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Nonlinear Differential Equations
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On very weak solutions of degenerate p-harmonic equations

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Abstract. We establish a regularity result for very weak solutions of some degenerate elliptic PDEs. The nonnegative function which measures the degree of degeneracy of ellipticity bounds is assumed to be exponentially integrable. We find that the scale of improved regularity is logarithmic.

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1 Introduction

In this paper we present estimates for nonlinear differential equations in which the ellipticity bounds degenerate.

We consider the following equation

$$\operatorname{div} A(x, Du) = 0 \qquad \text{in} \quad \Omega \subset \mathbb{R}^n \tag{1}$$

for a mapping $u : \Omega \to \mathbb{R}^m$, where Ω is a bounded open subset of $\mathbb{R}^n$. Following the lead of the familiar p-harmonic operator we suppose that $A : \Omega \times \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ satisfies the following assumptions, for almost every $x \in \Omega$, all $\xi, \eta \in \mathbb{R}^{n\times m}$ and all $\lambda \in \mathbb{R}$

$$|A(x,\xi) - A(x,\eta)| \le a(x)|\xi - \eta|(|\xi| + |\eta|)^{p-2}, \tag{2}$$

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle \ge b(x)|\xi - \eta|^2(|\xi| + |\eta|)^{p-2} \tag{3}$$

$$A(x,\lambda\xi) = |\lambda|^{p-2}\lambda A(x,\xi). \tag{4}$$

where $2 \leq p \leq n$, $0 < b(x) \leq a(x) < \infty$. The above three conditions imply the so-called distortion inequality

$$\frac{1}{p}|\xi|^p + \frac{p-1}{p}|A(x,\xi)|^{\frac{p}{p-1}} \leq K(x)\langle A(x,\xi),\xi\rangle\,. \tag{5}$$

The factor $K = K(x) \geq 1$ depends on $a(x)$ and $b(x)$ and is called distortion function for Equation (1).

In what follows $K(x)$ belongs to the exponential class $Exp(\Omega)$, defined via the Orlicz function $P(t) = e^t - 1$. Precisely, we assume that

$$\int_\Omega e^{\beta K(x)} dx \,<\, +\infty \tag{6}$$

for some $\beta > 0$.

Note that Equation (1) is the Euler - Lagrange equation of the variational integral

$$\mathcal{E}[u] = \int_\Omega \langle A(x,Du), Du\rangle dx \tag{7}$$

The natural setting for solutions of (1) pertains to the "finite energy" solutions, namely to the functions for which $\mathcal{E}[u]$ is finite. A simple use of Hölder inequality in Orlicz spaces implies that the gradient of a finite energy solution of the equation (1) lies in the Orlicz-Zygmund space $L^p \log^{-1} L_{loc}(\Omega, I\!R^{n\times m})$, which is contained in $L^q(\Omega, I\!R^{n\times m})$ for every $q < p$.

Recall that $L^p \log^\alpha L_{loc}(\Omega)$, $1 \leq p < +\infty$, $\alpha \in I\!R$, is the Orlicz space defined via the function $P(t) = t^p \log^\alpha(e+t)$, i.e. the space of all measurable functions f on Ω such that

$$\int_\Omega |f|^p \log^\alpha \left(e + \frac{|f|}{||f||_p}\right) dx \,< +\infty$$

A solution u of Equation (1) verifies the following integral identity

$$\int_\Omega \langle A(x,Du), D\Phi\rangle dx = 0$$

for all $\Phi \in C_0^\infty(\Omega; I\!R^m)$. It is worth pointing out that such identity remains valid also for functions not having "finite energy". This leads us to consider the so-called very weak solutions introduced by Iwaniec and Sbordone in [11].

Definition 1.1 A mapping $u \in W^{1,q}(\Omega; I\!R^m)$, for $q < p$, is a very weak solution of Equation (1) if

$$\int_\Omega \langle A(x,Du), D\Phi\rangle dx = 0$$

for all $\Phi \in C_0^\infty(\Omega; I\!R^m)$.

After [11], many papers have been devoted to the study of the regularity of such solutions (see for example [3], [7], [13]). All the previous results concern the case $K(x) \in L^\infty$ and show that very weak solutions are actually "finite energy" solutions provided the degree of the integrability of the gradient is not too far from the natural one.

As far as we are aware there have been no results concerning regularity of very weak solutions in the case $K(x)$ not bounded.

Here we fill this gap. Obviously, we have to confine ourselves to the study of mappings whose gradient is in a Zygmund class not too far from the natural one, i.e. the space $L^p \log^{-1} L_{loc}(\Omega; \mathbb{R}^{n\times m})$.

Theorem 1.2 *Let u be a very weak solution of (1) and assume that the distortion $K(x)$ satisfies the assumption in (6) for a fixed $\beta > 0$. There exists $\alpha = c_1(n)\beta > 0$ such that if*

$$Du \in L^p \log^{-\alpha-1} L_{loc}(\Omega; \mathbb{R}^{n\times m})$$

then u is a finite energy solution .

The techniques developed in previous papers do not seem to work here. Our new approach interplays between familiar results and classical tools such as Whitney cubes, maximal functions and an isoperimetric type inequality. In general, unfortunately, it is not possible to use test functions whose gradient is proportional to the gradient of the solution neither in treating very weak solutions nor in degenerate equations. In order to construct suitable test functions, we will need to adapt a well known technique due to Lewis ([12]), and later developed in [1], [3], [13], to the case of degenerate equations.

Recently, regularity properties of "finite energy" solutions have been investigated in [2], [9], [14], showing that the scale of improved degree of regularity is logarithmic.

Finally, it is worth pointing out that in the limit case $p = n$ our results give the same higher integrability property for the Jacobian of mappings with exponentially integrable distortion, recently proved in [5].

2 Preliminary results

This section is devoted to some results useful in the sequel. Let $g \in L^1(\mathbb{R}^n)$ and define the Hardy-Littlewood maximal function of g as

$$Mg(x) = \sup \left\{ \fint_Q |g| : x \in Q \subset \mathbb{R}^n \right\}$$

where Q is a cube with edges parallel to the coordinate axes. The following proposition holds:

Proposition 2.1 *Let $g \in L^1(\mathbb{R}^n)$. For all but countable number of parameters $t > 0$, we have*

$$\int_{|g|\geq t} |g| dx \leq Ct|\{x \in \mathbb{R}^n : \ Mg(x) > t\}| \leq C \int_{|g|>\frac{t}{2}} |g(x)| dx \tag{8}$$

The proof, based on Vitali's lemma and on the Calderon-Zygmund decomposition, is well known (see for example [16], [17]). Let us denote by $\chi_{g\leq t}$ the characteristic function of the level set $\{x \in U : g(x) \leq t\}$ and recall the following elementary fact from measure theory.

Lemma 2.2 *Let $g_k : U \to \mathbb{R}$ be measurable functions converging to g almost everywhere. Then for each regular value t of the limit function g, we have*

$$\chi_{g_k \leq t}(x) \to \chi_{g\leq t}(x) \qquad a.e. \ x \in U \tag{9}$$

Next Lemma reminds us Whitney's decomposition

Lemma 2.3 *Let $\mathcal{F}$ be a non-empty closed set in $\mathbb{R}^n$. There exists a disjoint collection of dyadic cubes $\{Q_1, Q_2, \dots\}$ such that*

$$\mathbb{R}^n \setminus \mathcal{F} = \bigcup_{i=1}^{\infty} Q_i$$

and

$$\mathrm{diam} Q_i \leq \mathrm{dist}(Q_i, \mathcal{F}) \leq 4 \, \mathrm{diam} Q_i \, .$$

Observe that, by triangle inequality, it is possible to get that $7nQ_i$ intersects $\mathcal{F}$ for $i = 1, \dots, n$. Moreover, for $Q_i^* = \frac{5}{4} Q_i$ for all $i = 1, \dots, n$ the following results hold ([16])

Proposition 2.4 *If Q_i^* and Q_k^* intersect, then*

$$\frac{1}{4} \mathrm{diam} Q_i \leq \mathrm{diam} Q_k \leq 4 \mathrm{diam} Q_i$$

Proposition 2.5 *For each fixed Q_i^*, there are at most 41^n cubes in the set $\{Q_1^*, Q_2^*, \dots\}$ which intersect Q_i^*.*

3 An Isoperimetric type Inequality

Our starting point is the following isoperimetric type inequality proved in [8] under more general assumptions.

Proposition 3.1 *Let E be a curl free matrix field in $L^q(\Omega, \mathbb{R}^{n\times m})$ and B a divergence free matrix field in $L^{q^*}(\Omega, \mathbb{R}^{n\times m})$ where $1 < q, q^* < \infty$ satisfy $\frac{1}{q} + \frac{1}{q^*} = 1 + \frac{1}{n}$. Suppose that $H = |E|^q + |B|^s$, $\frac{1}{q} + \frac{1}{s} = \frac{n}{n-1}$, is such that*

$$\liminf_{t\to\infty} t^{\frac{1}{n}} \int_{H>t} H(x)\, dx = 0 \tag{10}$$

Then

$$\left| \int_Q \langle B, E\rangle\, dx \right| \le C(n) \left(\int_{\partial Q} H\, d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}, \tag{11}$$

In the sequel we shall need to control the boundary integrals by the volume integrals. The next Lemma, which can be found in [6], is crucial to this aim.

Lemma 3.2 *Let us give a Whitney's decomposition $\Omega = \bigcup_{i=1}^{\infty} Q_i$ and f a function in $C_0^\infty(\Omega)$. There exist concentric cubes $Q_i \subset \Box_i \subset Q_i^* \subset 7nQ_i$ such that*

$$\left(\int_{\partial\Box_i} |f|\, d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}} \le C(n)\, |Q_i| \left(\fint_{7nQ_i} |f|\, dx \right)^{\frac{n}{n-1}} \tag{12}$$

for all $i = 1, 2, \ldots$

Finally we recall the following version of the Sobolev-Poincaré inequality (see [11]).

Lemma 3.3 *For each matrix field $A \in L^1_{loc}(\Omega; \mathbb{R}^{n\times m})$ with $\operatorname{div} A(x) \in L^r(\mathbb{R}^n)$, $1 < r < \infty$, there exists a divergence free matrix field $A_o \in L^1_{loc}(\Omega; \mathbb{R}^{n\times m})$ such that*

$$\left(\int_B |A(x) - A_o|^{\frac{nr}{n-r}}\, dx \right)^{\frac{n-r}{nr}} \le C(n,r) \left(\int_B |\operatorname{div} A(x)|^r\, dx \right)^{\frac{1}{r}} \tag{13}$$

for every ball B strictly contained in Ω.

4 The Main Estimate

This section is devoted to the following crucial estimate. The idea of the proof comes from [6], Theorem 9.1.

Theorem 4.1 *Let E be a curl free matrix field in $L^q(\Omega, \mathbb{R}^{n\times m})$ and B a divergence free matrix field in $L^s(\Omega, \mathbb{R}^{n\times m})$, $\frac{1}{s} + \frac{1}{q} = \frac{n}{n-1}$, with compact support. For all but countable number of parameters $t > 0$ we have*

$$\begin{aligned} \left| \int_{MH\le 2t} \langle B, E\rangle\, dx \right| &\le C(n)\, t^{\frac{n}{n-1}} \left| \{x \in \mathbb{R}^n : MH > 2t\} \right| \\ &\le C(n)\, t^{\frac{1}{n-1}} \int_{H>t} H(x)\, dx \end{aligned} \tag{14}$$

where $H = |B|^s + |E|^q$.

Proof. We need only to prove the first of the two inequalities in (14); the second, and the restriction on t, being a consequence of the maximal inequality stated in Proposition 2.1. We first prove it for smooth matrix fields and then we proceed by an approximation argument.

First Step: E, B *in* $C_0^\infty(\Omega; \mathbb{R}^{n\times m})$.

By a simple use of Divergence Theorem we have the following identity

$$\int_{\mathcal{F}} \langle B, E\rangle \; dx = -\int_{\mathbb{R}^n \setminus \mathcal{F}} \langle B, E\rangle \; dx \tag{15}$$

where $\mathcal{F}$ is a closed subset of $\mathbb{R}^n$. Choose $\mathcal{F}$ to be the level set of MH, namely

$$\mathcal{F} = \{x \in \mathbb{R}^n : MH(x) \leq 2t\}$$

and consider Whitney's decomposition of $\mathbb{R}^n \setminus \mathcal{F}$ into dyadic cubes

$$\mathbb{R}^n \setminus \mathcal{F} = \bigcup_{i=1}^{\infty} Q_i,$$

(see Lemma 2.3 and the subsequent Propositions 2.4 and 2.5). Using Lemma 3.2 we find concentric cubes $Q_i \subset \square_i \subset Q_i^* \subset 7nQ_i$, for which

$$\left(\int_{\partial \square_i} H\right)^{\frac{n}{n-1}} \leq 4\, C(n)\, |Q_i|\, t^{\frac{n}{n-1}} \tag{16}$$

This latter bound is due to the fact that the expanded cubes $7\,n\,Q_i$ intersect $\mathcal{F}$. Although the cubes $\square_i$, $i = 1, 2, \ldots$ may overlap slightly, there can exist at most $N = 41^n$ of such cubes having nonempty intersection. This is why we have the following finite expansion

$$\begin{aligned}\int_{\mathbb{R}^n \setminus F} \langle B, E\rangle \; dx \;=\; & \sum_{1\leq i_1} \int_{\square_{i_1}} \langle B, E\rangle \; dx - \sum_{1\leq i_1 < i_2} \int_{\square_{i_1 i_2}} \langle B, E\rangle \; dx \\ & + \sum_{1\leq i_1<i_2<i_3} \int_{\square_{i_1 i_2 i_3}} \langle B, E\rangle \; dx \\ & - - - - - - - - - - - \\ & (-1)^{N+1} \sum_{1\leq i_1 < \cdots < i_N} \int_{\square_{i_1 \cdots i_N}} \langle B, E\rangle \; dx \end{aligned} \tag{17}$$

where $\square_{i_1 \ldots i_N}$ stands for the rectangle $\square_{i_1} \cap \cdots \cap \square_{i_N}$. Now, applying Proposition 3.1 and using (16) we get

$$\left| \int_{\square_{i_1, \ldots, i_k}} \langle B, E\rangle \right| \leq C(n)\, t^{\frac{n}{n-1}} \;\left(|Q_{i_1}| + \cdots + |Q_{i_k}|\right) \tag{18}$$

for all $1 \leq k \leq 41^n$. The constant $C(n)$ may vary from line to line, but only depends on n. Obviously, this estimate plays no role if the rectangle $\Box_{i_1,\dots,i_k} = \Box_{i_1} \cap \dots \cap \Box_{i_k}$ is empty. Substituting (18) into (17) we obtain

$$\left| \int_{MH \leq 2t} \langle B, E \rangle \; dx \right| \leq C(n)\, t^{\frac{n}{n-1}} \sum_{i=1}^{\infty} c_i \, |Q_i| \tag{19}$$

Here each coefficient c_i indicates the number of times that the given cube $\Box_i$ is present in $\Box_{i_1} \cap \dots \cap \Box_{i_1} \neq \emptyset$ with $1 \leq i_1 < \dots < i_k$ and $k = 1, 2, \dots, N$. For a given $k = 1, 2, \dots, N$, there are at most N^{k-1} expressions $\Box_{i_1} \cap \dots \cap \Box_{i_k} \neq \emptyset$ in which $\Box_i$ appears as one of its terms, as $\Box_i$ may intersect at most 41^n of the cubes $\Box_1, \Box_2, \dots$ (see Proposition 2.5) . Consequently, we have the following uniform bound for these coefficients.

$$c_i \leq 1 + N + N^2 + \dots + N^{N-1} \leq N^N$$

Therefore inequality (19) reduces to

$$\begin{aligned} \left| \int_{MH \leq 2t} \langle B, E \rangle \; dx \right| \; &\leq \; C(n)\, t^{\frac{n}{n-1}} \sum_{i=1}^{\infty} |Q_i| = C(n)\, t^{\frac{n}{n-1}} \, |\mathbb{R}^n \backslash \mathcal{F}| \\ &= \; C(n)\, t^{\frac{n}{n-1}} \, |\{x \in \mathbb{R}^n : MH(x) > 2t\}| \end{aligned} \tag{20}$$

Second Step: *Approximation procedure*

By a standard mollification argument, we can consider two sequences E_k, $B_k \in C_0^\infty(\Omega, \mathbb{R}^{n\times m})$ such that $E_k \to E$ in $L^q(\Omega, \mathbb{R}^{n\times m})$ and $B_k \to B$ in $L^s(\Omega, \mathbb{R}^{n\times m})$.

We can certainly assume that $E_k \to E$, $B_k \to B$ and $\langle B_k, E_k \rangle \to \langle B, E \rangle$ at almost every point $x \in \mathbb{R}^n$. Since the functions $H_k = |B_k|^s + |E_k|^q$ converge to $H = |B|^s + |E|^q$ in $L^1(\mathbb{R}^n)$, we find that $MH_k \to MH$ weakly in $L^1(\mathbb{R}^n)$, thus in measure as well. In particular,

$$\lim_{k\to\infty} \left|\left\{x \in \mathbb{R}^n : \, MH_k(x) > 2t\right\}\right| = \left|\left\{x \in \mathbb{R}^n : \, MH(x) > 2t\right\}\right| \tag{21}$$

for every $t > 0$. We shall confine ourselves to a subsequence, again denoted by $\{MH_k\}$, so that $MH_k \to MH$ almost everywhere.

Since inequality (14) is valid in $C_0^\infty(\Omega; \mathbb{R}^{n\times m})$ we have

$$\left| \int_{MH_k \leq 2t} \langle B_k, E_k \rangle \, dx \right| \leq C(n)\, t^{\frac{n}{n-1}} \, \left|\left\{x \in \mathbb{R}^n : \, MH_k > 2t\right\}\right| \tag{22}$$

Passing to the limit in the right hand side is legitimate, by (21). Moreover the bounds

$$|\langle B_k, E_k \rangle| \leq H_k^{\frac{n}{n-1}} \leq |MH_k(x)|^{\frac{n}{n-1}}$$

imply that

$$\chi_{MH_k \le 2t}(x)\langle B_k, E_k\rangle \le (2t)^{\frac{n}{n-1}}\chi_\Omega(x) \tag{23}$$

Then Lemma 2.2 and (23) allow us to use the Lebesgue Dominated Convergence Theorem and to conclude that

$$\int_{MH_k\le 2t}\langle B_k, E_k\rangle\,dx - \int_{MH\le 2t}\langle B, E\rangle\,dx \;=\; \int_\Omega \big[\chi_{MH_k\le 2t}(x)\langle B_k, E_k\rangle - \chi_{MH\le 2t}(x)\langle B, E\rangle\big]\,dx \to 0$$

The proof of Theorem 4.1 is now complete. □

5 Proof of theorem 1.2

This section is devoted to the proof of our main theorem.

Proof. (of Theorem 1.2) Let us fix a ball B_0 strictly contained in Ω and a function $\varphi \in C_0^\infty(\Omega)$ such that $0 \le \varphi \le 1$ and supp$\varphi \subset B_0$. Consider $\tilde u = \varphi^{\frac{p}{p-1}} \cdot u$ and observe that it belongs to the Sobolev space $W^{1,q}(B_0; I\!R^m)$ for all $q < p$. To shorten the notation, we introduce the function

$$\mathcal{A}(x) = \varphi^p(x) A(x, Du) = A(x, \varphi^{\frac{p}{p-1}}\,Du)$$

The last identity is due to assumption (4). Equation (1) yields for all $q < p$

$$\mathrm{div}\mathcal{A}(x) = pA(x, Du)\varphi^{p-1}\nabla\varphi \in L^{\frac{q}{p-1}}(B_0; I\!R^m)$$

Applying the divergence operator, we obtain

$$\mathrm{div}A(x, \varphi^{\frac{p}{p-1}}\,Du) = \mathrm{div}(\mathcal{A}(x) - \mathcal{A}_o)$$

where $\mathcal{A}_o$ can be any divergence free matrix field. We use Lemma 3.3 to choose $\mathcal{A}_o$ such that (13) holds. Obviously

$$|\mathcal{A}(x) - \mathcal{A}_o| \in L^s(B_0) \text{ for every } s < \frac{nq}{n(p-1) - q}.$$

Then $\tilde u$ solves the following nonhomogeneous equation

$$\mathrm{div}A(x, D\tilde u) = \mathrm{div}g$$

where

$$g(x) = [\mathcal{A}(x) - \mathcal{A}_o] + [A(x, D\tilde u) - A(x, \varphi^{\frac{p}{p-1}}\,Du)]$$

belongs to $L^{\frac{q}{p-1}}(B_0; I\!R^{n\times m})$, for all $q < p$.

Now let us associate to $\tilde{u}$ the matrix fields $\tilde{B} = A(x, D\tilde{u}) - g$, $\tilde{E} = D\tilde{u}$ which are divergence and curl free, respectively. Setting $\tilde{H} = |\tilde{B}|^s + |\tilde{E}|^q$, with $q = \frac{p(n-1)}{n}$ and $s = \frac{p(n-1)}{(p-1)n}$, we can easily check that

$$\begin{aligned}
\int_{\tilde{H}\le 2t} \langle \tilde{B}, \tilde{E}\rangle \, dx &\le \int_{\{\tilde{H}\le 2t\}\cap\{M\tilde{H}\le 2t\}} \langle \tilde{B}, \tilde{E}\rangle \, dx + \int_{\{\tilde{H}\le 2t\}\cap\{M\tilde{H}> 2t\}} \langle \tilde{B}, \tilde{E}\rangle \, dx \\
&\le \int_{M\tilde{H}\le 2t} \langle \tilde{B}, \tilde{E}\rangle \, dx + c\int_{\{\tilde{H}\le 2t\}\cap\{M\tilde{H}> 2t\}} \tilde{H}^{\frac{n}{n-1}} \, dx \\
&\le \int_{M\tilde{H}\le 2t} \langle \tilde{B}, \tilde{E}\rangle \, dx + ct^{\frac{n}{n-1}} |\{M\tilde{H} > 2t\}| \\
&\le \int_{M\tilde{H}\le 2t} \langle \tilde{B}, \tilde{E}\rangle \, dx + ct^{\frac{1}{n-1}} \int_{\tilde{H}>t} \tilde{H} \, dx
\end{aligned}$$

where in the last inequality we used Proposition 2.1.

Therefore, applying Theorem 4.1, we deduce

$$\int_{\tilde{H}\le 2t} \langle \tilde{B}, \tilde{E}\rangle \, dx \le ct^{\frac{1}{n-1}} \int_{\tilde{H}>t} \tilde{H}(x) \, dx \tag{24}$$

Elementary calculations give

$$\langle \tilde{B}, \tilde{E}\rangle \;=\; \varphi^{p+p'} \langle B, E\rangle + \varphi^{p'} \langle \mathcal{A}_o(x) - \mathcal{A}(x), Du\rangle + p'\varphi^{p'-1}\nabla\varphi \langle \mathcal{A}_o(x), u\rangle$$

where p' denotes the Hölder conjugate exponent of p.

Setting $F = |\mathcal{A}_o - \mathcal{A}||Du| + |\nabla\varphi||\mathcal{A}_o||u|$ we have

$$\int_{\tilde{H}\le 2t} \varphi^{p+p'} \langle B, E\rangle \, dx \le \int_{\tilde{H}\le 2t} \langle \tilde{B}, \tilde{E}\rangle \, dx + c\int_{\tilde{H}\le 2t} F dx \tag{25}$$

and combining (24) and (25)

$$\int_{\tilde{H}\le 2t} \varphi^{p+p'} \langle B, E\rangle \, dx \le c\int_{\tilde{H}\le 2t} F dx + ct^{\frac{1}{n-1}} \int_{\tilde{H}>t} \tilde{H}(x) \, dx$$

Last inequality can be written as

$$\int_{\tilde{H}^{\frac{1}{n-1}}\le 2^{\frac{1}{n-1}}\lambda} \varphi^{p+p'} \langle B, E\rangle \, dx \le c\int_{\tilde{H}^{\frac{1}{n-1}}\le 2^{\frac{1}{n-1}}\lambda} F dx + c\lambda \int_{\tilde{H}^{\frac{1}{n-1}}>\lambda} \tilde{H}(x) \, dx \tag{26}$$

for $\lambda = t^{\frac{1}{n-1}}$. Consider the function

$$\Phi(\lambda) = \frac{1}{\lambda}(\log^{-\alpha-1}\lambda - (1+\alpha)\log^{-\alpha-2}\lambda) \tag{27}$$

where $\alpha > 0$ will be determined later.

Multiplying both sides of (26) by $\Phi(\lambda)$ and integrating with respect to λ between λ_0 and ∞ for $\lambda_0 > \max\{e^{1+\alpha}, e^{2\alpha}\}$ we get

$$\begin{aligned}
&\frac{1}{2\alpha}\int_{\tilde H^{\frac{1}{n-1}}<\lambda_0} \frac{\varphi^{p+p'}\langle B,E\rangle}{\log^\alpha \lambda_0}\,dx + \frac{1}{2\alpha}\int_{\tilde H^{\frac{1}{n-1}}>\lambda_0} \frac{\varphi^{p+p'}\langle B,E\rangle}{\log^\alpha \tilde H^{\frac{1}{n-1}}}\,dx \\
&\leq \frac{c(n)}{\alpha}\int_{B_0} \frac{F}{\log^\alpha \max\{\lambda_0, \tilde H^{\frac{1}{n-1}}\}}\,dx + \int_{\tilde H^{\frac{1}{n-1}}>\lambda_0} \frac{\tilde H^{\frac{n}{n-1}}}{\log^{1+\alpha}\tilde H^{\frac{1}{n-1}}}\,dx \qquad (28)
\end{aligned}$$

Note that for $\lambda > e^{1+\alpha}$ the function $\Phi(\lambda)$ is nonnegative. By the definition of $\tilde H$ one can easily check that

$$\tilde H^{\frac{n}{n-1}} \leq c\Big[\varphi^{p+p'} H^{\frac{n}{n-1}} + G\Big]$$

where $G = |\nabla\varphi|^p|u|^p + |\mathcal{A}-\mathcal{A}_o|^{p'}$. Using that the function $t^n\log^{-(1+\alpha)} t$ is increasing for $t > \lambda_0$ and the assumption in (5) we get

$$\begin{aligned}
\frac{\tilde H^{\frac{n}{n-1}}}{\log^{1+\alpha}\tilde H^{\frac{1}{n-1}}} &\leq n\frac{\varphi^{p+p'} H^{\frac{n}{n-1}} + G}{\log^{1+\alpha}\Big[\varphi^{p+p'} H^{\frac{n}{n-1}} + G\Big]} \\
&\leq n\frac{\varphi^{p+p'} K(x)\langle B,E\rangle + G}{\log^{1+\alpha}\Big[\varphi^{p+p'} H^{\frac{n}{n-1}} + G\Big]}
\end{aligned}$$

in the set where $\tilde H^{\frac{1}{n-1}} > \lambda_0$. The elementary inequality

$$ab \leq a\log(1+a) + e^b - 1$$

for non-negative real numbers, implies

$$\varphi^{p+p'}K(x)\langle B,E\rangle \leq \frac{2}{\beta}\left[exp\left(\frac{\beta}{2}K(x)\right) + \varphi^{p+p'}\langle B,E\rangle\log(1+\varphi^{p+p'}\langle B,E\rangle)\right]$$

Therefore, previous estimates yield

$$\begin{aligned}
\frac{\tilde H^{\frac{n}{n-1}}}{\log^{1+\alpha}\tilde H^{\frac{1}{n-1}}} \leq \frac{C(n)}{\beta}\Bigg[&\frac{exp\left(\frac{\beta}{2}K(x)\right)}{\log^{1+\alpha}\tilde H^{\frac{1}{n-1}}} + \frac{G}{\log^{1+\alpha}\tilde H^{\frac{1}{n-1}}} \\
&+ \frac{\varphi^{p+p'}K(x)\langle B,E\rangle\log(1+\varphi^{p+p'}\langle B,E\rangle)}{\log^{1+\alpha}\Big[\varphi^{p+p'}H^{\frac{n}{n-1}} + G\Big]}\Bigg] \\
\leq \frac{C(n)}{\beta}\Bigg[&\frac{exp\left(\frac{\beta}{2}K(x)\right)}{\log^{1+\alpha}\tilde H^{\frac{1}{n-1}}} + \frac{G}{\log^{1+\alpha}\tilde H^{\frac{1}{n-1}}} \\
&+ \frac{\varphi^{p+p'}K(x)\langle B,E\rangle}{\log^{\alpha}\Big[\varphi^{p+p'}H^{\frac{n}{n-1}} + G\Big]}\Bigg]
\end{aligned}$$

Integrating over the set where $\tilde{H}^{\frac{1}{n-1}} > \lambda_0$ we obtain

$$\begin{aligned}
&\int_{\tilde{H}^{\frac{1}{n-1}} > \lambda_0} \frac{\tilde{H}^{\frac{n}{n-1}}}{\log^{1+\alpha} \tilde{H}^{\frac{1}{n-1}}}\, dx \\
&\leq \frac{C(n)}{\beta} \int_{\tilde{H}^{\frac{1}{n-1}} > \lambda_0} \left[exp\left(\frac{\beta}{2} K(x)\right) + \frac{\varphi^{p+p'} \langle B, E\rangle}{\log^{\alpha} \tilde{H}^{\frac{1}{n-1}}} \right] dx \\
&\quad + C(n) \int_{\tilde{H}^{\frac{1}{n-1}} > \lambda_0} \frac{G}{\log^{1+\alpha} \tilde{H}^{\frac{1}{n-1}}}\, dx
\end{aligned}$$

Now combining the last inequality with (28), we have

$$\begin{aligned}
&\frac{1}{2\alpha \log^{\alpha} \lambda_0} \int_{\tilde{H}^{\frac{1}{n-1}} < \lambda_0} \varphi^{p+p'} \langle B, E\rangle\, dx \\
&\leq \left(\frac{c(n)}{\beta} - \frac{1}{2\alpha}\right) \int_{\tilde{H}^{\frac{1}{n-1}} > \lambda_0} \frac{\varphi^{p+p'} \langle B, E\rangle}{\log^{\alpha} \tilde{H}^{\frac{1}{n-1}}}\, dx \\
&+ \frac{c(n)}{\beta} \int_{\tilde{H}^{\frac{1}{n-1}} > \lambda_0} exp\left(\frac{\beta}{2} K(x)\right) dx + C(n) \int_{\tilde{H}^{\frac{1}{n-1}} > \lambda_0} \frac{G}{\log^{1+\alpha} \tilde{H}^{\frac{1}{n-1}}}\, dx \\
&+ \frac{C(n)}{\alpha} \int_{B_0} \frac{F}{\log^{\alpha} \max\{\lambda_0, \tilde{H}^{\frac{1}{n-1}}\}}\, dx
\end{aligned}$$

In order to have $\dfrac{C(n)}{\beta} - \dfrac{1}{2\alpha}$ negative, we choose $\alpha = \dfrac{\beta}{4C(n)}$, thus obtaining

$$\begin{aligned}
&\frac{1}{(n-1) 2\alpha \log^{\alpha} \lambda_0} \int_{\tilde{H}^{\frac{1}{n-1}} < \lambda_0} \varphi^{p+p'} \langle B, E\rangle\, dx \\
&\leq \frac{C(n)}{\beta} \int_{\tilde{H}^{\frac{1}{n-1}} > \lambda_0} exp\left(\frac{\beta}{2} K(x)\right) dx + C(n) \int_{\tilde{H}^{\frac{1}{n-1}} > \lambda_0} \frac{G}{\log^{1+\alpha} \tilde{H}^{\frac{1}{n-1}}}\, dx \\
&+ \frac{C(n)}{\alpha} \int_{B_0} \frac{F}{\log^{\alpha} \max\{\lambda_0, \tilde{H}^{\frac{1}{n-1}}\}}\, dx
\end{aligned}$$

Multiplying both sides of last inequality by $\log^{\alpha} \lambda_0$, letting λ_0 tend to infinity, and recalling the definition of F, it follows by monotone convergence theorem and Lebesgue dominated convergence theorem that

$$\int_{B_0} \langle B, E\rangle\, dx \leq C(n) \int_{B_0} \Big[|\nabla \varphi|\, |\mathcal{A}_o|\, |u| + |\mathcal{A} - \mathcal{A}_o|\, |Du| \Big]\, dx$$

since $exp(\frac{\beta}{2} K(x))$, $|u|^p$, $|\mathcal{A} - \mathcal{A}_o|^{p'}$, $|\mathcal{A}_o|\, |u|$ and $|\mathcal{A} - \mathcal{A}_o|\, |Du|$ are integrable. In conclusion $\langle B, E\rangle \in L^1_{loc}(\Omega)$ and then u has finite energy. $\square$

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