\kappa \in (0,1)$, depending on M but independent of ε, such that*

$$dist_{\mathbb{D}_\varepsilon^M}^{symm}\left(\mathcal{M}_0^M, \mathcal{M}_\varepsilon^M\right) \leq K\varepsilon^\kappa\,, \tag{2.26}$$

where $dist_{\mathbb{D}_\varepsilon^M}^{symm}(X,Y) := \max\{dist_{\mathbb{D}_\varepsilon^M}(X,Y), dist_{\mathbb{D}_\varepsilon^M}(Y,X)\}$ is the Hausdorff distance.

Remark 1. Let us recall that the only novelty here is property (*iv*) and its proof since the rest was already proven in [18, Thm. 4.2].

3. Estimates on the difference of solutions

The main goal of this section is to estimate the difference of the solutions to problems $\mathcal{P}_\varepsilon$ and $\mathcal{P}_0$. This is a very crucial step in order to prove property (*iv*) of Theorem 4 (cf. next section) and it has an interest on its own.

We argue as in [27] using the boundary layer technique devised in [37]. Thus we start by computing the first terms of the asymptotic expansions of the solution $(\phi(t), \psi(t), u(t))$ of problem $\mathcal{P}_\varepsilon$ as $\varepsilon \to 0$. This is done by introducing the fast variable $\tau := t/\varepsilon$ and expanding this solution as follows:

$$\begin{cases} \phi(t) = \phi_0(t,\tau) + \varepsilon\phi_1(t,\tau) + \cdots\,, \\ \psi(t) = \psi_0(t,\tau) + \varepsilon\psi_1(t,\tau) + \cdots\,, \\ u(t) = u_0(t,\tau) + \varepsilon u_1(t,\tau) + \cdots\,, \end{cases} \tag{3.1}$$

where $\phi_i(t,\tau)$, $\psi_i(t,\tau)$ and $u_i(t,\tau)$ are functions (independent of ε) of the form

$$\phi_i(t,\tau) = \overline{\phi}_i(t)+\widetilde{\phi}_i(\tau)\,, \quad \psi_i(t,\tau) = \overline{\psi}_i(t)+\widetilde{\psi}_i(\tau)\,, \quad u_i(t,\tau) = \overline{u}_i(t)+\widetilde{u}_i(\tau)\,, \tag{3.2}$$

with

$$\lim_{\tau\to+\infty}\widetilde{\phi}_i(\tau) = \lim_{\tau\to+\infty}\widetilde{\psi}_i(\tau) = \lim_{\tau\to+\infty}\widetilde{u}_i(\tau) = 0\,. \tag{3.3}$$

Inserting expansions (3.1) into equations (2.1)–(2.2), we obtain some equations for the terms $\phi_i(t,\tau)$, $\psi_i(t,\tau)$ and $u_i(t,\tau)$. Indeed, at order ε^{-1}, it follows from equation (2.1) that

$$\delta\partial_\tau\widetilde{\phi}_0(\tau) = 0\,, \quad \partial_\tau\widetilde{\psi}_0(\tau) = 0\,.$$

Consequently, by (3.3), we have $\widetilde{\phi}_0(\tau) = 0$, $\widetilde{\psi}_0(\tau) = 0$. At order ε, equations (2.1) yield

$$\begin{cases} \delta\partial_t\overline{\phi}_0(t) = \Delta\overline{\phi}_0(t) - f_1\big(\overline{\phi}_0(t)\big) + \lambda\overline{u}_0(t) + g_1\,, \\ \partial_t\overline{\psi}_0(t) = \alpha\Delta_\Gamma\overline{\psi}_0(t) - \partial_{\mathbf{n}}\overline{\phi}_0(t) - \beta\overline{\psi}_0(t) - f_2\big(\overline{\psi}_0(t)\big) + g_2\,. \end{cases} \tag{3.4}$$

On the other hand, we deduce from equation (2.2) that

$$\begin{cases} \lambda\partial_t\overline{\phi}_0(t) = \Delta\overline{u}_0(t)\,, \\ \partial_{\mathbf{n}}\overline{u}_0(t) = 0\,, \end{cases} \tag{3.5}$$

and

$$\begin{cases} \partial_\tau\widetilde{u}_0(\tau) - \Delta\widetilde{u}_0(\tau) = -\lambda\partial_\tau\widetilde{\phi}_1(\tau)\,, \\ \partial_{\mathbf{n}}\widetilde{u}_0(\tau) = 0\,, \end{cases} \tag{3.6}$$

with

$$\delta\partial_\tau\widetilde{\phi}_1(\tau) = \lambda\widetilde{u}_0(\tau)\,, \quad \partial_\tau\widetilde{\psi}_1(\tau) = 0\,.$$

Consequently, on account of (3.3), the remaining boundary layer terms are found from

$$\widetilde{\phi}_1(\tau) = \frac{\lambda}{\delta}\int_\tau^\infty \widetilde{u}_0(s)ds\,, \quad \widetilde{\psi}_1(\tau) = 0\,. \tag{3.7}$$

Expanding now the initial data, we have

$$\begin{cases} \overline{\phi}_1(0) + \widetilde{\phi}_1(0) = 0\,,\ \widetilde{\phi}_0(0) = 0\,,\ \overline{\phi}_0(0) = \phi(0)\,, \\ \widetilde{\psi}_0(0) = \overline{\psi}_1(0) = \widetilde{\psi}_1(0) = 0\,,\ \overline{\psi}_0(0) = \psi(0)\,, \end{cases} \tag{3.8}$$

and

$$\overline{u}_1(0) + \widetilde{u}_1(0) = 0\,, \quad \widetilde{u}_0(0) = u(0) - \overline{u}_0(0)\,.$$

Hence, the function $(\overline{\phi}_0(t), \overline{\psi}_0(t), \overline{u}_0(t))$ solves (3.4)–(3.5) with initial data $\overline{\phi}_0(0) = \phi(0)$, $\overline{\psi}_0(0) = \psi(0)$, that is,

$$\big(\overline{\phi}_0(t), \overline{\psi}_0(t), \overline{u}_0(t)\big) = S_t^0\Big(\phi(0), \psi(0), \mathcal{L}\big(\phi(0)\big)\Big)\,. \tag{3.9}$$

Recalling [27], we observe that it suffices to seek for a solution of problem $\mathcal{P}_\varepsilon$ of the form

$$\begin{cases} \phi(t) = \overline{\phi}_0(t) + \varepsilon\widetilde{\phi}(\tau) + \varepsilon\widehat{\phi}(t)\,, \\ \psi(t) = \overline{\psi}_0(t) + \varepsilon\widehat{\psi}(t)\,, \\ u(t) = \overline{u}_0(t) + \widetilde{u}(\tau) + \varepsilon\widehat{u}(t)\,, \end{cases} \tag{3.10}$$

where $(\overline{\phi}_0(t), \overline{\psi}_0(t), \overline{u}_0(t))$ is given by (3.9), the boundary layer term $\widetilde{u}(\tau)$ solves

$$\begin{cases} \partial_\tau\widetilde{u}(\tau) - \Delta\widetilde{u}(\tau) + \frac{\lambda^2}{\delta}\widetilde{u}(\tau) = 0\,, \\ \partial_{\mathbf{n}}\widetilde{u}(\tau) = 0\,, \\ \widetilde{u}(0) = u(0) - \mathcal{L}\big(\phi(0)\big)\,, \end{cases} \tag{3.11}$$

and the boundary layer terms $\widetilde{\phi}(\tau)$ and $\widetilde{\psi}(\tau)$ are defined by (3.7) (where $\widetilde{u}_0(\tau)$, $\widetilde{\phi}_1(\tau)$, $\widetilde{\psi}_1(\tau)$ are replaced by $\widetilde{u}(\tau)$, $\widetilde{\phi}(\tau)$ and $\widetilde{\psi}(\tau)$, respectively). Moreover, the boundary layer terms $\widehat{\phi}(t)$, $\widehat{\psi}(t)$ and $\widehat{u}(t)$ satisfy the following problems:

$$\begin{cases} \delta\partial_t\widehat{\phi}(t) = \Delta\widehat{\phi}(t) - \frac{1}{\varepsilon}\Big[f_1\big(\overline{\phi}_0(t) + \varepsilon\widetilde{\phi}(\tau) + \varepsilon\widehat{\phi}(t)\big) - f_1\big(\overline{\phi}_0(t)\big)\Big] \\ \qquad\qquad + \lambda\widehat{u}(t) + \Delta\widetilde{\phi}(\tau)\,, \\ \qquad\qquad \partial_t\widehat{\psi}(t) = \alpha\Delta_\Gamma\widehat{\psi}(t) - \partial_{\mathbf{n}}\widehat{\phi}(t) - \beta\widehat{\psi}(t) \\ \qquad\qquad - \frac{1}{\varepsilon}\Big[f_2\big(\overline{\psi}_0(t) + \varepsilon\widehat{\psi}(t)\big) - f_2(\overline{\psi}_0)\Big] - \partial_{\mathbf{n}}\widetilde{\phi}(\tau)\,, \\ \qquad\qquad \widehat{\phi}(0) = -\widetilde{\phi}(0)\,, \quad \widehat{\psi}(0) = 0\,, \end{cases} \tag{3.12}$$

and

$$\begin{cases} \varepsilon\partial_t\widehat{u}(t) = \Delta\widehat{u}(t) + \partial_t\widehat{\phi}(t) - \partial_t\overline{u}_0(t)\,, \\ \partial_{\mathbf{n}}\widehat{u}(t) = 0\,, \\ \widehat{u}(0) = 0\,. \end{cases} \tag{3.13}$$

The next lemma provides some estimates on the boundary layer terms $\widetilde{u}(\tau)$ and $\widetilde{\phi}(\tau)$, given by (3.11) and (3.7), respectively. The result is a straightforward consequence of standard energy estimates.

Lemma 5. *The following estimates hold:*

$$\|\widetilde{u}(\tau)\|_{H^2(\Omega)} + \|\partial_\tau\widetilde{u}(\tau)\|_2 \leq C\|\widetilde{u}(0)\|_{H^2(\Omega)}e^{-\rho\tau}\,, \tag{3.14}$$

$$\|\widetilde{\phi}(\tau)\|_{H^2(\Omega)} + \|\partial_\tau\widetilde{\phi}(\tau)\|_{H^2(\Omega)} \leq C\|\widetilde{u}(0)\|_{H^2(\Omega)}e^{-\rho\tau}\,, \tag{3.15}$$

where ρ, $C > 0$ are both independent of ε.

We can now estimate the remainder terms $\widehat{\phi}(t)$, $\widehat{\psi}(t)$ and $\widehat{u}(t)$ in expansion (3.10).

Lemma 6. *The following estimate holds:*

$$\big\|\big(\widehat{\phi}(t), \widehat{\psi}(t), \widehat{u}(t)\big)\big\|^2_{\mathbb{V}_2} + \|\partial_t\widehat{\phi}(t)\|_2^2 + \|\partial_t\widehat{\psi}(t)\|^2_{2,\Gamma} + \varepsilon\|\partial_t\widehat{u}(t)\|_2^2 \leq Ce^{Lt}\,, \tag{3.16}$$

where the positive constants C and L depend on $\|(\phi(0), \psi(0), u(0))\|_{\mathbb{V}_2}$, but are independent of ε.

Proof. We first note that the functions $\widetilde{\phi}(\tau)$, $\varepsilon\widehat{\phi}(t)$ and $\varepsilon\widehat{\psi}(t)$ are uniformly bounded with respect to ε in $H^2(\Omega)$, $H^2(\Omega)$ and $H^2(\Gamma)$, respectively. This easily follows from estimates (3.14)–(3.15), (2.9), (2.18). In particular, the initial datum $\widetilde{\phi}(0)$ is uniformly bounded in $H^2(\Omega)$ as $\varepsilon \to 0$. Observe preliminarily that all the constants C_i are independent of ε. Moreover, note that (2.7) entails

$$f_i'(y) \geq -K_i\,, \quad \forall\, y \in \mathbb{R}\,, \tag{3.17}$$

for some $K_i > 0$, $i = 1, 2$.

Multiplying the first equation of (3.12) by $\partial_t\widehat{\phi}(t)$, integrating over Ω, and using the second equation of (3.12), we have

$$\begin{aligned} \partial_t\big(\|\nabla\widehat{\phi}(t)\|_2^2 + \alpha\|\nabla_\Gamma\widehat{\psi}(t)\|^2_{2,\Gamma} + \beta\|\widehat{\psi}(t)\|^2_{2,\Gamma}\big) + 2\delta\|\partial_t\widehat{\phi}(t)\|_2^2 + 2\|\partial_t\widehat{\psi}(t)\|^2_{2,\Gamma} \\ \leq C_1\big(\|\widehat{\phi}(t)\|^2_{H^1(\Omega)} + \|\widehat{\psi}(t)\|^2_{2,\Gamma}\big) + C_2\big(\|\widetilde{\phi}(\tau)\|^2_{H^2(\Omega)} + \|\widehat{u}(t)\|_2^2\big)\,, \end{aligned} \tag{3.18}$$

where $C_1 > 0$ only depends on K_i and $C_2 > 0$ only depends on $\|\widetilde{\phi}(0)\|_{H^2(\Omega)}$. We now differentiate both equations of (3.12) with respect to t, multiply the first equation by $\partial_t\widehat{\phi}(t)$ and integrate over Ω to obtain

$$\begin{aligned}
&\partial_t\big(\delta\|\partial_t\widehat{\phi}(t)\|_2^2 + \|\partial_t\widehat{\psi}(t)\|_{2,\Gamma}^2\big) + 2\|\partial_t\widehat{\phi}(t)\|_{H^1(\Omega)}^2 + 2\|\partial_t\widehat{\psi}(t)\|_{H^1(\Gamma)}^2 \\
&\quad - 2\lambda\big\langle\partial_t\widehat{u}(t), \partial_t\widehat{\phi}(t)\big\rangle_2 \le 2K_1\|\partial_t\widehat{\phi}(t)\|_2^2 + 2K_2\|\partial_t\widehat{\psi}(t)\|_{2,\Gamma}^2 \\
&\quad - \frac{2}{\varepsilon}\Big[\Big\langle\Big(f_1'\big(\overline{\phi}_0(t) + \varepsilon\widetilde{\phi}(\tau) + \varepsilon\widehat{\phi}(t)\big) - f_1'\big(\overline{\phi}_0(t)\big)\Big)\partial_t\overline{\phi}_0(t), \partial_t\widehat{\phi}(t)\Big\rangle_2\Big] \\
&\quad - 2\Big[\Big\langle f_1'\big(\overline{\phi}_0(t) + \varepsilon\widetilde{\phi}(\tau) + \varepsilon\widehat{\phi}(t)\big)\partial_t\widetilde{\phi}(\tau), \partial_t\widehat{\phi}(t)\Big\rangle_2\Big] \\
&\quad - \frac{2}{\varepsilon}\Big[\Big\langle\Big(f_2'\big(\overline{\psi}_0(t) + \varepsilon\widehat{\psi}(t)\big) - f_2'(\overline{\psi}_0)\Big)\partial_t\overline{\psi}_0(t), \partial_t\widehat{\psi}(t)\Big\rangle_{2,\Gamma}\Big] \\
&\quad + \|\partial_t\Delta\widetilde{\phi}(\tau)\|_2\big(1 + \|\partial_t\widehat{\phi}(t)\|_2^2\big) + \|\partial_t\partial_{\mathbf{n}}\widetilde{\phi}(\tau)\|_{2,\Gamma}\big(1 + \|\partial_t\widehat{\psi}(t)\|_{2,\Gamma}^2\big)\,. \qquad (3.19)
\end{aligned}$$

Our aim is to estimate all the remaining terms on the right-hand side of (3.19). Since $\partial_t\overline{\phi}_0(t)$, $\partial_t\overline{\psi}_0(t)$ are bounded in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively, it follows, with the help of estimates (2.9) and (2.18), that

$$\begin{aligned}
&-\frac{2}{\varepsilon}\Big[\Big\langle\Big(f_1'\big(\overline{\phi}_0(t) + \varepsilon\widetilde{\phi}(\tau) + \varepsilon\widehat{\phi}(t)\big) - f_1'\big(\overline{\phi}_0(t)\big)\Big)\partial_t\overline{\phi}_0(t), \partial_t\widehat{\phi}(t)\Big\rangle_2\Big] \\
&\quad \le C_3\big\langle 1 + |\widehat{\phi}(t)||\partial_t\overline{\phi}_0(t)|, |\partial_t\widehat{\phi}(t)|\big\rangle_2 \\
&\quad \le C_4\big(1 + \|\partial_t\widehat{\phi}(t)\|_2^2 + \|\widehat{\phi}(t)\|_2^2\big) + \frac{1}{2}\|\widehat{\phi}(t)\|_{H^1(\Omega)}^2 + \|\partial_t\widehat{\phi}(t)\|_{H^1(\Omega)}^2\,, \qquad (3.20)
\end{aligned}$$

where the constants C_3, C_4 depend on the norm of the initial data $\phi(0)$, $\psi(0)$ and $u(0)$ in H^2. Similarly, we have

$$\begin{aligned}
&2\Big[\Big\langle f_1'\big(\overline{\phi}_0(t) + \varepsilon\widetilde{\phi}(\tau) + \varepsilon\widehat{\phi}(t)\big)\partial_t\widetilde{\phi}(\tau), \partial_t\widehat{\phi}(t)\Big\rangle_2\Big] \\
&\qquad \le C_5\|\partial_t\widetilde{\phi}(\tau)\|_{H^2(\Omega)}\big(1 + \|\partial_t\widehat{\phi}(t)\|_2^2\big)\,, \qquad (3.21)
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{2}{\varepsilon}\Big[\Big\langle\Big(f_2'\big(\overline{\psi}_0(t) + \varepsilon\widehat{\psi}(t)\big) - f_2'(\overline{\psi}_0)\Big)\partial_t\overline{\psi}_0(t), \partial_t\widehat{\psi}(t)\Big\rangle_{2,\Gamma}\Big] \\
&\quad \le C_6\big(1 + \|\partial_t\widehat{\psi}(t)\|_{2,\Gamma}^2 + \|\widehat{\psi}(t)\|_{2,\Gamma}^2\big) + \frac{1}{2}\|\widehat{\psi}(t)\|_{H^1(\Gamma)}^2 + \|\partial_t\widehat{\psi}(t)\|_{H^1(\Gamma)}^2\,. \qquad (3.22)
\end{aligned}$$

Multiplying now the first equation of (3.13) by $\partial_t\widehat{u}(t)$ and integrating over Ω, we obtain

$$\begin{aligned}
&\partial_t\Big(\|\nabla\widehat{u}(t)\|_2^2 + 2\big\langle\partial_t\overline{u}_0(t), \widehat{u}(t)\big\rangle_2\Big) + 2\varepsilon\|\partial_t\widehat{u}(t)\|_2^2 + 2\lambda\big\langle\partial_t\widehat{u}(t), \partial_t\widehat{\phi}(t)\big\rangle_2 \\
&\qquad = 2\big\langle\partial_t^2\overline{u}_0(t), \widehat{u}(t)\big\rangle_2 \le \|\partial_t^2\overline{u}_0(t)\|_{(H^1(\Omega))^*}\big(1 + \|\widehat{u}(t)\|_{H^1(\Omega)}^2\big)\,. \qquad (3.23)
\end{aligned}$$

Then, combining (3.18)–(3.22) and adding the resulting inequality to (3.23), we get

$$\frac{d}{dt}\Lambda(t) \le C_7\big(1+\|\partial_t^2\overline{u}_0(t)\|^2_{(H^1(\Omega))^*}+\|\partial_t\widetilde{\phi}(t/\varepsilon)\|_{H^2(\Omega)}+\|\partial_t\widetilde{u}(t/\varepsilon)\|_2\big)\Lambda(t)\,, \quad (3.24)$$

where we have set

$$\Lambda(t) := \delta\|\partial_t\widehat{\phi}(t)\|_2^2+\|\partial_t\widehat{\psi}(t)\|^2_{2,\Gamma}+\|\widehat{\phi}(t)\|^2_{H^1(\Omega)}+\|\widehat{\psi}(t)\|^2_{H^1(\Gamma)} + \|\nabla\widehat{u}(t)\|_2^2+2\big\langle\partial_t\overline{u}_0(t),\widehat{u}(t)\big\rangle_2+K\,.$$

The positive constant K will be chosen large enough such that $\Lambda(t)\ge 0$ for any $t\ge 0$, but we still need to estimate the term $\langle\widehat{u}(t)\rangle^2$. We integrate the first equation of (3.12) over Ω and, using the second equation, we find

$$\begin{aligned}\lambda\big\langle\widehat{u}(t)\big\rangle =&\,\delta\big\langle\partial_t\widehat{\phi}(t)\big\rangle+\big\langle\partial_t\widehat{\psi}(t)\big\rangle_\Gamma+\beta\big\langle\widehat{\psi}(t)\big\rangle_\Gamma\\ &+\frac{1}{\varepsilon}\Big\langle f_1\big(\overline{\phi}_0(t)+\varepsilon\widetilde{\phi}(\tau)+\varepsilon\widehat{\phi}(t)\big)-f_1\big(\overline{\phi}_0(t)\big)\Big\rangle\\ &+\frac{1}{\varepsilon}\Big\langle f_2\big(\overline{\psi}_0(t)+\varepsilon\widehat{\psi}(t)\big)-f_2(\overline{\psi}_0)\Big\rangle_\Gamma\,,\end{aligned} \quad (3.25)$$

where $\langle v\rangle_\Gamma$ stands for the average of v over Γ.

Recalling Theorems 1 and 3 and the fact that the functions $\phi(t)=\overline{\phi}_0(t)+\varepsilon\widetilde{\phi}(\tau)+\varepsilon\widehat{\phi}(t)$, $\psi(t)=\overline{\psi}_0(t)+\varepsilon\widehat{\psi}(t)$ and $\overline{\phi}_0(t)$, $\overline{\psi}_0(t)$ are uniformly bounded in L^∞ with respect to ε, it follows from (3.25) that

$$\lambda^2\big\langle\widehat{u}(t)\big\rangle^2 \le C_8\big(1+\|\partial_t\widehat{\phi}(t)\|_2^2+\|\partial_t\widehat{\psi}(t)\|^2_{2,\Gamma}+\|\widehat{\phi}(t)\|_2^2+\|\widehat{\psi}(t)\|^2_{2,\Gamma}\big)\,. \quad (3.26)$$

Combining (3.24) with (3.26), choosing K large enough and then applying Gronwall's inequality, taking into account the fact that (see (2.18))

$$\int_t^{t+1}\big(1+\|\partial_t^2\overline{u}_0(s)\|^2_{(H^1(\Omega))^*}+\|\partial_t\widetilde{\phi}(s/\varepsilon)\|_{H^2(\Omega)}+\|\partial_t\widetilde{u}(s/\varepsilon)\|_2\big)ds\le C_9\,,$$

we find

$$\begin{aligned}&\|\partial_t\widehat{\phi}(t)\|_2^2+\|\partial_t\widehat{\psi}(t)\|^2_{2,\Gamma}+\|\widehat{\phi}(t)\|^2_{H^1(\Omega)}\\ &\qquad+\|\widehat{\psi}(t)\|^2_{H^1(\Gamma)}+\|\widehat{u}(t)\|^2_{H^1(\Omega)}\le C_{10}e^{C_{11}t}\,,\end{aligned} \quad (3.27)$$

where the constants C_{10} and C_{11} depend on $\|(\phi(0),\psi(0),u(0))\|_{\mathbb{V}_2}$. Finally, estimate (3.16) follows from (3.27) (cf. [18,19] for more details). This finishes the proof of the lemma. □

On account of the asymptotic expansion (3.10), arguing as in [27], we deduce from estimates (3.14)–(3.16) that

Corollary 7. *Let* (ϕ, ψ, u) *be a solution to* $\mathcal{P}_\varepsilon$ *and* $(\overline{\phi}_0, \overline{\psi}_0, \overline{u}_0)$ *be a solution to* $\mathcal{P}_0$ *with initial data* $(\phi(0), \psi(0))$. *Then, there holds*

$$\begin{aligned} \|\phi(t)-\overline{\phi}_0(t)\|_{H^2(\Omega)} &+ \|\psi(t) - \overline{\psi}_0(t)\|_{H^2(\Gamma)} + \|u(t) - \overline{u}_0(t)\|_{H^2(\Omega)} \\ &+ \|\partial_t\phi(t) - \partial_t\overline{\phi}_0(t)\|_2 + \|\partial_t\psi(t) - \partial_t\overline{\psi}_0(t)\|_{2,\Gamma} \\ &+ \varepsilon\|\partial_t u(t) - \partial_t\overline{u}_0(t)\|_2 \\ &\leq C\Big(\big\|u(0) - \mathcal{L}\big(\phi(0)\big)\big\|_{H^2(\Omega)} e^{-\rho\frac{t}{\varepsilon}} + \varepsilon e^{Lt}\Big), \end{aligned} \tag{3.28}$$

where $\rho > 0$ *is a constant depending only on* Ω, Γ *and the positive constants* C, L *depend on* $\|(\phi(0), \psi(0), u(0))\|_{\mathbb{V}_2}$, *but are independent of* ε.

Corollary 8. *The following estimates hold:*

$$\begin{aligned} \|\partial_t u(t)\|_2 &\leq Q(\|(\phi(0), \psi(0), u(0))\|_{\mathbb{V}_2}) \\ &\quad \times \left[1 + \frac{1}{\varepsilon}\|u(0) - \mathcal{L}\big(\phi(0)\big)\|_{H^2(\Omega)} e^{-\rho\frac{t}{\varepsilon}}\right], \end{aligned} \tag{3.29}$$

$$\begin{aligned} \|u(t) - \mathcal{L}(\phi(t))\|_{H^2(\Omega)} &\leq Q\Big(\big\|\big(\phi(0), \psi(0), u(0)\big)\|_{\mathbb{V}_2}\Big) \\ &\quad \times \left[\varepsilon + \big\|u(0) - \mathcal{L}\big(\phi(0)\big)\big\|_{H^2(\Omega)} e^{-\rho\frac{t}{\varepsilon}}\right], \end{aligned} \tag{3.30}$$

where $\rho > 0$ *and the positive and monotone increasing function* Q *are independent of* ε.

We conclude this section with two estimates on the difference of two trajectories of $(S_t^\varepsilon, \mathbb{D}_\varepsilon^M)$ that are also necessary for the construction of exponential attractors in the next section. For the details of the proofs, the reader is referred to [18, Lemma 4.6] and [19, Lemma 17].

Lemma 9. *Let* $(\phi_i(t), \psi_i(t), u_i(t)) = S_t^\varepsilon(\phi_{0i}, \psi_{0i}, u_{0i})$, $i = 1, 2$. *Suppose that they belong to a bounded absorbing set in* $\mathbb{V}_2$, *of radius* $R > 0$. *Then, the following estimates hold:*

$$\begin{aligned} &\|(\phi_1 - \phi_2)(t)\|^2_{H^2(\Omega)} + \|(\psi_1 - \psi_2)(t)\|^2_{H^2(\Gamma)} + \|(u_1 - u_2)(t)\|^2_{H^2(\Omega)} \\ &\leq Ce^{Lt}\big(\|\phi_{01} - \phi_{02}\|^2_{H^2(\Omega)} + \|\psi_{01} - \psi_{02}\|^2_{H^2(\Gamma)} + \|u_{01} - u_{02}\|^2_{H^2(\Omega)}\big), \end{aligned} \tag{3.31}$$

$$\begin{aligned} &\|(\phi_1 - \phi_2)(t)\|^2_{H^3(\Omega)} + \|(\psi_1 - \psi_2)(t)\|^2_{H^3(\Gamma)} + \|(u_1 - u_2)(t)\|^2_{H^3(\Omega)} \\ &\leq C\frac{t+1}{t}e^{Lt}\big(\|\phi_{01} - \phi_{02}\|^2_{H^2(\Omega)} + \|\psi_{01} \\ &\quad - \psi_{02}\|^2_{H^2(\Gamma)} + \|u_{01} - u_{02}\|^2_{H^2(\Omega)}\big), \end{aligned} \tag{3.32}$$

for all $t > 0$, *where the constants* C, L *depend on* R, *but are independent of* ε.

4. Proof of Theorem 4

The proof follows from the application of an abstract result (see [27, Prop.3.2] and references therein) which is reported below for the reader's convenience.

Theorem 10. *Let $\mathbb{B}_\varepsilon \subset \Phi$, $\varepsilon \in [0,1]$, be a family of closed and bounded subsets of the Banach space Φ and let $\Sigma^\varepsilon : \mathbb{B}_\varepsilon \to \mathbb{B}_\varepsilon$ be a family of maps which satisfy the following properties:*

(1) *There exists another Banach space Φ_1, which is compactly embedded into Φ, such that, for every b_1^ε, $b_2^\varepsilon \in \mathbb{B}_\varepsilon$, the following estimate holds:*
$$\|\Sigma^\varepsilon b_1^\varepsilon - \Sigma^\varepsilon b_2^\varepsilon\|_{\Phi_1} \le C\|b_1^\varepsilon - b_2^\varepsilon\|_\Phi \,, \tag{4.1}$$
where the constant C is independent of ε.

(2) *There exist nonlinear "projectors" $\Pi_\varepsilon : \mathbb{B}_\varepsilon \to \mathbb{B}_0$ such that, for every $b^\varepsilon \in \mathbb{B}_\varepsilon$,*
$$\|\Sigma^\varepsilon_{(k)} b^\varepsilon - \Sigma^0_{(k)}\Pi_\varepsilon b^\varepsilon\|_\Phi \le \varepsilon L^k \,, \quad \forall\, k \in \mathbb{N} \,, \tag{4.2}$$
where $\Sigma^\varepsilon_{(k)}$ denotes the k-th iteration of Σ^ε and the constant L is independent of ε.

Then, the discrete semiflows $\Sigma^\varepsilon_{(k)}$ possess a family of exponential attractors $\mathcal{M}^d_\varepsilon$ which satisfy the discrete versions of (2.23)*–*(2.26)*.*

In our case, we set $\Phi = \mathbb{D}^M_\varepsilon$ and $\Phi_1 = \mathbb{V}_3$. Note that $\mathbb{V}_3$ is compactly embedded into $\mathbb{D}^M_\varepsilon$. Then, recalling (2.9), we define the sets $\mathbb{B}_\varepsilon \subset \mathbb{D}^M_\varepsilon$, for every $\varepsilon \in (0,1]$, in the following way:
$$\mathbb{B}_\varepsilon := \Big\{(\phi_0,\psi_0,u_0) \in \mathbb{D}^M_\varepsilon : \|(\phi_0,\psi_0,u_0)\|^2_{\mathbb{V}_2} \le 2Q_1\big(\|g_1\|_2^2 + \|g_1\|^2_{2,\Gamma}\big)\Big\} . \tag{4.3}$$
Similarly, on account of (2.18), we set
$$\mathbb{B}_0 := \Big\{(\phi_0,\psi_0,u_0) \in \mathbb{L}^M : \|(\phi_0,\psi_0)\|^2_{H^2(\Omega)\times H^2(\Gamma)} \le 2Q_2\big(\|g_1\|_2^2 + \|g_1\|^2_{2,\Gamma}\big)\Big\} . \tag{4.4}$$
These sets are uniform (with respect to ε) bounded absorbing sets for the semiflows (2.10) and (2.19) thanks to Theorems 1–2 and Theorem 3, respectively. Thus, there exists a time $T \ge 1$, independent of ε, such that
$$S^\varepsilon_T(\mathbb{B}_\varepsilon) \subseteq \mathbb{B}_\varepsilon \,, \quad \forall\, \varepsilon \in [0,1] \,. \tag{4.5}$$
Setting $\Sigma^\varepsilon = S^\varepsilon_T$, we easily realize that these maps satisfy the smoothing property (4.1), thanks to estimate (3.32). Besides, following [27], we define the projections $\Pi_\varepsilon : \mathbb{B}_\varepsilon \to \mathbb{B}_0$ by setting
$$\Pi_\varepsilon(\phi_0,\psi_0,u_0) := \big(\lambda\phi_0 + \varepsilon u_0, \psi_0, \mathcal{L}(\lambda\phi_0 + \varepsilon u_0)\big) \,, \tag{4.6}$$
which are well defined since $|\lambda\langle\phi_0\rangle + \varepsilon\langle u_0\rangle| = |I_\varepsilon(\phi_0,u_0)| \le M$. Thus, estimate (4.2) follows from estimate (3.28) and the obvious estimate
$$\big\|\Pi_\varepsilon(\phi_0,\psi_0,u_0) - \big(\phi_0,\psi_0,\mathcal{L}(\phi_0)\big)\big\|_{\mathbb{V}_2} \le C\varepsilon \,,$$
which holds for every $(\phi_0,\psi_0,u_0) \in \mathbb{B}_\varepsilon$, for some $C > 0$ independent of ε. Hence, Theorem 10 yields a discrete family of exponential attractors $\mathcal{M}^d_\varepsilon$ for the discrete semiflows Σ^ε_{nT} acting on the absorbing sets $\mathbb{B}_\varepsilon$. We now set
$$\mathcal{M}_\varepsilon := \cup_{t\in[T,T+1]} S^\varepsilon_t \mathcal{M}^d_\varepsilon \,.$$

Then, thanks to (3.28) and (3.30)–(3.32), we can argue as in [27] and deduce that $\mathcal{M}_\varepsilon$ fulfills assumptions (2.23), (2.25)–(2.26). Finally, property (2.24) follows from (3.29)–(3.30) (see also [27, Lemma 3.3]) and (3.31)–(3.32).

5. Global attractors revisited

In [19], the authors also establish the existence of the global attractor $\mathcal{A}_\varepsilon$ for $(S_t^\varepsilon, \mathbb{D}_\varepsilon^M)$ and its upper semicontinuity at $\varepsilon = 0$ (see [19, Thm. 4.3]). Here we wish to improve the latter result, by enlarging the phase space for the temperature u, according to the approach followed in [18].

Let us set $\mathbb{Z}_q := H^q(\Omega) \times H^q(\Gamma) \times H^{q-1}(\Omega)$, for any $q \geq 1$. Then, for any $\varepsilon \in (0, 1]$, we introduce the set

$$\mathbb{X}_\varepsilon^M := \left\{(\phi, \psi, u) \in H^2(\Omega) \times H^2(\Gamma) \times H^1(\Omega) : \psi = \phi_{|\Gamma},\ |I_\varepsilon| \leq M\right\}, \tag{5.1}$$

where $M \geq$ is given. This set is a complete metric space with the metric induced by the norm in $\mathbb{Z}_2$. From [18], we know that the semiflow S_t^ε can be defined on $\mathbb{X}_\varepsilon^M$. The existence of a bounded absorbing set has already been proven in [18, Lemma 4.3], but here we give a slightly different estimate which shows, in particular, the uniformity with respect to ε.

Lemma 11. *Let assumptions* (2.6)*–*(2.8) *be satisfied. Then, for any* $(\phi_0, \psi_0, u_0) \in \mathbb{X}_\varepsilon^M$, *the trajectory* $(\phi(t), \psi(t), u(t)) = S_t^\varepsilon(\phi_0, \psi_0, u_0)$ *satisfies the estimate*

$$\begin{aligned}
&\left\|\left(\phi(t), \psi(t), u(t)\right)\right\|_{\mathbb{Z}_2}^2 + \|\partial_t \phi(t)\|_2^2 + \|\partial_t \psi(t)\|_{2,\Gamma}^2 \\
&\quad + \int_t^{t+1} \left(\|\partial_t \phi(s)\|_{H^1(\Omega)}^2 + \|\partial_t \psi(s)\|_{H^1(\Gamma)}^2 + \varepsilon \|\partial_t u(s)\|_2^2\right) ds \\
&\leq Q\left(\|(\phi_0, \psi_0, u_0)\|_{\mathbb{Z}_2}^2\right) e^{-\rho t} + Q\left(\|g_1\|_2^2 + \|g_2\|_{2,\Gamma}^2\right),
\end{aligned} \tag{5.2}$$

for some $\rho > 0$ *and some positive increasing function* Q *which are both independent of* ε.

Proof. We proceed formally (see [18] for details on a rigorous argument). Consider system (2.1). Then, take the inner product in $L^2(\Omega)$ of the first equation with $\partial_t \phi(t)$ and the inner product in $L^2(\Gamma)$ of the second equation with $\partial_t \psi(t)$, respectively. Adding these relations together with the one obtained by taking the inner product of the first equation of (2.2) with $u(t)$, we deduce that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left[\|\nabla \phi(t)\|_2^2 + \alpha \|\nabla_\Gamma \psi(t)\|_{2,\Gamma}^2 + \beta \|\psi(t)\|_{2,\Gamma}^2 + \varepsilon \|u(t)\|_2^2\right] \\
&\quad + \frac{1}{2} \frac{d}{dt} \left[2 \left\langle F_1(\phi(t)), 1\right\rangle_2 + 2 \left\langle F_2(\psi(t)), 1\right\rangle_{2,\Gamma}\right] \\
&\quad + \|\nabla u(t)\|_2^2 + \delta \|\partial_t \phi(t)\|_2^2 + \|\partial_t \psi(t)\|_{2,\Gamma}^2 \\
&= \left\langle \partial_t \phi(t), g_1\right\rangle_2 + \left\langle \partial_t \psi(t), g_2\right\rangle_{2,\Gamma},
\end{aligned} \tag{5.3}$$

where $F_i(w) = \int_0^w f_i(y)dy$, $i = 1, 2$. Let us now take the inner product in $L^2(\Omega)$ of the first equation and second equation of (3.2) with $2\xi\phi(t)$, and $2\xi\psi(t)$, respectively, for some $\xi > 0$ to be fixed below. Combining the resulting relations with (5.3), we get, for some $0 < \kappa < \xi$,

$$\frac{d}{dt}E(t) + \kappa E(t) = \Lambda_1(t)\,, \tag{5.4}$$

where

$$\begin{aligned} E(t) :=\ & \|\nabla\phi(t)\|_2^2 + \alpha\|\nabla_\Gamma\psi(t)\|_{2,\Gamma}^2 + \beta\|\psi(t)\|_{2,\Gamma}^2 \\ & + 2\Big\langle F_1\big(\phi(t)\big), 1\Big\rangle_2 + 2\Big\langle F_2\big(\psi(t)\big), 1\Big\rangle_{2,\Gamma} \\ & + \varepsilon\|u(t)\|_2^2 + \xi\big[\delta\|\phi(t)\|_2^2 + \|\psi(t)\|_{2,\Gamma}^2\big] + E_0\,, \end{aligned}$$

for some $E_0 > 0$ such that $E(t)$ is nonnegative (cf. (2.7)) and

$$\begin{aligned} \Lambda_1(t) :=\ & 2\kappa\Big\langle F_1\big(\phi(t)\big) - f_1\big(\phi(t)\big)\phi(t), 1\Big\rangle_2 \\ & + 2\kappa\Big\langle F_2\big(\psi(t)\big) - f_2\big(\psi(t)\big)\psi(t), 1\Big\rangle_{2,\Gamma} \\ & - (2\xi - \kappa)\big(\|\nabla\phi(t)\|_2^2 + \alpha\|\nabla_\Gamma\psi(t)\|_{2,\Gamma}^2 + \beta\|\psi(t)\|_{2,\Gamma}^2\big) \\ & - 2(\xi - \kappa)\Big[\Big\langle f_1\big(\phi(t)\big), \phi(t)\Big\rangle_2 + \Big\langle f_2\big(\psi(t)\big), \psi(t)\Big\rangle_{2,\Gamma}\Big] \\ & - 2\big(\delta\|\partial_t\phi(t)\|_2^2 + \|\partial_t\psi(t)\|_{2,\Gamma}^2\big) + \big\langle\partial_t\phi(t), g_1\big\rangle_2 \\ & + \big\langle\partial_t\psi(t), g_2\big\rangle_{2,\Gamma} - 2\|\nabla u(t)\|_2^2 + 2\xi\kappa\big[\delta\|\phi(t)\|_2^2 + \|\psi(t)\|_{2,\Gamma}^2\big] \\ & + \kappa\varepsilon\|u(t)\|_2^2 + 2\xi\lambda\big\langle u(t), \phi(t)\big\rangle_2 + \kappa E_0\,. \end{aligned}$$

Observe now that

$$2\xi\lambda\langle u, \phi\rangle_2 = 2\xi\lambda\big\langle u - \langle u\rangle, \phi\big\rangle_2 + 2\xi|\Omega|I_\varepsilon\langle u\rangle - 2\xi\varepsilon|\Omega|\langle u\rangle^2\,. \tag{5.5}$$

On the other hand, we know that there exists a positive constant $\widehat{C}$ such that, for all $v \in H^1(\Omega)$,

$$\|v - \langle v\rangle\|_2^2 = \|v\|_2^2 - |\Omega|\langle v\rangle^2 \le \widehat{C}\|\nabla v\|_2^2\,. \tag{5.6}$$

Let us rewrite Λ_1 in the following way:

$$\begin{aligned} \Lambda_1(t) :=\ & 2\kappa\Big[\Big\langle F_1\big(\phi(t)\big) - f_1\big(\phi(t)\big)\phi(t), 1\Big\rangle_2 + \Big\langle F_2\big(\psi(t)\big) - f_2\big(\psi(t)\big)\psi(t), 1\Big\rangle_{2,\Gamma}\Big] \\ & - (2\xi - \kappa)\big(\|\nabla\phi(t)\|_2^2 + \alpha\|\nabla_\Gamma\psi(t)\|_{2,\Gamma}^2 + \beta\|\psi(t)\|_{2,\Gamma}^2\big) \\ & - 2(\xi - \kappa)\Big[\big\langle f_1\big(\phi(t)\big), \phi(t)\big\rangle_2 + \big\langle f_2\big(\psi(t)\big), \psi(t)\big\rangle_{2,\Gamma}\Big] \\ & - 2\xi\varepsilon|\Omega|\langle u\rangle^2 + \big\langle\partial_t\phi(t), g_1\big\rangle_2 + \big\langle\partial_t\psi(t), g_2\big\rangle_{2,\Gamma} \\ & - 2\big(\delta\|\partial_t\phi(t)\|_2^2 + \|\partial_t\psi(t)\|_{2,\Gamma}^2\big) - 2\|\nabla u(t)\|_2^2 \\ & + 2\xi|\Omega|I_\varepsilon\langle u\rangle + \kappa\varepsilon\|u(t)\|_2^2 - 2\kappa\varepsilon|\Omega|\langle u\rangle^2 \\ & + 2\xi\kappa\big[\delta\|\phi(t)\|_2^2 + \|\psi(t)\|_{2,\Gamma}^2\big] + 2\xi\lambda\big\langle u - \langle u\rangle, \phi\big\rangle_2\,. \end{aligned} \tag{5.7}$$

Then, recall that, owing to (2.10), we have

$$C_*|f_i(y)|(1+|y|) \leq 2f_i(y)y + C_{f_i}\,,$$
$$F_i(y) - f_i(y)y \leq C'_{f_i}|y|^2 + C''_{f_i}\,,$$

for any $y \in \mathbb{R}$ and $i = 1, 2$. Here C_{f_i}, C_*, C'_{f_i} and C''_{f_i} are positive, sufficiently large constants that depend on f_i only. Using these inequalities, we infer from (5.7) that

$$\begin{aligned} \Lambda_1(t) \leq & -\big(2\xi - \kappa - \kappa(C'_{f_1} + C'_{f_2}) - 2\xi\kappa\big)\left[\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)}\right] \\ & - \big(\delta\|\partial_t\phi(t)\|_2^2 + \|\partial_t\psi(t)\|^2_{2,\Gamma}\big) + C_\delta\big(1 + \|g_1\|_2^2 + \|g_2\|^2_{2,\Gamma}\big) \\ & - 2(\xi-\kappa)\left[\Big\langle f_1\big(\phi(t)\big), \phi(t)\Big\rangle_2 + \Big\langle f_2\big(\psi(t)\big), \psi(t)\Big\rangle_{2,\Gamma}\right] \\ & + 2\xi|\Omega| I_\varepsilon\langle u\rangle - 2(\xi-\kappa)\varepsilon|\Omega|\langle u\rangle^2 \\ & - 2(1 - \kappa\varepsilon\widehat{C})\|\nabla u(t)\|_2^2 + \xi^2\lambda^2\widehat{C}\|\phi(t)\|_2^2\,. \end{aligned} \tag{5.8}$$

Let us now estimate the average $\langle u\rangle$. Integrating equations (2.1) over Ω and Γ and adding the resulting identities, we obtain

$$\begin{aligned} \frac{d}{dt}\varepsilon\big\langle u(t)\big\rangle + \frac{\lambda^2}{\delta}\big\langle u(t)\big\rangle = & \frac{\lambda|\Gamma|}{\delta|\Omega|}\langle\partial_t\psi\rangle_\Gamma + \frac{\lambda\beta|\Gamma|}{\delta|\Omega|}\langle\psi\rangle_\Gamma \\ & + \frac{\lambda|\Gamma|}{\delta|\Omega|}\big\langle f_2(\psi)\big\rangle_\Gamma + \frac{1}{\delta}\big\langle f_1(\phi)\big\rangle_\Omega\,. \end{aligned} \tag{5.9}$$

We multiply (5.9) by

$$\zeta := \frac{2\xi\delta|\Omega| I_\varepsilon}{\lambda^2 - \kappa\delta\varepsilon}\,,$$

provided that $\kappa\delta\varepsilon < \lambda^2$, and we add the resulting relation to (5.4). Then, we derive the inequality

$$\begin{aligned} & \frac{d}{dt}\Big[E(t) + \zeta\varepsilon\big\langle u(t)\big\rangle\Big] + \kappa\Big[E(t) + \zeta\varepsilon\big\langle u(t)\big\rangle\Big] \\ & \quad \leq -\big(2\xi - \kappa - \kappa(C'_{f_1} + C'_{f_2}) - 2\xi\kappa - \xi^2\lambda^2\widehat{C}\big)\left[\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)}\right] \\ & \qquad - \big(\delta\|\partial_t\phi(t)\|_2^2 + \|\partial_t\psi(t)\|^2_{2,\Gamma}\big) + C_\delta\big(1 + \|g_1\|_2^2 + \|g_2\|^2_{2,\Gamma}\big) \\ & \qquad - 2(\xi-\kappa)\left[\Big\langle f_1\big(\phi(t)\big), \phi(t)\Big\rangle_2 + \Big\langle f_2\big(\psi(t)\big), \psi(t)\Big\rangle_{2,\Gamma}\right] - 2(\xi-\kappa)\varepsilon|\Omega|\langle u\rangle^2 \\ & \qquad - 2(1 - \kappa\varepsilon\widehat{C})\|\nabla u(t)\|_2^2 + \frac{\lambda|\Gamma|}{\delta|\Omega|}\langle\partial_t\psi\rangle_\Gamma + \frac{\lambda\beta|\Gamma|}{\delta|\Omega|}\langle\psi\rangle_\Gamma \\ & \qquad + \frac{\lambda|\Gamma|}{\delta|\Omega|}\big\langle f_2(\psi)\big\rangle_\Gamma + \frac{1}{\delta}\big\langle f_1(\phi)\big\rangle_\Omega\,. \end{aligned} \tag{5.10}$$

Due to assumption (2.11), we have that, for any $\eta > 0$, there exists $C_\eta > 0$ such that (cf. [27, (4.16)])

$$\Big|\Big\langle f_1\big(\phi(t)\big)\Big\rangle\Big| \leq \Big\langle\Big|f_1\big(\phi(t)\big)\Big|\Big\rangle \leq \eta\Big\langle f_1\big(\phi(t)\big), \phi(t)\Big\rangle_2 + C_\eta\,, \tag{5.11}$$

$$\Big|\Big\langle f_2\big(\psi(t)\big)\Big\rangle_\Gamma\Big| \leq \Big\langle\Big|f_2\big(\psi(t)\big)\Big|\Big\rangle_\Gamma \leq \eta\Big\langle f_2\big(\psi(t)\big), \psi(t)\Big\rangle_{2,\Gamma} + C_\eta\,. \tag{5.12}$$

Thus, thanks to the above inequalities, we deduce that (cf. also [19])

$$\begin{aligned}&\frac{d}{dt}\Big[E(t) + \zeta\varepsilon\langle u(t)\rangle] + \kappa\Big[E(t) + \zeta\varepsilon\langle u(t)\rangle\Big]\\ &\qquad + \kappa'\big(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + \delta\|\partial_t\phi(t)\|_2^2 + \|\partial_t\psi(t)\|^2_{2,\Gamma} + \|\nabla u(t)\|_2^2\big)\\ &\quad \leq C\big(1 + \|g_1\|_2^2 + \|g_2\|^2_{2,\Gamma}\big)\,,\end{aligned} \tag{5.13}$$

where κ, κ' and C are positive constants that are independent of ε. On the other hand, one can easily check that there exists a positive constant C independent of ε such that, for all $t \geq 0$,

$$\Lambda_2(t) \leq C\Big(E(t) + \zeta\varepsilon\langle u(t)\rangle\Big)\,, \tag{5.14}$$

where

$$\Lambda_2(t) = \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + \varepsilon\|u(t)\|_2^2\,.$$

Applying Gronwall's inequality to (5.13) and taking (5.14) into account, we obtain

$$\begin{aligned}&\Lambda_2(t) + \int_t^{t+1} \big(\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)} + \delta\|\partial_t\phi(s)\|_2^2\big)ds\\ &+ \int_t^{t+1} \big(\|\partial_t\psi(s)\|^2_{2,\Gamma} + \|\nabla u(s)\|_2^2\big)ds \leq C\big(E(0)e^{-\rho t} + 1 + \|g_1\|_2^2 + \|g_2\|^2_{2,\Gamma}\big)\,,\end{aligned} \tag{5.15}$$

for all $t \geq 0$, for some $\rho > 0$ and $C > 0$ independent of t and ε.

Hence, using (5.15) and arguing as in [18, Lemma 4.3, (4.20)–(4.23)], we deduce that

$$\begin{aligned}&\|\partial_t\phi(t)\|_2^2 + \|\partial_t\psi(t)\|^2_{2,\Gamma} + \|\nabla u(t)\|_2^2\\ &\qquad + \int_t^{t+1} \big(\|\partial_t\phi(s)\|^2_{H^1(\Omega)} + \|\partial_t\psi(s)\|^2_{H^1(\Gamma)} + \varepsilon\|\partial_t u(s)\|_2^2\big)ds\\ &\quad \leq Q\big(\|(\phi_0,\psi_0,u_0)\|^2_{\mathbb{X}^M_\varepsilon}\big)e^{-\rho t} + Q\big(\|g_1\|_2^2 + \|g_2\|^2_{2,\Gamma}\big)\,,\end{aligned} \tag{5.16}$$

and it follows from (2.1) that

$$\lambda\langle u(t)\rangle \leq Q\big(\|(\phi_0,\psi_0,u_0)\|^2_{\mathbb{X}^M_\varepsilon}\big)e^{-\rho t} + Q\big(\|g_1\|_2^2 + \|g_2\|^2_{2,\Gamma}\big)\,, \tag{5.17}$$

for some $\rho > 0$ and some positive increasing function Q, both independent of ε. Finally, (5.2) is derived from (5.15)–(5.17), arguing as in the final part of the proof of [18, Lemma 4.3]. □

The next lemma shows the existence of a compact absorbing set in $\mathbb{X}_\varepsilon^M$, namely,

Lemma 12. *Let assumptions* (2.6)–(2.8) *be satisfied. There exists a positive monotone nondecreasing function* Q (*independent of* ε) *and, for any* $R_0 > 0$, *there exists* $t_0 = t_0(R_0) > 0$ *such that*

$$\left\| \left(\phi(t), \psi(t), u(t)\right) \right\|_{\mathbb{Z}_3} \leq Q(R_0) , \quad \forall\, t \geq t_0 , \tag{5.18}$$

for any $(\phi_0, \psi_0, u_0) \in \mathcal{B}(R_0) \subset \mathbb{X}_\varepsilon^M$, *where* $\mathcal{B}(R_0)$ *is a ball of radius* R_0, *centered at* 0.

Proof. Following a formal argument (cf. [18, Lemma 4.4]), we differentiate the first two equations of (2.1) and the first equation of (2.2) with respect to time. Then, we multiply the resulting equations by $\partial_t^2 \phi(t)$, $\partial_t^2 \psi(t)$ and $\partial_t u(t)$, respectively. Integrating by parts and adding the resulting relations, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|\partial_t \phi(t)\|_{H^1(\Omega)}^2 + \|\partial_t \psi(t)\|_{H^1(\Gamma)}^2 + \varepsilon \|\partial_t u(t)\|_2^2\right] \\
&\qquad + \|\partial_t^2 \phi(t)\|_2^2 + \|\partial_t^2 \psi(t)\|_{2,\Gamma}^2 + \|\nabla \partial_t u(t)\|_2^2 \\
&\qquad = -\int_\Omega f_1'\left(\phi(t)\right) \partial_t \phi(t) \partial_t^2 \phi(t) dx - \int_\Gamma f_2'\left(\psi(t)\right) \partial_t \psi(t) \partial_t^2 \psi(t) dS .
\end{aligned} \tag{5.19}$$

Using Hölder's and Young's inequalities, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|\partial_t \phi(t)\|_{H^1(\Omega)}^2 + \|\partial_t \psi(t)\|_{H^1(\Gamma)}^2 + \varepsilon \|\partial_t u(t)\|_2^2\right) \\
&\qquad + \|\partial_t^2 \phi(t)\|_2^2 + \|\partial_t^2 \psi(t)\|_{2,\Gamma}^2 + \|\nabla \partial_t u(t)\|_2^2 \\
&\qquad \leq C\left(\left\| f_1'\left(\phi(t)\right) \partial_t \phi(t) \right\|_2^2 + \left\| f_2'\left(\psi(t)\right) \partial_t \psi(t) \right\|_{2,\Gamma}^2 \right) ,
\end{aligned} \tag{5.20}$$

for some $C > 0$ that is independent of ε and of the initial data. On account of the embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$, we can find a positive monotone increasing function Q, independent of ε, such that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\partial_t \phi(t)\|_{H^1(\Omega)}^2 + \|\partial_t \psi(t)\|_{H^1(\Gamma)}^2 + \varepsilon \|\partial_t u(t)\|_2^2\right) \\
&\qquad + \|\partial_t^2 \phi(t)\|_2^2 + \|\partial_t^2 \psi(t)\|_{2,\Gamma}^2 + \|\nabla \partial_t u(t)\|_2^2 \\
&\qquad \leq Q\left(\|\phi(t)\|_{H^2(\Omega)}\right) \|\partial_t \phi(t)\|_2^2 + Q\left(\|\psi(t)\|_{H^2(\Gamma)}\right) \|\partial_t \psi(t)\|_{2,\Gamma}^2 .
\end{aligned} \tag{5.21}$$

Recalling (5.2), we can apply the uniform Gronwall lemma to (5.21). This yields

$$\|\partial_t \phi(t)\|_{H^1(\Omega)}^2 + \|\partial_t \psi(t)\|_{H^1(\Gamma)}^2 + \varepsilon \|\partial_t u(t)\|_2^2 \leq Q(R_0) , \quad \forall\, t \geq t_0 , \tag{5.22}$$

where $t_0 = t_0(R_0) > 0$ is independent of ε. We can now use (5.2) and (5.22) to derive (5.18), arguing as in the proof of [18, Lemma 4.4]. □

We also recall the continuous dependence estimate (see [18, (3.42)]).

Lemma 13. *Let assumptions* (2.6)–(2.8) *be satisfied. Consider two trajectories originating from* $\mathbb{X}_\varepsilon^M$, *namely* $(\phi_i(t), \psi_i(t), u_i(t)) = S_t^\varepsilon(\psi_{0i}, \phi_{0i}, u_{0i})$, $i = 1, 2$. *Then, the following estimate holds:*

$$\begin{aligned} &\|(\phi_1 - \phi_2)(t)\|^2_{H^1(\Omega)} + \|(\psi_1 - \psi_2)(t)\|^2_{H^1(\Gamma)} + \varepsilon\|(u_1 - u_2)(t)\|_2^2 \\ &\qquad \le Ce^{Lt}\big(\|\phi_{01} - \phi_{02}\|^2_{H^1(\Omega)} + \|\psi_{01} - \psi_{02}\|^2_{H^1(\Gamma)} + \varepsilon\|u_{01} - u_{02}\|_2^2\big)\,, \end{aligned} \tag{5.23}$$

where C *and* L *are independent of* ε.

On account of (5.2), (5.18) and (5.23), we deduce (cf. [31]) the

Corollary 14. *Let assumptions* (2.6)–(2.8) *be satisfied. Then, for each* $\varepsilon \in (0, 1]$, *the dynamical system* $(S_t^\varepsilon, \mathbb{X}_\varepsilon^M)$ *has a connected global attractor* $\mathcal{A}_\varepsilon$ *that is bounded in* $\mathbb{Z}_3$.

Remark 2. It is worth noting that, in order to prove the result of Corollary 15, the nonlinearities f_i need only be in $C^1(\mathbb{R})$.

We also recall that, owing to Theorem 4, the dynamical system $(S_t^0, \mathbb{L}^M)$ has the global connected attractor $\mathcal{A}_0$ that is bounded in $\mathbb{V}_3 \subset \mathbb{Z}_3$. More precisely, recalling (2.22), this attractor is defined by

$$\mathcal{A}_0 = \big\{(\phi, \psi, u) \in \mathbb{L}_0^M : (\phi, \psi) \in \widehat{\mathcal{A}}_0,\ u = \mathcal{L}(\phi)\big\}\,,$$

where $\widehat{\mathcal{A}}_0$ is the global attractor, bounded in $H^3(\Omega) \times H^3(\Gamma)$, of the dynamical system $(\widehat{S}_t^0, \widehat{\mathbb{L}}^M)$ generated by the viscous Cahn–Hilliard equation with dynamic boundary conditions.

Using estimates (5.2) and (5.22), we can argue as in the proof of [19, Theorem 4.3] to get the

Theorem 15. *Let assumptions* (2.6)–(2.8) *be satisfied. Then, the family* $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ *is upper semicontinuous at* $\varepsilon = 0$, *that is,*

$$\lim_{\varepsilon\to 0^+} dist_{\mathbb{X}_\varepsilon^M}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0\,,$$

where $dist_{\mathbb{X}_\varepsilon^M}$ *denotes the Hausdorff semidistance in* $\mathbb{X}_\varepsilon^M$.

Acknowledgements

The second author was partially supported by the Italian PRIN Research Project 2006 *Problemi a frontiera libera, transizioni di fase e modelli di isteresi*.

References

[1] S. Aizicovici, E. Feireisl, F. Issard-Roch, Long time convergence of solutions to a phase-field system, Math. Methods Appl. Sci. 24 (2001), 277–287.

[2] P. W. Bates, S. Zheng, Inertial manifolds and inertial sets for the phase-field equations, J. Dynam. Differential Equations 4 (1992), 375–397.

[3] D. Brochet, X. Chen, D. Hilhorst, Finite dimensional exponential attractor for the phase-field model, Appl. Anal. 49 (1993), 197–212.

[4] D. Brochet, D. Hilhorst, Universal attractor and inertial sets for the phase-field model, Appl. Math. Lett. 4 (1991), 59–62.

[5] M. Brokate, J. Sprekels, Hysteresis and Phase Transitions, Springer, New York, 1996.

[6] J. W. Cahn, J. E. Hilliard, Free energy of a nonuniform system. I. Interfacial energy, J. Chem. Phys. 28 (1958), 258–267.

[7] G. Caginalp, An analysis of a phase field model of a free boundary, Arch. Ration. Mech. Anal. 92 (1986), 205–245.

[8] R. Chill, E. Fašangová, J. Prüss, Convergence to steady states of solutions of the Cahn–Hilliard and Caginalp equations with dynamic boundary conditions, Math. Nachr. 13 (2006), 1448–1462.

[9] L. Cherfils, S. Gatti, A. Miranville, Existence of global solutions to the Caginalp phase-field system with dynamic boundary conditions and singular potentials, J. Math. Anal. Appl. 343 (2008), 557–566.

[10] L. Cherfils, A. Miranville, Some remarks on the asymptotic behavior of the Caginalp system with singular potentials, Adv. Math. Sci. Appl. 16 (2007), 107–129.

[11] L. Cherfils, A. Miranville, On the Caginalp system with dynamic boundary conditions and singular potentials, Appl. Math., to appear.

[12] A. Damlamian, N. Kenmochi, N. Sato, Subdifferential operator approach to a class of nonlinear systems for Stefan problems with phase relaxation, Nonlinear Anal. 23 (1994), 115–142.

[13] C. Dupaix, A singularly perturbed phase field model with a logarithmic nonlinearity: upper semicontinuity of the attractor, Nonlinear Anal. 41 (2000), 725–744.

[14] C. Dupaix, D. Hilhorst, A singularly perturbed phase field problem: upper-semicontinuity of the attractor, in "Nonlinear analysis and applications (Warsaw, 1994)", GAKUTO Internat. Ser. Math. Sci. Appl. 7, 101–112, Gakkōtosho, Tokyo, 1996.

[15] C. Dupaix, D. Hilhorst, I. N. Kostin, The viscous Cahn–Hilliard equations as a limit of the phase field model: lower semicontinuity of the attractor, J. Dynam. Differential Equations 11 (1999), 333–353.

[16] C. Dupaix, D. Hilhorst, Ph. Laurençot, Upper semicontinuity of the attractor for a singularly perturbed phase field model, Adv. Math. Sci. Appl. 8 (1998), 115–143.

[17] C. M. Elliott, S. Zheng, Global existence and stability of solutions to the phase-field equations, in "Free boundary problems", Internat. Ser. Numer. Math. 95, 46–58, Birkhäuser Verlag, Basel, 1990.

[18] C. G. Gal, M. Grasselli, The nonisothermal Allen–Cahn equation with dynamic boundary conditions, Discrete Contin. Dyn. Syst. 22 (2008), 1009-1040.

[19] S. Gatti, A. Miranville, Asymptotic behavior of a phase-field system with dynamic boundary conditions, in "Differential Equations: Inverse and Direct Problems" (A. Favini, A. Lorenzi, Eds.), Ser. Lect. Notes Pure Appl. Math. 251, 149–170, Chapman & Hall/CRC, Boca Raton, 2006.

[20] M. Grasselli, H. Petzeltová, G. Schimperna, Long time behavior of solutions to the Caginalp system with singular potential, Z. Anal. Anwendungen 25 (2006), 51–72.

[21] A. Jiménez-Casas, A. Rodríguez-Bernal, Asymptotic behaviour for a phase field model in higher order Sobolev spaces, Rev. Mat. Complut. 15 (2002), 213–248.

[22] V. K. Kalantarov, On the minimal global attractor of a system of phase field equations (Russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 188 (1991), Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 22, 70–86, 186 [translation in J. Math. Sci. 70 (1994), no. 3, 1767–1777].

[23] O. V. Kapustyan, An attractor of a semiflow generated by a system of phase-field equations without uniqueness of the solution (Ukrainian), Ukraïn. Mat. Zh. 51 (1999), 1006–1009 [Translation in Ukrainian Math. J. 51 (1999), no.7, 1135–1139 (2000)].

[24] Ph. Laurençot, Degenerate Cahn–Hilliard equation as limit of the phase-field equation with non-constant thermal conductivity, in "Free boundary problems, theory and applications (Zakopane, 1995)", Pitman Res. Notes Math. Ser. 363, 135–144, Longman, Harlow, 1996.

[25] G. B. McFadden, Phase-field models of solidification, Contemp. Math. 306 (2002), 107–145.

[26] A. Miranville, S. Zelik, Exponential attractors for the Cahn–Hilliard equation with dynamic boundary conditions, Math. Models Appl. Sci. 28 (2005), 709–735.

[27] A. Miranville, S. Zelik, Robust exponential attractors for singularly perturbed phase-field type equations, Electron. J. Differential Equations 63 (2002), 1–28.

[28] A. Novick-Cohen, The Cahn–Hilliard equation: Mathematical and modeling perspectives, Adv. Math. Sci. Appl. 8 (1998), 965–985.

[29] A. Novick-Cohen, On the viscous Cahn–Hilliard equation, in "Material instabilities in continuum mechanics (Edinburgh, 1985–1986)", Oxford Sci. Publ., 329–342, Oxford Univ. Press, New York, 1988.

[30] J. Prüss, R. Racke, S. Zheng, Maximal regularity and asymptotic behavior of solutions for the Cahn–Hilliard equation with dynamic boundary conditions, Ann. Mat. Pura Appl. (4) 185 (2006), 627–648.

[31] V. Pata, S. Zelik, Global attractors for semigroups of closed operators, Comm. Pure Appl. Anal. 6 (2007), 481–486.

[32] R. Racke, S. Zheng, The Cahn–Hilliard equation with dynamical boundary conditions, Adv. Differential Equations 8 (2003), 83–110.

[33] R. Rossi, Asymptotic analysis of the Caginalp phase-field model for two vanishing time relaxation parameters, Adv. Math. Sci. Appl. 13 (2003), 249–271.

[34] N. Sato, T. Aiki, Phase field equations with constraints under nonlinear dynamic boundary conditions, Commun. Appl. Anal. 5 (2001), 215–234.

[35] G. Schimperna, Abstract approach to evolution equations of phase field type and applications, J. Differential Equations 164 (2000), 395–430.

[36] B. E. E. Stoth, The Cahn–Hilliard equation as degenerate limit of the phase-field equations, Quart. Appl. Math. 53 (1995), 695–700.

[37] M. I. Vishik, L. A. Lyusternik, Regular degeneration and boundary layer for linear differential equations with small parameter, Uspekhi Mat. Nauk. 2 (1957), 3–122.

[38] H. Wu, S. Zheng, Convergence to equilibrium for the Cahn–Hilliard equation with dynamic boundary conditions, J. Differential Equations 204 (2004), 511–531.

[39] Z. Zhang, Asymptotic behavior of solutions to the phase-field equations with Neumann boundary conditions, Commun. Pure Appl. Anal. 4 (2005), 683–693.

Ciprian G. Gal
Department of Mathematical Sciences
University of Missouri
Columbia, MO 65211
USA
e-mail: ciprian@math.missouri.edu

Maurizio Grasselli
Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano
I-20133 Milano
Italy
e-mail: maurizio.grasselli@polimi.it

Alain Miranville
Laboratoire de Mathèmatiques et Applications
Université de Poitiers
F-86962 Chasseneuil Futuroscope Cedex
France
e-mail: Alain.Miranville@math.univ-poitiers.fr

Received: 11 July 2007.
Accepted: 18 February 2008.