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BSDE on an infinite horizon and elliptic PDEs in infinite dimension

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Abstract. In this paper, we study the existence and uniqueness of mild solutions to a possibly degenerate elliptic partial differential equation $\mathcal{L}u(x) + \psi(x, u(x), \nabla u(x)G(x)) - \lambda u(x) = 0$ in Hilbert spaces. Our aim is, in the case in which $\psi(\cdot, 0, 0)$ is bounded, to drop the assumptions on the size of λ needed in [11]. The main tool will be existence, uniqueness and regular dependence on parameters of a bounded solution to a suitable backward stochastic differential equation with infinite horizon. Finally we apply the result to study an optimal control problem.

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1 Introduction

Let us consider the following elliptic partial differential equation in Hilbert space H,

$$\mathcal{L}u(x) + \psi(x, u(x), \nabla u(x)G(x)) - \lambda u(x) = 0, \quad x \in H, \tag{1.1}$$

where the second order operator $\mathcal{L}$ is:

$$\mathcal{L}\phi(x) = \frac{1}{2}Trace(G(x)G(x)^*\nabla^2\phi(x)) + \langle Ax, \nabla\phi(x)\rangle + \langle F(x), \nabla\phi(x)\rangle.$$

Here H and Ξ are two Hilbert spaces, A is the generator of a strongly continuous semigroup of bounded linear operators $(e^{tA})_{t\geq 0}$ in H, F and G are functions with values in H and $L(\Xi, H)$ respectively, satisfying appropriate Lipschitz conditions, ψ is a function from $H \times \mathbb{R} \times \Xi^*$ to $\mathbb{R}$, and $\lambda > 0$.

We are here concerned with the existence and uniqueness of the mild solution of the stationary PDE (1.1). Let us recall the definition of mild solution (we limit ourselves to bounded ones). Consider the solution X to the following stochastic evolution equation:

$$dX_s = AX_s ds + F(X_s)ds + G(X_s)dW_s, \; X_t = x, \tag{1.2}$$

where $t \geq 0$, $x \in H$ and W is a cylindrical Wiener process in the Hilbert space Ξ. The Markov process X defines a transition semigroup P acting on bounded measurable functions $\phi : H \to \mathbb{R}$ according to the formula:

$$P_{s-t}[\phi](x) = \mathbb{E}\phi(X_s^{t,x}), \; x \in H, \; s \geq t \geq 0.$$

Moreover $\mathcal{L}$, defined on regular enough functions, is the infinitesimal generator of $(P_t)_{t\geq 0}$.

Then a bounded function $u : H \to \mathbb{R}$, Gâteaux differentiable, is a mild solution of (1.1) if the equality

$$u(x) = e^{-\lambda T} P_T[u](x) + \int_0^T e^{-\lambda\tau} P_\tau[\psi(\cdot, u(\cdot, \nabla u(\cdot)G(\cdot))](x)d\tau \tag{1.3}$$

holds for all $x \in H$ and $T > 0$.

Existence and uniqueness of a mild solution of equation (1.1) in infinite dimensional spaces was recently studied by several authors employing different techniques. A first class of papers treats equation (1.3) by a fixed point argument, in a suitable function space (see [4], [12], [6] and [14]); the presence of the gradient of u in the nonlinear term obliges to require smoothing properties of the semigroup $(P_t)_{t\geq 0}$. This leads, in the above mentioned papers, to non-degeneracy conditions on G normally requiring that $e^{tA}H \subset Q_t H$, $\forall t > 0$ and $|Q_t^{-1/2} e^{tA}| \leq ct^{-\alpha}$ where $Q_t := \int_0^t e^{sA}GG^* e^{sA^*} ds$ and $c > 0$, $\alpha \in [0, 1[$ are suitable constants. We notice here that the above condition has been weakened in [14] to $e^{tA}GH \subset Q_t H$, for all $t > 0$ and $|Q_t^{-1/2} e^{tA} G| \leq ct^{-\alpha}$.

In [11] the use of backward stochastic differential equations allows to treat the case in which G depends on x and can be degenerate without restriction. Namely in [11] (following several papers dealing with finite dimensional situations, see, for instance [3], [7] and [15]) the solution of equation (1.3) is represented using a Markovian forward-backward system of equations

$$\begin{cases} dX_s = AX_s ds + F(X_s)ds + G(X_s)dW_s, & s \geq 0 \\ dY_s = \lambda Y_s ds - \psi(X_s, Y_s, Z_s)ds + Z_s dW_s, & s \geq 0 \\ X_0 = x \end{cases} \tag{1.4}$$

where the final condition for the second equation has been replaced by a suitable growth condition. The main drawback of the results in [11] is that the existence and uniqueness of the mild solution to equation (1.1) is proved only for λ large enough. This requirement is particularly unpleasant when we interpret equation (1.1) as a Hamilton-Jacobi-Bellman equation corresponding to infinite horizon control problem (see Section 5) since it imposes unnatural conditions on the treatable costs (see Remark 5.1).

Finally we mention here that viscosity solutions of equation (1.1) were also studied (see, for instance [13], [20], and their references). The point is that very few comparison results are available for viscosity solutions in infinite dimensions and all of them impose strong assumptions on operator G such as finite trace conditions, and on the class of possible solutions, such as continuity with respect to weak norms. Thus existence and uniqueness of viscosity solutions of equation (1.1) can be obtained only in very special situations.

Our aim here is to prove that, under suitable assumptions, the mild solution to (1.1) exists and is unique for all $\lambda > 0$ still allowing G to be degenerate or to depend on x. We choose the approach by infinite horizon backward stochastic differential equations, more precisely we require the solution to the backward stochastic differential equation in (1.4) to be bounded on the whole positive real semi-axis. We notice that existence and uniqueness of a bounded solution for the backward stochastic differential equation in (1.4) was established only recently, see [2] and [19]. The main technical point here will be proving differentiability of such bounded solution of the backward equation in system (1.4) with respect to the initial datum x of the forward equation. This will be done when either assuming that G is non-degenerate or assuming that G is constant and $A+\nabla F$ is dissipative. In the first case the result is an immediate consequence of a 'semi-linear' Bismut-Elworthy formula proved in [10]. In the second and more interesting case, the proof is slightly more complex and is based on an a-priori bound for suitable approximations of the equations for the gradient of Y with respect to x. We also notice here that in Remark 5.2 we show by a simple example taken from control theory that, in the degenerate case, if $A+\nabla F$ is not dissipative we can not expect to have differentiability.

The second aim of this paper is to apply the above result to an optimal control problem with state equation:

$$\begin{aligned} &dX^u_\tau = AX^u_\tau d\tau + F(X^u_\tau)d\tau + G(X^u_\tau)R(u_\tau)d\tau + G(X^u_\tau)dW_\tau,\\ &X^u_0 = x \in H, \end{aligned}$$

where u denotes the control process, taking values in a given subset $\mathcal{U}$ of a Banach space U, and R is a function with values in Ξ^*. Our purpose is to characterize the predictable control process u that minimizes an infinite horizon cost functional of the form

$$J(x,u) = \mathbb{E}\int_0^\infty e^{-\lambda\sigma} g(X_\sigma^u, u_\sigma)d\sigma,$$

where g is a given real bounded function and λ is any positive number. The results obtained on equation (1.1) allows to prove that the value function of the above problem is the unique mild solution of the corresponding Hamilton-Jacobi-Bellman equation (that has the same structure as (1.1). Moreover the optimal control is expressed in terms of a feedback that involves the gradient of that same solution to the Hamilton-Jacobi-Bellman equation. We comment here that similar kind of results where included in [11] but only for λ large enough; the acceptable size λ was related to the size of the set of admissible controls (see Remark 5.1).

The paper is organized as follows: in the next section, we recall some notations and known results; in Section 3, we study the SDE and BSDE associated with the PDE; in Section 4, we give our main result about the solution to PDE; and the last section is devoted to the application.

2 Preliminaries

2.1 Notations

The norm of an element x of a Banach space E will be denoted $|x|_E$ or simply $|x|$, if no confusion is possible. If F is another Banach space, $L(E,F)$ denotes the space of bounded linear operators from E to F, endowed with the usual operator norm.

The letters Ξ, H, K will always denote Hilbert spaces. Scalar product is denoted $\langle\cdot,\cdot\rangle$, with a subscript to specify the space, if necessary. All Hilbert spaces are assumed to be real and separable. $L_2(\Xi,K)$ is the space of Hilbert-Schmidt operators from Ξ to K, endowed with the Hilbert-Schmidt norm, that makes it a separable Hilbert space.

By a cylindrical Wiener process with values in a Hilbert space Ξ, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, we mean a family $\{W_t,\, t\geq 0\}$ of linear mappings $\Xi\to L^2(\Omega)$, denoted $\xi\mapsto\langle\xi,W_t\rangle$, such that

(i) for every $\xi\in\Xi$, $\{\langle\xi,W_t\rangle,\, t\geq 0\}$ is a real (continuous) Wiener process;

(ii) for every $\xi_1,\xi_2\in\Xi$ and $t\geq 0$, $\mathbb{E}\left(\langle\xi_1,W_t\rangle\cdot\langle\xi_2,W_t\rangle\right)\langle\xi_1,\xi_2\rangle_\Xi\, t$.

$(\mathcal{F}_t)_{t\geq 0}$ will denote, except in Section 5, the natural filtration of W, augmented with the family of $\mathbb{P}$-null sets. The filtration $(\mathcal{F}_t)$ satisfies the usual conditions. All the concepts of measurably for stochastic processes (e.g. predictability etc.) refer to this filtration. By $\mathcal{P}$ we denote the predictable σ-algebra and by $\mathcal{B}(\Lambda)$ the Borel σ-algebra of any topological space Λ.

Next we define some classes of stochastic processes with values in a Hilbert space K.

- $L^p_{\mathcal{P}}(\Omega; L^2(0,\zeta;K))$ defined for $\zeta \in]0,+\infty]$ and $p \in [1,\infty)$, denotes the space of equivalence classes of processes $Y : \Omega \times [0,\zeta[\to K$, admitting a predictable version and such that
$$|Y|^p_{L^p_{\mathcal{P}}(\Omega;L^2(0,\zeta;K))} = \mathbb{E}\left(\int_0^\zeta |Y_\tau|^2_K \, d\tau\right)^{p/2}.$$
Elements of $L^p_{\mathcal{P}}(\Omega; L^2(0,\zeta;K))$ are identified up to modification.
- $L^p_{\mathcal{P}}(\Omega; C(0,\zeta;K))$, defined for $\zeta \in]0,+\infty[$ and $p \in [1,\infty[$, denotes the space of predictable processes $\{Y_t,\, t \in [0,\zeta]\}$ with continuous paths in K, such that the norm
$$|Y|^p_{L^p_{\mathcal{P}}(\Omega;C([0,\zeta];K))} = \mathbb{E}\sup_{\tau\in[0,\zeta]} |Y_\tau|^p_K$$
is finite. Elements of $L^p_{\mathcal{P}}(\Omega; C(0,\zeta;K))$ are identified up to indistinguishability.

Moreover we give to the notations $L^p_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;K)$ and $L^p_{\mathcal{P},\mathrm{loc}}(\Omega; C(0,\infty;K))$ their obvious meaning.

We also recall notations and basic facts on a class of differentiable maps acting among Banach spaces, particularly suitable for our purposes (we refer the reader to [9] for details and properties). We notice that the use of Gâteaux differentiability in place of Fréchet differentiability is particularly suitable when dealing with evaluation (Nemitskii) type mappings on spaces of summable functions.

Let now X, Z, V denote Banach spaces. We say that a mapping $F : X \to V$ belongs to the class $\mathcal{G}^1(X,V)$ if it is continuous, Gâteaux differentiable on X, and its Gâteaux derivative $\nabla F : X \to L(X,V)$ is strongly continuous.

The last requirement is equivalent to the fact that for every $h \in X$ the map $\nabla F(\cdot)h : X \to V$ is continuous. Note that $\nabla F : X \to L(X,V)$ is not continuous in general if $L(X,V)$ is endowed with the norm operator topology; clearly, if this happens then F is Fréchet differentiable on X. It can be proved that if $F \in \mathcal{G}^1(X,V)$ then $(x,h) \mapsto \nabla F(x)h$ is continuous from $X \times X$ to V; if, in addition, G is in $\mathcal{G}^1(V,Z)$ then $G(F)$ belongs to $\mathcal{G}^1(X,Z)$ and the chain rule holds: $\nabla(G(F))(x) = \nabla G(F(x))\nabla F(x)$.

2.2 BSDEs with random terminal time

This section is devoted to recall a result on BSDEs on an infinite horizon, i.e., the following type of BSDE:
$$Y_\tau = Y_T + \int_\tau^T (\Psi(\sigma, Y_\sigma, Z_\sigma) - \lambda Y_\sigma)d\sigma - \int_\tau^T Z_\sigma dW_\sigma, \quad 0 \le \tau \le T < \infty. \tag{2.1}$$
Let $\lambda > 0$ be some real number. Consider $\Psi : \Omega \times [0,\infty) \times \mathbb{R} \times \Xi^* \to \mathbb{R}$ such that $\forall (y,z) \in \mathbb{R} \times \Xi^*$, $f(t,y,z))_{t\ge 0}$ is progressively measurable. We suppose the following:

Hypothesis 2.1

(i) Ψ *is uniformly Lipschitz in* z *with Lipschitz constant* K*:*

$$\forall t \geq 0, \forall y \in \mathbb{R}, \forall z, z' \in \Xi^*, \quad |\Psi(t,y,z) - \Psi(t,y,z')| \leq K|z - z'|,$$
$$\mathbb{P} - a.s.$$

(ii) $\forall t \geq 0, \forall z \in \Xi^*$, $y \to \Psi(t,y,z)$ *is continuous, there exists a continuous and increasing function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$*, such that*

$$\forall t \geq 0, \forall y \in \mathbb{R}, \forall z \in \Xi^*, \quad |\Psi(t,y,z)| \leq |\Psi(t,0,z)| + \varphi(|y|), \ \mathbb{P} - a.s.$$

(iii) $\lambda > 0$ *and* Ψ *is monotone in* y *in the following sense:*

$$\forall (t,y,y',z), \quad (y - y')(\Psi(t,y,z) - \Psi(t,y',z)) \leq 0, \ \mathbb{P} - a.s.$$

(iv) *There exists a constant* M *such that* $\forall t \geq 0, |\Psi(t,0,0)| \leq M, \ \mathbb{P} - a.s.$
We denote $\sup_{t\geq 0} |\Psi(t,0,0)|$ *by* M*.*

The existence and uniqueness of a solution to (2.1) under Hypothesis 2.1 was first studied by Briand and Hu in [2] and then generalized by Royer in [19]. They have established the following result when W is a finite dimensional Wiener process but the extension to the case in which W is a Hilbert-valued Wiener process is immediate.

Let us recall here their main result:

Lemma 2.1 *Let us suppose that Hypothesis 2.1 holds. Then we have:*

(i) *There exists a solution* (Y,Z) *to BSDE (2.1) such that* Y *is a continuous process bounded by* $\frac{M}{\lambda}$*, and* $Z \in L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi))$ *with* $\mathbb{E}\int_0^\infty e^{-2\lambda s} |Z_s|^2 ds < \infty$*. Moreover, the solution is unique in the class of processes* (Y,Z) *such that* Y *is continuous and uniformly bounded, and* Z *belongs to* $L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi))$*.*

(ii) *Denoting by* (Y^n, Z^n) *the unique solution of the following BSDE:*

$$Y^n_\tau = \int_\tau^n (\Psi(\sigma, Y^n_\sigma, Z^n_\sigma) - \lambda Y^n_\sigma) d\sigma - \int_\tau^n Z^n_\sigma dW_\sigma, \tag{2.2}$$

then $|Y^n_\tau| \leq \frac{M}{\lambda}$ *and the following convergence rate holds:*

$$|Y^n_\tau - Y_\tau| \leq \frac{M}{\lambda} \exp\{-\lambda(n - \tau)\}. \tag{2.3}$$

Moreover

$$\mathbb{E}\int_0^{+\infty} e^{-2\lambda\sigma} |Z^n_\sigma - Z_\sigma|^2 d\sigma \to 0.$$

3 The associated forward-backward system

3.1 The forward SDE

Now we consider the Itô stochastic equation for an unknown process $\{X_s, s \geq 0\}$ with values in a Hilbert space H:

$$X_s = e^{sA}x + \int_0^s e^{(s-\sigma)A}F(X_\sigma)d\sigma + \int_0^s e^{(s-\sigma)A}G(X_\sigma)dW_\sigma, \ s \geq 0. \tag{3.1}$$

Our assumptions will be the following:

Hypothesis 3.1 *(i) The operator A is the generator of a strongly continuous semigroup e^{tA}, $t \geq 0$, in a Hilbert space H. We denote by m and a two constants such that $|e^{tA}| \leq me^{at}$ for $t \geq 0$.*

(ii) $F : H \to H$ satisfies, for some constant $L > 0$,

$$|F(x) - F(y)| \leq L|x - y|, \ x, y \in H.$$

(iii) G denotes a mapping from H to $L(\Xi, H)$ such that for every $\xi \in \Xi$ the map $G(\cdot)\xi : H \to H$ is measurable, $e^{tA}G(x) \in L_2(\Xi, H)$ for every $t > 0$ and $x \in H$, and

$$|e^{tA}G(x)|_{L_2(\Xi,H)} \leq Lt^{-\gamma}e^{at}(1 + |x|), \tag{3.2}$$

$$|e^{tA}G(x) - e^{tA}G(y)|_{L_2(\Xi,H)} \leq Lt^{-\gamma}e^{at}|x - y|, \ t > 0, \ x, y \in H, \tag{3.3}$$

for some constants $L > 0$ and $\gamma \in [0, 1/2)$.

We start by recalling a well known result on solvability of equation (3.1) on a bounded interval, see e.g. [9].

Proposition 3.1 *Under the assumptions of Hypothesis 3.1, for every $p \in [2, \infty)$ and $T > 0$ there exists a unique process $X^x \in L^p_{\mathcal{P}}(\Omega; C(0, T; H))$ solution of (3.1). Moreover, for all fixed $T > 0$, the map $x \to X^x$ is continuous from H to $L^p_{\mathcal{P}}(\Omega; C(0, T; H))$.*

$$\mathbb{E} \sup_{\tau \in [0,T]} |X_\tau|^p \leq C(1 + |x|)^p, \tag{3.4}$$

for some constant C depending only on q, γ, T, L, a and m.

3.2 The backward SDE

The associated BSDE is:

$$Y^x_\tau = Y^x_T + \int_\tau^T (\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma) - \lambda Y^x_\sigma) d\sigma - \int_\tau^T Z^x_\sigma dW_\sigma, \quad 0 \leq \tau \leq T < \infty. \tag{3.5}$$

Here X^x is the unique mild solution to (3.1) starting from $X_0 = x$. Y is real valued and Z takes values in Ξ^*, $\psi : H \times \mathbb{R} \times \Xi^* \to \mathbb{R}$ is a given measurable function, $\lambda > 0$.

We will always assume the following on ψ:

Hypothesis 3.2 *(i)* *ψ is uniformly Lipschitz in z with Lipschitz constant K, that is $|\psi(x, y, z) - \psi(x, y, z')| \leq K|z - z'|$, $\forall x \in H$, $\forall y \in \mathbb{R}$, $\forall z, z' \in \Xi^*$.*

(ii) *$(x, y) \to \psi(x, y, z)$ is continuous for all $z \in \Xi^*$.*

(iii) *There exists a continuous and increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, such that $|\psi(x, y, z)| \leq |\psi(x, 0, z)| + \gamma(|y|)$, $\forall x \in H$, $\forall y \in \mathbb{R}$, $\forall z \in \Xi^*$.*

(iv) *$\lambda > 0$ and ψ is monotone in y in the following sense:*

$$\forall (x, y, y', z), \quad (y - y')(\psi(x, y, z) - \psi(x, y', z)) \leq 0.$$

(v) $\sup_{x \in H} |\psi(x, 0, 0)| := M < +\infty$.

Applying Lemma 2.1, we obtain:

Proposition 3.2 *Let us suppose that Hypotheses 3.1 and 3.2 hold. Then we have:*

(i) *For any $x \in H$, there exists a solution (Y^x, Z^x) to the BSDE (3.5) such that Y^x is a continuous process bounded by M/λ, and $Z \in L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0, \infty; \Xi))$ with $\mathbb{E} \int_0^\infty e^{-2\lambda s} |Z_s|^2 ds < \infty$. The solution is unique in the class of processes (Y, Z) such that Y is continuous and bounded, and Z belongs to $L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0, \infty; \Xi))$.*

(ii) *Denoting by $(Y^{n,x}, Z^{n,x})$ the unique solution of the following BSDE (with finite horizon):*

$$Y^{n,x}_\tau \int_\tau^n (\psi(X^x_\sigma, Y^{n,x}_\sigma, Z^{n,x}_\sigma) - \lambda Y^{n,x}_\sigma) d\sigma - \int_\tau^n Z^{n,x}_\sigma dW_\sigma, \tag{3.6}$$

then $|Y^{n,x}_\tau| \leq \frac{M}{\lambda}$ and the following convergence rate holds:

$$|Y^{n,x}_\tau - Y^x_\tau| \leq \frac{M}{\lambda} \exp\{-\lambda(n - \tau)\}. \tag{3.7}$$

Moreover,

$$\mathbb{E}\int_0^{+\infty} e^{-2\lambda\sigma}|Z_\sigma^{n,x} - Z_\sigma^x|^2 d\sigma \to 0. \tag{3.8}$$

(iii) *For all* $T > 0$ *and* $p \geq 1$, *the map* $x \to (Y^x\big|_{[0,T]}, Z^x\big|_{[0,T]})$ *is continuous from* H *to the space* $L^p_{\mathcal{P}}(\Omega; C(0,T;\mathbb{R})) \times L^p_{\mathcal{P}}(\Omega; L^2(0,T;\Xi))$.

Proof. Statements (i) and (ii) are immediate consequences of Lemma 2.1. Let us prove (iii). If $x'_m \to x$ as $m \to +\infty$ then

$$\begin{aligned}|Y_T^{x'_m} - Y_T^x| &\leq |Y_T^{x'_m} - Y_T^{n,x'_m}| + |Y_T^{n,x} - Y_T^x| + |Y_T^{n,x'_m} - Y_T^{n,x}| \\ &\leq 2\frac{M}{\lambda}\exp\{-\lambda(n-T)\} + |Y_T^{n,x'_m} - Y_T^{n,x}|.\end{aligned}$$

Moreover for fixed n, $Y_T^{n,x'_m} \to Y_T^{n,x}$ in $L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ by standard arguments, see, e.g. [19] or Proposition 4.3 in [9]. Thus $Y_T^{x'_m} \to Y_T^x$ in $L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Now we can notice that $(Y^x\big|_{[0,T]}, Z^x\big|_{[0,T]})$ is the unique solution of the following BSDE (with finite horizon):

$$Y_\tau^x = Y_T^x + \int_\tau^T (\psi(X_\sigma^x, Y_\sigma^x, Z_\sigma^x) - \lambda Y_\sigma^x) d\sigma - \int_\tau^T Z_\sigma^x dW_\sigma,$$

and the same holds for $(Y^{x'_m}\big|_{[0,T]}, Z^{x'_m}\big|_{[0,T]})$. So it is enough to apply standard parameter dependence arguments for finite horizon BSDEs, see, for instance [9], Proposition 4.3, to conclude that $(Y^{x'_m}\big|_{[0,T]}, Z^{x'_m}\big|_{[0,T]}) \to (Y^x\big|_{[0,T]}, Z^x\big|_{[0,T]})$ in $L^p_{\mathcal{P}}(\Omega; C(0,T;\mathbb{R})) \times L^p_{\mathcal{P}}(\Omega; L^2(0,T;\Xi))$.

□

We need to study the regularity of Y^x. More precisely, we would like to show that Y_0^x belongs to $\mathcal{G}^1(H,\mathbb{R})$. For this, we have to impose some additional assumptions. We propose two different sets of requirements. In the first we allow degeneracy of G but we have to impose that G is constant and $A + \nabla_x F(x)$ is dissipative. The second deals with the non-degenerate case and uses a Bismut-Elworthy type of formula.

3.3 Differentiability with respect to initial data. Degenerate diffusion case

In addition to Hypotheses (3.1) and (3.2) we assume:

Hypothesis 3.3

(i) *G does not depend on x (that is* $G \in L(\Xi, H)$ *with* $|e^{tA}G|_{L_2(\Xi,H)} \leq Lt^{-\gamma}e^{at}$, *for a suitable* $\gamma \in [0, 1/2)$*).*

(ii) *We have* $F(\cdot) \in \mathcal{G}^1(H, H)$.

(iii) *Operators* $A + F_x(x)$ *are dissipative (that is* $\langle Ay, y\rangle + \langle F_x(x)y, y\rangle \leq 0$ *for all* $x \in H$ *and* $y \in D(A)$*).*

(iv) $\psi(\cdot, \cdot, \cdot) \in \mathcal{G}^1(H \times \mathbb{R} \times \Xi^*, \mathbb{R})$ *and* $|\nabla_x \psi(x, y, z)|_{H^*} \leq c$, $|\nabla_y \psi(x, y, z)| \leq c$, $|\nabla_z \psi(x, y, z)| \leq c$, *for a suitable constant* $c > 0$ *and all* $x \in H$, $y \in \mathbb{R}$, $z \in \Xi^*$.

(v) $\nabla_y \psi(x, y, z) \leq 0$.

Remark 3.1 Assumption (*ii*), (*iv*) and (*v*) are clearly natural; indeed in the degenerate case we can not expect smoothing phenomena, thus if we require differentiable dependence on x, we have to assume differentiability of the coefficients. Assumption (*iii*) is natural as well if one thinks that we are asking differentiability of the value function of an infinite horizon control problem for a degenerate diffusion (see Remark 5.2). On the contrary Assumption (*i*) is apparently only due to technical reasons.

Lemma 3.1 *Under Assumptions 3.1 and 3.3 the map* $x \to X^x$ *is Gâteaux differentiable (that is belongs to* $\mathcal{G}(H, L^p_{\mathcal{P}}(\Omega, C(0, T; H)))$*). Moreover denoting by* $\nabla_x X^x$ *the partial Gâteaux derivative, then for every direction* $h \in H$*, the directional derivative process* $\nabla_x X^x h, \tau \in \mathbb{R}$*, solves,* $\mathbb{P} - a.s.$*, the equation*

$$\nabla_x X^x_\tau h = e^{\tau A}h + \int_0^\tau e^{\sigma A}\nabla_x F(X^x_\sigma)\nabla_x X^x_\sigma h\, d\sigma, \quad \tau \in \mathbb{R}^+. \tag{3.9}$$

Finally, $\mathbb{P}$*-a.s.,* $|\nabla_x X^x_\tau h| \leq |h|$*, for all* $\tau > 0$*.*

Proof. The first assertion and relation (3.9) are proved, for instance in, [9]. To prove the last assertion we proceed by a classical approximation argument (notice that the equation for ∇X has no stochastic integral term. Let $J(n, A) : n(nI - A)^{-1}$ for n large enough. As it is well known $J(n, A) \in L(H, D(A))$, $J(n, A)x \to x$ for all $x \in H$. Let $L^n_t = J(n, A)\nabla_x X^x_t h$, then, for all $T > 0$, $L^n \in L^p_{\mathcal{P}}(\Omega; C(0, T; D(A)))$ and satisfies

$$(L^n_t)' = AL^n_t + J(n, A)\nabla_x F(X^x_t)\nabla_x X^x_t h.$$

Computing $\frac{d}{dt}|L^n_t|^2$, by Hypothesis 3.3 (*iii*) we get:

$$\frac{d}{dt}|L^n_t|^2 \leq 2\langle L^n_t, (J(n, A)\nabla_x F(X^x_t)\nabla_x X^x_t h - \nabla_x F(X^x_t)J(n, A)\nabla_x X^x_t h)\rangle$$

and

$$|L_t^n|^2 \leq |J(n,A)h|^2 + 2\int_0^t \langle L_s^n, (J(n,A)\nabla_x F(X_s^x)\nabla_x X_s^x h - \nabla_x F(X_s^x) J(n,A)\nabla_x X_s^x h)\rangle ds$$

and the claim follows by passing to the limit as $n \to \infty$. □

The following is the main technical result of the paper.

Theorem 3.1 *Under Hypotheses 3.1, 3.3 and 3.2 the map $x \to Y_0^x$ belongs to $\mathcal{G}^1(H, \mathbb{R})$. Moreover $|Y_0^x| + |\nabla_x Y_0^x| \leq c$, for a suitable constant c.*

Proof. Fix $n \geq 1$, let us consider the solution $(Y^{n,x}, Z^{n,x})$ of (3.6). Then, see [9, Proposition 5.2], the map $x \to (Y^{n,x}(\cdot), Z^{n,x}(\cdot))$ is Gâteaux differentiable from H to $L^p_{\mathcal{P}}(\Omega,\ C(0,T;\mathbb{R})) \times L^p_{\mathcal{P}}(\Omega; L^2(0,T;\Xi^*))$, $\forall p \in [2,\infty)$. Denoting by $\nabla_x Y^{n,x} h, \nabla_x Z^{n,x} h$ the partial Gâteaux derivatives with respect to x in the direction $h \in H$, the processes $\{\nabla_x Y_\tau^{n,x} h, \nabla_x Z_\tau^{n,x} h, \tau \in [0,n]\}$ solves the equation, $\mathbb{P} - a.s.$,

$$\begin{aligned}
\nabla_x Y_\tau^{n,x} h \; = \; & \int_\tau^n \nabla_x \psi(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x X_\sigma^{n,x} h \, d\sigma \\
& + \int_\tau^n (-\lambda + \nabla_y \psi(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x})) \nabla_x Y_\sigma^{n,x} h \, d\sigma \\
& + \int_\tau^n \nabla_z \psi(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x Z_\sigma^{n,x} h \, d\sigma - \int_\tau^n \nabla_x Z_\sigma^{n,x} h \, dW_\sigma .
\end{aligned} \tag{3.10}$$

We notice that in the above formula, we are considering that $Z^{n,x}$, $\nabla_x Z^{n,x}$ have values in Ξ^* and $\nabla_z \psi$ has values in Ξ^{**}. So if we identify Ξ^{**} and Ξ we can assume that $\nabla_z \psi$ has values in Ξ and equation (3.10) can be rewritten as:

$$\begin{aligned}
\nabla_x Y_\tau^{n,x} h \; = \; & \int_\tau^n \nabla_x \psi(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x X_\sigma^{n,x} h \, d\sigma \\
& + \int_\tau^n (-\lambda + \nabla_y \psi(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x})) \nabla_x Y_\sigma^{n,x} h \, d\sigma \\
& + \int_\tau^n (\nabla_x Z_\sigma^{n,x} h) \left(\nabla_z \psi(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \, d\sigma - dW_\sigma \right).
\end{aligned}$$

By Hypotheses 3.2 and 3.3 and Lemma 3.1, we have that for all $x, h \in H$ the following holds $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ and all $\sigma \in [0,n]$:

$$\left| \nabla_x \psi(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x X_\sigma^x h \right| \leq c|h|,$$
$$\nabla_y \psi(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \leq 0, \qquad \left| \nabla_z \psi(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \right|_\Xi \leq c.$$

Therefore by the same argument based on Girsanov transform as in [2], Lemma 3.1., we obtain:

$$\sup_{\tau\in[0,n]} |\nabla_x Y^{n,x}_\tau| \le C|h|, \quad \mathbb{P} - \text{a.s.}; \tag{3.11}$$

and, again as in the proof of Lemma 3.1 in [2], applying Itô's formula to $e^{-2\lambda t}|\nabla_x Y^{n,x}_t h|^2$, we get:

$$\mathbb{E}\int_0^\infty e^{-2\lambda t}(|\nabla_x Y^{n,x}_t h|^2 + |\nabla_x Z^{n,x}_t h|^2)dt \le C|h|^2. \tag{3.12}$$

Let now $\mathcal{M}^{2,-2\lambda}$ be the Hilbert space of all couples of $(\mathcal{F}_t)_{t\ge0}$-adapted processes (y,z), where y has values in $\mathbb{R}$ and z in Ξ^*, such that

$$|(y,z)|^2_{\mathcal{M}^{2,-2\lambda}} := \mathbb{E}\int_0^\infty e^{-2\lambda t}(|y_t|^2+|z_t|^2)dt < +\infty$$

Fix $x, h \in H$, there exists a subsequence of $\{(\nabla_x Y^{n,x}h, \nabla_x Z^{n,x}h, \nabla_x Y^{n,x}_0 h) : n \in \mathbb{N}\}$ which we still denote by itself, such that $(\nabla_x Y^{n,x}h, \nabla_x Z^{n,x}h)$ converges weakly to $(U^1(x,h), V^1(x,h))$ in $\mathcal{M}^{2,-2\lambda}$ and $\nabla_x Y^{n,x}_0 h$ converges to $\xi(x,h) \in \mathbb{R}$.

We define now

$$\begin{aligned} U^2_\tau(x,h) &= \xi(x,h) - \int_0^\tau \nabla_x\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma)\nabla_x X^x_\sigma\, hd\sigma \\ &\quad - \int_0^\tau(-\lambda + \nabla_y\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma)U^1_\sigma(x,h)d\sigma \\ &\quad - \int_0^\tau \nabla_z\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma)V^1_\sigma(x,h)d\sigma + \int_0^\tau V^1_\sigma(x,h)dW_\sigma, \end{aligned} \tag{3.13}$$

where (Y^x, Z^x) is the unique bounded solution to the backward equation (3.5), see Proposition 3.2. Moreover we rewrite (3.10) as follows:

$$\begin{aligned} \nabla_x Y^{n,x}_\tau h &= \nabla_x Y^{n,x}_0 h - \int_0^\tau \nabla_x\psi(X^x_\sigma, Y^{n,x}_\sigma, Z^{n,x}_\sigma)\nabla_x X^x_\sigma h d\sigma \\ &\quad + \int_0^\tau(\lambda - \nabla_y\psi(X^x_\sigma, Y^{n,x}_\sigma, Z^{n,x}_\sigma))\nabla_x Y^{n,x}_\sigma h d\sigma \\ &\quad - \int_0^\tau \nabla_z\psi(X^x_\sigma, Y^{n,x}_\sigma, Z^{n,x}_\sigma)\nabla_x Z^{n,x}_\sigma h d\sigma + \int_0^\tau \nabla_x Z^{n,x}_\sigma h dW_\sigma. \end{aligned} \tag{3.14}$$

Since, in particular, $(Y^{n,x}, Z^{n,x}) \to (Y^x, Z^x)$ in measure $\mathbb{P}\times dt$; $\nabla_x\psi$, $\nabla_y\psi$, $\nabla_z\psi$ are bounded and finally $(\nabla_x Y^{n,x}h, \nabla_x Z^{n,x}h) \rightharpoonup (Y^x, Z^x)$ weakly in $\mathcal{M}^{2,-2\lambda}$ it is easy to show that $\nabla_x Y^{n,x}h$ converges to $U^2(x,h)$ weakly in $L^2_{\mathcal{P}}(0,T;\mathbb{R})$ for all $T>0$. Thus $U^2_t(x,h) = U^1_t(x,h)$, $\mathbb{P}$-a.s. for a.e. $t\in\mathbb{R}^+$ and $|U^2_t(x,h)| \le c|h|$, $\mathbb{P}$-a.s. for all $t\in\mathbb{R}^+$ (this last assertion follows from continuity of the trajectories of

$U^2(x,h)$ and from the fact that $|U^1_t(x,h)| \le c|h|$ $\mathbb{P}$-a.s. for almost every $t \in \mathbb{R}^+$). Therefore, coming back to equation (3.13), we have that $(U^2(x,h), V^1(x,h))$ is the unique bounded solution in $\mathbb{R}^+$ of the equation

$$\begin{aligned} U(\tau,x,h) \;=\; & U(0,x,h) - \int_0^\tau \nabla_x\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma)\nabla_x X^x_\sigma h d\sigma \\ & - \int_0^\tau (-\lambda + \nabla_y\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma))U(\tau,x,h)d\sigma \\ & - \int_0^\tau \nabla_z\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma)V(\sigma,x,h)d\sigma + \int_0^\tau V(\sigma,x,h)dW_\sigma. \end{aligned} \tag{3.15}$$

Notice that in particular $U(0,x,h) = \xi(x,h)$ is the limit of $\nabla_x Y^{n,x}_0 h$ (along the chosen subsequence). The uniqueness of the solution to (3.15) (see Lemma 2.1) implies that in reality $U(0,x,h) = \lim_{n\to\infty} \nabla_x Y^{n,x}_0 h$ along the original sequence.

Now let $x'_m \to x$. By (2.3), proceeding as in the proof of point (iii) in Proposition 3.2,

$$|U(0,x,h) - U(0,x'_m,h)| \le \frac{2c}{\lambda} e^{-\lambda n}|h| + |U_n(0,x,h) - U_n(0,x'_m,h)|, \tag{3.16}$$

where $(U_n(\cdot,x,h), V_n(\cdot,x,h)) \in L^p_{\mathcal{P}}(\Omega; C(0,T;\mathbb{R})) \times L^p_{\mathcal{P}}(\Omega; L^2_{\mathcal{P}}(0,T;\Xi))$ is the unique solution of the finite horizon BSDE:

$$\begin{aligned} U_n(\tau,x,h) \;=\; & \int_\tau^n \nabla_x\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma)\nabla_x X^x_\sigma h d\sigma \\ & + \int_\tau^n (-\lambda + \nabla_y\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma))U_n(\tau,x,h)d\sigma \\ & + \int_\tau^n \nabla_z\psi(X^x_\sigma, Y^x_\sigma, Z^x_\sigma)V_n(\sigma,x,h)d\sigma - \int_\tau^n V_n(\sigma,x,h)dW_\sigma, \end{aligned} \tag{3.17}$$

and similarly for $(U_n(\cdot,x'_m,h), V_n(\cdot,x'_m,h))$. We now notice that $\nabla_x\psi$, $\nabla_y\psi$, $\nabla_z\psi$ are, by assumptions, continuous and bounded. Moreover the following statements on continuous dependence on x hold:
maps $x \to X^x$, $x \to \nabla_x X^x h$ are continuous from $H \to L^p_{\mathcal{P}}(\Omega; C(0,T;H))$ (see [9] Proposition 3.3);
the map $x \to Y^x\big|_{[0,T]}$ is continuous from H to $L^p_{\mathcal{P}}(\Omega; C(0,T;\mathbb{R}))$ (see Proposition 3.2 here);
the map $x \to Z^x\big|_{[0,T]}$ is continuous from H to $L^p_{\mathcal{P}}(\Omega; L^2(0,T;\Xi))$ (see Proposition 3.2 here).

We can therefore apply to (3.17) the continuity result of [9 Proposition 4.3] to obtain in particular that $U_n(0,x'_m,h) \to U_n(0,x,h)$ for all fixed n as $m \to \infty$. And by (3.16) we can conclude that $U(0,x'_m,h) \to U(0,x,h)$ as $m \to \infty$.

Summarizing $U(0,x,h) = \lim_{n\to\infty} \nabla_x Y^{n,x}_0 h$ exists, moreover it is clearly linear in h and verifies $|U(0,x,h)| \le C|h|$, finally it is continuous in x for every h fixed.

Finally, for $t > 0$,

$$\lim_{t\searrow 0}\frac{1}{t}[Y_0^{x+th}-Y_0^x]=\lim_{t\searrow 0}\frac{1}{t}\lim_{n\to+\infty}[Y_0^{n,x+th}-Y_0^{n,x}]\lim_{t\searrow 0}\lim_{n\to+\infty}\int_0^1\nabla_x Y_0^{n,x+\theta th}hd\theta$$
$$=\lim_{t\searrow 0}\int_0^1 U(0,x+\theta th)hd\theta=U(0,x)h$$

and the claim is proved. □

3.4 Differentiability with respect to initial data. Non-Degenerate diffusion case

In addition to Hypothesis (3.1) and (3.2) we assume:

Hypothesis 3.4

(i) $|G(x)|_{L(\Xi,H)} \le L$, $\forall x \in H$ *and for a suitable constant* L.

(ii) $G(x)$ *has a bounded inverse:* $\exists B$ *such that* $|G(x)^{-1}|_{L(\Xi,H)} \le B$, $\forall x \in H$.

(iii) $F(\cdot) \in \mathcal{G}^1(H,H)$ *and for every* $t > 0$, $e^{tA}G(\cdot) \in \mathcal{G}^1(H, L_2(\Xi,H))$.

(iv) ψ *is uniformly Lipschitz in* y *with Lipschitz constant* L, *namely:*

$$\forall x \in H,\ \forall y,y' \in \mathbb{R},\ \forall z \in \Xi^*, \quad |\psi(x,y,z)-\psi(x,y',z)| \le L|y-y'|.$$

Under the above assumptions we still have Gâteaux differentiability of Y_0^x:

Theorem 3.2 *Under Hypotheses 3.1, 3.2 and 3.4 the map* $x \to Y_0^x$ *belongs to* $\mathcal{G}^1(H,\mathbb{R})$. *Moreover* $|Y_0^x| + |\nabla_x Y_0^x| \le c$ *for a suitable constant* c.

Proof. Let $u(x) = Y_0^x$. By Proposition 3.2, u is continuous and bounded. We need to consider the following equation, slightly more general than (3.1):

$$X_\tau = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}F(X_\sigma)\,d\sigma + \int_t^\tau e^{(\tau-\sigma)A}G(X_\sigma)\,dW_\sigma, \qquad (3.18)$$

for τ varying on an arbitrary time interval $[t,\infty) \subset [0,\infty)$. We set $X_\tau = x$ for $\tau \in [0,t)$ and we denote by $\{X_\tau^{t,x},\ \tau \ge 0\}$ the solution to indicate dependence on x and t.

Since the solution to equation (3.18) is unique and the coefficients of the equation (3.18) do not depend on time we have that the distribution of $X^x = X^{0,x}$ and the distribution of $X_{(t+\cdot)}^{t,x}$ in $C(0,\infty;H)$ are equal. Moreover $X_s^{0,x} = X_s^{t,X_t^{0,x}}$, $\mathbb{P}$-a.s. for every $s \ge t$. Since the (unique) bounded solution of the backward equation (3.5) is uniquely determined on an interval $[s,\infty)$ we immediately deduce that:

$$u(X_\sigma^x) = Y_\sigma^x, \quad \sigma \ge 0.$$

Using the above relation on a fixed $T > 0$ we deduce that the solution (Y^x, Z^x) of equation (3.5) is also the unique solution of the finite horizon BSDE:

$$Y_\tau^x = u(X_T^x) + \int_\tau^T (\psi(X_\sigma^x, Y_\sigma^x, Z_\sigma^x) - \lambda Y_\sigma^x) d\sigma - \int_\tau^T Z_\sigma^x dW_\sigma.$$

Then the claim immediately follows by Theorem 4.2 in [10]. □

4 Mild solution of the elliptic PDE

Now we can proceed as in [11]. We briefly report here definition and main result. Assuming that Hypothesis 3.1 holds, we define in the usual way the transition semigroup $(P_t)_{t\geq 0}$, associated to the process X:

$$P_t[\phi](x) = \mathbb{E}\, \phi(X_t^x), \qquad x \in H, \tag{4.1}$$

for every bounded measurable function $\phi : H \to \mathbb{R}$. Formally, the generator $\mathcal{L}$ of (P_t) is the operator

$$\mathcal{L}\phi(x) = \frac{1}{2}\text{Trace}\left(G(x)G(x)^* \nabla^2 \phi(x)\right) + \langle Ax + F(x), \nabla\phi(x)\rangle.$$

In this section we address solvability of the non linear stationary Kolmogorov equation:

$$\mathcal{L}u(x) - \lambda\, u(x) + \psi(x, u(x), \nabla u(x)\, G(x)) = 0, \qquad x \in H. \tag{4.2}$$

Note that, for $x \in H$, $\nabla u(x)$ belongs to H^*, so that $\nabla u(x)\, G(x)$ is in Ξ^*.

Definition 4.1 We say that a function $u : H \to \mathbb{R}$ is a mild solution of the non linear stationary Kolmogorov equation (4.2) if the following conditions hold:

(i) $u \in \mathcal{G}^1(H, \mathbb{R})$ and $\exists\, C > 0$ such that $|u(x)| \leq C$, $|\nabla_x u(x)h| \leq C\,|h|$, for all $x, h \in H$;

(ii) the following equality holds, for every $x \in H$ and $T \geq 0$:

$$u(x) = e^{-\lambda T}\; P_T[u](x) + \int_0^T e^{-\lambda\tau}\; P_\tau\Big[\psi\Big(\cdot, u(\cdot), \nabla u(\cdot)\, G(\cdot)\Big)\Big](x)\; d\tau. \tag{4.3}$$

Theorem 4.1 *Assume that Hypothesis 3.1, Hypothesis 3.2 and either Hypothesis 3.3 or Hypothesis 3.4 hold then equation (4.2) has a unique mild solution given by the formula*

$$u(x) = Y_0^x. \tag{4.4}$$

Moreover the following holds:

$$Y_\tau^x = u(X_\tau^x), \quad Z_\tau^x = \nabla u(X_\tau^x)\, G(X_\tau^x). \tag{4.5}$$

Proof. The proof is identical to the proof of Theorem 6.1 in [11]. We only notice that here thanks to Proposition 3.2 we can, for all $\lambda > 0$, construct the candidate bounded solution u. Moreover by Theorem 3.1 (in the degenerate case) or by Theorem 3.2 (in the non-degenerate case) we know in advance, again for all $\lambda > 0$, that $u \in \mathcal{G}^1(H)$ and has bounded derivative. □

Remark 4.1 In [11], even assuming boundedness of the nonlinearity ψ, authors were able to prove existence and uniqueness of a mild solution to equation (4.2) only for λ larger than $\frac{1}{2}(\mathrm{lip}_Z(\psi))^2$ where $\mathrm{lip}_Z(\psi) = \sup_{z,z' \in \Xi^*} |\psi(z) - \psi(z')|/|z - z'|$ is the optimal Lipschitz constant of ψ with respect to z, see Remark 5.2 in [11].

5 Application to optimal control

We wish to apply the above results to perform the synthesis of the optimal control for a general nonlinear control system on an infinite time horizon. To be able to use non-smooth feedbacks we settle the problem in the framework of weak control. Again we follow [11] with slight modifications. We report the argument for reader's convenience.

As above by H, Ξ we denote separable real Hilbert spaces and by U we denote a Banach space.

For fixed $x_0 \in H$ an *admissible control system* (a.c.s) is given by $(\Omega, \mathcal{E}, (F_t)_{t \geq 0}, \mathbb{P}, \{W_t, t \geq 0\}, u)$ where

- $(\Omega, \mathcal{E}, \mathbb{P})$ is a complete probability space and $(\mathcal{F}_t)_{t \geq 0}$ is a filtration on it satisfying the usual conditions.
- $\{W_t : t \geq 0\}$ is a Ξ-valued cylindrical Wiener process relatively to the filtration $(\mathcal{F}_t)$ and the probability $\mathbb{P}$.
- $u : \Omega \times [0, \infty[\to U$ is a predictable process (relatively to $(F_t)_{t \geq 0}$) that satisfies the constraint: $u_t \in \mathcal{U}$, $\mathbb{P}$-a.s. for a.e. $t \geq 0$, where $\mathcal{U}$ is a fixed closed subset of U.

To each a.c.s. we associate the mild solution $X^u \in L^r_{\mathcal{P}}(\Omega; C(0,T;H))$ (for arbitrary $T > 0$ and arbitrary $r \geq 1$) of the state equation:

$$\begin{cases} dX^u_\tau = (AX^u_\tau + F(X^u_\tau) + G(X^u_\tau)R(u_\tau))\ d\tau + G(X_\tau)\ dW_\tau, & \tau \geq 0, \\ X^u_0 = x_0 \in H, \end{cases} \tag{5.1}$$

and the cost:

$$J(x_0, u) = \mathbb{E} \int_0^{+\infty} e^{-\lambda\sigma} g(X^u_\sigma, u_\sigma)\ d\sigma, \tag{5.2}$$

where $g : H \times U \to \mathbb{R}$. Our purpose is to minimize the functional J over all a.c.s. Notice the occurrence of the operator G in the control term: this special structure of the state equation is imposed by our techniques.

We define in a classical way the Hamiltonian function relative to the above problem: for all $x \in H$, $z \in \Xi^*$,

$$\begin{aligned} &\psi(x,z) = \inf\{g(x,u) + zR(u) : u \in \mathcal{U}\} \\ &\Gamma(x,z) = \{u \in \mathcal{U} : g(x,u) + zu = \psi(x,z)\}. \end{aligned} \tag{5.3}$$

We will work under the following general setting:

Hypothesis 5.1 *The following holds:*

1. *A, F, G and ψ verify Hypothesis 3.1, Hypothesis 3.2 and either Hypothesis 3.3 or Hypothesis 3.4.*
2. *$R : \mathcal{U} \to \Xi$ is bounded.*
3. *$g : H \times U \to \mathbb{R}$ is continuous and bounded.*

Example 5.1 If $U = \Xi$, $\mathcal{U}$ is the ball $\{v \in U : |v|_\Xi \le r\}$ for some fixed $r > 0$, $R = \mathrm{Id}$ and $g(x,u)g_0(|u|^\alpha)+g_1(x)$ with $g_0 \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ convex, $g_0'(0) > 0$, $\alpha > 1$, $g_1 \in \mathcal{G}^1(H,\mathbb{R})$ with $|\nabla g_1(x)h| \le L|h|$ for suitable constant $L > 0$, and all $x, h \in H$ then by easy computations ψ is in $\mathcal{G}^1(H,\Xi^*)$ and is uniformly Lipschitz both in x and z (thus ψ verifies Hypothesis 3.4). Moreover $\Gamma(x,z) = \{-\nabla_z\psi(x,z)\}$ turns out to be always a singleton and a continuous function of z only.

We notice that for all $\lambda > 0$ the cost functional is well defined and $J(x_0,u) < \infty$ for all $x_0 \in H$ and all a.c.s.

By Theorem 4.1, for all $\lambda > 0$ the stationary Hamilton-Jacobi-Bellman equation relative to the above stated problem, namely:

$$\mathcal{L}v(x) = \lambda v(x) - \psi(x, \nabla v(x)G(x)), \qquad x \in H, \tag{5.4}$$

admits a unique mild solution, in the sense of Definition 4.1.

We are in a position to prove the main result of this section following [11]:

Theorem 5.1 *Assume Hypothesis 5.1 and suppose that $\lambda > 0$. Then the following holds*

1. *For all a.c.s. we have $J(x_0,u) \ge v(x_0)$.*
2. *The equality holds if and only if the following feedback law is verified by u and X^u:*

$$u_\tau \in \Gamma(X^u_\tau, \nabla v(X^u_\tau)G(X^u_\tau)), \quad \mathbb{P}-\text{a.s. for a.e. } \tau \ge 0. \tag{5.5}$$

3. *If $\Gamma(x,z)$ is non empty for all $x \in H$ and $z \in \Xi^*$ and $\gamma : H \times \Xi^* \to \mathcal{U}$ is a measurable selection of Γ (that always exists by the Filippov Theorem, see*

[1]) then there exists an a.c.s. for which the closed loop equation

$$\begin{cases} d\overline{X}_\tau &= A\overline{X}_\tau\, d\tau + G(\overline{X}_\tau)R(\gamma(\overline{X}_\tau, \nabla v(\overline{X}_\tau)G(\overline{X}_\tau))\, d\tau \\ & \quad +F(\overline{X}_\tau)\, d\tau + G(\overline{X}_\tau)\, dW_\tau,\ \tau \geq 0, \\ \overline{X}_0 &= x_0 \in H, \end{cases} \tag{5.6}$$

admits a solution. Moreover setting $\overline{u}_\tau = \gamma(\overline{X}_\tau, \nabla v(\overline{X}_\tau)G(\overline{X}_\tau))$ *then the couple* $(\overline{u}, \overline{X})$ *is optimal for the control problem.*

Proof. Denote by $\rho(T)$ the Girsanov density

$$\rho(T) = \exp\left(-\int_0^T \langle R(u_\sigma), dW_\sigma \rangle_\Xi - \frac{1}{2}\int_0^T |R(u_\sigma)|^2_\Xi\, d\sigma \right), \tag{5.7}$$

and let $\widetilde{\mathbb{P}}_T$ be the probability measure on $\mathcal{F}_T$ defined by $\widetilde{\mathbb{P}}_T = \rho(T)\ \mathbb{P}\Big|_{\mathcal{F}_T}$ and let $\tilde{\mathbb{E}}_T$ be the corresponding expectation. We notice that under $\widetilde{\mathbb{P}}_T$ the process

$$\widetilde{W}_\tau := \int_0^\tau R(u_\sigma)\, d\sigma + W_\tau, \qquad 0 \leq \tau \leq T, \tag{5.8}$$

is a cylindrical Wiener process. Relatively to $\widetilde{W}$ equation (5.1) can be written:

$$\begin{cases} dX^u_\tau = AX^u_\tau\, d\tau + F(X^u_\tau)\, d\tau + G(X_\tau)\, d\widetilde{W}_\tau, \qquad \tau \geq 0, \\ X_0 = x_0. \end{cases} \tag{5.9}$$

Let v be the unique mild solution of equation (5.4). Consider the following finite horizon Markovian forward-backward system (with respect to probability $\widetilde{P}_T$ and to the filtration generated by $\{\widetilde{W}_\tau : \tau \in [0,T]\}$).

$$\begin{cases} \widetilde{X}_\tau(x) = e^{\tau A}x + \displaystyle\int_0^\tau e^{(\tau-\sigma)A}F(\widetilde{X}_\sigma(x))\, d\sigma + \int_0^\tau e^{(\tau-\sigma)A}G(\widetilde{X}_\sigma(x))\, d\widetilde{W}_\sigma,\ \tau \geq 0, \\ \widetilde{Y}_\tau(x) - v(\widetilde{X}_T(x)) + \displaystyle\int_\tau^T \widetilde{Z}_\sigma(x)\, d\widetilde{W}_\sigma + \lambda \int_\tau^T \widetilde{Y}_\sigma(x)\, d\sigma \\ \qquad\qquad = \displaystyle\int_\tau^T \psi(\widetilde{X}_\sigma(x), \widetilde{Z}_\sigma(x))\, d\sigma,\ 0 \leq \tau \leq T, \end{cases} \tag{5.10}$$

and let $(\widetilde{X}(x), \widetilde{Y}(x), \widetilde{Z}(x))$ be its unique solution with the three processes predictable relatively to the filtration generated by $\{\widetilde{W}_\tau : \tau \in [0,T]\}$ and: $\tilde{\mathbb{E}}_T \sup_{t\in[0,T]} |\widetilde{X}_t(x)|^2 < +\infty$, $\widetilde{Y}(x)$ bounded and continuous, $\tilde{\mathbb{E}}_T \int_0^T |\widetilde{Z}_t(x)|^2 dt < +\infty$. Moreover, Theorem 4.1 and uniqueness of the solution of system (5.10), see [9], yields that

$$\widetilde{Y}_\tau(x) = v(\widetilde{X}_\tau(x)), \qquad \widetilde{Z}_\tau(x) = \nabla v(\widetilde{X}_\tau(x))G(\widetilde{X}_\tau(x)). \tag{5.11}$$

Comparing the forward equation in (5.10) with the state equation, rewritten as (5.9), and choosing $x = x_0$ we get $\widetilde{X}_t(x_0) = X_t^u$, $t \in [0,T]$, $\mathbb{P}$-a.s. Applying the Itô formula to $e^{-\lambda\tau}\widetilde{Y}_\tau(x_0)$, and restoring the original noise W we get

$$\begin{aligned}\widetilde{Y}_0(x_0) &+ \int_0^T e^{-\lambda\sigma}\widetilde{Z}_\sigma(x_0)\,dW_\sigma \\ &= \int_0^T e^{-\lambda\sigma}\left[\psi(X_\sigma^u, \widetilde{Z}_\sigma(x_0)) - \widetilde{Z}_\sigma(x_0)R(u_\sigma)\right]\,d\sigma + e^{-\lambda T}v(\widetilde{X}_T(x_0)).\end{aligned} \tag{5.12}$$

Using the identification in (5.11) and taking expectation with respect to $\mathbb{P}$, (5.12) yields

$$\begin{aligned}e^{-\lambda T}\mathbb{E}v(\widetilde{X}_T(x_0)) - v(x_0) \;=\; &-\mathbb{E}\int_0^T e^{-\lambda\sigma}\psi(X_\sigma^u, \nabla v(X_\sigma^u)G(X_\sigma^u))\,d\sigma \\ &+\mathbb{E}\int_0^T e^{-\lambda\sigma}\nabla v(X_\sigma^u)G(X_\sigma^u)R(u_\sigma)\,d\sigma.\end{aligned}$$

Recalling that v is bounded, letting $T \to \infty$, we conclude

$$\begin{aligned}J(x_0,u) = v(x_0) - \mathbb{E}\int_0^\infty e^{-\lambda\sigma}\Big[&\psi(X_\sigma^u, \nabla v(X_\sigma^u)G(X_\sigma^u)) \\ &- \nabla_x v(X_\sigma^u)G(X_\sigma^u)R(u_\sigma) - g(X_\sigma^u, u_\sigma)\Big]\,d\sigma.\end{aligned}$$

The above equality is known as the *fundamental relation* and immediately implies that $v(x_0) \le J(x_0,u)$ and that the equality holds if and only if (5.5) holds.

To conclude our argument it remains to prove existence of a weak solution to equation (5.6) in the whole $[0,+\infty[$, see also Section 4 in [8]. In order to do it we realize a "canonical"-Ξ-valued Wiener process. We choose a larger Hilbert space $\Xi' \supset \Xi$ in such a way that Ξ is continuously and densely embedded in Ξ' with Hilbert-Schmidt inclusion operator $\mathcal{J}$. By Ω we denote the space $C([0,\infty[,\Xi')$ of continuous functions $\omega : [0,\infty[\to \Xi'$ endowed with the standard locally convex topology and by $\mathcal{B}$ its Borel σ-field. Since $\mathcal{J}\mathcal{J}^*$ is nuclear on Ξ' we know (see [5]) that there exists a probability $\mathbb{P}$ on $\mathcal{B}$ such that $W_t'(\omega) := \omega(t)$ is a $\mathcal{J}\mathcal{J}^*$-Wiener process in Ξ' (that is $t \to \langle W_t', \xi'\rangle_{\Xi'}$ is a real valued Wiener process for all $\xi' \in \Xi'$ and $\mathbb{E}[\langle W_t', \xi'\rangle_{\Xi'}\langle W_s', \eta'\rangle_{\Xi'}] = \langle \mathcal{J}\mathcal{J}^*\xi', \eta'\rangle_{\Xi'}(t \wedge s)$ for all $\xi', \eta' \in \Xi'$, $t,s \in [0,\infty[$. We denote by $\mathcal{E}$ the $\mathbb{P}$-completion of $\mathcal{B}$ and by $\mathcal{F}_t$, $t \ge 0$, the $\mathbb{P}$-completion of $\mathcal{B}_t = \sigma\{W_s' : s \in [0,t]\}$.

The Ξ-valued cylindrical Wiener process $\{W_t^\xi : t \ge 0, \xi \in \Xi\}$ can now be defined as follows. For ξ in the image of $\mathcal{J}^*\mathcal{J}$ we take η such that $\xi = \mathcal{J}^*\mathcal{J}\eta$ and define $W_s^\xi = \langle W_s', \mathcal{J}\eta\rangle_{\Xi'}$. Then we notice that $\mathbb{E}|W_t^\xi|^2 = t|\mathcal{J}\eta|_{\Xi'}^2 = t|\xi|_\Xi^2$ and that $\mathcal{J}^*\mathcal{J}\Xi$ is dense in Ξ to deduce that the linear continuous mapping $\xi \to W_s^\xi$ (with values in $L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R})$) can be extended by continuity to the whole Ξ. An appropriate modification of $\{W_t^\xi : t \geq 0, \xi \in \Xi\}$ gives the required cylindrical Wiener process.

Now let $X \in L^p_{\mathcal{P},\mathrm{loc}}(\Omega, C(0,+\infty; H))$ be the mild solution of

$$\begin{cases} dX_\tau = AX_\tau \, d\tau + F(X_\tau)\, d\tau + G(X_\tau)\, dW_\tau \\ X_0 = x_0 \end{cases} \tag{5.13}$$

and let, $\forall T > 0$

$$\rho(T) = \exp\Bigg(- \int_0^T \langle R(\gamma(X_\sigma, \nabla v(X_\sigma) G(X_\sigma)), dW_\sigma\rangle_\Xi - \frac{1}{2}\int_0^T |R(\gamma(X_\sigma, \nabla v(X_\sigma)G(X_\sigma))|^2_\Xi \, d\sigma \Bigg). \tag{5.14}$$

Let $\widehat{\mathbb{P}}_T$ the probability on $\mathcal{F}_T$ admitting $\rho(T)$ as a density with respect to $\mathbb{P}$. Since Ξ' is a Polish space and $\widehat{\mathbb{P}}_{T+h}$ coincide with $\widehat{\mathbb{P}}_T$ on $\mathcal{B}_T$, $T, h \geq 0$, by known results (see [18], Chapter VIII, §1, Proposition (1.13)) there exists a probability $\widehat{\mathbb{P}}$ on $\mathcal{B}$ such that the restriction on $\mathcal{B}_T$ of $\widehat{\mathbb{P}}_T$ and that of $\widehat{\mathbb{P}}$ coincide, $T \geq 0$. Let $\widehat{\mathcal{E}}$ be the $\widehat{\mathbb{P}}$-completion of $\mathcal{B}$ and $\widehat{\mathcal{F}}_T$ be the $\widehat{\mathbb{P}}$-completion of $\mathcal{B}_T$. Moreover let

$$\widehat{W}_t := - \int_0^t R(\gamma(X_\sigma, \nabla v(X_\sigma)G(X_\sigma))\, d\sigma + W_t.$$

Since, for all $T > 0$, $\{\widehat{W}_t : t \in [0, T]\}$ is a Ξ-valued cylindrical Wiener process under $\widehat{\mathbb{P}}_T$ and the restriction of $\widehat{\mathbb{P}}_T$ and of $\widehat{\mathbb{P}}$ coincide on $\mathcal{B}_T$ modifying $\{\widehat{W}_t : t \geq 0\}$ in a suitable way on a $\widehat{\mathbb{P}}$-null probability set we can conclude that $(\Omega, \widehat{\mathcal{E}}, \{\widehat{\mathcal{F}}_t, t \geq 0\}, \widehat{\mathbb{P}}, \{\widehat{W}_t, t \geq 0\}, \gamma(X, \nabla v(X)G(X)))$ is an admissible control system. The above construction immediately ensures that, if we choose such an admissible control system, then (5.6) is satisfied. Indeed if we rewrite (5.13) in terms of $\{\widehat{W}_t : t \geq 0\}$ we get

$$\begin{cases} dX_\tau = AX_\tau \, d\tau + F(X_\tau)\, d\tau + G(X_\tau)\, [R(\gamma(X_\tau, \nabla v(X_\tau)G(X_\tau))) + d\widehat{W}_\tau] \\ X_0 = x_0 \end{cases}$$

and this concludes the proof. □

Remark 5.1 To compare the present result and the ones obtained in [11] we notice that, under assumption 5.1, Theorem 5.1 was proved in [11] only for $\lambda \geq \frac{1}{2}\sup_{u \in \mathcal{U}} |R(u)|^2$. On the other side in [11] it was possible to consider more general G and allow dependence of R on x.

Remark 5.2 To understand why Hypothesis 3.3 (iii) is necessary if we want a differentiable value function allowing G to be degenerate and λ to be any positive number consider the simple (one dimensional) deterministic control problem (with trivial control):

$$\begin{cases} X_t' = aX_t \\ X_0 = x_0 \end{cases} ; \qquad J(x_0) = \int_0^\infty e^{-\lambda s} g(X_s) ds,$$

where $a > 0$, $\lambda > 0$, and $g : \mathbb{R} \to \mathbb{R}$ is regular, positive and verifies $g(0) = 0$, $g(x) = 1$ for $|x| \geq 1$. Then by easy computations $J(x_0) \geq \lambda^{-1} x_0^{\lambda/a}$. Since $J(0) = 0$, J is not differentiable in 0 whenever $\lambda < a$.

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