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Existence and Multiplicity Results for the p-Laplacian with a p-Gradient Term

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Abstract. We study the existence and multiplicity of positive solutions to p-Laplace equations where the nonlinear term depends on a p-power of the gradient. For this purpose we combine Picone's identity, blow-up arguments, the strong maximum principle and Liouville-type theorems to obtain a priori estimates.

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1. Introduction

We study the existence and multiplicity of positive solutions of nonlinear elliptic equations of the form

$$\begin{cases} -\Delta_p u = |\nabla u|^p + \lambda f(x,u) & \text{in} \quad \Omega\,, \\ u = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{P_λ}$$

where λ is a positive parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain with $N > p > 1$, and $f : \Omega \times [0, +\infty) \to [0, +\infty)$ is a continuous function.

In the semilinear elliptic case ($p = 2$), the equation (P_λ) may be viewed as a perturbation of the stationary part of the equation

$$u_t - \Delta u = \varepsilon |\nabla u|^2\,.$$

Existence results for problems such as (P_λ) with $p = 2$ and $f(x,u) = f(x)$ start with the classical references [19, 20]. Later on, many authors have considered elliptic equations with first-order terms having quadratic growth with respect to the gradients (see [1, 5, 12, 14, 15, 22, 26] and references therein).

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We would like to mention that most of the works mentioned above deal with the case where the nonlinearity $f(x)$ belongs to an appropriate Lebesgue space while in this work we deal with functions $f(x,u)$ which lie between powers of u.

Concerning to equation (P_λ), it is worth mentioning that [24] and [17] have studied related problems. For instance, [24] studies the existence of positive solutions for the problem

$$\begin{cases} \Delta_m u + \psi(x,u,\nabla u) = 0\,, & x \in \Omega\,,\\ u(x) = 0\,, & x \in \partial\Omega\,, \end{cases} \tag{1.1}$$

where $1 < m < N$, and $\psi : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a non-negative continuous function satisfying

$$\left[\begin{array}{c} u^q - M|\eta|^\alpha \le \psi(x,u,\eta) \le a u^q + M|\eta|^\alpha\,,\\ \forall\,(x,u,\eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\,, \quad \text{where} \quad a \ge 1,\, M > 0\,,\\ q \in \left(m-1, \dfrac{N(m-1)}{N-m}\right) \quad \text{and} \quad \alpha \in \left(m-1, \dfrac{mq}{q+1}\right). \end{array} \right.$$

From these results a natural question arises, namely, are there exponents q and α ensuring the existence of positive solutions? We will give a partial answer to this question.

It is also interesting to mention that equations appearing in these types of problems do not have, in general, a variational structure, hence most of the classical tools to study partial differential equations fail. In our case performing an adequate change of variable we pass from the problem with a gradient term to a new one, but without a gradient term. However, for instance, the new nonlinearity does not verify the well-known Ambrosetti–Rabinowitz condition, which is a basic assumption needed to ensure compactness of the operator associated to the problem. In this work we will avoid this difficulty by using topological methods, Picone's identity, a blow-up argument and a Liouville-type theorem.

We note that, when f is a positive constant, the change of variable mentioned above transforms the equation in Problem (P_λ) into the equation $-\Delta_p v = \widetilde{\lambda}(v+1)^{p-1} = \widetilde{\lambda} g(v)$ in Ω with homogeneous Dirichlet boundary conditions, where $\widetilde{\lambda} = \lambda C$ for some positive constant C. This type of elliptic equation has been studied by several authors. For instance, when $p = 2$, existence of solutions was obtained in [6] when g is a C^1-convex, non-decreasing function. In [13], among other interesting results, the existence of weak non-negative solutions was obtained for $g(t) = (1+t)^q$ with $q > 0$ and $N > p > 1$. However, we emphasize that these particular cases are not considered here, because our nonlinearity g_λ is of mixed power-logarithmic type, and therefore the geometry associated to the problem is quite different (see Problem (2.2) below).

Throughout this work, we assume the following hypotheses:

(F_1) The nonlinearity $f : \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ is a Carathéodory function (i.e. $f(x,\cdot\,)$ is continuous for a.e. $x \in \Omega$ and $f(\,\cdot\,,s)$ is bounded if s belongs to bounded sets).

(F_2) There exist $q \in (0, +\infty)$ and positive constants c_0 and c_1 such that

$$c_0 t^q \le f(x,t) \le c_1 t^q$$

for all $(x,t) \in \overline{\Omega} \times [0,+\infty)$.

(F_3) If $q = p-1$ we have

$$\lim_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} = \inf_{t>0} \frac{f(x,t)}{t^{p-1}} = f_0\,,$$

uniformly for $x \in \overline{\Omega}$ for some positive constant f_0.

(F_4) If $0 < q < p-1$, there is a non-negative constant k such that the map

$$t \mapsto (t+1)^{p-1} f\big(x, (p-1)\ln(t+1)\big) + kt^{p-1}$$

is strictly increasing if $t \in [0,+\infty)$ for a.e. $x \in \Omega$.

Let us comment on the hypotheses above. The conditions (F_1) and (F_2), and also (F_3) are necessary to obtain existence and nonexistence results. But condition (F_4) is only a technical assumption in order to obtain multiplicity results (see the theorems below). More precisely, our main results are the following.

Theorem 1.1. *Let $q > p-1$ and assume that (F_1) and (F_2) hold. Then (P_λ) has at least one positive solution for all $\lambda > 0$.*

Theorem 1.2. *Let $q = p-1$ and assume that $(F_1), (F_2)$ and (F_3) hold. Then (P_λ) has at least one positive solution for $\lambda \in (0, \overline{\lambda})$ and no positive solutions for $\lambda \ge \overline{\lambda}$ where $\overline{\lambda} = \lambda_1(\Omega)/f_0$. Here $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$.*

Theorem 1.3. *Let $0 < q < p-1$ and assume that (F_1), (F_2) and (F_4) hold. Then there exists $\overline{\Lambda} > 0$ such that (P_λ) has at least two positive solutions for $0 < \lambda < \overline{\Lambda}$, at least one solution for $\lambda = \overline{\Lambda}$ and no solutions for $\lambda > \overline{\Lambda}$.*

The paper is organized as follows. In Section 2, we establish some basic facts that will be needed in the sequel. In particular, we will give a short proof of a Liouville-type theorem which depends only on Picone's identity. Section 3 is devoted to the proof of the Theorems stated above.

2. Preliminaries

Let u be a regular positive solution of (P_λ). Performing the change of variable $v = e^{\frac{u}{p-1}} - 1$ (see [18]) we get

$$\begin{cases} -\Delta_p v = g_\lambda(x,v) & \text{in} \quad \Omega\,, \\ v = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{2.2}$$

where $g_\lambda(x,v) = \lambda(\frac{v+1}{p-1})^{p-1} f(x,(p-1)\ln(v+1))$. Note that, for all $\lambda > 0$, the function g_λ satisfies:

(G_1) $g_\lambda : \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ is a Carathéodory function.

(G_2) For all $t \geq 0$,

$$\lambda c_0 (t+1)^{p-1} \big(\ln(t+1) \big)^q \leq \frac{g_\lambda(x,t)}{(p-1)^{q+1-p}} \leq \lambda c_1 (t+1)^{p-1} \big(\ln(t+1) \big)^q ,$$

for all $x \in \overline{\Omega}$.

(G_3) When $0 < q < p-1$, there exists a non-negative constant k such that the map $t \mapsto g_\lambda(x,t) + kt^{p-1}$ is strictly increasing if $t \in [0, +\infty)$ for a.e. $x \in \Omega$.

Note that, under the conditions stated above, the solutions to the problem (2.2) belong to $C^{1,\alpha}$ for some $\alpha \in (0,1)$ (see, for example, [16]). Therefore, if we are able to find a solution to the problem (2.2), we will obtain a regular solution to the original equation (P_λ).

The following result is the well-known Picone identity for the p-Laplacian (see e.g. [2, 11, 25]). This will be fundamental in our approach.

Theorem 2.1 (Picone's identity). *Let $u, v \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ be such that $u \geq 0$ and $v > 0$.*

Set

$$\mathcal{L}(u,v) = |\nabla u|^p + (p-1) \left(\frac{u}{v} \right)^p |\nabla v|^p - p \left(\frac{u}{v} \right)^{p-1} \nabla u |\nabla v|^{p-2} \nabla v,$$

$$\mathcal{R}(u,v) = |\nabla u|^p - \nabla \left(\frac{u^p}{v^{p-1}} \right) |\nabla v|^{p-2} \nabla v \,.$$

Then $\mathcal{L}(u,v) = \mathcal{R}(u,v) \geq 0$ a.e. in Ω. Moreover, $\mathcal{L}(u,v) = \mathcal{R}(u,v) = 0$ a.e. in Ω if and only if $u = cv$ in Ω for a constant $c > 0$.

Using Picone's identity we give a simple proof of the following Liouville-type result.

Lemma 2.1. *Let $c_0 > 0$, and let $p > 1$. Then the inequality*

$$c_0 w^{p-1} \leq -\Delta_p \, w \tag{2.3}$$

has no positive solution in $W^{1,p}_{loc}(G)$, where $G = \mathbb{R}^N$ or $G = \mathbb{R}^N_+$.

Proof. We argue by contradiction. Let w be a positive solution of (2.3). We may assume that $G = \mathbb{R}^N_+$. Take $R > 0$ and $x_0 \in \mathbb{R}^N_+$ such that $\overline{B_R(x_0)} \subset \mathbb{R}^N_+$ and $\lambda_1(B_R(x_0)) < c_0$. Denote by ϕ_1 the positive eigenfunction associated with $\lambda_1(B_R(x_0))$. Since $w > 0$ on $B_R(x_0)$, Hopf's Lemma ([27]) implies that $\phi_1^p / w^{p-1} \in W^{1,p}_0(B_R(x_0))$. Then, testing (2.3) by ϕ_1^p / w^{p-1} and integrating by parts, we get

$$\begin{aligned} c_0 \int_{B_R(x_0)} \phi_1^p &\leq \int_{B_R(x_0)} (-\Delta_p \, w) \frac{\phi_1^p}{w^{p-1}} \\ &= \int_{B_R(x_0)} p \, |\nabla w|^{p-2} \frac{\phi_1^{p-1}}{w^{p-1}} \nabla w \nabla \phi_1 - (p-1) \int_{B_R(x_0)} \frac{\phi_1^p}{w^p} \, |\nabla w|^p \,. \end{aligned}$$

The preceding inequality and Picone's identity implies that

$$c_0 \int_{B_R(x_0)} \phi_1^p - \int_{B_R(x_0)} |\nabla \phi_1|^p \leq - \int_{B_R(x_0)} \mathcal{L}(\phi_1, w) \leq 0 \,.$$

It follows that

$$c_0 \le \frac{\int_{B_R(x_0)} |\nabla \phi_1|^p}{\int_{B_R(x_0)} \phi_1^p} = \lambda_1\big(B_R(x_0)\big)$$

and this contradiction completes the proof. □

Since we use a homotopy argument, it is necessary to prove a priori bounds for solutions of a family of problems parameterized by $\tau \ge 0$, namely

$$\begin{cases} -\Delta_p\, v = g_\lambda(x,v) + \tau & \text{in} \quad \Omega\,, \\ v = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{$(A_\lambda)_\tau$}$$

where $\lambda > 0$ is fixed.

The following lemma allows us to control the parameter τ in the blow-up analysis.

Lemma 2.2. *Let v a positive solution of $(A_\lambda)_\tau$, where $\lambda \ge 0$. Then*

$$\tau \le \theta \left(\max_{\overline{\Omega}} v \right)^{p-1}$$

where the positive constant θ depends only on Ω.

Proof. Since v is a positive solution, the inequality holds when $\tau = 0$. Now, if $\tau > 0$, it (G_1) implies that

$$-\Delta_p\, v = g_\lambda(x,v) + \tau \ge \tau \quad \text{for all} \quad x \in \Omega.$$

Let e be the positive solution of

$$\begin{cases} -\Delta_p\, e = 1 & \text{in} \quad \Omega\,, \\ e = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.4}$$

and let $w = (\tau/2)^{\frac{1}{p-1}} e$ in Ω. Then it follows that $-\Delta_p\, w = \tau/2 < -\Delta_p\, v$ in Ω and $v = w$ on $\partial\Omega$. Thus, using the comparison lemma (see [9]), we obtain that $v \ge w$ in Ω. Therefore, there is a positive constant θ such that

$$\tau \le \theta\, v^{p-1}$$

at the maximum point of e. This completes the proof. □

The next proposition establishes a bound for the L^∞-norms of positive solutions of $(A_\lambda)_\tau$.

Proposition 2.1. *For any $\lambda > 0$, let $v \in C^1(\overline{\Omega})$ be a positive solution of $(A_\lambda)_\tau$. Then*

$$0 \le v(x) + \tau \le C \quad \textit{for all} \quad x \in \overline{\Omega}\,,$$

where C is a positive constant depending only on λ and Ω, but not on $\tau \ge 0$.

Proof. Suppose that the conclusion of Proposition 2.1 is false. Then there exists a sequence $\{(v_n, \tau_n)\}_{n\in\mathbb{N}}$ with v_n being a C^1-solution of $(A_\lambda)_{\tau_n}$ such that $\max_{\overline{\Omega}} v_n + \tau_n \xrightarrow[n\to\infty]{} \infty$. By Lemma 2.2, we may assume that there exists $x_n \in \Omega$ such that $v_n(x_n) = \max_{\overline{\Omega}} v_n = S_n \xrightarrow[n\to\infty]{} \infty$. Write $\delta_n = \operatorname{dist}(x_n, \partial\Omega)$ and define $w_n(y) = S_n^{-1} v_n(x)$, where $x = A_n y + x_n$ and $A_n = [\lambda(p-1)^{q+1-p}(\ln(S_n+1))^q]^{-\frac{1}{p}}$. The functions w_n are well-defined for y at least in $B(0, \delta_n A_n^{-1})$ and $w_n(0) = \max w_n = 1$. Straightforward computations show that

$$c_0 \Upsilon\big(S_n, w_n(y)\big) \le -\Delta_p\, w_n(y) - A_n^p \frac{\tau_n}{S_n^{p-1}} \le c_1 \Upsilon\big(S_n, w_n(y)\big)\,,$$

where $\Upsilon(S_n, w_n(y)) = (w_n(y) + S_n^{-1})^{p-1}[\frac{\ln(S_n w_n(y)+1)}{\ln(S_n+1)}]^q$. Now, by Lemma 2.2, $A_n^p \frac{\tau_n}{S_n^{p-1}}$ tends to zero as n tends to infinity. Then, applying regularity theorems for the p-Laplacian operator, we can obtain estimates for w_n such that (up to a subsequence) $w_n \to w$, locally uniformly, with w a C^1-function defined on $\mathbb{R}^N$ or on a half space, according to whether the limit of δ_n is positive or zero.

Thus, w is a solution of the problem

$$\begin{aligned} c_0 w^{p-1} \le -\Delta_p\, w \le c_1 w^{p-1}\,, \\ w > 0\,, \quad w(0) = \max w = 1 \end{aligned} \tag{2.5}$$

which contradicts Lemma 2.1. □

3. Proof of the main results

The following lemma is a variant of a result due to Rabinowitz [23] which was proved in [4].

Lemma 3.1. *Let $\mathbb{R}^+ = [0, +\infty)$ and $(E, \|\cdot\|)$ be a real Banach-space. Let $G : \mathbb{R}^+ \times E \to E$ be a continuous map which transforms bounded subsets into relatively compact ones. Moreover, suppose that G satisfies*

(a) $G(0,0) = 0$,
(b) *There exists $R > 0$ such that*
  (i) *$u \in E$, $\|u\| \le R$ and $u = G(0,u)$ implies $u = 0$,*
  (ii) *$deg(Id - G(0,\cdot), B(0,R), 0) = 1$.*

Let J denote the set of solutions of the problem

$$(\mathfrak{P}) u = G(t,u)$$

in $\mathbb{R}^+ \times E$. Further, let $\mathfrak{C}$ denote the component (i.e. maximal closed connected subset) of J containing $(0,0)$. If

$$\mathfrak{C} \cap \big(\{0\} \times E\big) = \{(0,0)\}\,,$$

then $\mathfrak{C}$ is unbounded on $\mathbb{R}^+ \times E$.

Now, we recall the following definition of lower and upper solutions of $(A_\lambda)_0$ (see, for example, [3]).

Definition 3.1. A function $\underline{v} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is said to be a lower-solution of $(A_\lambda)_0$ if

$$\begin{cases} \int_\Omega |\nabla \underline{v}|^{p-2} \nabla \underline{v}\, \nabla \phi \le \int_\Omega g_\lambda(x, \underline{v})\phi \\ \text{for each} \quad \phi \in W_0^{1,p}(\Omega) \quad \text{with} \quad \phi \ge 0\,, \\ \underline{v} \le 0 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

(A function $v \in W^{1,p}(\Omega)$ is said to be less than or equal to $w \in W^{1,p}(\Omega)$ on $\partial\Omega$ if $\max\{0, v - w\} \in W_0^{1,p}(\Omega)$).

Furthermore a function $\overline{v} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is said to be an upper-solution of $(A_\lambda)_0$ if

$$\begin{cases} \int_\Omega |\nabla \overline{v}|^{p-2} \nabla \overline{v}\, \nabla \phi \ge \int_\Omega g_\lambda(x, \overline{v})\phi \\ \text{for each} \quad \phi \in W_0^{1,p}(\Omega) \quad \text{with} \quad \phi \ge 0\,, \\ \overline{v} \ge 0 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

(A function $v \in W^{1,p}(\Omega)$ is said to be bigger than or equal to $w \in W^{1,p}(\Omega)$ on $\partial\Omega$ if $\max\{0, w - v\} \in W_0^{1,p}(\Omega)$). An upper-solution which is not a solution is said to be strict.

Proof of Theorem 1.1

We will prove a result which is slightly more general than Theorem 1.1.

Proposition 3.1. *Given $\lambda > 0$ there exists a positive constant $\tau^* = \tau^*(\lambda)$ such that $(A_\lambda)_\tau$ has at least one positive solution for $0 \le \tau < \tau^*$.*

Proof. Let $\lambda > 0$ and define $N : C^{1,\alpha}(\overline{\Omega}) \to L^\infty(\overline{\Omega})$ by $N(v) = g_\lambda(x, v)$. From the continuity of g_λ and the compactness of the inclusion $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow L^\infty(\overline{\Omega})$ we conclude that N is compact.

Also, we consider $T : L^\infty(\overline{\Omega}) \to C^{1,\alpha}(\overline{\Omega})$ as the unique weak solution $T(v)$ of the problem

$$\begin{cases} -\Delta_p T(v) = v \quad \text{in} \quad \Omega\,, \\ T(v) = 0 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

It is well-known that the function T is continuous and maps bounded sets into bounded sets (see Lemma 1.1 in [4], for example). Then $K = T \circ N : C^{1,\alpha}(\overline{\Omega}) \to C^{1,\alpha}(\overline{\Omega})$ is compact.

Let $G(t, v) = T(N(v) + t)$, then $G : \mathbb{R}^+ \times C^{1,\alpha}(\overline{\Omega}) \to C^{1,\alpha}(\overline{\Omega})$ is compact. Now, we will verify the hypotheses of Lemma 3.1. It is clear that $G(0, 0) = 0$. On the other hand, consider the compact homotopy $H(\mu, v) : [0, 1] \times C^{1,\alpha}(\overline{\Omega}) \to C^{1,\alpha}(\overline{\Omega})$ given by $H(\mu, v) = v - \mu\, G(0, v)$. We will show that

$$\text{if} \quad u \quad \text{is a nontrivial solution to} \quad H(\mu, v) = 0\,, \quad \text{then} \quad \|v\| > R > 0\,. \qquad (*)$$

This will implies that conditions (i) and (ii) of (b) hold.

In order to prove $(*)$, note that $H(\mu, v) = 0$ implies that v is a solution of the problem

$$\begin{cases} -\Delta_p\, v = \mu\, g_\lambda(x, v) & \text{in} \quad \Omega\,, \\ v = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{3.6}$$

where $\mu \in [0, 1]$. Now, from (G_2), given $\delta > 0$ small enough so that $p+\delta$ is less than the critical exponent $p^* = Np/(N-p)$, and given $\varepsilon > 0$, there exists a constant $c > 0$ (depending only on ε and δ) such that

$$g_\lambda(x, t) \le \lambda\, c_1 (c\,|t|^\delta + \varepsilon)|t|^{p-1}\,. \tag{3.7}$$

Multiplying the equation in (3.6) by v, integrating over Ω, using (3.7) and applying Hölder's and Poincare's inequalities, we have

$$\begin{aligned} \int_\Omega |\nabla v|^p &= \mu \int_\Omega g_\lambda(x, v)v \\ &\le \lambda\, c_1 \left(c \int_\Omega |v|^{\delta+p} + \varepsilon \int_\Omega |v|^p \right) \\ &\le \lambda C \left(\int_\Omega |\nabla v|^p \right)^{\frac{\delta+p}{p}} + \lambda C_1\, \varepsilon \int_\Omega |\nabla v|^p\,. \end{aligned}$$

Choosing ε small enough we obtain that $\|v\| > R$ for some positive constant $R = R(\lambda)$.

Proposition 2.1 implies that the component $\mathfrak{C}$ which contains $(0, 0)$ is bounded, so applying Lemma 3.1 we obtain that $\mathfrak{C} \cap (\{0\} \times C^{1,\alpha}(\overline{\Omega})) \neq \{(0, 0)\}$. Therefore, we have a positive solution of $(A_\lambda)_0$. Hence, defining $\tau^* = \sup\{\tau > 0 : (A_\lambda)_\tau$ has a positive solution v so that $(v, \tau) \in \mathfrak{C}\}$, we see that τ^* verifies the conclusion. This completes the proof. □

Proof of Theorem 1.2

First, note that $(A_\lambda)_0$ has no positive solutions for $\lambda > \overline{\lambda} = \frac{\lambda_1(\Omega)}{f_0}$. Indeed, if v is a positive solution of $(A_\lambda)_0$, then v satisfies

$$\begin{cases} -\Delta_p\, v = \lambda\, \widehat{g}\,(x, v) & \text{in} \quad \Omega\,, \\ v = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{$\star$}$$

where $\widehat{g}\,(x, v) = (\frac{v+1}{p-1})^{p-1} f(x, (p-1)\ln(v+1))$.

From (G_2) and (F_3) we have

$$\begin{aligned} \widehat{g}\,(x, v) &\ge f_0 \left(\frac{v+1}{p-1} \right)^{p-1} (p-1)^{p-1} \big(\ln(v+1)\big)^{p-1} \\ &= f_0 \big[(v+1)\ln(v+1)\big]^{p-1} \ge f_0\, v^{p-1}\,, \end{aligned}$$

then, using inequalities as in the proof of Lemma 2.1, we get

$$\big(\lambda f_0 - \lambda_1(\Omega)\big) \int_\Omega \phi_1^p \le \int_\Omega (-\Delta_p\, v) \frac{\phi_1^p}{v^{p-1}} - \int_\Omega |\nabla \phi_1|^p \le 0 \tag{3.8}$$

and this inequality implies that $\lambda \le \overline{\lambda}$.

Let $0 < \lambda < \overline{\lambda}$. Following the lines of the proof of Proposition 3.1, we will show that any nontrivial solution of the problem

$$\begin{cases} -\Delta_p\, w = \rho\, g_\lambda(x,w) & \text{in} \quad \Omega\,, \\ w = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases}$$

where $\rho \in [0,1]$, satisfies condition $(*)$.

To this end, choose $0 < \varepsilon < \frac{\overline{\lambda}}{\lambda} - 1$. Then there exist positive constants $c = c(\lambda, \varepsilon)$ and $0 < \delta < \frac{p^2}{N-p}$ such that

$$g_\lambda(x,t) \le \lambda f_0 (1+\varepsilon)\, t^{p-1} + c\, t^{p-1+\delta}\,.$$

Using the last inequality, together with the variational characterization of the first eigenvalue $\lambda_1(\Omega)$ and Poincare's inequality, we get

$$\begin{aligned} \|w\|^p = \int_\Omega |\nabla w|^p = \rho \int_\Omega g_\lambda(x,w) w \\ &\le \lambda f_0 (1+\varepsilon) \int_\Omega w^p + c \int_\Omega w^{p+\delta} \\ &\le \lambda(1+\varepsilon) \frac{f_0}{\lambda_1(\Omega)} \|w\|^p + C \|w\|^{p+\delta}\,. \end{aligned}$$

Hence $\|w\| > \tilde{c}$ for some positive constant $\tilde{c}$.

Therefore, using Lemma 3.1, we have shown the existence of a positive solution of $(A_\lambda)_0$ for $0 < \lambda < \overline{\lambda}$.

Finally, we need to prove that $(A_{\overline{\lambda}})_0$ has no positive solutions. Arguing by contradiction, we assume that there exists a positive solution v for $\lambda = \overline{\lambda}$. Then from (3.8) we arrive at

$$0 = \big(\lambda f_0 - \lambda_1(\Omega)\big) \int_\Omega \phi_1^p \le \int_\Omega (-\Delta_p\, v) \frac{\phi_1^p}{v^{p-1}} - \int_\Omega |\nabla \phi_1|^p = -\int_\Omega \mathcal{L}(\phi_1, v) \le 0\,.$$

Hence $\mathcal{L}(\phi_1, v) = 0$ and Picone's identity implies that there exists a $c > 0$ such that $v = c\phi_1$. This together with the comparison principle yields the contradiction.

Proof of Theorem 1.3

We define $\varphi : (0, +\infty) \to (0, +\infty)$ by

$$\varphi(t) = \frac{(t+1)^{p-1} (\ln(t+1))^q}{t^{p-1}}\,.$$

Since $0 < q < p-1$, there exists $t_0 > 0$ such that

$$\varphi(t_0) = \inf_{t>0} \varphi(t)\,.$$

With the same notation used in the proof of Theorem 1.2 we arrive at

$$\frac{\widehat{g}(x,t)}{t^{p-1}} \ge c_0 (p-1)^{q+1-p} \varphi(t) \quad \text{for all} \quad t > 0\,.$$

Hence $(A_\lambda)_0$ has no positive solutions if $\lambda > \Lambda = \lambda_1(\Omega) \frac{(p-1)^{p-1-q}}{c_0 \varphi(t_0)}$.

On the other hand, let $\tilde{\lambda}$ such that $\tilde{\lambda} c_1 (p-1)^{q+1-p} \varphi(t_0) = \|e\|_{L^\infty}^{1-p}$, where e is given in (2.4). Observe that, using a comparison argument, we get $\Lambda > \tilde{\lambda}$. Then, for any $0 < \lambda \leq \tilde{\lambda}$, we have that $\overline{v}_\lambda = Me$, is an upper-solution of $(A_\lambda)_0$ where $M = t_0/\|e\|_{L^\infty}$. In effect, we have

$$\begin{aligned}\int_\Omega |\nabla \overline{v}_\lambda|^{p-2}\, \nabla \overline{v}_\lambda \nabla \phi = M^{p-1} \int_\Omega |\nabla e|^{p-2} \nabla e \nabla \phi \\ = \int_\Omega t_0^{p-1} \varphi(t_0) \tilde{\lambda} c_1 (p-1)^{q+1-p} \phi \\ \geq \int_\Omega (Me)^{p-1} \varphi(Me) \tilde{\lambda} c_1 (p-1)^{q+1-p} \phi \\ \geq \int_\Omega \lambda \widehat{g}(x, Me) \phi = \int_\Omega g_\lambda(x, \overline{v}_\lambda) \phi \,.\end{aligned}$$

Thus $\overline{v}_\lambda$ is an upper-solution of $(A_\lambda)_0$. Therefore, applying Theorem 1.3 of [7], we obtain a solution of $(A_\lambda)_0$ for any $0 < \lambda < \tilde{\lambda}$.

Let

$$\overline{\Lambda} = \sup \left\{ \lambda > 0 : \ (A_\lambda)_0 \text{ has a positive solution} \right\},$$

then $\tilde{\lambda} \leq \overline{\Lambda} \leq \Lambda$.

Lemma 3.2. $(A_{\overline{\Lambda}})_0$ *has at least one positive solution.*

Proof. As in the proof of Theorem 1.1 of [8], we consider $\{\lambda_n\}_{n\in\mathbb{N}}$ a strictly increasing sequence with $\lim_{n\to\infty} \lambda_n = \overline{\Lambda}$, $\lambda_n \in [\overline{\Lambda}/2, \overline{\Lambda})$, and write v_{λ_n} for a positive solution of $(A_{\lambda_n})_0$ for $n = 1, 2, \ldots$. Note that the behavior of φ near zero implies that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\lambda_1(\Omega) + \varepsilon \leq \frac{\overline{\Lambda}}{2}\, c_0 (p-1)^{q+1-p} \varphi(s) \quad \text{for all} \quad s \in (0, \delta)\,. \tag{3.9}$$

Setting $\varepsilon = 1$ we choose $\delta_0 > 0$ so that (3.9) holds for any $s \in (0, \delta_0)$. Using the above we will show that

$$\|v_{\lambda_n}\|_\infty \geq \delta_0 \quad \text{for} \quad n = 1, 2, \ldots\,. \tag{3.10}$$

Indeed, if we assume that $0 < v_{\lambda_n} < \delta_0$ in Ω, then Picone's identity implies that

$$\begin{aligned}\int_\Omega \phi_1^p &= \int_\Omega \big(\lambda_1(\Omega) + 1\big) \phi_1^p - \int_\Omega \lambda_1(\Omega) \phi_1^p \\ &\leq \frac{\overline{\Lambda}}{2} \int_\Omega \widehat{g}_{\lambda_n}(x, v_{\lambda_n}) \frac{\phi_1^p}{v_{\lambda_n}^{p-1}} - \int_\Omega |\nabla \phi_1|^p \\ &\leq \lambda_n \int_\Omega \widehat{g}_{\lambda_n}(x, v_{\lambda_n}) \frac{\phi_1^p}{v_{\lambda_n}^{p-1}} - \int_\Omega |\nabla \phi_1|^p \\ &= \int_\Omega (-\Delta_p\, v_{\lambda_n}) \frac{\phi_1^p}{v_{\lambda_n}^{p-1}} - \int_\Omega |\nabla \phi_1|^p \leq 0\,.\end{aligned}$$

This is a contradiction. Hence we obtain (3.10).

As in Proposition 2.1 we conclude that the functions v_{λ_n} are equibounded on $C^{1,\alpha}(\overline{\Omega})$. Thus by the Arzela–Ascoli Theorem, there exists a subsequence of $\{v_{\lambda_n}\}$ which converges in $C_0^1(\overline{\Omega})$ to a function $v \in C_0^1(\overline{\Omega})$ that may be identified easily as a solution of $(A_{\overline{\Lambda}})_0$. Moreover (3.10), shows that

$$\|v\|_\infty \geq \delta_0 \,.$$

This proves the lemma. □

Let $0 < \lambda < \overline{\Lambda}$ be fixed. The following lemma was proved by de Figueiredo and Lions [10] in the particular case $p = 2$ and $g_\lambda(x,u) = g(u)$, although slight modifications of that proof allow us to deal with the problem $(A_\lambda)_0$.

Lemma 3.3. *In addition to (G_1), (G_2) and (G_3), assume that $(A_\lambda)_0$ has a strict upper-solution w_λ. Then $(A_\lambda)_0$ has two solutions $0 < u_1 < u_2$, and u_1 is the minimal solution.*

Proof. Let us denote by X the Banach space of C^1-functions on $\overline{\Omega}$ which are 0 on $\partial\Omega$, endowed with the usual C^1-norm. Define g_λ for negative s as $g_\lambda(x,s) = g_\lambda(x,0) = 0$ for all $x \in \Omega$. The solutions of $(A_\lambda)_0$ with this extended g_λ are the same as the original problem $(A_\lambda)_0$. For in either case the solutions are all positive. Let us define the one-parameter family of functions

$$h_\mu(x,s) = \mu g_\lambda(x,s) + (1-\mu)\xi s_+^{p-1}\,, \quad 0 \leq \mu \leq 1$$

where $\xi > \lambda_1(\Omega)$ is a fixed number and $s_+ = \max\{0,s\}$. In view of our assumptions, there is a constant $c_1 > 0$ such that

$$\|u\|_X \leq c_1 \tag{3.11}$$

for all possible solutions of the problems

$$\begin{cases} -\Delta_p\, u = h_\mu(x,u) & \text{in} \quad \Omega\,, \quad 0 \leq \mu \leq 1\,, \\ u = 0 \quad \text{on} & \partial\Omega\,. \end{cases} \tag{$(Q)_\mu$}$$

Indeed, assume that there exist $\{(u_n,\mu_n)\}$ such that $\|u_n\|_X \to +\infty$ and $\mu_n \in [0,1]$. If $0 \in \overline{\{\mu_n\}_{n\in\mathbb{N}}}$, i.e., 0 is a cluster point of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$, then without loss of generality we may assume that $\mu_n \to 0$. Thus, for n large enough we have that $\xi(1-\mu_n) > \lambda_1(\Omega)$. Now, using an argument similar to one used in the proof of Lemma 2.1, we get

$$\big(\xi(1-\mu_n) - \lambda_1(\Omega)\big)\int_\Omega \phi_1^p \leq 0\,.$$

But this is impossible. Therefore, there exists $0 < \mu_0 \leq 1$ such that $\mu_0 \leq \mu_n$ for all $n \in \mathbb{N}$. Now, using Lieberman's estimates [21], we may assume that $\|u_n\|_\infty \to +\infty$. Let $x_n \in \Omega$ be such that $u_n(x_n) = \|u_n\|_\infty = S_n$, $d_n = \operatorname{dist}(x_n,\partial\Omega)$ and $A_n = [\mu_n\lambda(p-1)^{q+1-p}(\ln(S_n+1))^q]^{-\frac{1}{p}}$. In this way, defining $w_n(y) = S_n^{-1}u_n(A_n y + x_n)$, we see that the functions w_n are well-defined at least on $B(0, d_n A_n^{-1})$ and $w_n(0) = \|w_n\|_\infty = 1$. Now, straightforward computations show that

$$c_0\Upsilon\big(S_n, w_n(y)\big) \leq -\Delta_p\, w_n(y) - (1-\mu_n)\xi A_n^p\big(w_n(y)\big)^{p-1} \leq c_1\Upsilon\big(S_n, w_n(y)\big)\,,$$

where $\Upsilon(S_n, w_n(y))$ is as in the proof of Proposition 2.1. Now, applying regularity theorems for the p-Laplacian operator, we can obtain $C^{1,\alpha}(\overline{\Omega})$ estimates for w_n such that for a subsequence $w_n \to w$, locally uniformly, with w a C^1-function defined on $\mathbb{R}^N$ or on a halfspace. This implies that w is a solution of $-\Delta_p w \geq c_0 w^{p-1}$, $w \geq 0, w(0) = \max w = 1$, which is a contradiction.

Therefore, there exist $c_1 > 0$ such that $\|u\|_X \leq c_1$ for all possible positive solutions u of $(Q)_\mu$ for $0 \leq \mu \leq 1$.

By (G_3) there exists $k > 0$ such that $g_\lambda(x, s) + ks^{p-1}$ is increasing for $s \in [0, \|w_\lambda\|_{L^\infty}]$ for a.e. $x \in \Omega$. Let $K_\mu = (-\Delta_p + k)^{-1}(h_\mu(x, \cdot\,) + k\,\cdot\,)$. More precisely, let us define $K_\mu : X \to X$ as follows: $K_\mu v = u$, where u is the solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u + ku^{p-1} = h_\mu(x, v) + kv^{p-1} & \text{in} \quad \Omega\,, \\ u = 0 \quad \text{on} \quad \partial\Omega\,. \end{cases}$$

The mapping K_μ so defined is compact. From the Schauder estimates (see for instance [21]) it follows that there exists a constant $c_2 > 0$ such that

$$\|K_\mu v\|_X \leq c_2\,, \quad \forall\, v \in X\,, \quad 0 \leq v \leq w_\lambda\,. \tag{3.12}$$

Since $0 < q < p - 1$, we see that there is an $\varepsilon_\lambda > 0$ such that $\varepsilon_\lambda \phi_1$ is a lower-solution for all problems $(Q)_\mu$. Also we may take $\varepsilon_\lambda \phi_1 < w_\lambda$ on Ω. It follows from the maximum principle that any solution u of $(Q)_\mu$ such that $u \geq \varepsilon_\lambda \phi_1$ on Ω satisfies the strict inequalities $u > \varepsilon_\lambda \phi_1$ on Ω and $\partial u/\partial\nu < \varepsilon_\lambda(\partial\phi_1/\partial\nu)$ on $\partial\Omega$. Now consider the bounded open set

$$\mathcal{O} = \left\{ u \in X :\, \|u\|_X < c_1 + c_2 + 1,\, u > \varepsilon_\lambda \phi_1 \text{ in } \Omega,\, \frac{\partial u}{\partial \nu} < \varepsilon_\lambda \frac{\partial \phi_1}{\partial \nu} \text{ on } \partial\Omega \right\},$$

where c_1 and c_2 are the constants defined in (3.11) and (3.12), respectively, and $\varepsilon_\lambda \phi_1$ is the lower-solution defined above. By the above remarks, it follows that $0 \notin (I - K_\mu)(\partial\mathcal{O})$. So the degree $\deg(I - K_\mu, \mathcal{O}, 0)$ is independent of $\mu \in [0, 1]$. Clearly the degree $\deg(I - K_0, \mathcal{O}, 0) = 0$ since $(Q)_0$ has no solution. Hence

$$\deg(I - K_1, \mathcal{O}, 0) = 0\,.$$

Now let us consider the following open subset of $\mathcal{O}$:

$$\mathcal{O}' = \left\{ u \in \mathcal{O} :\, u < w_\lambda \text{ in } \Omega,\, \frac{\partial u}{\partial \nu} > \frac{\partial w_\lambda}{\partial \nu} \text{ on } \partial\Omega \right\},$$

and we claim that $\deg(I - K_1, \mathcal{O}', 0) = 1$. Once this is proved, it follows that $\deg(I - K_\mu, \mathcal{O} \setminus \overline{\mathcal{O}'}, 0) = -1$. So $(Q)_1$, which is the same as $(A_\lambda)_0$, has two solutions $\tilde{u}_1 \in \mathcal{O}'$ and $u_2 \in \mathcal{O} \setminus \overline{\mathcal{O}'}$, which are not necessarily ordered. So we have to proceed further in order to complete the proof of the theorem. Let $v = \min\{w_\lambda, u_2\}$. It follows that v is in $W_0^{1,\infty}(\Omega)$, $\varepsilon_\lambda \phi_1 \leq v$ and

$$-\Delta_p v \geq g_\lambda(x, v)\,, \quad \text{in} \quad \mathcal{D}'(\Omega).$$

So the monotone iteration method yields the existence of a solution u_1 of $(A_\lambda)_0$, with $0 < u_1 < u_2$ and u_1 is the minimal solution of $(A_\lambda)_0$. To complete the proof we have to prove the above claim. To do so observe that K_1 maps $\mathcal{O}'$ into $\mathcal{O}'$. Let $u_0 \in \mathcal{O}'$ and consider the constant mapping $C : \mathcal{O}' \to \mathcal{O}'$ defined by $C(u) = u_0$. By the convexity of $\mathcal{O}'$, it follows that $I - K_1$ is homotopic to $I - C$ on $\mathcal{O}'$ and $\deg(I - K_1, \mathcal{O}', 0) = \deg(I - C, \mathcal{O}', 0)$. But this last degree is trivially equal to 1. The proof is complete. □

Using Lemma 3.3 above, we only need to construct a strict upper-solution $\overline{u}_\lambda$. It is easy to see that, setting $\overline{u}_\lambda = u_{\overline{\Lambda}}$, where $u_{\overline{\Lambda}}$ is a positive solution of $(A_{\overline{\Lambda}})_0$, $\overline{u}_\lambda$ is a strict upper-solution to the problem $(A_\lambda)_0$ for any $0 < \lambda < \overline{\Lambda}$.

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