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Cantor families of periodic solutions of wave equations with C^k nonlinearities

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Abstract. We prove bifurcation of Cantor families of periodic solutions for wave equations with nonlinearities of class C^k. It requires a modified Nash-Moser iteration scheme with interpolation estimates for the inverse of the linearized operators and for the composition operators.

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1 Introduction

The aim of this paper is to prove the bifurcation of "large" Cantor families of small amplitude periodic solutions for wave equations like

$$\begin{cases} u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.1}$$

where the nonlinearity f vanishes at $u = 0$, f is just C^k with respect to u, for some k large enough, and H^1 with respect to x. The frequencies of these periodic solutions form a set of asymptotically full measure.

If $f(x, 0) = 0$ then $(u, u_t) = (0, 0)$ is an elliptic equilibrium of the infinite dimensional Hamiltonian system associated to (1.1).

Bifurcation of small amplitude periodic and quasi-periodic solutions of (1.1) has been first proved, for analytic nonlinearities, by Kuksin [14] and Wayne [22] via KAM theory. Further extensions of the KAM techniques have been developed e.g. in [16], [8] [15], and, more recently, in [11], [23].

In the early nineties Craig-Wayne [10] introduced the Lyapunov-Schmidt reduction method to find periodic solutions of nonlinear wave equations in the case of periodic boundary conditions, solving the small divisor problem present in the range equation via an analytic Nash-Moser type technique, see also [9]. Subsequently Bourgain [5]-[6]-[7] developed this method to find also quasi-periodic solutions (in Gevrey class).

Another approach to overcome the small divisor problem for finding analytic periodic solutions of (1.1) has been proposed by Gentile-Matropietro-Procesi [13] via the Lindsted series power expansion method.

We underline that in all these papers the nonlinearity f is required to be analytic in both (x, u), fact which is exploited in the structure and in the estimates of the corresponding recursive schemes. In [20], adapting the method of [10], Su proved existence of periodic solutions of (1.1) which are Sobolev functions in both (t, x) for the specific nonlinearity $f = -m^2 u + g(x)u^3$ where $g(x)$ belongs to some Sobolev space (and is even). Since such an f is analytic with respect to u, one could expect existence of periodic solutions analytic in time.

Actually, in [4], for nonlinearities analytic with respect to u but just H^1 with respect to x, existence of periodic solutions of (1.1) which are analytic in time and valued in $H^3(0, \pi) \cap H_0^1(0, \pi)$ was proved.

On the other hand, for nonlinearities f just of class C^k with respect to u one could expect existence of periodic solutions which are differentiable up to some finite order in time, and no more. However the iterative Nash-Moser scheme of [4] (and [10]) has to be deeply modified. This is done in this paper.

Let us explain the main difficulties. In [10]–[4] analyticity plays an important role at several stages. However, as in any Nash-Moser scheme, the most delicate step is the inversion of the linearized operators obtained at each iteration. The method developed in [10] is based on the Frölich-Spencer estimates [12] and exploits analyticity in the exponentially fast decay off-the diagonal of the coefficients of the matrices representing the linear operators.

On the other hand, the new method in [4] dealt with nonlinearities only H^1 with respect to x and, what is more important, it did not use analyticity with respect to time for the inversion of the linearized operators.

Once this key property of invertibility is obtained in Sobolev spaces, a Nirenberg - Moser type interpolation estimate for the inverse operator in high Sobolev norms is needed (see property (P5) proved in section 5). In particular, (4.9) is necessary for the convergence of the iterative scheme, see remark 4.2.

Next, also other steps of the Nash-Moser iteration scheme of [10]–[4] have to be modified. In doing this we adapt ideas of [17], [18], [19]. We describe the main changes, which are based on functional inequalities and a-priori estimates, in section 4, see remarks 4.1, 4.3, 4.4.

In order to focus the attention on the main issue -namely the solution of the range equation via a differentiable Nash-Moser iteration scheme- we consider the completely resonant nonlinear wave equations of [4], without making any attempt to deal with nonlinearities which would require to solve differently the bifurcation equation.

Finally we mention that, for zero measure sets of frequencies, existence of periodic solutions of wave equations can be proved just using a standard implicit function theorem, see [1], [2], [3].

1.1 Main result

We first introduce, for any integer $k \geq 1$, the set of nonlinearities

$$\mathcal{F}_k := \Big\{ f \in C([0,\pi] \times \mathbf{R}, \mathbf{R}) \ : \ u \mapsto f(\cdot, u) \text{ is in } C^k(\mathbf{R}, H^1(0,\pi)) \Big\} .$$

Since $H^1(0,\pi)$ is continuously embedded in $C([0,\pi], \mathbf{R})$, if $f \in \mathcal{F}_k$ then $\partial_u^l f \in C([0,\pi] \times \mathbf{R}, \mathbf{R})$, $\forall 0 \leq l \leq k$.

Remark 1.1 If $\partial_u^l f(x,u)$, $\partial_x \partial_u^l f(x,u)$ exist and are continuous for all $0 \leq l \leq k$, then f belongs to $\mathcal{F}_k$. These assumptions are not necessarily satisfied by nonlinearities like $f(x,u) = a_p(x) u^p$, $p \in \mathbf{N}$, where $a_p \in H^1(0,\pi)$, which, however, belong to $\mathcal{F}_k$, $\forall k$.

Remark 1.2 Since we look for small amplitude solutions, we could more generally consider nonlinearities $f : [0,\pi] \times (-\rho, \rho) \to \mathbf{R}$ defined locally in some neighborhood $(-\rho,\rho)$ of $u = 0$. However we prefer to avoid technicalities in the definition of $\mathcal{F}_k$.

Normalizing the period to 2π, we look for solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.2}$$

where the nonlinearity $f \in \mathcal{F}_k$ satisfies, for some $p \in \mathbf{N}$, $2 \leq p \leq k$,

$(\mathbf{H})_p$ $\quad f(x,u) = a_p(x) u^p + r(x,u)$ with $a_p \in H^1(0,\pi)$ and
$r(x,0) = (\partial_u r)(x,0) = \ldots = (\partial_u^p r)(x,0) = 0$.

We look for solutions of (1.2) in the Sobolev space

$$\begin{aligned} H^s \ &:= \ \Big\{ u = \textstyle\sum_{l \in \mathbf{Z}} \exp{(\mathrm{i}lt)}\ u_l(x) \ \Big| \ u_l \in H_0^1((0,\pi), \mathbf{R}), u_l = u_{-l} \ \forall l \in \mathbf{Z}, \\ &\qquad\qquad \|u\|_s^2 := \textstyle\sum_{l \in \mathbf{Z}} (l^{2s} + 1) \|u_l\|_{H^1}^2 < +\infty \Big\} \\ &= \ H^s_{\mathrm{even}}(\mathbf{T}, H_0^1(0,\pi)) \end{aligned}$$

of even 2π-periodic in time functions with values in $H_0^1(0,\pi)$.

We fix for the sequel the constant $s > 1/2$ so that H^s is a multiplicative Banach algebra

$$\|u_1\, u_2\|_s \le C(s) \|u_1\|_s \|u_2\|_s\,, \qquad \forall u_1, u_2 \in H^s\,,$$

and

$$\|u\|_{L^\infty(\mathbf{T}, H_0^1(0,\pi))} \le C(s)\|u\|_s\,, \qquad \forall u \in H^s\,. \tag{1.3}$$

After the rescaling $u \to \delta u$, $\delta > 0$, equation (1.2) takes the form

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta^{p-1} g(\delta, x, u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.4}$$

where

$$g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^p} = a_p(x) u^p + \frac{r(x, \delta u)}{\delta^p}\,.$$

To find solutions of (1.4) we implement the Lyapunov-Schmidt reduction according to the orthogonal decomposition

$$H^s = (V \cap H^s) \oplus (W \cap H^s)$$

where

$$W := \Big\{ w = \sum_{l \in \mathbf{Z}} \exp(\mathrm{i} l t)\; w_l(x) \in H^0 \;\mid\; \int_0^\pi w_l(x) \sin(lx)\, dx = 0,\ \forall l \in \mathbf{Z} \Big\},$$

and

$$V := \Big\{ v = \sum_{l \ge 1} 2\cos(lt) u_l \sin(lx) \;\Big|\; u_l \in \mathbf{R}\,,\ \sum_{l\ge 1} l^2 |u_l|^2 < +\infty \Big\}$$

is the space of the solutions of $v_{tt} - v_{xx} = 0$ that belong to $H_0^1(\mathbf{T} \times (0,\pi), \mathbf{R})$ and are even in time.

Projecting (1.4), setting $u = v + w$, $v \in V$, $w \in W$, and imposing the "frequency-amplitude" relation

$$\frac{\omega^2 - 1}{2} = s^* \varepsilon\,, \qquad \varepsilon := \delta^{p-1}\,, \tag{1.5}$$

with $s^* = \pm 1$ to be chosen later (see (1.8)), yields

$$\begin{cases} -\Delta v = s^* \Pi_V g(\delta, x, v + w) & (Q) \\ L_\omega w = \varepsilon\, \Pi_W g(\delta, x, v + w) & (P) \end{cases} \tag{1.6}$$

where

$$\Delta v := v_{xx} + v_{tt}\,, \qquad\qquad L_\omega := -\omega^2 \partial_t^2 + \partial_x^2$$

and Π_V, Π_W denote the projectors respectively on V and W.

For $\delta = 0$ the (Q)-equation reduces to

$$-\Delta v = s^* \Pi_V (a_p(x) v^p) \,. \tag{1.7}$$

We assume for simplicity that

$$\Pi_V (a_p(x) v^p) \not\equiv 0,$$

which is equivalent to

$$\exists v \in V \quad \text{such that} \quad \int_\Omega a_p(x) v^{p+1} \neq 0 \,, \qquad \text{with} \quad \Omega := \mathbf{T} \times (0, \pi) \,.$$

Taking

$$s^* := \begin{cases} 1 & \text{if } \exists\, v \in V \text{ such that } \int_\Omega a_p(x) v^{p+1} > 0 \\ -1 & \text{if } \exists\, v \in V \text{ such that } \int_\Omega a_p(x) v^{p+1} < 0 \end{cases} \tag{1.8}$$

equation (1.7) possesses at least one solution $\bar v \in V$ (in fact infinitely many) which can be seen (cfr. [4]) as a critical point of "mountain pass" type for the functional

$$\Phi_0(v) := \int_\Omega \frac{|\nabla v|^2}{2} - s^* a_p(x) \frac{v^{p+1}}{p+1} \, dt \, dx \,.$$

We assume the following nondegeneracy condition (of KAM type) which can be verified on several examples, see [4].

(ND) There exists a nondegenerate solution $\bar v \in V \setminus \{0\}$ of equation (1.7), namely $h = 0$ is the unique solution of the linearized equation

$$-\Delta h = s^* \Pi_V (p \, a_p(x) \bar v^{p-1} h) \ , \ h \in V.$$

We can state our main existence result.

Theorem 1.1 *Let $f(x, u)$ satisfy assumptions* $(\mathbf{H})_p$ *with $p \geq 2$ and* **(ND)**. *Fix $s > 1/2$. There is $k := k(s, p) \in \mathbf{N}$ such that if $f \in \mathcal{F}_k$, there exists $\delta_0 > 0$, a Cantor like set $\mathcal{C} \subset [0, \delta_0)$ of asymptotically full measure, i.e. satisfying*

$$\lim_{\eta \to 0^+} \frac{\text{meas}(\mathcal{C} \cap (0, \eta))}{\eta} = 1 \,,$$

and a C^1-curve $[0, \delta_0) \ni \delta \to u(\delta) \in H^s$ such that:

- *(i)* $\|u(\delta) - \delta \bar v\|_s = O(\delta^2)$
- *(ii) $\forall\ \delta \in \mathcal{C}$, $u(\delta)$ is a 2π-periodic solution of (1.2), with $\omega = \omega(\delta)$ given by (1.5).*

As a consequence, $\forall \delta \in \mathcal{C}$, $\widetilde{u}(\delta)(t,x) := u(\delta)(\omega(\delta)t, x)$ is a $2\pi/\omega(\delta)$-periodic solution of (1.1).

The order k of differentiability of the nonlinearity $f \in \mathcal{F}_k$ has to satisfy

$$k \geq \max\{p+3, s+\beta+3\} \tag{1.9}$$

where $\beta > 0$ is defined in (4.27), see also the comments after (4.28). The function $u(\delta)$ defines a C^1 Whitney extension of the family of periodic solutions of (1.2) constructed in the above theorem, see [4].

Remark 1.3 The condition $k \geq \max\{p+3, s+3\}$ is assumed for the regularity of the composition operator $g(\delta, x, u)$ in Lemma 2.6, and $k \geq s+\beta+3$ is used for the convergence of the Nash-Moser scheme, see remark 4.3.

The main changes to be introduced to prove Theorem 1.1 with respect to the method of [4], regard the solution of the range equation through a differentiable Nash-Moser iterative scheme. This is done in sections 4 and 5, see remarks 4.1, 4.2, 4.3, 4.4.

On the other hand, we follow [4] to overcome the problem posed by an infinite dimensional bifurcation (Q)-equation performing a further finite dimensional Lyapunov-Schmidt reduction. Let us introduce the decomposition

$$V = V_1 \oplus V_2$$

where

$$\begin{cases} V_1 := \Big\{ v \in V \mid v = \sum_{l=1}^{\bar{N}} \cos(lt) u_l \sin(lx) \Big\} \\ V_2 := \Big\{ v \in V \mid v = \sum_{l > \bar{N}} \cos(lt) u_l \sin(lx) \Big\}, \end{cases} \tag{1.10}$$

and $\bar{N}$ is defined in Lemma 3.1. Setting $v := v_1 + v_2$, $v_1 \in V_1, v_2 \in V_2$, system (1.6) is equivalent to

$$\begin{cases} -\Delta v_1 = s^* \Pi_{V_1} g(\delta, x, v_1 + v_2 + w) & (Q1) \\ -\Delta v_2 = s^* \Pi_{V_2} g(\delta, x, v_1 + v_2 + w) & (Q2) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v_1 + v_2 + w) & (P) \end{cases}$$

where Π_{V_i} $(i = 1, 2)$ denote the projectors on V_i.

As in [4] we solve first the $(Q2)$ equation. This requires minor modifications for Lemma 3.1 and the new Lemma 3.2 below. Finally, we can solve the $(Q1)$ equation exactly as in [4].

2 Preliminaries on composition operators

We first recall some interpolation estimates and smoothness results for the composition operators, which we shall often use in the sequel.

The positive constants $C(s)$, $C(s')$ can assume different values from line to line.

Lemma 2.1 (Moser-Nirenberg) *Let $s' \geq 0$ and $s > 1/2$. $\forall u_1, u_2 \in H^{s'} \cap H^s$, we have*

$$\|u_1 u_2\|_{s'} \leq C(s')\Big(\|u_1\|_{L^\infty(\mathbf{T},H^1(0,\pi))}\|u_2\|_{s'} + \|u_1\|_{s'}\|u_2\|_{L^\infty(\mathbf{T},H^1(0,\pi))}\Big) \quad (2.1)$$

$$\leq C(s')\Big(\|u_1\|_s\|u_2\|_{s'} + \|u_1\|_{s'}\|u_2\|_s\Big)\,. \quad (2.2)$$

Proof. For the proof of (2.1) when $s' \in \mathbf{N}$ see e.g. [21, Proposition 3.7]. The same estimates work also for maps valued in $H^1_0(0,\pi)$, because $H^1_0(0,\pi)$ is an algebra. Next (2.2) follows by (1.3) since $s > 1/2$. □

Lemma 2.2 (Logarithmic convexity) *Let $0 \leq \alpha \leq a \leq b \leq \beta$ with $a + b = \alpha + \beta$. Then $\forall u_1, u_2 \in H^\beta$*

$$\|u_1\|_a\|u_2\|_b \leq \lambda\|u_1\|_\alpha\|u_2\|_\beta + (1-\lambda)\|u_2\|_\alpha\|u_1\|_\beta \quad \text{with} \quad \lambda := \frac{\beta - a}{\beta - \alpha}\,. \quad (2.3)$$

In particular

$$\|u\|_a\|u\|_b \leq \|u\|_\alpha\|u\|_\beta\,, \qquad \forall u \in H^\beta\,. \quad (2.4)$$

Proof. By the convexity of the map $\sigma \mapsto \log\|u\|_\sigma$ (see e.g. [18] chapter 1),

$$\begin{cases} \|u_1\|_a \leq \|u_1\|_\alpha^\lambda\|u_2\|_\beta^{1-\lambda} & \text{with} \quad a := \lambda\alpha + (1-\lambda)\beta \\ \|u_2\|_b \leq \|u_2\|_\alpha^{1-\lambda}\|u_2\|_\beta^\lambda & \text{with} \quad b := (1-\lambda)\alpha + \lambda\beta\,. \end{cases}$$

Then (2.3) follows by Young inequality. □

Lemma 2.3 *Let $f \in \mathcal{F}_1$. Then the composition operator $y(x) \mapsto f(x, y(x))$ is in $C(H^1(0,\pi), H^1(0,\pi))$ and satisfies, with the notation $M := \|y\|_{L^\infty(0,\pi)}$,*

$$\|f(x,y(x))\|_{H^1} \leq C\Big(\max_{u\in[-M,M]}\|f(\cdot,u)\|_{H^1} + \max_{u\in[-M,M]}\|\partial_u f(\cdot,u)\|_{H^1}\|y\|_{H^1}\Big)\,. \quad (2.5)$$

Proof. Let $y(x) \in H^1(0,\pi)$, $M = \|y\|_{L^\infty}$, $z(x) = f(x,y(x))$. We shall use for convenience the notations

$$A(f,M) := \max_{u\in[-M,M]}\|f(\cdot,u)\|_{H^1}\,, \qquad B(f,M) := \max_{u\in[-M,M]}\|\partial_u f(\cdot,u)\|_{H^1}\,.$$

$A(f,M)$ and $B(f,M)$ are finite because $f \in \mathcal{F}_1$ and so $u \mapsto \|f(\cdot,u)\|_{H^1}$ and $u \mapsto \|\partial_u f(\cdot,u)\|_{H^1}$ are continuous.

First, f being continuous, it is clear that we have $z \in L^\infty(0,\pi) \subset L^2(0,\pi)$ and

$$\|z\|_{L^2} \le C\|z\|_{L^\infty} \le C \max_{u\in[-M,M]} \|f(\cdot,u)\|_{L^\infty} \le CA(f,M)\,. \tag{2.6}$$

In the case when f is smooth (say C^2), it is clear as well that we have $z \in H^1(0,\pi)$ and

$$z'(x) = \partial_u f(x,y(x))y'(x) + \partial_x f(x,y(x)) = a(x) + b(x) + c(x) \tag{2.7}$$

with

$$\begin{gathered} a(x) := \partial_u f(x,y(x))y'(x), \quad b(x) := \partial_x f(x,0), \\ c(x) := \int_0^{y(x)} \partial_x\partial_u f(x,u)\; du\,. \end{gathered} \tag{2.8}$$

Using that $\partial_u f$ is continuous, we obtain, as in (2.6),

$$\begin{aligned} \|a\|_{L^2} + \|b\|_{L^2} &\le \|\partial_u f(x,y(x))\|_{L^\infty}\|y'\|_{L^2} + \|\partial_x f(\cdot,0)\|_{L^2} \\ &\le CB(f,M)\|y'\|_{L^2} + A(f,M). \end{aligned} \tag{2.9}$$

Moreover

$$\|c\|_{L^2}^2 \le 2M \int_0^\pi \Big(\int_{-M}^M |\partial_x\partial_u f(x,u)|^2\; du\Big)\, dx$$

by the Cauchy-Schwarz inequality and because $|y(x)| \le M,\ \forall x \in (0,\pi)$. Hence, by Fubini theorem,

$$\begin{aligned} \|c\|_{L^2}^2 &\le 2M \int_{-M}^M \Big(\int_0^\pi |\partial_x\partial_u f(x,u)|^2\; dx\Big)\, du \\ &\le 2M \int_{-M}^M \|\partial_u f(\cdot,u)\|_{H^1}^2\; du \le 4M^2 B(f,M)\,. \end{aligned} \tag{2.10}$$

(2.5) is a consequence of (2.6)–(2.10) and the fact that $M \le C\|y\|_{H^1}$.

If we remove the smoothness assumption on f and assume only $f \in \mathcal{F}_1$, we can consider some regularizing sequence of smooth maps (f_N) such that $A(f_N - f, M) + B(f_N - f, M) \to 0$ as $N \to \infty$ and introduce $z_N(x) := f_N(x,y(x))$. It is clear that $\|z_N - z\|_{L^\infty} \to 0$. Writing (2.7) for $z_N(x)$ and passing to the limit with the help of (2.9)–(2.10), we obtain that (2.7) is still true for z (note in particular that the last integral in (2.8) is well defined for almost all $x \in (0,\pi)$), and that $z \in H^1(0,\pi)$ satisfies (2.5).

Finally, (2.7), the properties of f and $\partial_u f$, the embedding $H^1(0,\pi) \subset C([0,\pi])$ and the Lebesgue convergence theorem imply the continuity of the map $y(x) \mapsto f(x,y(x))$ from $H^1(0,\pi)$ to $H^1(0,\pi)$. $\square$

Lemma 2.4 *Let $f \in \mathcal{F}_{p+1}$ satisfy $(\mathbf{H})_p$ for some $p \geq 0$. Then the composition operator*

$$y(x) \mapsto g(\delta, x, y(x)) = \delta^{-p} f(x, \delta y(x))$$

maps $H^1(0,\pi)$ into $H^1(0,\pi)$ and satisfies

$$\|g(\delta, x, y(x))\|_{H^1} \leq C(\|y\|_{H^1}) \,. \tag{2.11}$$

Moreover the map $(\delta, y) \mapsto g(\delta, x, y(x))$ is in $C([0,\delta_0) \times H^1(0,\pi), H^1(0,\pi))$.

Proof. Since f satisfies $(H)_p$ we have by the Taylor formula,

$$g(\delta, x, u) = u^p \int_0^1 (\partial_u^p f)(x, \delta s u) \frac{(1-s)^{p-1}}{(p-1)!} \, ds \,.$$

Then, applying Lemma 2.3 with $\partial_u^p f \in \mathcal{F}_1$, and since $H^1(0,\pi)$ is an algebra, $\forall y \in H^1(0,\pi)$,

$$g(\delta, x, y(x)) = (y(x))^p \int_0^1 (\partial_u^p f)(x, \delta s y(x)) \frac{(1-s)^{p-1}}{(p-1)!} \, ds \ \in \ H^1(0,\pi)$$

and (2.11) holds.

Moreover, still by Lemma 2.3, the map $(\delta, y, s) \mapsto (\partial_u^p f)(x, \delta s y(x))$ is in $C([0,\delta_0) \times H^1(0,\pi) \times [0,1], H^1(0,\pi))$. Hence, since $y(x) \mapsto (y(x))^p$ belongs to $C^\infty(H^1(0,\pi),\, H^1(0,\pi))$, also the map $(\delta, y) \mapsto g(\delta, x, y(x))$ belongs to $C([0,\delta_0) \times H^1(0,\pi), H^1(0,\pi))$. ☐

Lemma 2.5 (Composition operator) *Let $f \in \mathcal{F}_k$ satisfy $(\mathbf{H})_p$ for some $p \geq 0$ and $k \geq p+1$. Then the composition operator*

$$u(t,x) \mapsto g(\delta, x, u(t,x)) = \delta^{-p} f(x, \delta u)$$

maps $H^{s'} \cap H^s$ into $H^{s'}$ for any $0 \leq s' \leq k-1$, and satisfies

$$\|g(\delta, x, u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_{s'}) \,. \tag{2.12}$$

Moreover the map $(\delta, u) \mapsto g(\delta, x, u)$ is in $C([0,\delta_0) \times (H^{s'} \cap H^s), H^{s'})$.

Proof. We prove the lemma when $s' = l$ is an integer (for $s' \notin \mathbf{N}$ it could be proved using Fourier dyadic decomposition). We show, by iteration, that for $l \in \mathbf{N}$, $l \leq k-1$,

$$\|g(\delta, x, u)\|_l \leq C(l, \|u\|_s)(1 + \|u\|_l) \,, \quad \forall u \in H^l \cap H^s$$

and that if $u_n \to u$ in $H^s \cap H^l$ and $\delta_n \to \delta$, then $g(\delta_n, x, u_n) \to g(\delta, x, u)$ in H^l.

For $l = 0$, applying Lemma 2.4,

$$\begin{aligned} \|g(\delta, x, u)\|_0 &\leq C \max_{t \in \mathbf{T}} \|g(\delta, x, u(t, \cdot))\|_{H^1(0,\pi)} \leq C(\max_{t \in \mathbf{T}} \|u(t, \cdot)\|_{H^1(0,\pi)}) \\ &\overset{(1.3)}{\leq} C(\|u\|_s) \,. \end{aligned}$$

Moreover, if $u_n \to u$ in H^s and $\delta_n \to \delta$ then $\max_{t \in \mathbf{T}} \|u_n(t, \cdot) - u(t, \cdot)\|_{H^1(0,\pi)} \to 0$. Hence, by the continuity property in Lemma 2.4 and the compactness of $\mathbf{T}$, $\max_{t \in \mathbf{T}} \|g(\delta_n, x, u_n(t, \cdot)) - g(\delta, x, u(t, \cdot))\|_{H^1(0,\pi)} \to 0$, whence $g(\delta_n, x, u_n) \to g(\delta, x, u)$ in H^0.

Now assume that the above property holds up to order l, with $l + 1 \leq k - 1$. We have

$$\|g(\delta, x, u)\|_{l+1} \leq \|g(\delta, x, u)\|_0 + \|\partial_t \, g(\delta, x, u)\|_l = \|g(\delta, x, u)\|_0 + \|\partial_u g(\delta, x, u)\partial_t u\|_l,$$

where $\partial_u g(\delta, x, u) = \delta^{-(p-1)} \partial_u f(x, \delta u)$, $\partial_u f \in \mathcal{F}_{k-1}$ and $\partial_u f$ satisfies $(\mathbf{H})_{\max(0,p-1)}$. As a consequence, since $k - 1 \geq \max(p, l + 1)$, by the induction assumption, we have

$$\|\partial_u g(\delta, x, u)\|_l \leq C(l, \|u\|_s)(1 + \|u\|_l) \,, \qquad \forall u \in H^l \cap H^s \tag{2.13}$$

and, by Lemma 2.4,

$$\|\partial_u g(\delta, x, u)\|_{L^\infty(\mathbf{T}; H^1(0,\pi))} \leq C(\|u\|_s) \,. \tag{2.14}$$

For $l = 0$, we get

$$\|g(\delta, x, u)\|_1 \leq C(\|u\|_s) + \|\partial_u g(\delta, x, u)\|_{L^\infty(\mathbf{T}; H^1(0,\pi))} \|u\|_1 \leq C(\|u\|_s)(1 + \|u\|_1).$$

For $l \geq 1$, applying (2.1), (2.14), (1.3), for any $\tilde{s} \in (1/2, \min(1, s))$, (2.13),

$$\begin{aligned} \|g(\delta, x, u)\|_{l+1} \leq C(\|u\|_s) &+ C(l) \Big(\|\partial_u g(\delta, x, u)\|_{L^\infty(\mathbf{T}; H^1(0,\pi))} \|\partial_t u\|_l \\ &+ \|\partial_u g(\delta, x, u)\|_l \|\partial_t u\|_{L^\infty(\mathbf{T}; H^1(0,\pi))} \Big) \\ &\leq C(l, \|u\|_s)(1 + \|u\|_{l+1}) + C(l) \|\partial_u g(\delta, x, u)\|_l \|\partial_t u\|_{\tilde{s}} \\ &\leq C(l, \|u\|_s)(1 + \|u\|_{l+1} + \|u\|_l \|u\|_{\tilde{s}+1}) \,. \end{aligned} \tag{2.15}$$

Now, by the interpolation inequality (2.4), since $\max(l + 1, \tilde{s}) \geq \max(l, \tilde{s} + 1)$,

$$\|u\|_l \|u\|_{\tilde{s}+1} \leq \|u\|_{l+1} \|u\|_{\tilde{s}} \leq \|u\|_{l+1} \|u\|_s \,,$$

and, by (2.15), we get (2.12) for $s' = l + 1$. The continuity property is obtained in the same way. $\square$

Lemma 2.6 (Regularity) *Let $f \in \mathcal{F}_k$ satisfy $(\mathbf{H})_p$ for some $p \geq 0$ and $k \geq p+3$. Then, provided $0 \leq s' \leq k-3$, the map*

$$G : \begin{cases} [0, \delta_0) \times (H^s \cap H^{s'}) \to H^{s'} \\ (\delta, u) \mapsto g(\delta, x, u) \end{cases}$$

is C^1, and $D_u^2 G$ exists everywhere and is continuous in $[0, \delta_0) \times (H^s \cap H^{s'})$. Moreover

$$D_u G(\delta, u)[h] = \partial_u g(\delta, x, u) h \,, \quad D_u^2 G(\delta, u)[h, h] = (\partial_u^2 g)(\delta, x, u) h^2 \,, \ \ \forall h \in H^s \cap H^{s'},$$

and

$$\begin{cases} \|\partial_u g(\delta, x, u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_{s'}) \\ \|\partial_u^2 g(\delta, x, u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_{s'}) \,. \end{cases} \tag{2.16}$$

Proof. First of all,

$$\partial_u g(\delta, x, u) = \delta^{-(p-1)} (\partial_u f)(x, \delta u) \,, \quad \partial_u^2 g(\delta, x, u) = \delta^{-(p-2)} (\partial_u^2 f)(x, \delta u)$$

where $\partial_u f \in \mathcal{F}_{k-1}$ satisfies $(H)_{\max(0,p-1)}$ and $\partial_u^2 f \in \mathcal{F}_{k-2}$ satisfies $(H)_{\max(0,p-2)}$. Applying Lemma 2.5 to the composition operators induced by $\partial_u f$, $\partial_u^2 f$, we get the bounds (2.16).

To prove that G is differentiable with respect to u, write

$$\|g(\delta, x, u+h) - g(\delta, x, u) - \partial_u g(\delta, x, u) h\|_{s'} =$$

$$\left\| h \int_0^1 (\partial_u g(\delta, x, u + \sigma h) - \partial_u g(\delta, x, u)) \, d\sigma \right\|_{s'} \leq$$

$$C(s') \|h\|_{\max(s,s')} \max_{\sigma \in [0,1]} \left\| \partial_u g(\delta, x, u + \sigma h) - \partial_u g(\delta, x, u) \right\|_{\max(s,s')} = o(\|h\|_{\max(s,s')})$$

by the continuity property of $u \mapsto \partial_u g(\delta, x, u)$ derived from Lemma 2.5. Hence $D_u G(\delta, u)[h] = \partial_u g(\delta, x, u) h$, $\forall h \in H^s \cap H^{s'}$, and $(\delta, u) \mapsto D_u G(\delta, u)$ is continuous, still by Lemma 2.5. Similarly G is twice differentiable in u and $D_u^2 G$ is continuous.

At last, to prove that G is differentiable with respect to δ, write

$$\partial_\delta g(\delta, x, u) = \delta^{-(p+1)} \widetilde{f}(x, \delta u) \qquad \text{where} \qquad \widetilde{f}(x, u) := u \partial_u f(x, u) - p f(x, u) \,.$$

We have $\widetilde{f} \in \mathcal{F}_{k-1}$ and $\widetilde{f}$ satisfies $(\mathbf{H})_{p+1}$. Since $k \geq \max\{p+3, s'+3\}$ we deduce that G is differentiable with respect to δ, with continuous derivative, again by Lemma 2.5. $\square$

3 Solution of the (Q2)-equation

By a direct bootstrap argument the solution $\bar{v} \in V$ of the zero order bifurcation equation (1.7) satisfies $\bar{v} \in V \cap C^{\infty}(\Omega)$. In particular there exists $R > 0$ such that

$$\|\bar{v}\|_s < R\,.$$

For the sequel $R > 0$ *is a fixed constant.*

We look for a solution of

$$v_2 = s^*(-\Delta)^{-1}\Pi_{V_2} g(\delta, x, v_1 + v_2 + w)\,. \tag{3.1}$$

We recall that V_2 is defined in (1.10). In what follows, $B(r; E)$ will denote the open ball of center 0 and radius r in the normed vectorspace E.

Lemma 3.1 (Solution of the $(Q2)$-equation) *There exists $\bar{N} := \bar{N}(R) \in \mathbf{N}_+$, $\delta_0 > 0$ such that:*

a) *$\forall \|v_1\|_s \le 2R$, $\forall \|w\|_s \le 1$, $\forall \delta \in (0, \delta_0)$, there exists a unique $v_2 = v_2(\delta, v_1, w) \in V_2 \cap H^{s+2}$ with $\|v_2(\delta, v_1, w)\|_s \le 1$ which solves the $(Q2)$-equation.*

b) *$v_2(\cdot,\cdot,\cdot) \in C^1\Big([0,\delta_0) \times B(2R; V_1) \times B(1; W \cap H^s), V_2 \cap H^{s+2}\Big)$ and $D^2_w v_2$ exists everywhere, is continuous and bounded in $[0,\delta_0) \times B(2R; V_1) \times B(1; W \cap H^s)$.*

Proof. The proof is as in Lemma 2.1 of [4], using Lemma 2.5. **b)** is a consequence of Lemma 2.6. □

For the differentiable Nash-Moser scheme that we shall use to solve the range equation we have to prove that $\|v_2(\delta, v_1, w)\|_{s'}$ depends in a linear way on $\|w\|_{s'}$. We shall need also estimates on the differentials (up to order 2) of v_2 with respect to w, $\forall s \le s' \le k - 3$. This requires to improve lemma 2.1-d) of [4].

Lemma 3.2 *Suppose $w \in W \cap H^{s'} \cap H^s$, $0 \le s' \le k-3$ and $\|w\|_s \le 1$. Then $v_2(\delta, v_1, w) \in V_2 \cap H^{s'+2}$ and, $\forall \delta \in (0, \delta_0)$, $\forall \|v_1\|_s \le 2R$,*

$$\begin{cases} \|v_2(\delta, v_1, w)\|_{s'+2} \le K\Big(1 + \|w\|_{s'}\Big) \\ \|D_w v_2(\delta, v_1, w)[h]\|_{s'+2} \le K\Big[(1 + \|w\|_{s'})\|h\|_s + \|h\|_{s'}\Big] \\ \|D^2_w v_2(\delta, v_1, w)[h,h]\|_{s'+2} \le K\Big[(1 + \|w\|_{s'})\|h\|^2_s + \|h\|_s\|h\|_{s'}\Big] \end{cases} \tag{3.2}$$

for some $K := K(s') > 0$. Moreover, for $s' = 0$, we have

$$\|D_w v_2(\delta, v_1, w)[h]\|_2 \le K\|h\|_0\,. \tag{3.3}$$

Proof. By a bootstrap argument like in lemma 4.1-d) of [4], using the regularizing properties of $(-\Delta)^{-1}$, $v_2(w) := v_2(\delta, v_1, w) \in V \cap H^{s'+2}$. Next, since $v_2(w)$ solves

(3.1), we get

$$\begin{aligned} \|v_2(w)\|_{s'+2} &= \Big\|(-\Delta)^{-1}\Pi_{V_2} g(\delta, x, v_1 + v_2(w) + w)\Big\|_{s'+2} \\ &\le \Big\|\Pi_{V_2} g(\delta, x, v_1 + v_2(w) + w)\Big\|_{s'} \\ &\overset{(2.12)}{\le} C(s', \|v_1 + v_2(w) + w\|_s)\Big(1 + \|v_1 + v_2(w) + w\|_{s'}\Big) \\ &\le K_0(s', R, \|w\|_s)\Big(1 + \|v_1\|_{s'} + \|w\|_{s'} + \|v_2(w)\|_{s'}\Big) \\ &\le K(s', R)\Big(1 + 2R\bar{N}^{s'-s} + \|w\|_{s'} + \|v_2(w)\|_{s'}\Big). \end{aligned}$$

Iterating still n times the above estimate, where n is the smallest integer such that $s' - 2n \le s$, we finally obtain the first bound in (3.2) (we recall that $\bar{N}$ depends only on R and f).

The two other bounds of (3.2) can be proved in the same way by the expressions

$$D_w v_2(w)[h] = (-\Delta)^{-1}\Pi_{V_2} \partial_u g(\delta, x, v_1 + v_2 + w)\Big(h + D_w v_2(w)[h]\Big) \tag{3.4}$$

and

$$\begin{aligned} D_w^2 v_2(w)[h,h] = (-\Delta)^{-1}\Pi_{V_2}\Big(&\partial_u^2 g(\delta, x, v_1 + v_2 + w)(h + D_w v_2(w)[h])^2 \\ &+ \partial_u g(\delta, x, v_1 + v_2 + w) D_w^2 v_2(w)[h,h]\Big). \end{aligned}$$

At last, by (3.4) and the regularizing property of $(-\Delta)^{-1}$,

$$\|D_w v_2(w)[h]\|_2 \le C\|\partial_u g(\delta, x, v_1 + v_2 + w)\|_s \Big\|h + D_w v_2(w)[h]\Big\|_0.$$

By (2.16), $\|\partial_u g(\delta, x, v_1 + v_2 + w)\|_s \le C$, $\forall \|v_1\|_s \le 2R$, $\|v_2\|_s \le 1$, $\|w\|_s \le 1$. Finally, using (4.6) because $D_w v_2(w)[h] \in V_2$,

$$\|D_w v_2(w)[h]\|_2 \le C\Big(\|h\|_0 + \frac{1}{\bar{N}^2}\Big\|D_w v_2(w)[h]\Big\|_2\Big)$$

which implies (3.3), provided that $\bar{N}$ has been chosen large enough. □

4 Solution of the (P)-equation

We have to solve the (P)-equation with $v_2 = v_2(\delta, v_1, w)$, namely

$$L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w) \tag{4.1}$$

where

$$\Gamma(\delta, v_1, w) := g(\delta, x, v_1 + w + v_2(\delta, v_1, w)). \tag{4.2}$$

The solution $w(\delta, v_1)$ of the (P)-equation (4.1) is obtained by means of a Nash-Moser Implicit Function Theorem for (δ, v_1) belonging to a Cantor-like set of parameters.

We consider the orthogonal splitting

$$W = W^{(N_n)} \oplus W^{(N_n)\perp}$$

where

$$W^{(N_n)} = \Big\{ w \in W \ \Big| \ w = \sum_{|l| \le N_n} \exp(\mathrm{i}lt)\ w_l(x) \Big\},$$

$$W^{(N_n)\perp} = \Big\{ w \in W \ \Big| \ w = \sum_{|l| > N_n} \exp(\mathrm{i}lt)\ w_l(x) \Big\},$$

with[1]

$$N_n := [e^{\lambda \chi^n}], \tag{4.3}$$

$\lambda = \ln N_0$ and $\chi > 1$. We denote by

$$P_{N_n} : W \to W^{(N_n)} \qquad \text{and} \qquad P^{\perp}_{N_n} : W \to W^{(N_n)\perp}$$

the orthogonal projectors onto $W^{(N_n)}$ and $W^{(N_n)\perp}$.

Remark 4.1 One major difference with respect to the analytic case is making the sequence of finite dimensional truncations N_n increase super-exponentially fast like in (4.3) (in [10]–[4] we had $N_n = N_0 2^n$). This is useful to prove the smallness of the remainder r_n defined in (4.14), see remark 4.3.

The convergence of the recursive scheme is based on properties $(P1)$-$(P2)$-$(P3)$-$(P4)$-$(P5)$ below.

- **(P1)** (**Regularity**) Γ is in $C^1((0, \delta_0) \times B(2R; V_1) \times B(1; W \cap H^s); H^s)$. Moreover $D^2_w \Gamma$ exists everywhere, is continous and bounded on $[0, \delta_0) \times B(2R; V_1) \times B(1; W \cap H^s)$.

$(P1)$ is a consequence of the regularity of the composition operator induced by $g(\delta, x, u)$ on H^s (see Lemma 2.6), and the regularity of the map $v_2(\cdot, \cdot, \cdot)$ of Lemma 3.1.

[1] The symbol $[a]$ denotes the integer part of $a \in \mathbf{R}$.

- **(P2)** (**Tame**) For all $\|v_1\|_s \le 2R$, $\|w\|_s \le 1$, $\forall s \le s' \le k-1$,

$$\|\Gamma(\delta, v_1, w)\|_{s'} \le C(s')(1 + \|w\|_{s'})\,.$$

$(P2)$ is a consequence of (2.12) and (3.2) (where we need only $s' \le k-1$ in the first bound).

- **(P3)** (**Taylor Tame**) For all $\|v_1\|_s \le 2R$, $\|w\|_s \le 1$, $\forall s \le s' \le k-3$,

$$\begin{aligned}&\Big\|\Gamma(\delta, v_1, w+h) - \Gamma(\delta, v_1, w) - D_w\Gamma(\delta, v_1, w)[h]\Big\|_{s'}\\ &\quad \le C(s')\Big(\|w\|_{s'}\|h\|_s^2 + \|h\|_s\|h\|_{s'}\Big)\end{aligned}$$

for all $h \in H^{s'}$. In particular, for $s' = s$,

$$\Big\|\Gamma(\delta, v_1, w+h) - \Gamma(\delta, v_1, w) - D_w\Gamma(\delta, v_1, w)[h]\Big\|_s \le C\|h\|_s^2\,. \tag{4.4}$$

Since

$$\begin{aligned} D_w^2\Gamma(\delta, v_1, w)[h,h] \;&=\; (\partial_u^2 g)(\delta, x, v_1 + w + v_2(\delta, v_1, w))\Big(h + D_w v_2(\delta, v_1, w)[h]\Big)^2 \\ &\quad +\; (\partial_u g)(\delta, x, v_1 + w + v_2(\delta, v_1, w)) D_w^2 v_2(\delta, v_1, w)[h,h], \end{aligned}$$

$(P3)$ can be derived from the bound on $\|D_w^2\Gamma(\delta, v_1, w)[h,h]\|_{s'}$ provided by (2.16), (3.2) and (2.2).

- **(P4)** (**Smoothing**) $\forall N \in \mathbf{N}$,

$$\|P_N u\|_{s+r} \le N^r \|u\|_s\,, \qquad \forall u \in H^s \tag{4.5}$$

$$\|P_N^{\perp} u\|_s \le\; N^{-r}\|u\|_{s+r}\,, \qquad \forall u \in H^{s+r}\,. \tag{4.6}$$

The next property $(P5)$ is an *invertibility property* of the linearized operator

$$\mathcal{L}_N(\delta, v_1, w)[h] := L_\omega h - \varepsilon P_N \Pi_W D_w\Gamma(\delta, v_1, w)[h]\,, \qquad \forall h \in W^{(N)}\,.$$

Definition 4.1 (Melnikov Non-resonance conditions) We define

$$\begin{aligned}\Delta_N^{\gamma,\tau}(v_1, w) := \Big\{\delta \in [0, \delta_0) \;\Big|\; &|\omega l - j| \ge \tfrac{\gamma}{(l+j)^\tau}\,,\; \Big|\omega l - j - \varepsilon\tfrac{M(\delta, v_1, w)}{2j}\Big| \ge \tfrac{\gamma}{(l+j)^\tau} \\ &\forall l \in \mathbf{N},\; j \ge 1,\; l \ne j,\; \tfrac{1}{3|\varepsilon|} < l,\; l \le N,\; j \le 2N\Big\}\end{aligned}$$

for some constant $0 < \gamma < 1$, $1 < \tau < 2$, where

$$M(\delta, v_1, w) := \frac{1}{|\Omega|}\int_\Omega \partial_u g\Big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)\Big)\, dtdx\,, \tag{4.7}$$

$\Omega := \mathbf{T} \times (0, \pi)$ and ω, ε are related to δ in (1.5).

The next property is proved in section 5.

- **(P5) (Invertibility of $\mathcal{L}_N$)** Fix $1 < \tau < 2$, $0 < \gamma < 1$. Suppose

$$\|w\|_{s+\sigma} \le 1 \qquad \text{where} \qquad \sigma := \frac{\tau(\tau-1)}{2-\tau}\,. \tag{4.8}$$

There exists $\delta_0 := \delta_0(\gamma, \tau, R, f) > 0$ such that,

$$\forall \|v_1\|_s \le 2R\,, \quad \forall \delta \in \Delta_N^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)\,,$$

$\mathcal{L}_N(\delta, v_1, w)$ is invertible and $\mathcal{L}_N^{-1}(\delta, v_1, w) : W^{(N)} \to W^{(N)}$ satisfies $\forall s' \ge s$

$$\Big\|\mathcal{L}_N^{-1}(\delta, v_1, w)[h]\Big\|_{s'} \le K(s')\frac{N^{\tau-1}}{\gamma}\Big(\|h\|_{s'} + \|w\|_{s'+\sigma}\|h\|_s\Big) \tag{4.9}$$

for some positive constant $K(s')$. In particular, by (4.9) and (4.8),

$$\Big\|\mathcal{L}_N^{-1}(\delta, v_1, w)[h]\Big\|_s \le \frac{K}{\gamma}N^{\tau-1}\|h\|_s\,. \tag{4.10}$$

Note that, under assumption (4.8), we have already established in [4] the invertibility of $\mathcal{L}_N$ (in [4] analyticity was not used at this step). The new problem here is to prove the interpolation type estimate (4.9) in high Sobolev norms.

Remark 4.2 The importance of (4.9) is that the big norm $\|w\|_{s'+\sigma}$ will be compensated by the small norm $\|h\|_s$ along the Nash-Moser iteration. This is sufficient to imply convergence. It is used in Lemma 4.2.

4.1 The Nash-Moser scheme

We define inductively the sequence $\{w_n\}_{n\ge 0}$.

Proposition 4.1 (Induction) *Let $A_0 := \{(\delta, v_1) \mid \delta \in [0, \delta_0),\ \|v_1\|_s \le 2R\}$. N_n being defined in (4.3), $\exists\ \varepsilon_0 := \varepsilon_0(\gamma, \tau, s, R, f) > 0$, such that for $\delta_0^{p-1} < \varepsilon_0$, there exists a sequence $w_n := w_n(\delta, v_1) \in W^{(N_n)}$ of solutions of the equation*

$$(P - N_n) \qquad\qquad L_\omega w_n - \varepsilon P_{N_n}\Pi_W \Gamma(\delta, v_1, w_n) = 0\,,$$

defined inductively for $(\delta, v_1) \in A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0$ where

$$A_n := \Big\{(\delta, v_1) \in A_{n-1} \mid \delta \in \Delta_{N_n}^{\gamma,\tau}(v_1, w_{n-1})\Big\} \subseteq A_{n-1}\,,$$

satisfying

$$\|w_n\|_{s+\sigma} \le 1 \qquad \text{where} \qquad \sigma := \frac{\tau(\tau-1)}{2-\tau}\,, \tag{4.11}$$

$w_n = \sum_{i=0}^n h_i$ with $h_i \in W^{(N_i)}$ such that

$$\|h_0\|_s \le |\varepsilon|K_0\,, \qquad \|h_i\|_s \le \bar{K}|\varepsilon|\gamma^{-1}N_i^{-\sigma-1} \qquad \forall 1 \le i \le n \tag{4.12}$$

for some constants K_0 , $\bar{K} > 0$.

Proof. In the proof K, K' denote absolute constants depending on f, R, γ, τ, s, s' at most. We shall possibly indicate only the dependence on the index $s' \geq s$.

First step: initialization. Let $N_0 := [e^\lambda]$ be given. If

$$|\omega - 1|N_0 \leq |\omega - 1|e^\lambda \leq \frac{1}{2}$$

then $L_{\omega|W^{(N_0)}}$ is invertible and $\|L_\omega^{-1}h\|_s \leq 2\|h\|_s$, $\forall h \in W^{(N_0)}$. This is straightforward, noticing that the eigenvalues $\omega^2 l^2 - j^2$ $(0 \leq l \leq N_0,\ j \geq 1,\ j \neq l)$ of $L_{\omega|W^{(N_0)}}$ satisfy

$$|\omega^2 l^2 - j^2| = |\omega l - j|(\omega l + j) \geq \Big(|l - j| - |\omega - 1|N_0\Big)(\omega l + j) \geq \Big(1 - \frac{1}{2}\Big).$$

By the Contraction Mapping Theorem, using Property $(P1)$, there exist $\varepsilon_1 > 0$, $K_0 > 0$, such that, $\forall \|v_1\|_s \leq 2R$, $\forall |\varepsilon| < \min\{\varepsilon_1, e^{-\lambda}/4\}$, the equation

$$(P - N_0) \qquad L_\omega w_0 - \varepsilon P_{N_0} \Pi_W \Gamma(\delta, v_1, w_0) = 0$$

has a unique solution $w_0(\delta, v_1)$ satisfying $\|w_0(\delta, v_1)\|_s \leq K_0|\varepsilon|$.

Second step: iteration. Suppose we have already defined a solution $w_n \in W^{(N_n)}$ of equation $(P - N_n)$ satisfying the properties stated in the proposition. We want to find a solution

$$w_{n+1} = w_n + h_{n+1}, \quad h_{n+1} \in W^{(N_{n+1})}$$

of the equation

$$(P - N_{n+1}) \qquad L_\omega w_{n+1} - \varepsilon P_{N_{n+1}} \Pi_W \Gamma(\delta, v_1, w_{n+1}) = 0\,.$$

Develop

$$\begin{aligned} &L_\omega(w_n + h) - \varepsilon P_{N_{n+1}} \Pi_W \Gamma(\delta, v_1, w_n + h) \\ &\quad = L_\omega w_n - \varepsilon P_{N_{n+1}} \Pi_W \Gamma(\delta, v_1, w_n) + L_\omega h \\ &\qquad - \varepsilon P_{N_{n+1}} \Pi_W D_w \Gamma(\delta, v_1, w_n)[h] + R_n(h) \\ &\quad = r_n + \mathcal{L}_{N_{n+1}}(\delta, v_1, w_n)[h] + R_n(h) \end{aligned} \tag{4.13}$$

where, since w_n solves equation $(P - N_n)$, $r_n \in W^{(N_n)\perp}$,

$$\begin{cases} r_n := L_\omega w_n - \varepsilon P_{N_{n+1}} \Pi_W \Gamma(\delta, v_1, w_n) = -\varepsilon P_{N_n}^\perp P_{N_{n+1}} \Pi_W \Gamma(\delta, v_1, w_n) \\ R_n(h) := \ -\varepsilon P_{N_{n+1}} \Pi_W \Big(\Gamma(\delta, v_1, w_n + h) - \Gamma(\delta, v_1, w_n) - D_w \Gamma(\delta, v_1, w_n)[h]\Big). \end{cases} \tag{4.14}$$

Inversion of $\mathcal{L}_{N_{n+1}}(\delta, v_1, w_n)$. By property (P5), $\mathcal{L}_{N_{n+1}}(\delta, v_1, w_n)$ is invertible because (4.11) holds, and for (δ, v_1) restricted to the set of parameters

$$A_{n+1} := \Big\{(\delta, v_1) \in A_n \mid \delta \in \Delta^{\gamma,\tau}_{N_{n+1}}(v_1, w_n)\Big\} \subseteq A_n\,.$$

The inverse operator satisfies, by (4.10),

$$\Big\|\mathcal{L}_{N_{n+1}}(\delta,v_1,w_n)^{-1}h\Big\|_s \le \frac{K}{\gamma}N_{n+1}^{\tau-1}\|h\|_s\,, \qquad \forall h\in W^{(N_{n+1})}\,. \tag{4.15}$$

By (4.13), solving equation $(P-N_{n+1})$ means finding a solution $h\in W^{(N_{n+1})}$ of

$$h = -\mathcal{L}_{N_{n+1}}(\delta,v_1,w_n)^{-1}\Big(r_n+R_n(h)\Big)\,,$$

namely a fixed point of the map

$$\mathcal{G}_{n+1}: W^{(N_{n+1})}\to W^{(N_{n+1})}\,, \qquad \mathcal{G}_{n+1}(h) := -\mathcal{L}_{N_{n+1}}(\delta,v_1,w_n)^{-1}\Big(r_n+R_n(h)\Big).$$

Lemma 4.1 (Contraction) *Given $\beta>0$ (to be specified later), we set*

$$B_n := 1+\|w_n\|_{s+\beta}\,.$$

There exists $K_1:=K_1(\beta)>0$ such that $\mathcal{G}_{n+1}$ is contraction in the ball

$$\mathcal{B}_{n+1}(s) := \Big\{h\in W^{(N_{n+1})} \ \ \Big|\ \ \|h\|_s\le \rho_{n+1} := 2\frac{|\varepsilon|}{\gamma}K_1N_{n+1}^{\tau-1}N_n^{-\beta}B_n\Big\}$$

provided

$$K_1\frac{|\varepsilon|}{\gamma}N_{n+1}^{\tau-1}\rho_{n+1} = 2K_1^2\frac{|\varepsilon|^2}{\gamma^2}N_{n+1}^{2(\tau-1)}N_n^{-\beta}B_n \le \frac{1}{2}\,. \tag{4.16}$$

Proof. By (4.15)

$$\begin{aligned}\Big\|\mathcal{G}_{n+1}(h)\Big\|_s &= \Big\|\mathcal{L}_{N_{n+1}}(\delta,v_1,w_n)^{-1}\Big(r_n+R_n(h)\Big)\Big\|_s\\ &\le \frac{K}{\gamma}N_{n+1}^{\tau-1}\Big(\|r_n\|_s+\|R_n(h)\|_s\Big)\\ &\le \frac{K'}{\gamma}N_{n+1}^{\tau-1}\Big(|\varepsilon|\,N_n^{-\beta}\|P_{N_{n+1}}\Pi_W\Gamma(w_n)\|_{s+\beta}+|\varepsilon|\ \|h\|_s^2\Big) \end{aligned} \tag{4.17}$$

using the smoothing estimate (P4) since $r_n\in W^{(N_n)\perp}$, and (4.4). Here, for brevity, $\Gamma(w_n):=\Gamma(\delta,v_1,w_n)$.

Now, the tame estimate (P2) entails

$$\|P_{N_{n+1}}\Pi_W\Gamma(w_n)\|_{s+\beta} \ \le\ \bar{K}(\beta)(1+\|w_n\|_{s+\beta}) = \bar{K}(\beta)B_n \tag{4.18}$$

because $\|w_n\|_s\le 1$ by (4.11). By (4.17) and (4.18) we get

$$\Big\|\mathcal{G}_{n+1}(h)\Big\|_s \le \frac{|\varepsilon|}{\gamma}K_1N_{n+1}^{\tau-1}\Big(N_n^{-\beta}B_n+\|h\|_s^2\Big) \tag{4.19}$$

for some $K_1 := K_1(\beta) > 0$. If $\|h\|_s \leq \rho_{n+1}$ then

$$\Big\|\mathcal{G}_{n+1}(h)\Big\|_s \leq \frac{|\varepsilon|}{\gamma} K_1 N_{n+1}^{\tau-1} N_n^{-\beta} B_n + \frac{|\varepsilon|}{\gamma} K_1 N_{n+1}^{\tau-1}\, \rho_{n+1}^2$$

and therefore $\mathcal{G}_{n+1}$ maps $\mathcal{B}_{n+1}(s)$ into $\mathcal{B}_{n+1}(s)$ if

$$\frac{|\varepsilon|}{\gamma} K_1 N_{n+1}^{\tau-1} N_n^{-\beta} B_n + \frac{|\varepsilon|}{\gamma} K_1\, N_{n+1}^{\tau-1}\, \rho_{n+1}^2 \leq \rho_{n+1}\,.$$

By the definition of ρ_{n+1}, the previous condition holds, if (4.16) is verified.

The condition to prove that $\mathcal{G}_{n+1}$ is a contraction is similar. □

Remark 4.3 By (4.17)–(4.18) the term r_n is estimated like $\|r_n\|_s \leq K|\varepsilon| N_n^{-\beta} B_n$. In the next Lemma 4.3 we obtain an a priori-estimate on the growth of the B_n independently of β, implying the super-exponential smallness of $\|r_n\|_s$ for β large enough, see (4.27). Here the tame estimates are deeply exploited.

In [10]–[4] the smallness of r_n was proved exploiting analyticity.

Now the main task is to estimate the growth of the B_n i.e. to control the divergence of the high norms $\|w_n\|_{s+\beta}$ of the approximate solutions w_n.

Lemma 4.2 *There exists $K_2 := K_2(\beta) > 0$ such that, if*

$$K_2 \frac{|\varepsilon|}{\gamma} N_n^{\tau-1} \rho_n = 2K_2 K_1 \frac{\varepsilon^2}{\gamma^2} N_n^{2(\tau-1)} N_{n-1}^{-\beta} B_{n-1} \leq \frac{1}{2} \tag{4.20}$$

and $|\varepsilon| 2K_2 \gamma^{-1} \leq 1$, then

$$B_n \leq (1 + N_n^{\tau-1+\sigma}) B_{n-1}\,. \tag{4.21}$$

Proof. We have

$$B_n := 1 + \|w_n\|_{s+\beta} \leq 1 + \|w_{n-1}\|_{s+\beta} + \|h_n\|_{s+\beta} = B_{n-1} + \|h_n\|_{s+\beta}\,. \tag{4.22}$$

Now we use the estimate for the inverse $\mathcal{L}_{N_n}(\delta, v_1, w_{n-1})^{-1}$ in high Sobolev norm given in property (P5).

Since $h_n = -\mathcal{L}_{N_n}(\delta, v_1, w_{n-1})^{-1}(r_{n-1} + R_{n-1}(h_n))$, we obtain

$$\begin{aligned}
\|h_n\|_{s+\beta} &\overset{(P5)}{\leq} K(\beta) \frac{N_n^{\tau-1}}{\gamma} \Big(\|r_{n-1}\|_{s+\beta} + \|R_{n-1}(h_n)\|_{s+\beta} \\
&\quad + \|w_{n-1}\|_{s+\beta+\sigma} (\|r_{n-1}\|_s + \|R_{n-1}(h_n)\|_s) \Big) \\
&\overset{(P4)}{\leq} K'(\beta) \frac{N_n^{\tau-1}}{\gamma} \Big(\|r_{n-1}\|_{s+\beta} + |\varepsilon| \|Q_{n-1}(h_n)\|_{s+\beta} \Big) \\
&\quad + K'(\beta) \frac{N_n^{\tau-1+\sigma}}{\gamma} \|w_{n-1}\|_{s+\beta} \Big(\|r_{n-1}\|_s + |\varepsilon| \|Q_{n-1}(h_n)\|_s \Big)
\end{aligned} \tag{4.23}$$

because $w_{n-1} \in W^{(N_{n-1})}$, and having defined

$$Q_{n-1}(h) := \Gamma(\delta, v_1, w_{n-1} + h) - \Gamma(\delta, v_1, w_{n-1}) - D_w\Gamma(\delta, v_1, w_{n-1})[h] \, .$$

Now we use the tame estimate

$$\begin{aligned} \|r_{n-1}\|_{s+\beta} &= |\varepsilon| \|P^{\perp}_{N_{n-1}} P_{N_n} \Pi_W \Gamma(w_{n-1})\|_{s+\beta} \leq K(\beta)|\varepsilon| \|\Gamma(w_{n-1})\|_{s+\beta} \\ &\overset{(P2)}{\leq} |\varepsilon| K'(\beta)(1 + \|w_{n-1}\|_{s+\beta}) = |\varepsilon| K'(\beta) B_{n-1} \end{aligned} \tag{4.24}$$

and the Taylor tame estimate (P3)

$$\begin{aligned} \|Q_{n-1}(h_n)\|_{s+\beta} &\leq K(\beta)\Big(\|w_{n-1}\|_{s+\beta}\|h_n\|_s^2 + \|h_n\|_s\|h_n\|_{s+\beta}\Big) \\ &\leq K(\beta)\Big(B_{n-1}\rho_n^2 + \rho_n\|h_n\|_{s+\beta}\Big) \end{aligned} \tag{4.25}$$

because $\|h_n\|_s \leq \rho_n$.

We insert in (4.23) the estimates (4.24), (4.25), $\|r_{n-1}\|_s \leq |\varepsilon| C$ and $\|Q_{n-1}(h_n)\|_s \leq C$, to obtain

$$\|h_n\|_{s+\beta} \leq K_2 \frac{|\varepsilon|}{\gamma} N_n^{\tau-1+\sigma} B_{n-1} + \Big(K_2 \frac{|\varepsilon|}{\gamma} N_n^{\tau-1} \rho_n\Big) \|h_n\|_{s+\beta}$$

for some positive $K_2 := K_2(\beta)$. By (4.20)

$$\|h_n\|_{s+\beta} \leq K_2 \frac{|\varepsilon|}{\gamma} N_n^{\tau-1+\sigma} B_{n-1} + \frac{1}{2}\|h_n\|_{s+\beta}$$

whence

$$\|h_n\|_{s+\beta} \leq 2K_2 \frac{|\varepsilon|}{\gamma} N_n^{\tau-1+\sigma} B_{n-1} \leq N_n^{\tau-1+\sigma} B_{n-1} \tag{4.26}$$

for $2K_2|\varepsilon|\gamma^{-1} \leq 1$. By (4.22) and (4.26) we get (4.21). $\square$

The recurrence inequality (4.21) is very important because it proves a bound for the divergence of the B_n *independent of* β, provided that δ (or ε) is smaller than some positive constant which depends on β.

Lemma 4.3 (Bound of B_n) *There exists $C = C(\chi, \lambda, \tau, \sigma) > 0$ such that $B_n \leq C B_0 N_n^{\frac{\chi}{\chi-1}(\tau-1+\sigma)}$.*

Proof. Set for brevity $\alpha := \tau - 1 + \sigma$. Iterating (4.21) and using $N_n \leq e^{\lambda\chi^n} < N_n + 1 < 2N_n$ we get

$$\begin{aligned} B_n &\leq B_0 \Pi_{i=1}^n (1 + N_i^\alpha) \leq B_0 \Pi_{i=1}^n (1 + e^{\alpha\lambda\chi^i}) = B_0 \Pi_{i=1}^n (e^{-\alpha\lambda\chi^i} + 1) e^{\alpha\lambda\chi\frac{\chi^n - 1}{\chi - 1}} \\ &\leq C' B_0 e^{\alpha\lambda\frac{\chi^{n+1}}{\chi-1}} \leq C B_0 N_n^{\frac{\chi}{\chi-1}\alpha}, \end{aligned}$$

where $C = 2^{\frac{\chi}{\chi-1}\alpha} C' = 2^{\frac{\chi}{\chi-1}\alpha} \Pi_{i=1}^{+\infty}(e^{-\alpha\lambda\chi^i} + 1)$ is finite and independent of n. $\square$

There remains to check condition (4.16). We choose

$$\beta := \chi(\tau+\sigma) + \frac{\chi}{\chi-1}(\tau-1+\sigma) \qquad \text{where} \qquad \sigma := \frac{\tau(\tau-1)}{2-\tau}\,. \tag{4.27}$$

Lemma 4.4 *For $\varepsilon\gamma^{-1}$ small enough (independently of n) condition (4.16) hold.*

Proof. By Lemma 4.3, condition (4.16) to have the contraction holds if

$$c_n := 2K_1^2 \frac{\varepsilon^2}{\gamma^2} N_{n+1}^{2(\tau-1)} N_n^{-\beta} C B_0 N_n^{\frac{\chi}{\chi-1}(\tau-1+\sigma)} \le \frac{1}{2}\,.$$

Now, since $N_{n+1} \le e^{\lambda\chi^{n+1}} \le (N_n+1)^{\chi}$, and by the definition of β in (4.27),

$$\begin{aligned} c_n &\le 2K_1^2 \frac{\varepsilon^2}{\gamma^2} C B_0 (N_n+1)^{2\chi(\tau-1)} N_n^{-\beta+\frac{\chi}{\chi-1}(\tau-1+\sigma)} \\ &= 2K_1^2 \frac{\varepsilon^2}{\gamma^2} C B_0 (1+N_n^{-1})^{2\chi(\tau-1)} N_n^{\chi(\tau-2-\sigma)} \\ &\le \bar{K}\frac{\varepsilon^2}{\gamma^2} \end{aligned}$$

since $\tau < 2$ and $\sigma > 0$. Condition (4.16), holds for $\varepsilon\gamma^{-1} > 0$ small enough. $\square$

Next we check that estimate (4.12) holds for h_{n+1}. By the estimates for ρ_{n+1} in Lemma 4.1, for B_n in Lemma 4.3, and by the definition of β in (4.27),

$$\begin{aligned} \|h_{n+1}\|_s &\le 2|\varepsilon|\gamma^{-1} K_1 N_{n+1}^{\tau-1} N_n^{-\beta} B_n \\ &\le 2|\varepsilon|\gamma^{-1} K_1 N_{n+1}^{\tau-1} C B_0 N_n^{-\chi(\tau+\sigma)} \\ &\le 2|\varepsilon|\gamma^{-1} K_1 C B_0 N_{n+1}^{-(1+\sigma)} 2^{\chi(\tau+\sigma)} \le \bar{K}|\varepsilon|\gamma^{-1} N_{n+1}^{-(\sigma+1)}, \end{aligned}$$

for some $\bar{K} > 0$. We used that $N_n^{\chi} \ge N_{n+1}(1+N_n^{-1})^{-\chi} \ge N_{n+1}2^{-\chi}$.

To complete the inductive argument and the proof of Proposition 4.1, we have still to verify that (4.20) holds when n is replaced by $n+1$ and that condition (4.11) holds for w_{n+1}.

Condition (4.20) (at rank $n+1$) is exactly of the same type as condition (4.16) and it holds true in the same way, for $\varepsilon\gamma^{-1} > 0$ small enough.

Finally condition (4.11) at rank $n+1$ holds true because, by the induction hypothesis

$$\begin{aligned} \|w_{n+1}\|_{s+\sigma} &\le \sum_{i=0}^{n+1} \|h_i\|_{s+\sigma} \overset{(P4)}{\le} \sum_{i=0}^{n+1} N_i^{\sigma}\|h_i\|_s \overset{(4.12)}{\le} \sum_{i=0}^{n+1} \bar{K}\frac{|\varepsilon|}{\gamma} N_i^{\sigma} N_i^{-\sigma-1} \\ &\le \bar{K}\frac{|\varepsilon|}{\gamma}\sum_{i=0}^{+\infty} N_i^{-1} \le 1 \end{aligned}$$

for $\varepsilon\gamma^{-1}$ small enough, independently on n. $\square$

Remark 4.4 We have never used $\chi < 2$ as it is customary in several Nash-Moser theorems, whence the convergence of this Nash-Moser scheme is arbitrarily fast! This is due to the fact that, for semilinear f, the very strong Taylor tame estimate (P3) holds. It is used in Lemma 4.2.

Now, choosing $\chi > 1$, we can optimize the choice of β which reflects on the regularity assumptions on the nonlinearity $f \in \mathcal{F}_k$, see (1.9). For each $\tau \in (1,2)$ the minimal value of β defined in (4.27) is

$$\bar{\beta}(\tau) = \min_{\chi > 1}\Big(\chi(\tau+\sigma) + \frac{\chi}{\chi - 1}(\tau - 1 + \sigma)\Big) = \Big(\sqrt{\tau+\sigma} + \sqrt{\tau+\sigma-1}\Big)^2 .$$

The function $\tau \mapsto \tau + \sigma = \tau + \frac{\tau(\tau-1)}{2-\tau}$ is increasing for $\tau \in (1,2)$, as well as the function $\tau \mapsto \bar{\beta}(\tau)$. For $\tau \to 1$,

$$\bar{\beta}(\tau) \to \inf_{\tau \in (1,2)} \bar{\beta}(\tau) = 1 . \tag{4.28}$$

Hence we can take for instance $\beta = 2$ for τ close to 1. Note that for $\tau = 1$ the existence of periodic solutions can be achieved using just the classical implicit function theorem and not the Nash-Moser one, see [2], [3].

By the regularity property $(P1)$ we can prove as in section 3.2 of [4] the existence of a C^1 Whitney extension $w(\delta, v_1)$ for all $(\delta, v_1) \in (0, \delta_0) \times B(2R; V_1)$.

Finally, once the (P)-equation has been solved, the part concerning the solution of the $(Q1)$-equation and the measure estimate for the Cantor set remains the same as in section 5 of [4]. We had just used that the path of solutions of the bifurcation equation is C^1, see Proposition 3.2 in [4]. The conclusions of Theorem 1.1 follow like in [4].

5 Analysis of the linearized problem: proof of (P5)

Recalling (4.2), the operator $\mathcal{L}_N(\delta, v_1, w)$ can be written as

$$\begin{aligned}
&\mathcal{L}_N(\delta, v_1, w)[h] \\
&\quad := L_\omega h - \varepsilon P_N \Pi_W D_w \Gamma(\delta, v_1, w)[h] \\
&\quad = L_\omega h - \varepsilon P_N \Pi_W \Big(\partial_u g(\delta, x, v_1 + w + v_2(\delta, v_1, w))\Big(h + D_w v_2(\delta, v_1, w)[h]\Big)\Big) \\
&\quad = L_\omega h - \varepsilon P_N \Pi_W \Big(a(t,x)\ h\Big) - \varepsilon P_N \Pi_W \Big(a(t,x)\ D_w v_2(\delta, v_1, w)[h]\Big)
\end{aligned}$$

where, for brevity,

$$a(t,x) := \partial_u g(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)) .$$

We decompose

$$\mathcal{L}_N(\delta, v_1, w) = D - \mathcal{M}_1 - \mathcal{M}_2$$

where

$$\begin{cases} Dh := L_\omega h - \varepsilon P_N \Pi_W (a_0(x)\ h) \\ \mathcal{M}_1 h := \varepsilon P_N \Pi_W (\bar{a}(t,x)\ h) \\ \mathcal{M}_2 h := \varepsilon P_N \Pi_W (a(t,x)\ D_w v_2[h]) \end{cases}$$

and

$$\bar{a}(t,x) := a(t,x) - a_0(x)\,, \qquad a_0(x) := \frac{1}{2\pi}\int_0^{2\pi} a(t,x)\,dt\,.$$

We deduce from (2.16) and (3.2) that, $\forall \|v_1\|_s \le 2R$, $\forall \|w\|_{s+\sigma} \le 1$, $\forall \delta \in [0,\delta_0)$

$$\|a\|_{s'} = \Big\|(\partial_u g)(\delta, x, v_1 + w + v_2(\delta, v_1, w))\Big\|_{s'} \le C(s')(1 + \|w\|_{s'})\,. \tag{5.1}$$

In particular,

$$\|a\|_s \le \|a\|_{s+\sigma} \le C\,. \tag{5.2}$$

5.1 Inversion of D

$\forall\, h \in W^{(N)}$, the k^{th} time Fourier coefficient of Dh is

$$(Dh)_k = (\omega^2 k^2 + \partial_x^2) h_k - \varepsilon \pi_k (a_0(x) h_k) \equiv D_k h_k$$

where $D_k : \mathcal{D}(D_k) \subset F_k \to F_k$ is the operator

$$D_k h = \omega^2 k^2 h + \partial_x^2 h - \varepsilon \pi_k (a_0(x)\ h)$$

with

$$F_k := \Big\{ f \in H_0^1((0,\pi);\mathbf{R}) \ \mid \ \int_0^\pi f(x)\sin(kx)\,dx = 0 \Big\} = \langle \sin(kx) \rangle^\perp,$$

$\pi_k : H_0^1((0,\pi);\mathbf{R}) \to F_k$ being the L^2-orthogonal projector.

Lemma 5.1 (Diagonalization of D) *There exists a Hilbert basis of $W^{(N)}$ consisting of eigenvectors $\{\cos(kt) v_{k,j}\}_{k,j}$ of D with eigenvalues $\{\omega^2 k^2 - \lambda_{k,j}\}$ verifying*

$$\lambda_{k,j} = \lambda_{k,j}(\delta, v_1, w) = j^2 + \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon \|a_0\|_{H^1}}{j}\Big)$$

where $M(\delta, v_1, w)$ is defined in (4.7).

Proof. Lemma 4.1 of [4]. □

If all the eigenvalues of D are different from zero we can define

$$|D|^{-1/2} h := \sum_{|k| \le N} \exp(\mathrm{i}kt) |D_k|^{-1/2} h_k\,, \qquad \forall h = \sum_{|k|\le N} \exp(\mathrm{i}kt) h_k \in W^{(N)}$$

where $|D_k|^{-1/2} : F_k \to F_k$ is the diagonal operator defined by

$$|D_k|^{-1/2} v_{k,j} := \frac{v_{k,j}}{\sqrt{|\omega^2 k^2 - \lambda_{k,j}|}}, \qquad \forall j \ge 1,\ j \ne |k|\,.$$

Lemma 5.2 *If $\delta \in \Delta_N^{\gamma;\tau}(v_1, w)$ then $\forall 0 < k \le N$,*

$$\alpha_k := \min_{j \ne |k|} |\omega^2 k^2 - \lambda_{k,j}| \ge \frac{c\gamma}{|k|^{\tau-1}} > 0 \ , \quad \alpha_0 \ge \frac{1}{2} \, ,$$

and

$$\Big\| |D_k|^{-1/2} u \Big\|_{H^1} \le \frac{2}{\sqrt{\alpha_k}} \|u\|_{H^1} \, . \tag{5.3}$$

Proof. Lemma 4.3 and 4.2 of [4]. □

Lemma 5.3 (Estimate of $|D|^{-1/2}$) *Let $\delta \in \Delta_N^{\gamma;\tau}(v_1, w)$. $\forall s' \ge 0$*

$$\Big\| |D|^{-1/2} h \Big\|_{s'} \le \frac{C}{\sqrt{\gamma}} \|h\|_{s' + \frac{\tau-1}{2}} \, , \qquad \forall h \in W^{(N)} \tag{5.4}$$

whence, by (4.5),

$$\Big\| |D|^{-1/2} h \Big\|_{s'} \le C \frac{N^{\frac{\tau-1}{2}}}{\sqrt{\gamma}} \|h\|_{s'} \, , \qquad \forall h \in W^{(N)}. \tag{5.5}$$

Proof. Corollary 4.2 of [4]. □

5.2 Inversion of $\mathcal{L}_N$

Write

$$\mathcal{L}_N = D - \mathcal{M}_1 - \mathcal{M}_2 = |D|^{1/2} U \Big(I - U^{-1} \mathcal{R}_1 - U^{-1} \mathcal{R}_2 \Big) |D|^{1/2}$$

where

$$U := |D|^{-1/2} D |D|^{-1/2} \qquad \text{and} \qquad \mathcal{R}_i := |D|^{-1/2} \mathcal{M}_i |D|^{-1/2}, \quad i = 1, 2 \, .$$

Lemma 5.4 *U is invertible and $\forall s' \ge s$,*

$$\Big\| U^{-1} h \Big\|_{s'} = \|h\|_{s'} \Big(1 + O(\varepsilon \|a_0\|_{H^1}) \Big) \qquad \forall \ h \in W^{(N)}. \tag{5.6}$$

Proof. Lemma 4.4 of [4]. □

Lemma 5.5 (Analysis of the Small Divisors) *Let $\delta \in \Delta_N^{\gamma;\tau}(v_1, w)$. There exists $C > 0$ such that, $\forall l \ne k$,*

$$\frac{1}{\alpha_k \alpha_l} \le C \frac{|k - l|^{2\sigma}}{\gamma^2 |\varepsilon|^{\tau-1}} \, , \qquad \sigma := \frac{\tau(\tau-1)}{2 - \tau} \, . \tag{5.7}$$

Proof. Lemma 4.5 of [4]. □

In Lemma 4.8 of [4] we had shown that the operator $\mathcal{R}_1$ acts somehow as a multiplication operator for a function with higher regularity, with σ more derivatives. It is therefore natural to expect also a Moser type estimate for $\mathcal{R}_1$ like (2.2). This is the content of the next lemma.

Lemma 5.6 (Bound of $\mathcal{R}_1$) $\forall s' \geq s$

$$\Big\|\mathcal{R}_1 h\Big\|_{s'} \leq K_1 \gamma^{-1} |\varepsilon|^{\frac{3-\tau}{2}} \Big(\|h\|_{s'} + \|w\|_{s'+\sigma} \|h\|_s \Big) , \qquad \forall h \in W^{(N)} \tag{5.8}$$

for some constant $K_1 := K_1(s') > 0$.

Proof. For $h \in W^{(N)}$,

$$\mathcal{R}_1 h = \sum_{|k| \leq N} (\mathcal{R}_1 h)_k \exp(\mathrm{i}kt) = \varepsilon\, |D|^{-1/2} P_N \Pi_W \Big(\bar{a}(t,x)\ |D|^{-1/2} h \Big)$$

with

$$(\mathcal{R}_1 h)_k = \varepsilon\, |D_k|^{-1/2} \pi_k \Big(\bar{a}\ |D|^{-1/2} h \Big)_k = \varepsilon\, |D_k|^{-1/2} \pi_k \Big[\sum_{|l| \leq N} \bar{a}_{k-l} |D_l|^{-1/2} h_l \Big] .$$

Set $A_m := \|\bar{a}_m(x)\|_{H^1}$. From (5.9), (5.3) and (5.7), using that $A_0 = 0$,

$$\Big\|(\mathcal{R}_1 h)_k\Big\|_{H^1} \leq |\varepsilon| C \sum_{|l| \leq N, l \neq k} \frac{A_{k-l}}{\sqrt{\alpha_k}\sqrt{\alpha_l}} \|h_l\|_{H^1} \overset{(5.7)}{\leq} \frac{C'}{\gamma} |\varepsilon|^{\frac{3-\tau}{2}} \sum_{|l| \leq N} A_{k-l} |k-l|^{\sigma} \|h_l\|_{H^1}$$

whence

$$\begin{aligned} \Big\|\mathcal{R}_1 h\Big\|_{s'}^2 &= \sum_{|k| \leq N} (k^{2s'} + 1) \|(\mathcal{R}_1 h)_k\|_{H^1}^2 \leq \frac{C^2}{\gamma^2} |\varepsilon|^{3-\tau} \sum_{|k| \leq N} (k^{2s'} + 1) s_k^2 \\ &= \frac{C^2}{\gamma^2} |\varepsilon|^{3-\tau} \|\widetilde{s}\|_{s'}^2, \end{aligned} \tag{5.9}$$

where we have set

$$\widetilde{s}(t) := \sum_{|k| \leq N} s_k \exp(\mathrm{i}kt) \qquad \text{with} \qquad s_k := \sum_{|l| \leq N} A_{k-l} |k-l|^{\sigma} \|h_l\|_{H^1} ,\ s_{-k} = s_k .$$

It turns out that $\widetilde{s} = P_N(\widetilde{a}\widetilde{c})$ where $\widetilde{a}(t) := \sum_{l \in \mathbf{Z}} |l|^{\sigma} A_l \exp(\mathrm{i}lt)$ and $\widetilde{c}(t) := \sum_{|l| \leq N} \|h_l\|_{H^1} \exp(\mathrm{i}lt)$. Therefore, by (5.9) and since $s, s' > 1/2$, applying the estimates on the product of Lemma 2.1,

$$\begin{aligned} \Big\|\mathcal{R}_1 h\Big\|_{s'} &\leq C\gamma^{-1} |\varepsilon|^{\frac{3-\tau}{2}} \|\widetilde{a}\widetilde{c}\|_{s'} \\ &\leq C(s')\gamma^{-1} |\varepsilon|^{\frac{3-\tau}{2}} \Big(\|\widetilde{a}\|_s \|\widetilde{c}\|_{s'} + \|\widetilde{a}\|_{s'} \|\widetilde{c}\|_s \Big) \\ &\leq C(s')\gamma^{-1} |\varepsilon|^{\frac{3-\tau}{2}} \Big(\|a\|_{s+\sigma} \|h\|_{s'} + \|a\|_{s'+\sigma} \|h\|_s \Big) \end{aligned}$$

since $\|\widetilde{a}\|_{s'} \leq \|a\|_{s'+\sigma}$ and $\|\widetilde{c}\|_{s'} = \|h\|_{s'}$, $\forall s'$. By (5.1) and (5.2) we deduce (5.8). $\square$

Remark 5.1 The loss of σ derivatives in the previous Lemma is due to the small divisors estimate (5.7) which is absorbed by the polynomial decay of the coefficients A_m.

Lemma 5.7 (Bound of $\mathcal{R}_2$) $\forall s' \geq s$

$$\left\|\mathcal{R}_2 h\right\|_{s'} \leq K_2 \frac{|\varepsilon|}{\gamma}\Big(\|h\|_{s'} + \|w\|_{s'+\sigma}\|h\|_s\Big), \qquad \forall h \in W^{(N)} \tag{5.10}$$

for some $K_2 := K_2(s') > 0$.

Proof. Fix some $\tilde{s} \in (1/2, \min(1, s))$. Recalling that

$$\mathcal{R}_2 h := \varepsilon\, |D|^{-1/2} P_N \Pi_W \Big(a(t,x)\, D_w v_2\Big[\ |D|^{-1/2} h\Big]\Big)$$

we have

$$\begin{aligned}
\left\|\mathcal{R}_2 h\right\|_{s'} &\overset{(5.4)}{\leq} C\frac{|\varepsilon|}{\sqrt{\gamma}}\ \left\|P_N \Pi_W\Big(a\ D_w v_2\Big[|D|^{-1/2}h\Big]\Big)\right\|_{s'+\frac{\tau-1}{2}} \\
&\leq C\frac{|\varepsilon|}{\sqrt{\gamma}}\ \left\|a\, D_w v_2\Big[|D|^{-1/2}h\Big]\right\|_{s'+\frac{\tau-1}{2}} \\
&\overset{(2.2)}{\leq} C\frac{|\varepsilon|}{\sqrt{\gamma}}\Big(\|a\|_{s'+\frac{\tau-1}{2}}\left\|D_w v_2\Big[|D|^{-1/2}h\Big]\right\|_{\tilde{s}} \\
&\qquad + \|a\|_{\tilde{s}}\left\|D_w v_2\Big[|D|^{-1/2}h\Big]\right\|_{s'+\frac{\tau-1}{2}}\Big).
\end{aligned} \tag{5.11}$$

By (3.3) and since $(\tau-1)/2 < \sigma$, $\tilde{s} \leq 2$,

$$\begin{aligned}
\|a\|_{s'+\frac{\tau-1}{2}}\left\|D_w v_2\Big[|D|^{-1/2}h\Big]\right\|_{\tilde{s}} &\leq C\|a\|_{s'+\sigma}\left\||D|^{-1/2}h\right\|_0 \\
&\leq \frac{C(s'+\sigma)}{\sqrt{\gamma}}(1+\|w\|_{s'+\sigma})\|h\|_s
\end{aligned} \tag{5.12}$$

by (5.1), (5.4) and because $(\tau-1)/2 < 1/2 < s$. We then distinguish two cases. *1st case :* $s' + (\tau-1)/2 \leq 2$. Then, by (5.2), (3.3) and the inequality $\tilde{s} \leq s$,

$$\|a\|_{\tilde{s}}\left\|D_w v_2\Big[|D|^{-1/2}h\Big]\right\|_{s'+\frac{\tau-1}{2}} \leq C\left\||D|^{-1/2}h\right\|_0 \overset{(5.4)}{\leq} \frac{C}{\sqrt{\gamma}}\|h\|_{\frac{\tau-1}{2}} \leq \frac{C}{\sqrt{\gamma}}\|h\|_{s'}. \tag{5.13}$$

2nd case : $s' + (\tau-1)/2 > 2$. In particular, $s' > 3/2 > \tilde{s} + 1/2$. By (5.2) and (3.2), which is still valid when s is replaced with $\tilde{s}$,

$$\begin{aligned}\|a\|_{\tilde s}\Big\|D_w v_2\Big[|D|^{-1/2}h\Big]\Big\|_{s'+\frac{\tau-1}{2}} &\le C\Big(1+\|w\|_{s'+\frac{\tau-1}{2}-2}\Big)\Big\||D|^{-1/2}h\Big\|_{\tilde s}\\ &\quad + C\Big\||D|^{-1/2}h\Big\|_{s'+\frac{\tau-1}{2}-2}\\ &\overset{(5.4)}{\le} \frac{C}{\sqrt{\gamma}}\Big((1+\|w\|_{s'-1/2})\|h\|_{\tilde s+1/2}+\|h\|_{s'+\tau-3}\Big)\\ &\overset{(2.3)}{\le} \frac{C}{\sqrt{\gamma}}\Big(\|w\|_{s'}\|h\|_{\tilde s}+\|h\|_{s'}\Big) \end{aligned}\qquad (5.14)$$

using $\tau-3\le 0$, $\tilde s+1/2\le s'$ and $\|w\|_{\tilde s}\le\|w\|_{s+\sigma}\le 1$ as assumed in (4.8). Estimate (5.10) is a consequence of (5.11), (5.12), and (5.13) or (5.14). □

Next define

$$\mathcal{R} := U^{-1}\mathcal{R}_1 + U^{-1}\mathcal{R}_2\,.$$

Lemma 5.8 *$\forall p\in\mathbb{N}$, $\forall s'\ge s$,*

$$\Big\|\mathcal{R}^p h\Big\|_{s'} \le \Big(K_3|\varepsilon|^{\frac12}\gamma^{-1}\Big)^p\Big(\|h\|_{s'}+p\|w\|_{s'+\sigma}\|h\|_s\Big)\,,\qquad \forall h\in W^{(N)} \qquad (5.15)$$

where $K_3 = 4(K_1(s')+K_2(s'))$.

Proof. By induction. Statement (5.15) is true for $p=1$ because, by Lemma 5.4, 5.6 and 5.7, for ε small enough, $\forall s'\ge s$, $\tau\in(1,2)$,

$$\|\mathcal{R}h\|_{s'}\le\|U^{-1}\mathcal{R}_1h\|_{s'}+\|U^{-1}\mathcal{R}_2h\|_{s'}\le K'\frac{|\varepsilon|^{1/2}}{\gamma}\Big(\|h\|_{s'}+\|w\|_{s'+\sigma}\|h\|_s\Big)\quad (5.16)$$

with $K' := 2(K_1(s')+K_2(s'))$.

Next, suppose (5.15) holds for p. We want to prove it for $p+1$. By induction

$$\begin{aligned}\|\mathcal{R}^{p+1}h\|_{s'} &= \Big\|\mathcal{R}^p(\mathcal{R}h)\Big\|_{s'}\\ &\overset{(5.15)}{\le} \Big(K_3\gamma^{-1}|\varepsilon|^{\frac12}\Big)^p\Big(\|\mathcal{R}h\|_{s'}+p\|w\|_{s'+\sigma}\|\mathcal{R}h\|_s\Big)\\ &\overset{(5.16)}{\le} \Big(K_3\gamma^{-1}|\varepsilon|^{\frac12}\Big)^p\Big(K'\gamma^{-1}|\varepsilon|^{\frac12}\Big(\|h\|_{s'}+\|w\|_{s'+\sigma}\|h\|_s\Big)\\ &\quad\overset{(4.8)}{+}\ p\|w\|_{s'+\sigma}\,2K'\gamma^{-1}|\varepsilon|^{\frac12}\|h\|_s\Big)\\ &= K_3^p\Big(\gamma^{-1}|\varepsilon|^{\frac12}\Big)^{p+1}K'\Big(\|h\|_{s'}+(1+2p)\|w\|_{s'+\sigma}\|h\|_s\Big)\\ &\le K_3^p\Big(\gamma^{-1}|\varepsilon|^{\frac12}\Big)^{p+1}2K'\Big(\|h\|_{s'}+(1+p)\|w\|_{s'+\sigma}\|h\|_s\Big)\\ &= \Big(K_3\gamma^{-1}|\varepsilon|^{\frac12}\Big)^{p+1}\Big(\|h\|_{s'}+(1+p)\|w\|_{s'+\sigma}\|h\|_s\Big)\end{aligned}$$

because $K_3 = 2K'$. This proves the statement. □

Finally we consider the Neumann-power series for the inverse

$$\Big(I - \mathcal{R}\Big)^{-1} = I + \sum_{p \geq 1} \mathcal{R}^p \,.$$

Lemma 5.9 (Neumann series) *There is* $\mu(s') > 0$ *such that* $\forall s' \geq s$, $\forall |\varepsilon|^{\frac{1}{2}}\gamma^{-1} \leq \mu(s')$,

$$\Big\|(I - \mathcal{R})^{-1} h\Big\|_{s'} \leq 2\Big(\|h\|_{s'} + \|w\|_{s'+\sigma}\|h\|_s\Big)\,. \tag{5.17}$$

Proof. Using (5.15)

$$\begin{aligned}
\|(I-\mathcal{R})^{-1}h\|_{s'} &\leq \|h\|_{s'} + \sum_{p\geq 1}\|\mathcal{R}^p h\|_{s'} \\
&\leq \|h\|_{s'} + \sum_{p\geq 1}\Big(K_3\gamma^{-1}|\varepsilon|^{\frac{1}{2}}\Big)^p\Big(\|h\|_{s'} + p\|w\|_{s'+\sigma}\|h\|_s\Big) \\
&\leq 2\|h\|_{s'} + 2\|w\|_{s'+\sigma}\|h\|_s,
\end{aligned}$$

taking $K_3|\varepsilon|^{\frac{1}{2}}\gamma^{-1}$ small enough. □

Proof of property (P5) completed. We have

$$\mathcal{L}_N^{-1} = |D|^{-1/2}\Big(I - \mathcal{R}\Big)^{-1}U^{-1}|D|^{-1/2}$$

whence

$$\begin{aligned}
\|\mathcal{L}_N^{-1}h\|_{s'} &\overset{(5.5)}{\leq} C\frac{N^{\frac{\tau-1}{2}}}{\sqrt{\gamma}}\Big\|(I-\mathcal{R})^{-1}U^{-1}|D|^{-1/2}h\Big\|_{s'} \\
&\overset{(5.17)}{\leq} C\frac{N^{\frac{\tau-1}{2}}}{\sqrt{\gamma}}\Big(\|U^{-1}|D|^{-1/2}h\|_{s'} + \|w\|_{s'+\sigma}\|U^{-1}|D|^{-1/2}h\|_s\Big) \\
&\overset{(5.6)}{\leq} C\frac{N^{\frac{\tau-1}{2}}}{\sqrt{\gamma}}\Big(\||D|^{-1/2}h\|_{s'} + \|w\|_{s'+\sigma}\||D|^{-1/2}h\|_s\Big) \\
&\overset{(5.5)}{\leq} C\frac{N^{\tau-1}}{\gamma}\Big(\|h\|_{s'} + \|w\|_{s'+\sigma}\|h\|_s\Big),
\end{aligned}$$

completing the proof of property $(P5)$.

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