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Symmetric and asymmetric multiple clusters in a reaction-diffusion system

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Abstract. We consider the following Gierer-Meinhardt system in R:

$$\begin{cases} \epsilon^2 A^{''} - A + \frac{A^2}{H} = 0, & x \in (-1,1), \\ DH^{''} - H + A^2 = 0, & x \in (-1,1), \\ A^{'}(-1) = A^{'}(1) = H^{'}(-1) = H^{'}(1) = 0, \end{cases}$$

where $\epsilon > 0$ is a small parameter and $D > 0$ is a constant independent of ϵ.

A **cluster** is a combination of several spikes concentrating at the same point. In this paper, we rigorously show the existence of symmetric and asymmetric multiple clusters. This result is new for systems and seems not to occur for single equations. We reduce the problem to the computation of two matrices which depend on the coefficient D as well as the number of different clusters and the number of spikes within each cluster.

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1 Introduction

Since the work of Turing [21] in 1952, many models have been derived and investigated to explore the so-called Turing instability [21]. One of the most famous models in biological pattern formation is the Gierer-Meinhardt system [10], [15], [16], which in one dimension can be stated as follows:

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^p}{H^q}, \quad x \in (-1,1),\, t > 0, \\ \tau H_t = D\Delta H - H + \frac{A^r}{H^s}, \quad x \in (-1,1),\, t > 0, \\ A^{'}(\pm, t) = H^{'}(\pm, t) = 0, \end{cases} \tag{1.1}$$

where (p, q, r, s) satisfy

$$1 < \frac{qr}{(s+1)(p-1)} < +\infty, \quad 1 < p < +\infty,$$

$$\text{and where} \quad \epsilon << 1, \quad 0 < D < \infty, \quad \tau \geq 0,$$

$$D \text{ and } \tau \text{ are constants which are independent of } \epsilon.$$

In this paper, we consider the steady-state problem of (1.1) and further assume that $(p, q, r, s) = (2, 1, 2, 0)$. Namely we consider the following elliptic system

$$\begin{cases} \epsilon^2 A^{''} - A + \frac{A^2}{H} = 0, \quad x \in (-1,1), \\ DH^{''} - H + A^2 = 0, \quad x \in (-1,1), \\ A(x) > 0, H(x) > 0, \quad x \in (-1,1), \\ A^{'}(-1) = A^{'}(1) = H^{'}(-1) = H^{'}(1) = 0. \end{cases} \tag{1.2}$$

We remark that our results for (1.2) can be easily generalized to more general (p, q, r, s) cases. The main difficulty in studying (1.2) is that there is **no** variational structure. On the other hand, (1.2) represents a typical activator-inhibitor in biological pattern formation.

Problem (1.2) has been studied by numerous authors. Let us mention several important existence results on multiple spike (also called multiple peak) solutions which are related to our present paper.

1) **(Existence of symmetric $N-$peaked steady-state Solutions)**

I. Takagi [20] first established the existence of N-peaked steady-state solutions with peaks centered at

$$x_j = -1 + \frac{2j-1}{N}, \quad j = 1, \ldots, N,$$

for $\epsilon << 1$, $\frac{\epsilon}{\sqrt{D}} << 1$.

Such solutions are symmetric and they are obtained from a single spike by reflection. We call them **symmetric** $N-$peaked solutions since all the peaks have the same heights. Takagi's proof is based on symmetry and the implicit function theorem.

2) (**Existence of asymmetric** $N-$**peaked solutions**)

By using matched asymptotic analysis, M. Ward and the first author in [22] showed by asymptotic expansions that for $D < D_N$, where D_N is given explicitly, problem (1.2) has **asymmetric** $N-$peaked steady-state solutions. Such asymmetric solutions are generated by two types of peaks – called type **A** and type **B**, respectively. Type **A** and type **B** peaks have *different heights.* They can be arranged in any given order

$$\mathbf{ABAABBB} \ldots \mathbf{ABBBA} \ldots \mathbf{B}$$

to form an $N-$peaked solution. The existence of such solutions is surprising. It shows that the solution structure of (1.2) is much more complicated than one would first expect. The stability of such asymmetric $N-$peaked solutions is also studied in [22], through a formal approach. The stability issue of symmetric and asymmetric N-peaked solutions is addressed in [13] and [22].

We remark that asymmetric patterns can also be obtained for the Gierer-Meinhardt system on the real line by a dynamical systems approach, see [7].

In [26], we gave a rigorous and unified theoretic foundation for the existence and stability of general $N-$peaked (symmetric or asymmetric) solutions. In particular, the results of [13] and [22] were rigorously established. Moreover, it was shown that if the N peaks are separated, then they are generated by peaks of type **A** and type **B**, respectively. This implies that there are only two kinds of N-peaked patterns: the symmetric $N-$peaked solutions constructed in [20] and the asymmetric $N-$peaked patterns constructed in [22].

3) (**Existence of a single cluster on the real line**)

Doelman, Kaper and H. van der Ploeg, [7], and independently Chen, del Pino and M. Kowalczyk [2] considered the Gierer-Meinhardt system on the real line. They constructed multiple-spike solutions concentrating at a single point on the real line. It turns out that the distance between neighbouring spikes is of the order $O(\epsilon \log \frac{1}{\epsilon})$. We call such solution a single cluster. In other words, a cluster is a collection of **multiple** spikes concentrating at a single point.

Similar results in R^2 were obtained in [3]. There the geometry of the spike locations can be very complex.

The existence of a single cluster or multiple clusters in a higher dimensional bounded domain has been proved in [12], [4] for a singularly perturbed Neumann problem. It is proved that given nondegenerate local minimum points of the mean curvature of the boundary there exist (multiple) clusters concentrating at these point(s). In [14] for the nonlinear Schrodinger equation an analogous result is proved for (nondegenerate local) maximum points of the potential. To obtain multiple clusters for single equations, we must either have nontrivial geometry of the domain or nontrivial critical points of the potential.

The results in this paper imply that a reaction diffusion system can generate multiple clusters even when the domain is **trivial** and in the absence of a potential. Moreover, we will show that there are both **symmetric** and **asymmetric** multiple clusters. The locations of these clusters are determined by three ingredients: the number of clusters, the number of spikes within each cluster, and the order of clusters.

Before we state our main results in Section 2, we introduce some notation. Let $L^2(-1,1)$ and $H^2(-1,1)$ be the usual Lebesgue and Sobolev spaces. With the variable w we denote the unique solution of the following problem:

$$\begin{cases} w'' - w + w^2 = 0, \quad y \in R, \\ w > 0, \quad w(0) = \max_{y\in R} w(y), \\ w(y) \to 0 \quad \text{as } |y| \to \infty. \end{cases} \tag{1.3}$$

In fact, it is easy to see that $w(y)$ can be written explicitly:

$$w(y) = \frac{3}{2}\text{sech}^2\left(\frac{y}{2}\right). \tag{1.4}$$

Let

$$I := (-1,1). \tag{1.5}$$

For $z \in (-1,1)$, let $G_D(x,z)$ be the Green function given by

$$\begin{cases} DG''_D(x,z) - G_D(x,z) + \delta_z(x) = 0, \quad x \in (-1,1), \\ G'_D(-1,z) = G'_D(1,z) = 0. \end{cases} \tag{1.6}$$

We can calculate explicitly

$$G_D(x,z) = \begin{cases} \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1+x)]\cosh[\theta(1-z)], & -1 < x < z, \\ \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1-x)]\cosh[\theta(1+z)], & z < x < 1, \end{cases} \tag{1.7}$$

where

$$\theta = D^{-1/2}. \tag{1.8}$$

We decompose $G_D(x,z)$ into a singular part and a regular part:

$$G_D(x,z) = K_D(|x-z|) - H_D(x,z), \tag{1.9}$$

where

$$K_D(|x-z|) = \frac{1}{2\sqrt{D}} e^{-\frac{1}{\sqrt{D}}|x-z|} \tag{1.10}$$

is the singular part of $G_D(x,z)$ and H_D is the regular part H_D of G_D. Note that H_D is C^∞ in both x and z. Moreover,

$$H_D(x,x) = \frac{1}{2\sqrt{D}} - \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1+x)] \cosh[\theta(1-x)]. \tag{1.11}$$

We use the notation $e.s.t$ to denote an exponentially small term of order the $O(e^{-d/\epsilon})$ for some $d > 0$ in the corresponding norm. By C we denote a generic constant which may change from line to line.

This paper has the following structure: In Section 2 we introduce our three main hypotheses, (H1)–(H3) and state our two main results, Theorem 2.1 and Theorem 2.2. In Section 3, we provide some preliminary results. In Sections 4–6, we construct suitable approximate solutions and give some calulcations for them, namely about the space dependence of the heights (Section 5) and the error terms (Section 6). In Sections 7–9, we prove the existence of multiple-clustered solutions: In Section 7, we use the Liapunov-Schmidt method to reduce the existence of solutions to (1.2) to a finite dimensional problem; in Section 8 we solve this finite-dimensional problem and complete the proof of Theorem 2.2. In Section 9, we prove Theorem 2.1.

2 Main Results: Existence of Symmetric and Asymmetric Multiple Clusters

Let $-1 < x_1^0 < \cdots < x_j^0 < \cdots < x_N^0 < 1$ be N points in $(-1,1)$ and let w be the unique solution of (1.3).

We introduce several matrices for later use: For $\mathbf{x} = (x_1, \ldots, x_N) \in (-1,1)^N$, let

$$\mathcal{G}_D(\mathbf{x}) = (G_D(x_i, x_j)). \tag{2.1}$$

Recall that

$$G_D(x_i, x_j) = K_D(|x_i - x_j|) - H_D(x_i, x_j).$$

Let us denote $\frac{\partial}{\partial x_i}$ as ∇_{x_i}. When $i \neq j$, we can define $\nabla_{x_i} G(x_i, x_j)$ in the classical way. When $i = j$, $K_D(|x_i - x_j|) = K_D(0) = \frac{1}{2\sqrt{D}}$ is a constant and we define

$$\nabla_{x_i} G_D(x_i, x_i) := -\frac{1}{2} \left.\frac{d}{dx}\right|_{x=x_i} H_D(x,x).$$

Similarly, we define

$$\nabla_{x_i} \nabla_{x_j} G_D(x_i, x_j) = \begin{cases} -\frac{1}{2} \frac{d}{dx}|_{x=x_i} \frac{\partial}{\partial x}|_{x=x_i} H_D(x,x), & \text{if } \ i = j, \\ \nabla_{x_i} \nabla_{x_j} G_D(x_i, x_j), & \text{if } \ i \neq j. \end{cases} \tag{2.2}$$

Now the derivatives of the matrix $\mathcal{G}_D$ are defined as follows:

$$\nabla \mathcal{G}_D(\mathbf{x}) = (\nabla_{x_i} G_D(x_i, x_j)), \tag{2.3}$$

$$\nabla^2 \mathcal{G}_D(\mathbf{x}) = (\nabla_{x_i} \nabla_{x_j} G_D(x_i, x_j)). \tag{2.4}$$

By definition, it is easy to compute that

$$\mathcal{G}_D = \frac{\theta}{\sinh(2\theta)}(a_{ij}),\ \nabla \mathcal{G}_D = \frac{\theta^2}{\sinh(2\theta)}(b_{ij}),\ \nabla^2 \mathcal{G}_D = \frac{\theta^3}{\sinh(2\theta)}(c_{ij}),$$

where

$$a_{ij} = \begin{cases} \cosh(\theta(1+x_i))\cosh(\theta(1-x_j)), \text{ if } \ i \le j; \\ \cosh(\theta(1-x_i))\cosh(\theta(1+x_j)), \text{ if } \ i > j, \end{cases} \tag{2.5}$$

$$b_{ij} = \begin{cases} \sinh(\theta(1+x_i))\cosh(\theta(1-x_j)), \text{ if } \ i < j; \\ \frac{1}{2}\sinh(2\theta x_i), \text{ if } i = j; \\ -\sinh(\theta(1-x_i))\cosh(\theta(1+x_j)), \text{ if } \ i > j, \end{cases} \tag{2.6}$$

and

$$c_{ij} = \begin{cases} -\sinh(\theta(1+x_i))\sinh(\theta(1-x_j)), \text{ if } \ i < j; \\ \cosh(2\theta x_i), \quad \text{if } \ i = j; \\ -\sinh(\theta(1-x_i))\sinh(\theta(1+x_j)), \text{ if } \ i > j. \end{cases} \tag{2.7}$$

We now have our first assumption:
(H1) There exists a solution $(\hat{\xi}_1^0, \ldots, \hat{\xi}_N^0)$ of the following equation

$$\sum_{j=1}^{N} G_D(x_m^0, x_j^0) n_j (\hat{\xi}_j^0)^2 = \hat{\xi}_m^0, \quad m = 1, \ldots, N. \tag{2.8}$$

Next we introduce the following matrix

$$b_{ij} = G_D(x_i^0, x_j^0) n_j (\hat{\xi}_j^0), \quad \mathcal{B} = (b_{ij}). \tag{2.9}$$

Our second assumption is the following:
(H2) It holds that

$$\frac{1}{2} \notin \sigma(\mathcal{B}), \tag{2.10}$$

where $\sigma(\mathcal{B})$ is the set of eigenvalues of $\mathcal{B}$.

Remark 2.1 Since the matrix $\mathcal{B}$ is of the form $\mathcal{G}_D\mathcal{D}$, where $\mathcal{G}_D$ is symmetric and $\mathcal{D}$ is a diagonal matrix, it is easy to see that the eigenvalues of $\mathcal{B}$ are real.

By the assumption (H2) and the implicit function theorem, for $\mathbf{x} = (x_1, \dots, x_N)$ near $\mathbf{x}_0 = (x_1^0, \dots, x_N^0)$, there exists a unique solution $\hat{\xi}(\mathbf{x}) = (\hat{\xi}_1(\mathbf{x}), \dots, \hat{\xi}_N(\mathbf{x}))$ for the following equation

$$\sum_{j=1}^{N} G_D(x_i, x_j) n_j \hat{\xi}_j^{\,2} = \hat{\xi}_i, \quad i = 1, \dots, N. \tag{2.11}$$

Set

$$\mathcal{H}(\mathbf{x}) = (\hat{\xi}_i(\mathbf{x}) \delta_{ij}), \tag{2.12}$$

$$\mathcal{N} = (n_i \delta_{ij}). \tag{2.13}$$

We define the following vector field:

$$F(\mathbf{x}) = (F_1(\mathbf{x}), \dots, F_N(\mathbf{x})),$$

where

$$F_i(\mathbf{x}) = \sum_{l=1}^{N} \nabla_{x_i} G_D(x_i, x_l) n_l \hat{\xi}_l^2 \tag{2.14}$$

$$= -\nabla_{x_i} H_D(x_i, x_i) n_i \hat{\xi}_i^2 + \sum_{l \neq i} \nabla_{x_i} G_D(x_i, x_l) n_l \hat{\xi}_l^2, \quad i = 1, \dots, N.$$

Set

$$\mathcal{M}(\mathbf{x}) = (\nabla_{x_j} F_i(\mathbf{x})). \tag{2.15}$$

Our final assumption concerns the vector field $F(\mathbf{x})$.
(H3) We assume that at $\mathbf{x}^0 = (x_1^0, \dots, x_N^0)$:

$$F(\mathbf{x}^0) = 0, \tag{2.16}$$

$$\det\,(\mathcal{M}(\mathbf{x}^0)) \neq 0. \tag{2.17}$$

Let us now calculate $\mathcal{M}(\mathbf{x}^0)$: Therefore we first compute the derivatives of $\hat{\xi}$. It is easy to see that $\hat{\xi}(\mathbf{x})$ is C^1 in $\mathbf{x}$ and from (2.8) we can calculate:

$$\nabla_{x_j} \hat{\xi}_i = 2 \sum_{l=1}^{N} G_D(x_i, x_l) n_l \hat{\xi}_l \nabla_{x_j} \hat{\xi}_l + \sum_{l=1}^{N} \nabla_{x_j} (G_D(x_i, x_l)) n_l \hat{\xi}_l^2.$$

For $i \neq j$, we have

$$\nabla_{x_j}\hat{\xi}_i = 2\sum_{l=1}^{N} G_D(x_i, x_l) n_l \hat{\xi}_l \nabla_{x_j}\hat{\xi}_l + \nabla_{x_j}(G_D(x_i, x_j)) n_j \hat{\xi}_j^2.$$

For $i = j$, we have

$$\nabla_{x_i}\hat{\xi}_i = 2\sum_{l=1}^{N} G_D(x_i, x_l) n_l \hat{\xi}_l \nabla_{x_i}\hat{\xi}_l + \sum_{l=1}^{N} \frac{\partial}{\partial x_i}(G_D(x_i, x_l)) n_l \hat{\xi}_l^2$$

$$= 2\sum_{l=1}^{N} G_D(x_i, x_l) n_l \hat{\xi}_l \nabla_{x_i}\hat{\xi}_l + \nabla_{x_i}(G_D(x_i, x_i)) n_i \hat{\xi}_i^2 + \sum_{l=i}^{N} \nabla_{x_i}(G_D(x_i, x_l)) n_l \hat{\xi}_l^2,$$

since $\frac{1}{2}\frac{d}{dx_i}G_D(x_i, x_i) = \nabla_{x_i} G_D(x_i, x_i)$.

Note that

$$(\nabla_{x_j} G_D(x_i, x_j)) = (\nabla \mathcal{G}_D)^T.$$

Therefore, if we denote

$$\nabla \xi = (\nabla_{x_j}\hat{\xi}_i), \tag{2.18}$$

then we have

$$\nabla \xi(\mathbf{x}) = (id - 2\mathcal{G}_D \mathcal{N}\mathcal{H})^{-1}[(\nabla \mathcal{G}_D)^T \mathcal{N}\mathcal{H}^2 + (\nabla \mathcal{G}_D) E \mathcal{N}\mathcal{H}^2], \tag{2.19}$$

where id is the identity matrix and E is the matrix whose elements are all equal to 1.

We can compute $\mathcal{M}(\mathbf{x}^0)$ by using (2.19) and definition (2.2):

$$\mathcal{M}(\mathbf{x}^0) = \nabla^2 \mathcal{G}_D \mathcal{N}\mathcal{H}^2 \tag{2.20}$$

$$+2\nabla \mathcal{G}_D \mathcal{N}\mathcal{H}(id - 2\mathcal{G}_D \mathcal{N}\mathcal{H})^{-1}[(\nabla \mathcal{G}_D)^T \mathcal{N}\mathcal{H}^2 + (\nabla \mathcal{G}_D) E \mathcal{N}\mathcal{H}^2].$$

Our first result is about the existence of **symmetric** multiple cluster solution which generalizes the results of I. Takagi [20].

Theorem 2.1 *(Existence of symmetric multiple clusters)*
Let N and n be two positive integers and

$$x_j^0 = -1 + \frac{2j-1}{N}, \quad j = 1, \ldots, N.$$

Then, for $\epsilon << 1$, problem (1.2) has a solution with N equidistant clusters which concentrate at $x_1^0, \dots, x_N^0$ and each of which consists of n spikes. More precisely, it can be said that

$$A_\epsilon(x) \sim \sum_{j=1}^{N} \sum_{k=1}^{n} \xi_\epsilon \hat{\xi}^0 w\left(\frac{x - x_{j,k}^\epsilon}{\epsilon}\right), \tag{2.21}$$

$$H_\epsilon(x_{j,k}^\epsilon) \sim \xi_\epsilon \hat{\xi}^0, \quad j = 1, \dots, N,\ k = 1, \dots, n, \tag{2.22}$$

$$x_{j,k}^\epsilon \to x_j^0, \quad j = 1, \dots, N,\ k = 1, \dots, n, \tag{2.23}$$

where

$$\xi_\epsilon := \left(\epsilon \int_R w^2(z)\, dz\right)^{-1}. \tag{2.24}$$

Furthermore,

$$x_{j,s}^\epsilon - x_{j,s-1}^\epsilon = \epsilon \log \frac{1}{\epsilon} - \epsilon \log[\frac{\hat{\xi}_0}{2D}(s-1)(n+1-s)] + o(\epsilon), \tag{2.25}$$

$j = 1, \dots, N,\ s = 2, \dots, n,$ *and*

$$\hat{\xi}^0 = \frac{2tanh\ \frac{\theta}{N}}{n\theta}. \tag{2.26}$$

Remark 2.2 If $n = 1$, this recovers the results of [20]. Theorem 2.1 also generalizes the results of [2] and [7] to a bounded interval.

Our next result concerns the existence of **asymmetric** multiple clusters.

Theorem 2.2 *(Existence of asymmetric multiple clusters)*

Let $N, n_1, \dots, n_N$ be $N + 1$ positive integers.

Assume that for $(x_1^0, \dots, x_N^0) \in (-1, 1)^N$ with $x_1^0 < x_1^0 < \dots < x_N^0$ assumptions (H1), (H2) and (H3) are satisfied. Let $(\xi_1^0, \dots, \xi_N^0)$ be given by (H1). Then for $\epsilon << 1$, problem (1.2) has a solution with N clusters which concentrate at $x_1^\epsilon, \dots, x_N^\epsilon$, or more precisely:

$$A_\epsilon(x) \sim \sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_\epsilon \hat{\xi}_j^0 w\left(\frac{x - x_{j,k}^\epsilon}{\epsilon}\right), \tag{2.27}$$

$$H_\epsilon(x_{j,k}^\epsilon) \sim \xi_\epsilon \hat{\xi}_j^0, \quad j = 1, \dots, N,\ k = 1, \dots, n_j, \tag{2.28}$$

$$x_{j,k}^\epsilon \to x_j^0, \quad j = 1, \dots, N,\ k = 1, \dots, n_j, \tag{2.29}$$

$$x_{j,s}^\epsilon - x_{j,s-1}^\epsilon = \epsilon \log \frac{1}{\epsilon} - \epsilon \log\left[\frac{\hat{\xi}_j^0}{2D}(s-1)(n_j+1-s)\right] + o(\epsilon), \tag{2.30}$$

$j = 1, \dots, N,\ s = 2, \dots, n_j.$

Remark 2.3 Equation (2.30) expresses the fact that we have **two different** scalings in the spike locations: the distance between the centers of clusters which is of the order $O(1)$ and the distance between spikes within each cluster which is of the order $O(\epsilon \log \frac{1}{\epsilon})$.

Let us now comment on how to check assumptions (H1)–(H3).

It is difficult to check (H1) directly. Instead, we note that $\mathcal{G}_D^{-1}$ is a tridiagonal matrix. (See [13] and [22].) More precisely, we calculate

$$\mathcal{G}_D^{-1} = (g_{ij}) = 2\sqrt{D} \begin{pmatrix} \gamma_1 & \beta_1 & 0 & & \ddots & \ddots & 0 \\ \beta_1 & \gamma_2 & \beta_2 & & \ddots & \ddots & \\ \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots \\ \ddots & \ddots & \beta_{j-1} & & \gamma_j & \beta_j & 0 \\ \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & & 0 & \beta_{N-1} & \gamma_N \end{pmatrix},$$

where

$$\gamma_1 = \coth(\theta_1 + \theta_2) + \tanh(\theta_1),$$

$$\gamma_j = \coth(\theta_{j-1} + \theta_j) + \coth(\theta_j + \theta_{j+1}), \quad j = 2, \ldots, N-1,$$

$$\gamma_N = \coth(\theta_{N-1} + \theta_N) + \tanh(\theta_N),$$

$$\beta_j = -\mathrm{csch}(\theta_j + \theta_{j+1}), \quad j = 1, \ldots, N-1$$

and θ_j is given by

$$\theta_j = \theta\left(x_j^0 - x_{j-1}^0\right). \tag{2.31}$$

(Recall that θ was defined in (1.8).)

Note that

$$g_{ij} = 2\sqrt{D}(\beta_{j-1}\delta_{i(j-1)} + \gamma_j\delta_{ij} + \beta_j\delta_{i(j+1)}). \tag{2.32}$$

Verifying (2.8) amounts to checking the following identity

$$\sum_{j=1}^{N} g_{ij}\hat{\xi}_j^0 = n_i(\hat{\xi}_i^0)^2, \quad i = 1, \ldots, N, \tag{2.33}$$

which is an easy exercise.

Condition (2.16) prescribes the locations $\mathbf{x}_0 = (x_1^0, \dots, x_N^0)$ of the clusters. Condition (2.17) is a nondegeneracy condition. Combining (2.8) and (2.16), we see that at $\mathbf{x}_0$ we must solve the following ODE:

$$\begin{cases} Dh^{''}(x) + h(x) + \sum_{j=1}^N n_j (\xi_j^0)^2 \delta_{x_j^0} = 0, \quad -1 < x < 1, \\ h(x_j^0) = \xi_j^0, \quad j = 1, \dots, N, \\ h^{'}(x_j^0+) + h^{'}(x_j^0-) = 0, \quad j = 1, \dots, N, \\ h^{'}(-1) = h^{'}(1) = 0. \end{cases} \tag{2.34}$$

The derivation of (2.34) is similar to Section 7 of [26]. From (2.34), we obtain the following: Given a set of positive integers $(n_1, \dots, n_N)$, we can compute the locations of $\mathbf{x}_0 = (x_1^0, \dots, x_N^0)$ explicitly. Then we can compute the matrices $\mathcal{B}$ and $\mathcal{M}$.

To verify (H2) and (H3), we need to know the eigenvalues of $\mathcal{B}$ and $\mathcal{M}$. In the same way as for the matrix $\mathcal{G}_D$, one can show that $\mathcal{B}^{-1}$ is a tridiagonal matrix. Even with this piece of information, it is almost impossible to obtain an explicit formula for the eigenvalues. Numerical software for solving eigenvalue problems of large matrices is indispensable. Numerical computations do suggest that assumptions (H2) and (H3) are always satisfied for D small.

The main idea in proving Theorem 2.2 consists of the following steps: We first rewrite (1.2) as a single nonlocal equation:

$$\mathcal{S}_\epsilon[A] = \epsilon^2 A^{''} - A + \frac{A^2}{\mathcal{T}[A]} = 0,$$

where $H = \mathcal{T}[A]$ satisfies

$$DH^{''} - H + A^2 = 0, \quad H^{'}(-1) = H^{'}(1) = 0.$$

Step 1: We choose good approximate solutions.

$$A \sim w_{\epsilon,\underline{\mathbf{x}}} = \sum_{j=1}^N \sum_{k=1}^{n_j} \xi_\epsilon \xi_{j,k} w\left(\frac{x - x_{j,k}}{\epsilon}\right), \quad j = 1, \dots, N,\ k = 1, \dots, n_N,$$

where $\xi_{j,k}$ and $x_{j,k}$ will have to be chosen carefully. More precisely, we first choose $x_{j,k}$ such that

$$x_{1,1} < x_{1,2} < \ldots < x_{1,n_1} < x_{2,1} < \ldots < x_{2,n_2} < \ldots < x_{N,1} < \ldots < x_{N,n_N},$$

$$x_{j,l} - x_{j,l-1} \sim \epsilon \log \frac{1}{\epsilon} - \epsilon \log \left[\frac{\hat{\xi}_j^0}{2D} (l-1)(n_j + 1 - l) \right], \ j = 1, 2, \dots, n,\ l = 2, \dots, n_j,$$

$$\left| \frac{\sum\limits_{k=1}^{n_j} x_{j,k}}{n_j} - x_j^0 \right| \le \eta \epsilon^{3/4}, \quad j = 1, \dots, N,$$

where $\eta > 0$ is a suitably chosen small constant.

Next we choose $\xi_{j,k}$ so that they will solve a system of algebraic equations. This is done in Section 4.

Step 2: The error terms.

We then compute the space dependence of the heights, $\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}](x_{j,k}+\epsilon y)-\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}](x_{j,k})$ and the error term $S[w_{\epsilon,\underline{\mathbf{x}}}]$. This is done in Section 5 and Section 6, respectively.

Step 3: The Liapunov-Schmidt reduction method.

By using the Liapunov-Schmidt reduction method we solve the following equation

$$\mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}+\phi]=\sum_{j,k}\alpha_{j,k}\frac{dw_{\epsilon,\underline{\mathbf{x}}}}{dx_{j,k}},$$

$$\int_I\phi\frac{dw_{\epsilon,\underline{\mathbf{x}}}}{dx_{j,k}}\,dx=0,\quad j=1,\ldots,N,\ k=1,\ldots,n_j,$$

where $\alpha_{j,k}=\alpha_{j,k}(\underline{\mathbf{x}})$ are some scalar functions depending on $\underline{\mathbf{x}}$.

This is done in Section 7.

Step 4: The reduced problem.

Finally, we solve the following reduced problem:

$$\alpha_{j,k}(\underline{\mathbf{x}}^\epsilon)=0,\quad j=1,\ldots,N,\quad k=1,\ldots,n_j.$$

This is done in Section 8.

A natural question is the following: Are all $N-$cluster solutions generated by two types of clusters as is the case for spikes? We believe that this should be true but the proof may be complicated and is left to a future study.

3 Some preliminaries

In this section, we consider a system of nonlocal linear operators. We first recall from [26]:

Theorem 3.1 *Consider the following nonlocal differential operator*

$$L\phi=\phi''-\phi+2w\phi-\gamma\frac{\int_R w\phi}{\int_R w^2}w^2=\alpha\phi. \tag{3.1}$$

If $\gamma\neq 1$, *then*

$$Ker(L)=span\{w'\}.$$

Next, we consider the following system of nonlocal operators

$$\begin{aligned}L\Phi &:= \Delta\Phi-\Phi+2w\Phi\\ &-2\left(\int_R w\,\mathcal{C}\Phi\,dy\right)\left(\int_R w^2\,dy\right)^{-1}w^2,\end{aligned} \tag{3.2}$$

where

$$\Phi = \begin{pmatrix} \phi_{1,1} \\ \vdots \\ \phi_{1,n_1} \\ \vdots \\ \phi_{N,1} \\ \vdots \\ \phi_{N,n_N} \end{pmatrix} \in (H^2(R))^{|\mathbf{n}|},$$

$\mathbf{n} = (n_1, n_2, \ldots, n_N)$, $|\mathbf{n}| = n_1 + n_2 + \ldots + n_N$;

$$\mathcal{C} = (c_{j,k;m,s}), \quad c_{j,k;m,s} = G_D(x_j, x_m)\hat{\xi}_m$$
$$\text{for } j, m = 1, \ldots, N,\ k = 1, \ldots, n_j,\ s = 1, \ldots, n_m,$$

Remark 3.1 The matrix $\mathcal{C}$ is the product of a symmetric matrix and a diagonal matrix. It therefore has only real eigenvalues.

Lemma 3.2 *Suppose that (H2) holds. Then*

$$\frac{1}{2} \notin \sigma(\mathcal{C}). \tag{3.3}$$

Proof. Let $\eta = (\eta_{1,1}, \ldots, \eta_{1,n_1}, \ldots, \eta_{N,1}, \ldots, \eta_{N,n_N})$ be an eigenvector of $\mathcal{C}$ with eigenvalue λ. Then we have

$$\sum_{m,s} c_{j,k;m,s}\eta_{m,s} = \lambda\eta_{j,k}.$$

This can be rewritten as

$$\sum_m G_D(x_j, x_m)\hat{\xi}_m \sum_s \eta_{j,s} = \lambda\eta_{j,k}. \tag{3.4}$$

Summing over k, we obtain

$$\sum_m G_D(x_j, x_m) n_j \hat{\xi}_m \sum_k \eta_{j,k} = \lambda \sum_k \eta_{j,k}.$$

So $\sum_k \eta_{j,k}$ is an eigenvector of $\mathcal{B}$. Thus, by (H2), either $\lambda \neq \frac{1}{2}$ or $\sum_k \eta_{j,k} = 0$ for $j = 1, \ldots, N$. In the latter case, we then have from (3.4) that $\lambda\eta_{j,k} = 0$ and hence $\lambda = 0$. In any case, we obtain $\lambda \neq \frac{1}{2}$. $\square$

Assumption (H2) and Lemma 3.2 imply that

(H2') It holds that

$$\frac{1}{2} \notin \sigma(\mathcal{C}). \tag{3.5}$$

For later use, we set

$$L_0 u := u^{''} - u + 2wu, \tag{3.6}$$

where $u \in H^2(R)$.

The conjugate operator of L under the scalar product in $L^2(R)$ is

$$\begin{aligned} L^*\Psi = \Psi^{''} - \Psi + 2w\Psi \\ -2\mathcal{C}^T \left(\int_R w^2 \Psi \, dy \right) \left(\int_R w^2 \, dy \right)^{-1} w, \end{aligned} \tag{3.7}$$

where

$$\Psi = \begin{pmatrix} \psi_{1,1} \\ \vdots \\ \psi_{1,n_1} \\ \vdots \\ \psi_{N,1} \\ \vdots \\ \psi_{N,n_N} \end{pmatrix} \in (H^2(R))^{|\mathbf{n}|}.$$

We obtain the following

Lemma 3.3 *Assume that assumption (H2) holds. Then*

$$Ker(L) = X_0 \oplus X_0 \oplus \cdots \oplus X_0, \tag{3.8}$$

where

$$X_0 = span \left\{ w^{'}(y) \right\}$$

and

$$Ker(L^*) = X_0 \oplus X_0 \oplus \cdots \oplus X_0. \tag{3.9}$$

Here the number of factors is $|\mathbf{n}|$.

Proof. Let us first prove (3.8). Suppose

$$L\Phi = 0.$$

Let us diagonalize $\mathcal{C}$ such that

$$P^{-1}\mathcal{C}P = J,$$

where P is an orthogonal matrix and by Remark 3.1 J has diagonal form, i.e.,

$$J = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_{|\mathbf{n}|} \end{pmatrix}$$

with suitable real numbers $\sigma_j, \quad j = 1, 2, \ldots, |\mathbf{n}|$.

Defining

$$\Phi = P\tilde{\Phi}$$

we have

$$\tilde{\Phi}'' - \tilde{\Phi} + 2w\tilde{\Phi} - 2\left(\int_R w^2\, dy\right)^{-1} \int_R wJ\tilde{\Phi}\, dy w^2 = 0. \tag{3.10}$$

For $l = 1, 2, \ldots, |\mathbf{n}|$ we consider the l-th equation of system (3.10):

$$\begin{aligned} &\tilde{\Phi}_l'' - \tilde{\Phi}_l + 2w\tilde{\Phi}_l \\ &-2\sigma_l \left(\int_R w^2\right)^{-1} \int_R w\tilde{\Phi}_l\, dy w^2 = 0. \end{aligned} \tag{3.11}$$

By Theorem 3.1, (3.11) tells us that

$$\tilde{\Phi}_l \in X_0. \tag{3.12}$$

(since by assumption (H2') we know that $\sigma_l \neq 1/2$).

Continuing in the same way for $l = 1, \ldots, N$, we have

$$\tilde{\Phi}_l \in X_0,\ l = 1, \ldots, |\mathbf{n}|. \tag{3.13}$$

(3.8) is thus proved.

To prove (3.9), we proceed similarly for L^*.

Using $\sigma(\mathcal{C}) = \sigma(\mathcal{C}^T)$, the l-th equation of the diagonalized system is as follows:

$$\begin{aligned} &\tilde{\Psi}_l'' - \tilde{\Psi}_l + 2w\tilde{\Psi}_l \\ &-2\left(\int_R w\, dy\right)^{-1} \sigma_l \int_R w^2\tilde{\Psi}_l\, dy w = 0. \end{aligned} \tag{3.14}$$

Multiplying (3.14) by w and integrating over the real line, we obtain

$$(1-2\sigma_l)\int_R w^2\tilde{\Psi}_l\,dy = 0,$$

which implies that

$$\int_R w^2\tilde{\Psi}_l\,dy = 0,$$

since $2\sigma_l \neq 1$.

Thus all the nonlocal terms vanish and we have

$$L_0\tilde{\Psi}_l = 0, \quad l = 1,\ldots,|\mathbf{n}|. \tag{3.15}$$

This implies that $\tilde{\Psi}_l \in X_0$ for $l = 1,\ldots,|\mathbf{n}|$. □

As a consequence of Lemma 3.3, we have

Lemma 3.4 *The operator*

$$L : (H^2(R))^{|\mathbf{n}|} \to (L^2(R))^{|\mathbf{n}|}$$

is invertible if it is restricted as follows

$$L : (X_0 \oplus \cdots \oplus X_0)^{\perp} \cap (H^2(R))^{|\mathbf{n}|} \to (X_0 \oplus \cdots \oplus X_0)^{\perp} \cap (L^2(R))^{|\mathbf{n}|}.$$

Moreover, L^{-1} is bounded.

Proof. This follows from the Fredholm Alternatives Theorem and Lemma 3.3. □

4 Computations I: The approximate solutions

Let $-1 < x_1^0 < \cdots < x_j^0 < \cdots x_N^0 < 1$ be N points satisfying the assumptions (H1) – (H3). Let

$$\mathbf{x}^0 = (x_1^0,\ldots,x_N^0). \tag{4.1}$$

In this section, we now construct an approximate solution to (1.2) with N clusters concentrating at these prescribed N points. As we shall see, these approximate solutions are to be valid in $O(\epsilon^{3/4})$.

Let $-1 < x_{1,1} < \cdots < x_{1,n_1} < x_{2,1} < \cdots < x_{2,n_2} < \cdots < x_{N,1} < \cdots < x_{N,n_N} < 1$ be such that

$$\epsilon\log\tfrac{1}{\epsilon} - \epsilon\log[\tfrac{\hat{\xi}_j^0}{2D}(l-1)(n_j+1-l)] - \eta\epsilon \le x_{j,l} - x_{j,l-1} \tag{4.2}$$
$$\le \epsilon\log\tfrac{1}{\epsilon} - \epsilon\log[\tfrac{\hat{\xi}_j^0}{2D}(l-1)(n_j+1-l)] + \eta\epsilon, \quad j = 1,\ldots,N,\ l = 2,\ldots,n_j,$$

and

$$\left| \frac{\sum_{l=1}^{n_j} x_{j,l}}{n_j} - x_j^0 \right| \leq \eta \epsilon^{3/4}, \tag{4.3}$$

where $\eta > 0$ is a small number which will be chosen in Section 7. The reason why we assume (4.2) will become clear in Section 8.

We use $\otimes_\eta$ to denote the set of all $\underline{\mathbf{x}} = (x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{N,1}, \ldots, x_{N,n_N})$ satisfying (4.2) and (4.3). We further denote

$$\underline{\mathbf{x}}^0 = (x_{1,1}^0, \ldots, x_{1,n_1}^0, \ldots, x_{N,1}^0, \ldots, x_{N,n_N}^0) \tag{4.4}$$

and we set

$$\otimes_0 = \{\underline{\mathbf{x}}^0\}. \tag{4.5}$$

To simplify our notation, for $\underline{\mathbf{x}} \in \Omega_\eta$ we set

$$w_{j,k}(x) = w\left(\frac{x - x_{j,k}}{\epsilon}\right) \cdot \chi\left(\left|\frac{x - x_{j,k}}{\delta}\right|\right), \tag{4.6}$$

where χ is a smooth cut-off function which satisfies the following conditions:

$$\chi(x) = 1, \text{ for } |x| < \frac{1}{2}, \quad \chi(x) = 0, \text{ for } |x| > \frac{3}{4}, \quad \chi \in C_0^\infty(R), \tag{4.7}$$

and

$$\delta = \frac{1}{10}\epsilon \log \frac{1}{\epsilon}.$$

From (4.2), using that $w(y) \sim e^{-|y|}$ as $|y| \to \infty$, we derive that

$$\int_{-1}^{1} w_{j,k} w_{m,s}\, dx = \begin{cases} e.s.t., & \text{if } j \neq m, \\ O(\epsilon^3), & \text{if } j = m,\ |k - s| \geq 2, \\ O(\epsilon^2), & \text{if } j = m, |k - s| = 1, \end{cases} \tag{4.8}$$

$$\begin{aligned} & w\left(\frac{x_{j,l} - x_{j,l-1}}{\epsilon}\right) \\ & = \frac{\epsilon}{2D}\hat{\xi}_j^0[(l-1)(n_j + 1 - l)] + O(\eta\epsilon),\ j = 1, \ldots, N,\ l = 2, \ldots, n_j, \end{aligned} \tag{4.9}$$

where $\eta > 0$ is a small number.

For $\underline{\mathbf{x}} \in \Omega_\eta$,

$$w_{\epsilon,\mathbf{x}}(x) = \xi_\epsilon \sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_{j,k} w_{j,k}(x), \tag{4.10}$$

where ξ_ϵ is defined in (2.24) and the numbers $\xi_{j,k} > 0$ will be chosen at the end of this section. By rescaling $\hat{A} = \xi_\epsilon A$, $\hat{H} = \xi_\epsilon H$, we obtain that (1.2) is equivalent to the following system for the rescaled functions $\hat{A}, \hat{H}$:

$$\begin{cases} \epsilon^2 \hat{A}'' - \hat{A} + \frac{\hat{A}^2}{\hat{H}} = 0 \quad \text{in } (-1,1), \\ D\hat{H}'' - \hat{H} + \xi_\epsilon \hat{A}^2 = 0 \quad \text{in } (-1,1), \\ \hat{A}(x) > 0, \hat{H}(x) > 0 \quad \text{in } (-1,1), \\ \hat{A}'(-1) = \hat{A}'(1) = \hat{H}'(-1) = \hat{H}'(1) = 0. \end{cases} \tag{4.11}$$

From now on, we shall work with (4.11) and drop the hats. We first rewrite (4.11) as a single equation with a nonlocal term.

For a function $A \in H^2(-1,1)$, we define $\mathcal{T}[A]$ to be the solution of

$$\begin{cases} D(\mathcal{T}[A])'' - \mathcal{T}[A] + \xi_\epsilon \frac{A^2}{\mathcal{T}[A]} = 0, \ -1 < x < 1, \\ (\mathcal{T}[A])'(-1) = (\mathcal{T}[A])'(1) = 0. \end{cases} \tag{4.12}$$

It is easy to see that the solution $\mathcal{T}[A]$ is unique and positive. Then (4.11) becomes

$$\mathcal{S}_\epsilon[A] := \epsilon^2 A'' - A + \frac{A^2}{\mathcal{T}[A]} = 0, \quad A > 0, \quad A'(-1) = A'(1) = 0. \tag{4.13}$$

Let $A = w_{\epsilon,\underline{\mathbf{x}}}$, where $\underline{\mathbf{x}} \in \otimes_\eta$. We are now going to choose $\xi_{j,k}$.

Let us first compute

$$\tau_{m,s} := \mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}](x_{m,s}). \tag{4.14}$$

From (4.12), we have

$$\begin{aligned} \tau_{m,s} &= \xi_\epsilon \int_{-1}^{1} G_D(x_{m,s}, z) w^2_{\epsilon,\underline{\mathbf{x}}}(z)\, dz + e.s.t. \\ &= \xi_\epsilon \int_{-1}^{1} G_D(x_{m,s}, z) \left[\sum_{k=1}^{n_m} \xi^2_{m,k} w^2_{m,k}(z) + \sum_{k \neq l} \xi_{m,k}\xi_{m,l} w_{m,k}(z) w_{m,l}(z) \right] dz \\ &\quad + \xi_\epsilon \int_{-1}^{1} G_D(x_{m,s}, z) \\ &\qquad \sum_{j \neq m} \left[\sum_{k=1}^{n_j} \xi^2_{j,k} w^2_{j,k}(z) + \sum_{k \neq l} \xi_{j,k}\xi_{j,l} w_{j,k}(z) w_{j,l}(z) \right] dz + e.s.t. \\ &= I_1 + I_2 + e.s.t., \end{aligned} \tag{4.15}$$

where I_1 and I_2 are defined by the last equality.

The integral I_2 is easy to compute:

$$
\begin{aligned}
I_2 &= \xi_\epsilon \sum_{j\neq m}\sum_{k=1}^{n_j} \xi_{j,k}^2 \left[G_D(x_{m,s},x_{j,k})\epsilon \int_R w^2(y)\,dy + O(\epsilon^2)\right] \\
&\quad + \xi_\epsilon \sum_{j\neq m}\sum_{k\neq l}\left[\xi_{j,k}\xi_{j,l}\int_I G_D(x_{m,s},z)w_{j,k}(z)w_{j,l}(z)\,dz + O(\epsilon^2)\right] \\
&= \xi_\epsilon \epsilon \int_{R^2} w^2(y)\,dy \left[\sum_{j\neq m}\sum_{k=1}^{n_j}\xi_{j,k}^2 G_D(x_{m,s},x_{j,k}) + O(\epsilon)\right], \qquad (4.16)
\end{aligned}
$$

using the estimate (4.8).

For I_1, we have

$$
\begin{aligned}
I_1 &= \xi_\epsilon \sum_{k=1}^{n_m} \xi_{m,k}^2 \int_I G_D(x_{m,s},z)w_{m,k}^2(z)\,dz \\
&\quad + \xi_\epsilon \sum_{k\neq l}\xi_{m,k}\xi_{m,l}\int_I G_D(x_{m,s},z)w_{m,k}(z)w_{m,l}(z)\,dz. \qquad (4.17)
\end{aligned}
$$

Let us now compute

$$
\int_I G_D(x_{m,s},z)w_{m,k}^2(z)\,dz = \int_I G_D(x_{m,s},z)\left(w\left(\frac{z-x_{m,k}}{\epsilon}\right)\right)^2 dz.
$$

If $k\neq s$, we have

$$
\int_I G_D(x_{m,s},z)w_{m,k}^2\,dz = G_D(x_{m,s},x_{m,k})\left(\epsilon\int_R w(y)^2\,dy + O(\epsilon^2)\right). \qquad (4.18)
$$

If $k = s$, we have

$$
\begin{aligned}
&\int_I G_D(x_{m,s},z)w_{m,s}^2(z)\,dz \\
&= \int_I \left[\frac{1}{2\sqrt{D}}e^{-|x_{m,s}-z|/\sqrt{D}} - H(x_{m,s},z)\right] w_{m,s}^2(z)\,dz \\
&= \left[\frac{1}{2\sqrt{D}} - H(x_{m,s},x_{m,s})\right]\left(\epsilon\int_R w^2(y)\,dy + O(\epsilon^2)\right) \\
&= G_D(x_{m,s},x_{m,s})\left(\epsilon\int_R w^2(y)\,dy + O(\epsilon^2)\right). \qquad (4.19)
\end{aligned}
$$

In conclusion, we have

$$
\int_I G_D(x_{m,s},z)w_{m,k}^2(z)\,dz = G_D(x_{m,s},x_{m,k})\left(\epsilon\int_R w^2(y)\,dy + O(\epsilon^2)\right). \qquad (4.20)
$$

Next, for $k \neq l$,

$$\int_I G_D(x_{m,s}, z) w_{m,k}(z) w_{m,l}(z)\, dz = O\left(\epsilon^2 \int_R w^2(y)\, dy\right) \tag{4.21}$$

by (4.8).

Combining (4.20) and (4.21), we have

$$I_1 = \xi_\epsilon \epsilon \int_R w^2(y)\, dy \left[\sum_{k=1}^{n_m} \xi_{m,k}^2 G_D(x_{m,s}, x_{m,k}) + O(\epsilon)\right]. \tag{4.22}$$

Substituting (4.16) and (4.22) into (4.15), we conclude that

$$\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}](x_{m,s}) = \tau_{m,s} = \sum_{j,k} \xi_{m,s}\xi_{j,k} G_D(x_{m,s}, x_{j,k}) + O(\epsilon). \tag{4.23}$$

We now choose $\xi_{j,k}$ such that

$$\xi_{m,s} = \sum_{j,k} \xi_{m,s}\xi_{j,k} G_D(x_{m,s}, x_{j,k}). \tag{4.24}$$

To see that (4.24) has a unique solution, we note that in the limit $\epsilon \to 0$ (4.24) becomes

$$\xi_{m,s}^0 = \sum_{j=1}^{N} \left(\sum_{k=1}^{n_j} (\xi_{j,k}^0)^2\right) G_D(x_m^0, x_j^0). \tag{4.25}$$

By (H1), (4.25) has a solution with $\xi_{m,s}^0 = \xi_m^0$. By (H2), Lemma 3.2, and the implicit function theorem, (4.24) has a solution. From (4.23) and (4.24), for this solution it follows that

$$\tau_{m,s} = \xi_{m,s} + O(\epsilon). \tag{4.26}$$

This concludes the construction of our approximate solutions.

5 Computations II: The space dependence of the heights

In this section, we compute the space dependence of the heights which is given by the difference $\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}](x) - \mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}](x_{m,s})$ for $\mathbf{x} \in \Omega_\eta$ and $|x - x_{m,s}| < \delta$. This is an important step in determining the spike and cluster locations. To simplify our notation, we let

$$H_{\epsilon,\underline{\mathbf{x}}} = \mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}]. \tag{5.1}$$

Let $x = x_{m,s} + \epsilon y$. Similar to Section 4, we calculate

$$\begin{aligned}
&H_{\epsilon,\underline{\mathbf{x}}}(x_{m,s} + \epsilon y) - H_{\epsilon,\underline{\mathbf{x}}}(x_{m,s}) \\
&\quad = \xi_\epsilon \int_I [G_D(x_{m,s} + \epsilon y, z) - G_D(x_{m,s}, z)] \\
&\qquad \times \left(\sum_{k=1}^{n_m} \xi_{m,k}^2 w_{m,k}^2(z) + \sum_{k \neq l} \xi_{m,k}\xi_{m,l} w_{m,k}(z) w_{m,l}(z) \right) dz \\
&\qquad + \xi_\epsilon \int_I [G_D(x_{m,s} + \epsilon y, z) - G_D(x_{m,s}, z)] \\
&\qquad \times \left(\sum_{j \neq m} \sum_{k=1}^{n_j} \xi_{j,k}^2 w_{j,k}^2(z) + \sum_{j \neq m} \sum_{k \neq l} \xi_{j,k}\xi_{j,l} w_{j,k}(z) w_{j,l}(z) \right) dz \\
&\quad = J_1 + J_2,
\end{aligned} \tag{5.2}$$

where J_1 and J_2 are defined by the last equality.

We first calculate J_2:

$$\begin{aligned}
J_2 &= \xi_\epsilon \int_I [G_D(x_{m,s} + \epsilon y, z) - G_D(x_{m,s}, z)] \\
&\quad \times \left(\sum_{j \neq m} \sum_{k=1}^{n_j} \xi_{j,k}^2 w_{j,k}^2(z) + \sum_{j \neq m} \sum_{k \neq l} \xi_{j,k}\xi_{j,l} w_{j,k}(z) w_{j,l}(z) \right) dz \\
&= \xi_\epsilon \int_I \left[\nabla_{x_{m,s}} G_D(x_{m,s}, z) + O(\epsilon|y|) \right] \epsilon y \\
&\quad \times \left(\sum_{j \neq m} \sum_{k=1}^{n_j} \xi_{j,k}^2 w_{j,k}^2(z) + \sum_{j \neq m} \sum_{k \neq l} \xi_{j,k}\xi_{j,l} w_{j,k}(z) w_{j,l}(z) \right) dz \\
&= \left[\sum_{j \neq m} \sum_{k=1}^{n_j} \xi_{j,k}^2 \nabla_{x_{m,s}} G_D(x_{m,s}, x_{j,k}) + O(\epsilon|y|) \right] \epsilon y
\end{aligned} \tag{5.3}$$

by (4.8). For J_1, we have

$$\begin{aligned}
J_1 &= \xi_\epsilon \int_I [G_D(x_{m,s} + \epsilon y, z) - G_D(x_{m,s}, z)] \\
&\quad \times \left(\sum_{k=1}^{n_m} \xi_{m,k}^2 w_{m,k}^2(z) + \sum_{k \neq l} \xi_{m,k}\xi_{m,l} w_{m,k}(z) w_{m,l}(z) \right) dz \\
&= \xi_\epsilon \sum_{k=1}^{n_m} \xi_{m,k}^2 \int_I [G_D(x_{m,s} + \epsilon y, z) - G_D(x_{m,s}, z)]\, w_{m,k}^2(z)\, dz
\end{aligned}$$

$$+\,\xi_\epsilon \sum_{k\neq l} \xi_{m,k}\xi_{m,l} \int_I \left[G_D(x_{m,s}+\epsilon y, z) - G_D(x_{m,s}, z)\right] w_{m,k}(z) w_{m,l}(z)\, dz$$

$$= \xi_\epsilon \sum_{k=1}^{n_m} \xi_{m,k}^2 \int_I \left[G_D(x,z) - G_D(x_{m,s},z)\right] w_{m,k}^2 \, dz + O(\epsilon^2 y^2) \tag{5.4}$$

by (4.8). Note that

$$\begin{aligned}
&\int_I \left[G_D(x,z) - G_D(x_{m,s},z)\right] w^2\left(\frac{z-x_{m,k}}{\epsilon}\right) dz \\
&= \int_I \left[\frac{1}{2\sqrt{D}}\left(e^{-|x-z|/\sqrt{D}} - e^{-|x_{m,s}-z|/\sqrt{D}}\right) - \left(H_D(x,z) - H_D(x_{m,s},z)\right)\right] \\
&\quad \times w^2\left(\frac{z-x_{m,k}}{\epsilon}\right) dz \\
&= \frac{1}{2\sqrt{D}} \int_I \left(e^{-|x-z|/\sqrt{D}} - e^{-|x_{m,s}-z|/\sqrt{D}}\right) w^2\left(\frac{z-x_{m,k}}{\epsilon}\right) dz \\
&\quad - \nabla_{x_{m,s}} H(x_{m,s}, x_{m,k}) \left(\epsilon^2 y \int_R w^2(z)\, dz\right) + O(\epsilon^3 y^2).
\end{aligned} \tag{5.5}$$

Let $z = x_{m,k} + \epsilon\tilde{z}$. If $k = s$, we have

$$\begin{aligned}
&\frac{1}{2\sqrt{D}} \int_I \left(e^{-|x-z|/\sqrt{D}} - e^{-|x_{m,s}-z|/\sqrt{D}}\right) w^2\left(\frac{z-x_{m,k}}{\epsilon}\right) dz \\
&\quad = \frac{1}{2\sqrt{D}}\epsilon \int_R \left(e^{-\epsilon|y-\tilde{z}|/\sqrt{D}} - e^{-\epsilon|\tilde{z}|/\sqrt{D}}\right) w^2(\tilde{z})\; d\tilde{z}\,(1 + O(\epsilon|y|)) \\
&\quad = \frac{1}{2\sqrt{D}}\epsilon \left[\epsilon \int_R (|\tilde{z}| - |y-\tilde{z}|)\, w^2(\tilde{z})\; d\tilde{z} + O(\epsilon^2 y^2)\right] \\
&\quad = \frac{1}{2\sqrt{D}}\epsilon \left[\epsilon T_0(y) + O(\epsilon^2 y^2)\right],
\end{aligned} \tag{5.6}$$

where

$$T_0(y) = \int_R (|\tilde{z}| - |y-\tilde{z}|)\, w^2(\tilde{z})\, d\tilde{z} \tag{5.7}$$

is an even function. If $k \neq s$, then

$$\begin{aligned}
&\frac{1}{2\sqrt{D}} \int_I \left(e^{-|x-z|/\sqrt{D}} - e^{-|x_{m,s}-z|/\sqrt{D}}\right) w^2\left(\frac{z-x_{m,k}}{\epsilon}\right) dz \\
&\quad = \frac{\epsilon}{2\sqrt{D}} \int_R \left(e^{-|x_{m,s}-x_{m,k}+\epsilon(y-\tilde{z})|/\sqrt{D}} - e^{-|x_{m,s}-x_{m,k}-\epsilon\tilde{z}|/\sqrt{D}}\right) w^2(\tilde{z})\; d\tilde{z}
\end{aligned}$$

$$= \frac{\epsilon}{2\sqrt{D}} \int_R \left[\frac{1}{\sqrt{D}} \left(|x_{m,s} - x_{m,k} - \epsilon\tilde{z}| - |x_{m,s} - x_{m,k} + \epsilon(y - \tilde{z})| \right) + O(|x_{m,s} - x_{m,k}|^2) \right] w^2(\tilde{z})\, d\tilde{z}$$

$$= \frac{\epsilon}{2\sqrt{D}} \left[\frac{1}{\sqrt{D}} \left(- \frac{x_{m,s} - x_{m,k}}{|x_{m,s} - x_{m,k}|} \right) \epsilon y + O\left(\epsilon^2 \log^2 \frac{1}{\epsilon} \right) \right] \int_R w^2(y) dy. \tag{5.8}$$

Combining (5.3), (5.6), and (5.8), we have

$$\begin{aligned} &H_{\epsilon,\underline{\mathbf{x}}}(x_{m,s} + \epsilon y) - H_{\epsilon,\underline{\mathbf{x}}}(x_{m,s}) \\ &= \frac{\epsilon}{2\sqrt{D} \int_R w^2(y)\, dy} T_0(|y|) \xi_{m,s}^2 + \frac{\epsilon}{2D} \sum_{k \neq s} \xi_{m,k}^2 \left(- \frac{x_{m,s} - x_{m,k}}{|x_{m,s} - x_{m,k}|} \right) y \\ &\quad - \sum_{k=1}^{n_m} \xi_{m,k}^2 \nabla_{x_{m,s}} H_D(x_{m,s}, x_{m,k}) \epsilon y \\ &\quad + \sum_{j \neq m} \sum_{k=1}^{n_j} \xi_{j,k}^2 \nabla_{x_{m,s}} G_D(x_{m,s}, x_{j,k})\, \epsilon y + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_{j,k}^2 (\epsilon^2 |y|^2). \end{aligned} \tag{5.9}$$

6 Computations III: The error terms

In this section, we compute the error terms.

Recall from (4.12) that

$$\mathcal{S}_\epsilon[A] := \epsilon^2 A'' - A + \frac{A^2}{\mathcal{T}[A]}, \tag{6.1}$$

where $\mathcal{T}[A]$ is defined by (4.12). We now compute the error term

$$\begin{aligned} \mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}] &= \mathcal{S}_\epsilon \left[\sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_{j,k} w_{j,k} \right] \\ &= \epsilon^2 \Delta \left(\sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_{j,k} w_{j,k} \right) - \sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_{j,k} w_{j,k} \\ &\quad + \frac{\left(\sum\limits_{j=1}^{N} \sum\limits_{k=1}^{n_j} \xi_{j,k} w_{j,k} \right)^2}{H_{\epsilon,\underline{\mathbf{x}}}} \\ &= \frac{\sum_{j=1}^{N} \left(\sum_{k=1}^{n_j} \xi_{j,k} w_{j,k} \right)^2 + e.s.t.}{H_{\epsilon,\underline{\mathbf{x}}}} - \sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_{j,k} w_{j,k}^2 \end{aligned}$$

$$= \sum_{j=1}^{N} \left(\sum_{k=1}^{n_j} \xi_{j,k}^2 \left(\frac{1}{H_{\epsilon,\underline{\mathbf{x}}}} - \frac{1}{\xi_{j,k}} \right) w_{j,k}^2 \right)$$
$$+ \sum_{j=1}^{N} \sum_{k \neq l} \xi_{j,k} \xi_{j,l} w_{j,k} w_{j,l} \frac{1}{H_{\epsilon,\underline{\mathbf{x}}}} + e.s.t.$$
$$= \sum_{j=1}^{N} \left(\sum_{j=1}^{n_j} \xi_{j,k} \frac{(\xi_{j,k} - \tau_{j,k}) + (\tau_{j,k} - H_{\epsilon,\underline{\mathbf{x}}})}{H_{\epsilon,\underline{\mathbf{x}}}} w_{j,k}^2 \right)$$
$$+ \sum_{j=1}^{N} \sum_{k \neq l} \xi_{j,k} \xi_{j,l} w_{j,k} w_{j,l} \frac{1}{H_{\epsilon,\underline{\mathbf{x}}}} + e.s.t. \tag{6.2}$$

By (4.9), (4.26) and (5.9), this implies that

$$\|S_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}]\|_{L^2(-\frac{1}{\epsilon},\frac{1}{\epsilon})} = O(\epsilon). \tag{6.3}$$

The estimates derived in this section provide an important step that will make our approach work in the rest of the paper.

7 The Liapunov-Schmidt Reduction Method

In this section, we use the Liapunov-Schmidt reduction method to solve the problem

$$\mathcal{S}_\epsilon \left[\sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_{j,k} w_{j,k} + v \right] = \sum_{j=1}^{N} \sum_{k=1}^{n_j} \beta_{j,k} \frac{d\tilde{w}_{j,k}}{dx} \tag{7.1}$$

for real constants $\beta_{j,k}$ and a function $v \in H^2(-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ which is small in the corresponding norm, where $\xi_{j,k}$ is given by (4.24), $w_{j,k}$ is defined by (4.6), and $\underline{\mathbf{x}} = (x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{N,1}, \ldots, x_{N,n_N}) \in \otimes_\eta$.

To this end, we need to study the linearized operator

$$\tilde{L}_{\epsilon,\underline{\mathbf{x}}} : H^2(I_\epsilon) \to L^2(I_\epsilon)$$

defined by

$$\tilde{L}_{\epsilon,\underline{\mathbf{x}}} := \mathcal{S}_\epsilon^{'}[A]\phi = \epsilon^2 \phi^{''} - \phi + \frac{2A\phi}{T[A]} - \frac{A^2}{(T[A])^2}(T^{'}[A]\phi),$$

where $A = \sum_{j=1}^{N} \sum_{k=1}^{n_j} \xi_{j,k} w_{j,k}$, $I_\epsilon = (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$, and for a given $\phi \in L^2(I)$ we introduce $T^{'}[A]\phi$ as the unique solution of

$$\begin{cases} D(T^{'}[A]\phi)^{''} - (T^{'}[A]\phi) + 2\xi_\epsilon A\phi = 0, & -1 < x < 1, \\ (T^{'}[A]\phi)^{'}(-1) = (T^{'}[A]\phi)^{'}(1) = 0. \end{cases} \tag{7.2}$$

We define the approximate kernel and co-kernel, respectively, as follows:

$$\mathcal{K}_{\epsilon,\underline{\mathbf{x}}} := \text{span}\left\{ \frac{dw_{j,k}}{dx} \middle| j = 1, \ldots, N,\ k = 1, \ldots, n_j \right\} \subset H^2(I_\epsilon),$$

$$\mathcal{C}_{\epsilon,\underline{\mathbf{x}}} := \text{span}\left\{ \frac{dw_{j,k}}{dx} \middle| j = 1, \ldots, N,\ k = 1, \ldots, n_j \right\} \subset L^2(I_\epsilon).$$

Recall the definition of the following system of linear operators from (3.2):

$$L\Phi := \Delta\Phi - \Phi + 2w\Phi$$

$$-2\left(\int_R w\mathcal{C}\Phi\,dy\right)\left(\int_R w^2\,dy\right)^{-1} w^2,$$

where

$$\Phi = \begin{pmatrix} \phi_{1,1} \\ \vdots \\ \phi_{1,n_1} \\ \vdots \\ \phi_{N,1} \\ \vdots \\ \phi_{N,n_N} \end{pmatrix} \in (H^2(R))^{|\mathbf{n}|}.$$

By Lemma 3.4, we know that

$$L : (X_0 \oplus \cdots \oplus X_0)^\perp \cap (H^2(R))^{|\mathbf{n}|} \to (X_0 \oplus \cdots \oplus X_0)^\perp \cap (L^2(R))^{|\mathbf{n}|}$$

is invertible with a bounded inverse.

We also introduce the projection $\pi^\perp_{\epsilon,\underline{\mathbf{x}}} : L^2(I_\epsilon) \to \mathcal{C}^\perp_{\epsilon,\underline{\mathbf{x}}}$ and study the operator $L_{\epsilon,\underline{\mathbf{x}}} := \pi^\perp_{\epsilon,\underline{\mathbf{x}}} \circ \tilde{L}_{\epsilon,\underline{\mathbf{x}}}$. By letting $\epsilon \to 0$, we will show that $L_{\epsilon,\underline{\mathbf{x}}} : \mathcal{K}^\perp_{\epsilon,\underline{\mathbf{x}}} \to \mathcal{C}^\perp_{\epsilon,\underline{\mathbf{x}}}$ is invertible with a bounded inverse provided ϵ is small enough. For this we will use the fact that the operator L is the limit of the operator $L_{\epsilon,\underline{\mathbf{x}}}$ as $\epsilon \to 0$.

This statement is contained in the following proposition.

Proposition 7.1 *There exist positive constants $\bar{\epsilon}, \eta, \lambda$ such that for all $\epsilon \in (0, \bar{\epsilon})$, $\underline{\mathbf{x}} \in \otimes_\eta$, we have*

$$\|L_{\epsilon,\underline{\mathbf{x}}}\phi\|_{L^2(\Omega_\epsilon)} \geq \lambda \|\phi\|_{H^2(I_\epsilon)}. \tag{7.3}$$

Furthermore, the map

$$L_{\epsilon,\underline{\mathbf{x}}} = \pi^\perp_{\epsilon,\underline{\mathbf{x}}} \circ \tilde{L}_{\epsilon,\underline{\mathbf{x}}} : \mathcal{K}^\perp_{\epsilon,\underline{\mathbf{x}}} \to \mathcal{C}^\perp_{\epsilon,\underline{\mathbf{x}}}$$

is surjective.

Proof of Proposition 7.1: This proof follows the method of Liapunov-Schmidt reduction which was also used in [1], [11], [12], [9], [18], [19], and [26].

Suppose that (7.3) is false. Then there exist sequences $\{\epsilon_k\}$, $\{\underline{\mathbf{x}}^k\}$, $\{\phi^k\}$ with $\epsilon_k \to 0$, $\underline{\mathbf{x}}^k \in \otimes_\eta$, such that

$$\|L_{\epsilon_k,\underline{\mathbf{x}}^k}\phi^k\|_{L^2(I_{\epsilon_k})} \to 0, \qquad \text{as } k \to \infty, \tag{7.4}$$

$$\|\phi^k\|_{H^2(I_{\epsilon_k})} = 1, \qquad k = 1,\, 2,\, \dots . \tag{7.5}$$

We define $\phi_{\epsilon,j,k}$, $j = 1, 2, \dots, N, l = 1, \dots, n_j$ and $\phi_{\epsilon,|\mathbf{n}|+1}$ as follows:

$$\phi_{\epsilon,j,l}(x) = \phi_\epsilon(x)\chi\left(\frac{x - x_{j,l}}{\delta}\right), \quad x \in I, \tag{7.6}$$

$$\phi_{\epsilon,|\mathbf{n}|+1}(x) = \phi_\epsilon(x) - \sum_{j,l}\phi_{\epsilon,j,l}(x), \quad x \in I.$$

At first (after rescaling) $\phi_{\epsilon,i}$ are only defined on I_ϵ. However, by a standard result they can be extended to R such that their norm in $H^2(R)$ is still bounded by a constant independent of ϵ_k and $\underline{\mathbf{x}}^k$ for ϵ small enough. In the following, we will study this extension. For simplicity, we keep the same notation for the extension. Since for $j = 1, 2, \dots, N, l = 1, \dots, n_j$ each sequence $\{\phi_i^k\} := \{\phi_{\epsilon_k,j,l}\}$ $(k = 1, 2, \dots)$ is bounded in $H^2_{loc}(R)$, it has a weak limit in $H^2_{loc}(R)$, and therefore also a strong limit in $L^2_{loc}(R)$ and $L^\infty_{loc}(R)$.

Call these limits ϕ_i. Then $\Phi = \begin{pmatrix} \phi_{1,1} \\ \vdots \\ \phi_{1,n_1} \\ \vdots \\ \phi_{N,1} \\ \vdots \\ \phi_{N,n_N} \end{pmatrix}$ solves the system

$$L\Phi = 0.$$

By Lemma 3.3, $\Phi \in \mathrm{Ker}(L) = X_0 \oplus \cdots \oplus X_0$. Since $\phi^k \in K^\perp_{\epsilon_k,\underline{\mathbf{x}}^k}$ by taking $k \to \infty$ we get $\Phi \in \mathrm{Ker}(L)^\perp$. Together, these two statements give $\Phi = 0$.

By elliptic estimates, we get $\|\phi_{\epsilon_k,j,l}\|_{H^2(R)} \to 0$ as $k \to \infty$ for $j = 1, 2, \dots, N, l = 1, \dots, n_j$.

Furthermore, $\phi_{\epsilon,|\mathbf{n}|+1} \to \phi_{|\mathbf{n}|+1}$ in $H^2(R)$, where $\Phi_{|\mathbf{n}|+1}$ satisfies

$$\Delta\phi_{|\mathbf{n}|+1} - \phi_{|\mathbf{n}|+1} = 0 \quad \text{in } R.$$

Therefore, we conclude that $\phi_{|\mathbf{n}|+1} = 0$ and $\|\phi^k_{|\mathbf{n}|+1}\|_{H^2(R)} \to 0$ as $k \to \infty$.

This contradicts $\|\phi^k\|_{H^2(I_{\epsilon_k})} = 1$. To complete the proof of Proposition 7.1, we just need to show that the operator which is conjugate to $L_{\epsilon,\underline{\mathbf{x}}}$ (denoted by $L^*_{\epsilon,\underline{\mathbf{x}}}$) is injective from $\mathcal{K}^\perp_{\epsilon,\underline{\mathbf{x}}}$ to $\mathcal{C}^\perp_{\epsilon,\underline{\mathbf{x}}}$. Note that $L^*_{\epsilon,\underline{\mathbf{x}}}\psi = \pi_{\epsilon,\underline{\mathbf{x}}} \circ \tilde{L}^*_{\epsilon,\underline{\mathbf{x}}}$ with

$$\tilde{L}^*_{\epsilon,\underline{\mathbf{x}}}\psi = \epsilon^2\Delta\psi - \psi + \frac{2A\psi}{T[A]} - T^{'}[A]\psi\,\frac{A^2}{(T[A])^2}.$$

The proof for $L^*_{\epsilon,\underline{\mathbf{x}}}$ follows along the same line as the proof for $L_{\epsilon,\underline{\mathbf{x}}}$ and is therefore omitted. □

Now we are in a position to solve the equation

$$\pi^\perp_{\epsilon,\underline{\mathbf{x}}} \circ \mathcal{S}_\epsilon(w_{\epsilon,\underline{\mathbf{x}}} + \phi) = 0. \tag{7.7}$$

Since $L_{\epsilon,\underline{\mathbf{x}}}|_{K^\perp_{\epsilon,\underline{\mathbf{x}}}}$ is invertible (call the inverse $L^{-1}_{\epsilon,\underline{\mathbf{x}}}$) we can rewrite this as

$$\phi = -(L^{-1}_{\epsilon,\underline{\mathbf{x}}} \circ \pi^\perp_{\epsilon,\underline{\mathbf{x}}} \circ \mathcal{S}_\epsilon(w_{\epsilon,\underline{\mathbf{x}}})) - (L^{-1}_{\epsilon,\underline{\mathbf{x}}} \circ \pi^\perp_{\epsilon,\underline{\mathbf{x}}} \circ N_{\epsilon,\underline{\mathbf{x}}}(\phi)) \equiv M_{\epsilon,\underline{\mathbf{x}}}(\phi), \tag{7.8}$$

where

$$N_{\epsilon,\underline{\mathbf{x}}}(\phi) = \mathcal{S}_\epsilon(w_{\epsilon,\underline{\mathbf{x}}} + \phi) - \mathcal{S}_\epsilon(w_{\epsilon,\underline{\mathbf{x}}}) - \mathcal{S}^{'}_\epsilon(w_{\epsilon,\underline{\mathbf{x}}})\phi \tag{7.9}$$

and the operator $M_{\epsilon,\underline{\mathbf{x}}}$ is defined by (7.8) for $\phi \in H^2(I_\epsilon)$. We are going to show that the operator $M_{\epsilon,\underline{\mathbf{x}}}$ is a contraction on

$$B_{\epsilon,r_0} \equiv \{\phi \in H^2(I_\epsilon) | \|\phi\|_{H^2(I_\epsilon)} < r_0\}$$

if r_0 and ϵ are small enough. We have by (6.3) and Proposition 7.1

$$\begin{aligned}\|M_{\epsilon,\underline{\mathbf{x}}}(\phi)\|_{H^2(I_\epsilon)} &\le \lambda^{-1}\big(\|\pi^\perp_{\epsilon,\underline{\mathbf{x}}} \circ N_{\epsilon,\underline{\mathbf{x}}}(\phi)\|_{L^2(I_\epsilon)} \\ &\quad + \left\|\pi^\perp_{\epsilon,\underline{\mathbf{x}}} \circ \mathcal{S}_\epsilon(w_{\epsilon,\underline{\mathbf{x}}})\right\|_{L^2(I_\epsilon)}\big) \\ &\le \lambda^{-1}C(c(r_0)r_0 + \epsilon),\end{aligned}$$

where $\lambda > 0$ is independent of $r_0 > 0$, $\epsilon > 0$ and $c(r_0) \to 0$ as $r_0 \to 0$. Similarly, we show

$$\begin{aligned}&\|M_{\epsilon,\underline{\mathbf{x}}}(\phi) - M_{\epsilon,\underline{\mathbf{x}}}(\phi^{'})\|_{H^2(I_\epsilon)} \\ &\le \lambda^{-1}C(c(r_0)r_0)\|\phi - \phi^{'}\|_{H^2(I_\epsilon)},\end{aligned}$$

where $c(r_0) \to 0$ as $r_0 \to 0$. If we choose $r_0 = \epsilon^\alpha$ for $\alpha < 1$ and ϵ small enough, then $M_{\epsilon,\underline{\mathbf{x}}}$ is a contraction on B_{ϵ,r_0}. The existence of a fixed point $\phi_{\epsilon,\underline{\mathbf{x}}}$ now follows from the standard contraction mapping principle and $\phi_{\epsilon,\underline{\mathbf{x}}}$ is a solution of (7.8).

We have thus proved

Lemma 7.2 *There exist $\overline{\epsilon} > 0$ $\eta > 0$ such that for every pair of $\epsilon, \underline{\mathbf{x}}$ with $0 < \epsilon < \overline{\epsilon}$ and $\underline{\mathbf{x}} \in \otimes_\eta$, there is a unique $\phi_{\epsilon,\underline{\mathbf{x}}} \in K^\perp_{\epsilon,\underline{\mathbf{x}}}$ satisfying $\mathcal{S}_\epsilon(w_{\epsilon,\underline{\mathbf{x}}} + \phi_{\epsilon,\underline{\mathbf{x}}}) \in \mathcal{C}_{\epsilon,\underline{\mathbf{x}}}$. Furthermore, we have the estimate*

$$\|\phi_{\epsilon,\underline{\mathbf{x}}}\|_{H^2(I_\epsilon)} \leq C\epsilon^\alpha, \tag{7.10}$$

where $\alpha < 1$.

Remark 7.3 By one more iteration, it can actually be shown that

$$\|\phi_{\epsilon,\underline{\mathbf{x}}}\|_{H^2(I_\epsilon)} \leq C\epsilon. \tag{7.11}$$

8 The reduced problem

In this section, we solve the reduced problem. This complete the proof of our main existence result given by Theorem 2.2.

By Lemma 7.2, for every $\underline{\mathbf{x}} \in \otimes_\eta$, there exists a unique solution $\phi_{\epsilon,\underline{\mathbf{x}}} \in \mathcal{K}^\perp_{\epsilon,\underline{\mathbf{x}}}$ such that

$$\mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}} + \phi_{\epsilon,\underline{\mathbf{x}}}] = v_{\epsilon,\underline{\mathbf{x}}} \in \mathcal{C}_{\epsilon,\underline{\mathbf{x}}}. \tag{8.1}$$

Our idea is to find $\underline{\mathbf{x}}^\epsilon = (x^\epsilon_{1,1}, \ldots, x^\epsilon_{1,n_1}, \ldots, x^\epsilon_{N,1}, \ldots, x^\epsilon_{N,n_N}) \in \Omega_\eta$ near

$$\underline{\mathbf{x}}^0 = (x^0_{1,1}, \ldots, x^0_{1,n_1}, \ldots, x^0_{N,1}, \ldots, x^0_{N,n_N}) \in \otimes_0$$

such that also

$$\mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}^\epsilon} + \phi_{\epsilon,\underline{\mathbf{x}}^\epsilon}] \perp \mathcal{C}_{\epsilon,\underline{\mathbf{x}}^\epsilon} \tag{8.2}$$

and therefore $\mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}^\epsilon} + \phi_{\epsilon,\underline{\mathbf{x}}^\epsilon}] = 0$.

(Recall that $\otimes_0$ contains only one point.)

To this end, we let

$$W_{\epsilon,m,s}(\underline{\mathbf{x}}) := \epsilon^{-1} \int_I \mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}} + \phi_{\epsilon,\underline{\mathbf{x}}}] \frac{dw_{m,s}}{dx}\, dx,$$

$$W_\epsilon(\underline{\mathbf{x}}) := (W_{\epsilon,1,1}(\underline{\mathbf{x}}), \ldots, W_{\epsilon,N,n_N}(\underline{\mathbf{x}})) : \otimes_\eta \to R^{|\mathbf{n}|}.$$

Then $W_\epsilon(\underline{\mathbf{x}})$ is a map which is continuous in $\underline{\mathbf{x}}$ and our problem is reduced to finding a zero of the vector field $W_\epsilon(\underline{\mathbf{x}})$.

We note that

$$\begin{aligned}
&\epsilon^{-1} \int_I \mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}} + \phi_{\epsilon,\underline{\mathbf{x}}}] \frac{dw_{m,s}}{dx}\, dx \\
&\quad = \epsilon^{-1} \int_I \left[\mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}] + \mathcal{S}'_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}](\phi_{\epsilon,\underline{\mathbf{x}}}) + O(\|\phi_{\epsilon,\underline{\mathbf{x}}}\|^2_{H^2(I_\epsilon)}) \right] \frac{dw_{m,s}}{dx}\, dx \\
&\quad = \epsilon^{-1} \int_I \mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}] \frac{dw_{m,s}}{dx}\, dx + O(\epsilon)
\end{aligned}$$

since

$$
\begin{aligned}
&\epsilon^{-1}\int_I \mathcal{S}_\epsilon'[w_{\epsilon,\underline{\mathbf{x}}}](\phi_{\epsilon,\underline{\mathbf{x}}})\frac{dw_{m,s}}{dx}\,dx\\
&= \epsilon^{-1}\int_I\left[\epsilon^2\phi_{\epsilon,\underline{\mathbf{x}}}^{''}-\phi_{\epsilon,\underline{\mathbf{x}}}+\frac{2w_{\epsilon,\underline{\mathbf{x}}}}{\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}]}\phi_{\epsilon,\underline{\mathbf{x}}}-\frac{w_{\epsilon,\underline{\mathbf{x}}}^2}{(\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}])^2}(\mathcal{T}'[w_{\epsilon,\underline{\mathbf{x}}}]\phi_{\epsilon,\underline{\mathbf{x}}})\right]\frac{dw_{m,s}}{dx}\,dx\\
&= \epsilon^{-1}\int_I\left[[\frac{1}{\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}]}-\frac{1}{\xi_{j,k}}]2w_{\epsilon,\underline{\mathbf{x}}}\phi_{\epsilon,\underline{\mathbf{x}}}-\frac{w_{\epsilon,\underline{\mathbf{x}}}^2}{(\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}])^2}(\mathcal{T}'[w_{\epsilon,\underline{\mathbf{x}}}]\phi_{\epsilon,\underline{\mathbf{x}}})\right]\frac{dw_{m,s}}{dx}\,dx\\
&= \epsilon^{-1}\int_I\left[\frac{(\xi_{j,k}-\tau_{j,k})+(\tau_{j,k}-\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}]}{\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}]\xi_{j,k}}2\phi_{\epsilon,\underline{\mathbf{x}}}\right.\\
&\qquad\left.-\frac{w_{\epsilon,\underline{\mathbf{x}}}^2}{(\mathcal{T}[w_{\epsilon,\underline{\mathbf{x}}}])^2}(\mathcal{T}'[w_{\epsilon,\underline{\mathbf{x}}}]\phi_{\epsilon,\underline{\mathbf{x}}})\right]\frac{dw_{m,s}}{dx}\,dx\\
&= O(\epsilon^{-1}\|\phi_{\epsilon,\underline{\mathbf{x}}}\|_{H^2(I_\epsilon)}\epsilon)\\
&= O(S_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}]_{L^2(I_\epsilon)})=O(\epsilon).
\end{aligned}
$$

Thus it remains to compute

$$
\frac{1}{\epsilon}\int_I \mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}]\frac{dw_{m,s}}{dx}\,dx = c_{m,s}. \tag{8.3}
$$

Let $x=x_{m,s}+\epsilon y$. By (5.9), we have

$$
\frac{1}{\epsilon}\int_I \mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}]\frac{dw_{m,s}}{dx}\,dx=\frac{1}{\epsilon}\int_R \mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}]w'(y)\,dy+O(\epsilon),
$$

where

$$
w'(y)=\frac{dw(y)}{dy}.
$$

For clarity, we set

$$
w_{m,s}'(y)=w'(y)\quad\text{since } x=x_{m,s}+\epsilon y.
$$

We calculate by (5.9) and (6.2)

$$
\begin{aligned}
c_{m,s} &= \frac{1}{\epsilon}\int_R \mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}]w'(y)\,dy\\
&= \frac{1}{\epsilon}\sum_{j=1}^N\sum_{k=1}^{n_j}\xi_{j,k}\int_R\frac{\xi_{j,k}-H_{\epsilon,\underline{\mathbf{x}}}}{H_{\epsilon,\underline{\mathbf{x}}}}w_{j,k}^2w_{m,s}'\,dy\\
&\quad+\frac{1}{\epsilon}\sum_{j=1}^N\sum_{k\neq l}\xi_{j,k}\xi_{j,l}\int_R\frac{w_{j,k}w_{j,l}}{H_{\epsilon,\underline{\mathbf{x}}}}w_{m,s}'\,dy+O(\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\epsilon}\sum_{k=1}^{n_m}\xi_{m,k}\int_R \frac{\xi_{m,k}-H_{\epsilon,\underline{\mathbf{x}}}}{H_{\epsilon,\underline{\mathbf{x}}}} w_{m,k}^2 w_{m,s}'\,dy \\
&\quad + \frac{1}{\epsilon}\sum_{k\neq l}\xi_{m,k}\xi_{m,l}\int_R \frac{w_{m,k}w_{m,l}}{H_{\epsilon,\underline{\mathbf{x}}}} w_{m,s}'\,dy + O(\epsilon) \\
&= \frac{1}{\epsilon}\xi_{m,s}\int_R \frac{\xi_{m,s}-H_{\epsilon,\underline{\mathbf{x}}}}{H_{\epsilon,\underline{\mathbf{x}}}} w_{m,s}^2 w_{m,s}'\,dy \\
&\quad + \frac{1}{\epsilon}\sum_{k\neq s}\xi_{m,k}\int_R \frac{\xi_{m,k}-H_{\epsilon,\underline{\mathbf{x}}}}{H_{\epsilon,\underline{\mathbf{x}}}} w_{m,k}^2 w_{m,s}'\,dy \\
&\quad + \frac{1}{\epsilon}\sum_{k\neq l}\xi_{m,k}\xi_{m,l}\int_R \frac{w_{m,k}w_{m,l}}{H_{\epsilon,\underline{\mathbf{x}}}} w_{m,s}'\,dy + O(\epsilon) \\
&= E_1 + E_2 + E_3
\end{aligned}
\tag{8.4}
$$

where E_1, E_2, and E_3 are defined by the last equality.

By (5.9), we have

$$
\begin{aligned}
E_1 = \Bigg[&\sum_{k\neq s}\frac{1}{2D}\frac{x_{m,s}-x_{m,k}}{|x_{m,s}-x_{m,k}|}\xi_{m,k}^2 - \sum_{k=1}^{n_m}\xi_{m,k}^2\nabla_{x_{m,s}}H(x_{m,s},x_{m,k}) \\
&+ \sum_{j\neq m}\sum_{k=1}^{n_j}\xi_{j,k}^2\nabla_{x_{m,s}}G_D(x_{m,s},x_{j,k})\Bigg]\frac{1}{3}\int_R w^3(y)\,dy,
\end{aligned}
\tag{8.5}
$$

$$
E_2 = \sum_{k\neq s}\xi_{m,k}^2 O(\epsilon),
$$

$$
\begin{aligned}
E_3 &= \frac{1}{\epsilon}\sum_{k\neq s}\xi_{m,k}\xi_{m,s}\int_R \frac{w_{m,k}w_{m,s}w_{m,s}'}{H_{\epsilon,\underline{\mathbf{x}}}}\,dy + O(\epsilon) \\
&= \frac{1}{\epsilon}\sum_{k\neq s}\xi_{m,k}\int_R w_{m,s}^2 w_{m,k}'\,dy + O(\epsilon) \\
&= \frac{1}{\epsilon}\sum_{k\neq s}\xi_{m,k}\int_R w^2(y)w'\left(y+\frac{x_{m,s}-x_{m,k}}{\epsilon}\right)dy + O(\epsilon) \\
&= \frac{1}{\epsilon}\sum_{k\neq s}\xi_{m,k}w\left(\frac{x_{m,s}-x_{m,k}}{\epsilon}\right)\frac{x_{m,s}-x_{m,k}}{|x_{m,s}-x_{m,k}|} \\
&\quad \frac{1}{3}\int_R w^3(y)\,dy + O(\epsilon).
\end{aligned}
\tag{8.6}
$$

In summary, we obtain the following vector field

$$\begin{aligned}
&c_{m,s}\left(\frac{1}{3}\int_R w^3(y)\,dy\right)^{-1}\\
&=\frac{1}{\epsilon}\left(\frac{1}{3}\int_R w^3(y)\,dy\right)^{-1}\int_I \mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}}]\frac{dw_{m,s}}{dx_{m,s}}\,dx\\
&=\sum_{k\neq s}\left(\frac{1}{2D}\xi_{m,k}^2-\frac{1}{\epsilon}\xi_{m,k}w\left(\frac{x_{m,s}-x_{m,k}}{\epsilon}\right)\right)\frac{x_{m,s}-x_{m,k}}{|x_{m,s}-x_{m,k}|}\\
&\quad+\sum_{j\neq m}\sum_{k=1}^{n_j}\xi_{j,k}^2\nabla_{x_{m,s}}G_D(x_{m,s},x_{j,k})\\
&\quad-\sum_{k=1}^{n_m}\xi_{m,k}^2\nabla_{x_{m,s}}H(x_{m,s},x_{m,k})+O(\epsilon)\\
&=\sum_{k\neq s}\left(\frac{1}{2D}\xi_{m,k}^2-\frac{1}{\epsilon}\xi_{m,k}w\left(\frac{x_{m,s}-x_{m,k}}{\epsilon}\right)\right)\frac{x_{m,s}-x_{m,k}}{|x_{m,s}-x_{m,k}|}\\
&\quad+\sum_{j\neq m}\nabla_{x_m}G_D(x_m,x_j)n_j\xi_j^2\\
&\quad-\nabla_{x_m}H(x_m,x_m)n_m\xi_m^2+O(\epsilon^{3/4}).
\end{aligned}\tag{8.7}$$

Note that when $\underline{\mathbf{x}}=\underline{\mathbf{x}}^0=(x_{1,1}^0,\dots,x_{1,n_1}^0,\dots,x_{N,1}^0,\dots,x_{N,n_N}^0)\in\otimes_0$ we have

$$\sum_{k\neq s}\left(\frac{1}{2D}\xi_{m,k}^2(\underline{\mathbf{x}}^0)-\frac{1}{\epsilon}\xi_{m,k}(\underline{\mathbf{x}}^0)w\left(\frac{x_{m,s}^0-x_{m,k}^0}{\epsilon}\right)\right)\frac{x_{m,s}^0-x_{m,k}^0}{|x_{m,s}^0-x_{m,k}^0|}$$

$$=O\left(\sum_{m,k}|\xi_{m,k}(\underline{\mathbf{x}}^0)-\xi_m^0|\right)=O(\epsilon^{3/4})$$

since

$$\begin{aligned}
&\sum_{k\neq s}\left(\frac{1}{2D}(\xi_m^0)^2-\frac{1}{\epsilon}\xi_m^0 w\left(\frac{x_{m,s}^0-x_{m,k}}{\epsilon}\right)\right)\frac{x_{m,s}^0-x_{m,k}^0}{|x_{m,s}^0-x_{m,k}^0|}+O(\epsilon)\\
&=\frac{1}{2D}(\xi_m^0)^2(2s-1-n_m)\\
&\quad+\frac{1}{\epsilon}\xi_m^0\left(w\left(\frac{x_{m,s+1}^0-x_{m,s-1}^0}{\epsilon}\right)-w\left(\frac{x_{m,s}^0-x_{m,s-1}^0}{\epsilon}\right)\right)+O(\epsilon)\\
&=O(\epsilon)
\end{aligned}$$

since

$$w\left(\frac{x^0_{m,s+1}-x^0_{m,s}}{\epsilon}\right)-w\left(\frac{x^0_{m,s}-x^0_{m,s-1}}{\epsilon}\right)=\frac{\epsilon}{2D}\xi^0_m(n_m-2s+1)+O(\epsilon^2),$$

$$w\left(\frac{x^0_{m,2}-x^0_{m,1}}{\epsilon}\right)=\frac{\epsilon}{2D}\xi^0_m(n_m-1)+O(\epsilon^2),$$

$$w\left(\frac{x^0_{m,s}-x^0_{m,s-1}}{\epsilon}\right)=\frac{\epsilon}{2D}\xi^0_m(n_m-1)+O(\epsilon^2).$$

Furthermore,

$$\sum_{j\neq m}\nabla_{x^0_m}G_D(x^0_m,x^0_j)n_j\xi_j^2-\nabla_{x^0_m}H(x^0_m,x^0_m)n_m\xi_m^2=O(\epsilon^{3/4})$$

by assumption (H3).

Let

$$F_{1,m,s}(\underline{\mathbf{x}})=\sum_{k\neq s}\left(\frac{1}{2D}\xi^2_{m,k}-\frac{1}{\epsilon}\xi_{m,k}w\left(\frac{x_{m,s}-x_{m,k}}{\epsilon}\right)\right)\frac{x_{m,s}-x_{m,k}}{|x_{m,s}-x_{m,k}|}$$
$$+\sum_{j\neq m}\nabla_{x_m}G_D(x_m,x_j)n_j\xi_j^2-\nabla_{x_m}H(x_m,x_m)n_m\xi_m^2. \tag{8.8}$$

Then we have

$$W_{\epsilon,j,k}=F_{1,j,k}(\underline{\mathbf{x}})+O(\epsilon^{3/4}) \tag{8.9}$$

and

$$W_{\epsilon,j,k}(\underline{\mathbf{x}}^0)=O(\epsilon^{3/4}). \tag{8.10}$$

We need the following lemma.

Lemma 8.1 *Let*

$$F_m(\underline{\mathbf{x}})=\sum_{j\neq m}\nabla_{x_m}G_D(x_m,x_j)n_j\xi_j^2-\nabla_{x_m}H(x_m,x_m)n_m\xi_m^2$$

and $F_{1,m,s}(\underline{\mathbf{x}})$ *be given by (8.8).*

Suppose that

$$det(\nabla_{x_i}F_j(\underline{\mathbf{x}}^0))\neq 0. \tag{8.11}$$

Then

$$det(\nabla_{x_{j,k}}F_{1,m,s}(\underline{\mathbf{x}}^0))\neq 0. \tag{8.12}$$

Proof. We denote

$$(\nabla_{x_i} F_j(\underline{\mathbf{x}}^0)) = (m_{ij}).$$

Note that m_{ij} is the (i,j)-th element of the matrix $\mathcal{M}$ defined by (2.15).

Then, by definition, it is easy to see that

$$\begin{aligned}
w\left(\frac{x^0_{m,s}-x^0_{m,s-1}}{\epsilon}\right) &= \frac{\epsilon}{2D}\xi^0_m[(s-1)(n_m+1-s)] + o(\epsilon),\ s=2,\dots,n_m,\\
w\left(\frac{x^0_{m,s}-x^0_{m,k}}{\epsilon}\right) &= O(\epsilon^2),\ |s-k|\geq 2,\\
\nabla_{x_{j,t}} F_{1,i,s} &= \begin{cases} m_{ij}, \text{ if } i\neq j,\\ c_0(i)a^i_{st} + m_{ii}, \text{ if } i=j,\end{cases}
\end{aligned}$$

where $c_0(i) = \frac{\xi^0_i}{2D} > 0$ and a^i_{st} is the (s,t)-th element of the following $(n_i \times n_i)$ matrix

$$\mathcal{A}^i = (a^i_{st}) = \begin{pmatrix} a^i_{11} & a^i_{12} & 0 & \ddots & \ddots & 0\\ a^i_{21} & a^i_{22} & a^i_{23} & \ddots & \ddots & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\ 0 & \ddots & \ddots & 0 & a^i_{n_i(n_i-1)} & a^i_{n_in_i}\end{pmatrix}, \tag{8.13}$$

where

$$\begin{aligned}
a^i_{11} &= (n_i-1), a^i_{12} = -(n_i-1),\\
a^i_{s(s-1)} &= -(s-1)(n_i+1-s),\\
a^i_{ss} &= -(n_i+1-2s),\\
a^i_{s(s+1)} &= s(n_i-s), s=2,\dots,n_i-1,\\
a^i_{n_i(n_i-1)} &= -(n_i-1), a^i_{n_in_i} = n_i-1.
\end{aligned}$$

Observe that

$$\sum_{t=1}^{n_i} a^i_{st} = \sum_{s=1}^{n_i}(-1)^s a^i_{st} = 0 \tag{8.14}$$

and zero is a simple eigenvalue of $\mathcal{A}^i$. (See [2].)

Suppose that we have

$$\sum_{i=1}^{N}\sum_{s=1}^{n_i} \nabla_{x_{i,s}} F_{1,j,t}\eta_{i,s} = 0, \quad j=1,\dots,N,\ t=1,\dots,n_j.$$

This implies

$$0=\sum_{i=1}^{N}\sum_{s=1}^{n_i}\nabla_{x_{j,t}}F_{1,i,s}\eta_{i,s}=\sum_{i=1}^{N}m_{ij}\sum_{s=1}^{n_i}\eta_{i,s}+\sum_{i=1}^{N}c_0(i)\sum_{s=1}^{n_i}a^i_{st}\eta_{i,s}.$$

By (8.14),

$$\begin{aligned}0&=\sum_{t=1}^{n_j}\sum_{i=1}^{N}m_{ij}\sum_{s=1}^{n_i}\eta_{i,s}+\sum_{i=1}^{N}c_0(i)\sum_{s=1}^{n_i}\left(\sum_{t=1}^{n_j}a^i_{st}\right)\eta_{i,s}\\&=\sum_{i=1}^{N}m_{ij}n_j\sum_{s=1}^{n_i}\eta_{i,s}.\end{aligned}$$

By assumption (8.11),

$$\sum_{s=1}^{n_i}\eta_{i,s}=0.$$

Hence, we have

$$\sum_{s=1}^{n_i}a^i_{st}\eta_{i,s}=0,\quad \sum_{s=1}^{n_i}\eta_{i,s}=0. \tag{8.15}$$

This implies that

$$\eta_{i,t}=0$$

by (8.14) and since zero is a simple eigenvalue of $\mathcal{A}^i$. This proves (8.12). □

By Lemma 8.1, at $\underline{\mathbf{x}}^0$, we have $F_1(\underline{\mathbf{x}}^0)=O(\epsilon^{3/4})$ and

$$\det(\nabla_{\underline{\mathbf{x}}^0}F_1(\underline{\mathbf{x}}^0))\neq 0.$$

Therefore we may write W_ϵ as

$$\begin{aligned}W_\epsilon(\underline{\mathbf{x}})&=F_1(\underline{\mathbf{x}})+O(\epsilon^{3/4})\\&=\nabla F_1(\underline{\mathbf{x}}^0)(\underline{\mathbf{x}}-\underline{\mathbf{x}}^0)+O(|\underline{\mathbf{x}}-\underline{\mathbf{x}}^0|^2)+O(\epsilon^{3/4}).\end{aligned}$$

By Lemma 8.1 and Brouwer's fixed point theorem it follows that for $\epsilon<<1$ there exists a $\underline{\mathbf{x}}^\epsilon\in\otimes_\eta$ such that $W_\epsilon(\underline{\mathbf{x}}^\epsilon)=0$.

Thus we have proved the following proposition.

Proposition 8.2 *For ϵ sufficiently small there exist points $\underline{\mathbf{x}}^\epsilon$ with $\underline{\mathbf{x}}^\epsilon\to\underline{\mathbf{x}}^0$ such that $W_\epsilon(\underline{\mathbf{x}}^\epsilon)=0$.*

Finally, we prove Theorem 2.2.

Proof of Theorem 2.2: By Proposition 8.2, there exists $\underline{\mathbf{x}}^\epsilon\to\underline{\mathbf{x}}^0$ such that $W_\epsilon(\underline{\mathbf{x}}^\epsilon)=0$. In other words, $\mathcal{S}_\epsilon[w_{\epsilon,\underline{\mathbf{x}}^\epsilon}+\phi_{\epsilon,\underline{\mathbf{x}}^\epsilon}]=0$. Let $A_\epsilon=\xi_\epsilon(w_{\epsilon,\underline{\mathbf{x}}^\epsilon}+\phi_{\epsilon,\underline{\mathbf{x}}^\epsilon}), H_\epsilon=\xi_\epsilon T[w_{\epsilon,\underline{\mathbf{x}}^\epsilon}+\phi_{\epsilon,\underline{\mathbf{x}}^\epsilon}]$. By the Maximum Principle, $A_\epsilon>0, H_\epsilon>0$. Moreover (A_ϵ,H_ϵ) satisfies all the properties of Theorem 2.2. □

9 Proof of Theorem 2.1

In this section, we show how Theorem 2.1 can be proved easily without any assumption on D. In fact, by reflection, we may assume that $N = 1$. We may further assume that $A(-x) = A(x), H(-x) = H(x)$. There are two cases to be considered: n is even or n is odd. We choose $x_1 < x_2 < \ldots < x_n$ to be such that

$$\begin{aligned} \epsilon \log \frac{1}{\epsilon} - \epsilon \log \left[\frac{\hat{\xi}_l^0}{2D}(l-1)(n+1-l) \right] - \eta\epsilon \leq x_l - x_{l-1} \\ \leq \epsilon \log \frac{1}{\epsilon} - \epsilon \log \left[\frac{\hat{\xi}_l^0}{2D}(l-1)(n+1-l) \right] + \eta\epsilon \end{aligned} \tag{9.1}$$

and

$$\sum_{j=1}^{n} x_j = 0 \tag{9.2}$$

Thus we have $(n-1)$ independent variables from $(x_1, \ldots, x_n)$. On the other hand, the matrix $\mathcal{A}^i$ with $n_i = n$ has exactly $(n-1)$ nonzero eigenvalues and one zero eigenvalue. So if we proceed as in Section 8, we have nondegeneracy. Similar arguments as in Section 8 give the conclusion of Theorem 2.1.

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