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# A remark on the junction in a thin multi-domain: the non convex case

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**Abstract.** Our aim consists of studying, in the spirit of Gamma convergence, a dimension reduction problem for a multi-domain filled of either an hyperelastic material or a non simple grade-two one. We derive asymptotically the limit energy density starting from a sample described trough non convex bulk energy densities, depending either on the first or second order derivative of the displacement.

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## 1 Introduction

The purpose of this article is to derive the energy density of a multi-domain constituted either by non simple grade two material or by a hyperelastic material, both described through non convex bulk energy densities, by making use of the basic principles of the Γ convergence method. It is a sequel of previous works investigating junction conditions in thin multi-domains modelled either by convex second order bulk energies or by convex energies depending on the first order derivatives of the displacement, see [17] and [13], respectively. The bulk of presented proofs is mainly concerned about the second order case, and we present the result in the gradient case just in the last section, the motivation being that, though not a corollary the gradient case can be given in a more classical framework.

The model problem can be described through

$$\int_{\Omega_n} W(D^s U_n)dx \qquad (s = 1, 2) \tag{1.1}$$

where $\Omega_n$ stands for the thin multi-domain, as in figure

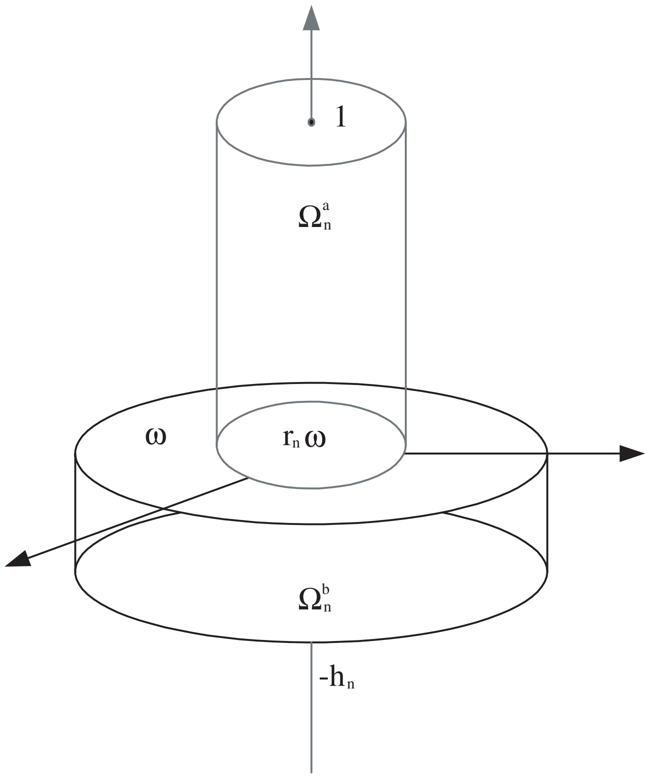


Figure 1 The thin multidomain.

$W$ is a continuous energy density with $p$ growth ($p > 1$) from above and below, that may be defined either on triples of symmetric $3\times3$ matrices or on $3\times3$ matrices, in dependence on the fact that the model describes non simple grade two materials or takes into account 'classical' nonlinear elastic energies, and $U_n$ is the deformation defined on $\Omega_n$ subject to suitable boundary condition, while $D^s$ stands for the derivatives of order $s = 1, 2$. In the sequel, $x = (x_1, x_2, x_3) = (x', x_3)$ denotes the generic point of $\mathbb{R}^3$, $\mathbb{M}^{m\times N}$, the set of $m\times N$ matrices, $Sym(\mathbb{R}^{k\times k})$ (for $k = 3, 2$) the set of the triples of completely symmetric $k\times k$- bilinear forms. We denote by $D$ and $D^2$ the gradient and the hessian tensor, while, $D_{x'}$ and $D^2_{x'}$, $D_{x_3}$ and $D^2_{x_3}$ stand for the gradient and the hessian tensor with respect to the first 2 variables, for the first and the second derivative with respect to the last variable, respectively. Then, according to these notations, $D^2_{x',x_3}$ stands for $D_{x'}(D_{x_3})$.

Let $s = 2$, let $\omega \subset \mathbb{R}^2$ be a bounded open connected and simply connected set such that the origin in $\mathbb{R}^2$, denoted by $0'$, belongs to $\omega$ and $\mathcal{L}^2(\omega) = 1$, and let $\{r_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}} \subset ]0, 1[$ be two sequences such that

$$\lim_n h_n = 0 = \lim_n r_n. \tag{1.2}$$

For every $n \in \mathbb{N}$, consider the thin multi-domain $\Omega_n = \Omega_n^a \cup \Omega_n^b$ ($a$ for "above", $b$ for "below") union of two vertical cylinders, one placed upon the other: a "wire" $\Omega_n^a = r_n\omega \times [0,1[$ with small cross section $r_n\omega$ and constant height, a " thin film" $\Omega_n^b = \omega \times ]-h_n, 0[$ with small thickness $h_n$ and constant cross section. Moreover, set $\Omega = \omega \times ]-1,1[$.

For every $n \in \mathbb{N}$, consider the transformation fields $\overline{U}_n$ corresponding to equilibria, namely those fields which minimize or almost minimize the energy

$$U_n \longrightarrow \int_{\Omega_n} W\left(D^2 U_n\right) dx = \int_{\Omega_n} W \begin{pmatrix} D^2_{x'} U_n & \left(D^2_{x',x_3} U_n\right)^T \\ D^2_{x',x_3} U_n & D^2_{x_3} U_n \end{pmatrix} dx, \tag{1.3}$$

among all the functions $U_n \in W^{2,p}(\Omega_n;\mathbb{R}^3)$ realizing the Dirichlet boundary condition $c^a + d^a \cdot x'$ (the $\cdot$ here represents the row column product) on the top of $\Omega_n^a$, and $f^b + g^b x_3$ on the lateral surface of $\Omega_n^b$, for some $c^a \in \mathbb{R}^3$, $d^a \in \mathbb{R}^{3\times 2}$ and $f^b, g^b \in W^{2,p}(\omega;\mathbb{R}^3)$. As it is usual, one tries to reformulate the problem on a fixed domain through appropriate re-scalings which map $\Omega_n$ into $\Omega := \Omega^a \cup \Omega^b$. Namely, by setting

$$u_n(x) = \begin{cases} u_n^a(x', x_3) = U_n(r_n x', x_3), & (x', x_3) \text{ a.e. in } \Omega^a = \omega \times [0,1[; \\ \\ u_n^b(x', x_3) = U_n(x', h_n x_3), & (x', x_3) \text{ a.e. in } \Omega^b = \omega \times ]-1, 0[; \end{cases} \tag{1.4}$$

it is easily seen that $u_n^a \in W^{2,p}(\Omega^a;\mathbb{R}^3)$ assumes the re-scaled Dirichlet boundary condition $c^a + r_n d^a \cdot x'$ on the top of $\Omega^a$, $u_n^b \in W^{2,p}(\Omega^b;\mathbb{R}^3)$ assumes the re-scaled Dirichlet boundary condition $f^b + h_n g^b x_3$ on the lateral boundary of $\Omega^b$. Moreover, $u_n = (u_n^a, u_n^b)$ satisfies the following junction conditions:

$$\begin{cases} u_n^a(x', 0) = u_n^b(r_n x', 0), & x' \text{ a.e. in } \omega; \\ \\ \dfrac{1}{r_n} D_{x'} u_n^a(x', 0) = (D_{x'} u_n^b)(r_n x', 0), & x' \text{ a.e. in } \omega; \\ \\ D_{x_3} u_n^a(x', 0) = \dfrac{1}{h_n} D_{x_3} u_n^b(r_n x', 0), & x' \text{ a.e. in } \omega; \end{cases}$$

and is an admissible field for the re-scaled energy:

$$u_n = (u_n^a, u_n^b) \longrightarrow \int_{\Omega^a} W \begin{pmatrix} \dfrac{1}{r_n^2} D^2_{x'} u_n^a & \left(\dfrac{1}{r_n} D^2_{x',x_3} u_n^a\right)^T \\ \dfrac{1}{r_n} D^2_{x',x_3} u_n^a & D^2_{x_3} u_n^a \end{pmatrix} dx + \\ \frac{h_n}{r_n^2} \int_{\Omega^b} W \begin{pmatrix} D^2_{x'} u_n^b & \left(\dfrac{1}{h_n} D^2_{x',x_3} u_n^b\right)^T \\ \dfrac{1}{h_n} D^2_{x',x_3} u_n^b & \dfrac{1}{h_n^2} D^2_{x_3} u_n^b \end{pmatrix} dx, \tag{1.5}$$

among all the functions $u_n$ subject to the same conditions of $\overline{u}_n$. Our aim consists of describing the limit energy in (1.5), as $n \to +\infty$, when the volumes of $\Omega_n^a$ and $\Omega_n^b$ tend to zero with the same rate, i.e.

$$\lim_n \frac{h_n}{r_n^2} = q \in ]0, +\infty[. \tag{1.6}$$

Define

$$K_n^a : u^a \in W^{2,p}(\Omega^a; \mathbb{R}^3) \longrightarrow \int_{\Omega^a} W \begin{pmatrix} \frac{1}{r_n^2} D^2_{x'} u^a & \left(\frac{1}{r_n} D^2_{x',x_3} u^a\right)^T \\ \frac{1}{r_n} D^2_{x',x_3} u^a & D^2_{x_3} u^a \end{pmatrix} dx, \tag{1.7}$$

$$K_n^b : u^b \in W^{2,p}(\Omega^b; \mathbb{R}^3) \longrightarrow \int_{\Omega^b} W \begin{pmatrix} D^2_{x'} u^b & \left(\frac{1}{h_n} D^2_{x',x_3} u^b\right)^T \\ \frac{1}{h_n} D^2_{x',x_3} u^b & \frac{1}{h_n^2} D^2_{x_3} u^b \end{pmatrix} dx, \tag{1.8}$$

and

$$\begin{aligned} \mathcal{U}_n = \Big\{ & (u^a, u^b) \in \left(c^a + r_n d^a \cdot x' + W_a^{2,p}(\Omega^a; \mathbb{R}^3)\right) \times \left(f^b + h_n g^b x_3 + W_b^{2,p}(\Omega^b; \mathbb{R}^3)\right) : \\ & u^a(x', 0) = u^b(r_n x', 0), \qquad x' \text{ a.e. in } \omega; \\ & \frac{1}{r_n} D_{x'} u^a(x', 0) = (D_{x'} u^b)(r_n x', 0), \quad x' \text{ a.e. in } \omega; \\ & D_{x_3} u^a(x', 0) = \frac{1}{h_n} D_{x_3} u^b(r_n x', 0), \quad x' \text{ a.e. in } \omega \Big\}, \end{aligned} \tag{1.9}$$

with $r_n$, $h_n$, $c^a$, $d^a$, $f^b$ and $g^b$ as defined before, $W_a^{2,p}(\Omega^a; \mathbb{R}^3)$ the closure, with respect to $W^{2,p}$-norm, of $\{u^a \in C^\infty(\overline{\Omega^a}; \mathbb{R}^3) : u^a = 0 \text{ in a neighbourhood of } \omega \times \{1\}\}$ and $W_b^{2,p}(\Omega^b; \mathbb{R}^3)$ the closure, with respect to $W^{2,p}$-norm, of $\{u^b \in C^\infty(\overline{\Omega^b}; \mathbb{R}^3) : u^b = 0 \text{ in a neighbourhood of } \partial\omega \times ]-1, 0[\}$. Without loss of generality, we may assume that

$$f^b = 0 = g^b \text{ a.e. in } B, \tag{1.10}$$

for some bi-dimensional ball $B$ such that $0' \in B \subset\subset \omega$ . We seek to determine the asymptotic behavior of

$$\inf \left\{ \begin{array}{l} \liminf_n \left( K_n^a(u_n^a) + \frac{h_n}{r_n^2} K_n^b(u_n^b) \right) : (u_n^a, u_n^b) \in \mathcal{U}_n, \\ (u_n^a, u_n^b) \rightharpoonup (u^a, u^b) \text{in } W^{2,p}, \left( \frac{1}{r_n} D_{x'} u_n^a, \frac{1}{h_n} D_{x_3} u_n^b \right) \rightharpoonup (\xi^a, \xi^b) \text{ in } W^{1,p} \end{array} \right\}, \tag{1.11}$$

with $u^a, u^b, \xi^a, \xi^b$ limit functions, $u^a$ and $\xi^a$ just depending on $x_3$ and $u^b$ and $\xi^b$ just depending on $x'$. We also emphasize that $u^a$ and $u^b$ represent the limit deformations 'above' and 'below' of the limit multi-structure, while $\xi^a$ and $\xi^b$, whose appearance comes from the presence of a second order energy, can be interpreted as Cosserat vectors for the limit multi-domain, namely $\xi^a$ (a pair of triples) which does not describe neither torsion or bending effect of the limit wire, but just keeps memory of the deformation of the cross section of the original $3-D$ wire, while $\xi^b$, is a vector taking into account the limit deformation of the middle surface of the $3-D$ film and not describing bending effects of the limit film below. In other words these vectors (two above in $\xi^a$ and one below $\xi^b$) keep memory of the unit normals to the middle curve (above) and surface (below) in the original thick multi-domain. As regards the limit multi-structure, one can also say that $(\xi^a, D_{x_3}u^a)$ constitutes the Cosserat triple for the 'limit wire', while $(D_{x'}u^b, \xi^b)$ is the analogous one for the limit film. The choice of the topology in (1.11) will be made clear in the sequel. The limit energy will be still represented through an integral functional, namely it will be given, (on the class $V^p$ in (2.3) below) by

$$\int_0^1 \hat{W}^{**}(D^2_{x_3}u^a, D_{x_3}\xi^a)dx_3 + q\int_\omega Q_{\mathcal{A}^2}W_0(D^2_{x'}u^b, D_{x'}\xi^b)dx', \tag{1.12}$$

where $Q_{\mathcal{A}^2}W_0$ and $\hat{W}^{**}$ are suitable 'convexifications' of the original energy density.

We point out that, as proven in [17], the limit problem is partially coupled by the junction condition:

$$u^a(0) = u^b(0') \text{ if } 1 < p \leq 2,$$

and coupled by the previous junction condition for $u$ and by the junction conditions:

$$\xi^a(0) = D_{x'}u^b(0'), \ \ D_{x_3}u^a(0) = \xi^b(0') \text{ if } 2 < p.$$

Moveover the minimizers of the limit problem depend also on the limit of the ratio between the volumes of $\Omega^a_n$ and $\Omega^b_n$.

The structure of the paper is the following. In section 2 some preliminaries are established, dealing essentially with $\Gamma$ convergence and convexity properties. Since the case of non simple grade two materials is primarily considered, section 3 is devoted to establish compactness and semicontinuity theorems to be exploited in the sequel. Section 4 concerns the $\Gamma$ convergence result. Finally the fifth section considers the case of hyperelastic energy density, i.e. the gradient case. Some remarks, essentially dealing with multi-structures in $\mathbb{R}^N$, for $N > 3$, both for gradient and hessian cases, are given in the last section. There is also a final remark showing that arguing as in Proposition 4.1 the limit energy can be better described also in the convex case, through a fewer number of limit functions than it has been done in [13] and [17].

# 2 Notations and Preliminaries

In the thin multi-domain we introduce a bulk energy density of the kind $W(D^s U)$ (s=1,2). First we consider the case $s = 2$, eventually we present the result for the case $s = 1$. We recall that by $Sym(\mathbb{R}^{3\times 3})$ we denote the set of triples of completely symmetric bilinear forms on $\mathbb{R}^3$. Precisely, let

$$W : \mathcal{M} \in Sym(\mathbb{R}^{3\times 3}) \longrightarrow W(\mathcal{M}) \in \mathbb{R} \tag{2.1}$$

be a continuous function satisfying the following assumptions:

$$\begin{aligned} &a + \alpha|\mathcal{M}|^p \leq W(\mathcal{M}) \leq b + \beta|\mathcal{M}|^p, \quad \forall \mathcal{M} \in Sym(\mathbb{R}^{3\times 3}); \\ &\text{for some } a, b \in \mathbb{R},\ \alpha, \beta \in ]0, +\infty[ \text{ and } 1 < p < +\infty. \end{aligned} \tag{2.2}$$

Moreover, in the sequel, for a given $A \in Sym(\mathbb{R}^{2\times 2})$, $B \in \mathbb{R}^{3\times 2}$ and $C \in \mathbb{R}^3$, $W\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$ means $W(\mathcal{M})$, where $\mathcal{M} = (m_{i,j})^{k=1,2,3}_{i,j=1,2,3}$ and $(m_{i,j})^{k=1,2}_{i,j=1,2} = A$, $(m_{3,j})^{k=1,2,3}_{j=1,2} = B$, $(m_{i,3})^{k=1,2,3}_{i=1,2} = B^T$ and $m^{k=1,2,3}_{3,3} = C$. To describe the limit energy of the sequence in (1.11), as $n \to +\infty$, when the volumes of $\Omega^a_n$ and $\Omega^b_n$ tend to zero with the same rate, (cf. (1.2) and (1.6)) we introduce the limit spaces $V^p$ (we point out the strong dependence on $p$ for the limit junction conditions):

$$V^p = \begin{cases} \left\{((u^a, u^b), (\xi^a, \xi^b)) \in \mathcal{U}\times\Xi : u^a(0) = u^b(0')\right\}, & \text{if } 1 < p \leq 2; \\ \left\{ \begin{array}{l} ((u^a, u^b), (\xi^a, \xi^b)) \in \mathcal{U}\times\Xi : u^a(0) = u^b(0'), \\ \xi^a(0) = D_{x'}u^b(0'), \quad D_{x_3}u^a(0) = \xi^b(0') \end{array} \right\}, & \text{if } 2 < p; \end{cases} \tag{2.3}$$

where

$$\mathcal{U} = \left(c^a + W^{2,p}_a(]0,1[;\mathbb{R}^3)\right) \times \left(f^b + W^{2,p}_0(\omega;\mathbb{R}^3)\right),$$

$$\Xi = \left(d^a + \left(W^{1,p}_a]0,1[;\mathbb{R}^{3\times 2}\right)\right) \times (g^b + W^{1,p}_0(\omega;\mathbb{R}^3)),$$

($W^{2,p}_a(]0,1[;\mathbb{R}^3)$ is the closure, with respect to $W^{2,p}$-norm, of $\{u^a \in C^\infty([0,1];\mathbb{R}^3) : u^a = 0$ in a neighbourhood of $1\}$.)

Introduce the auxiliary space:

$$V = \left\{ \begin{array}{l} (u^a, u^b, \xi^a, \xi^b) \in \left(c^a + C^\infty_a([0,1];\mathbb{R}^3)\right) \times \left(f^b + C^\infty_0(\omega;\mathbb{R}^3)\right) \times \\ \left(d^a + \left(C^\infty_a([0,1];\mathbb{R}^{3\times 2})\right)\right) \times \left(g^b + C^\infty_0(\omega;\mathbb{R}^3)\right) : \\ u^a(0) = u^b(0'), \quad D_{x'}u^b(0') = \xi^a(0), \quad D_{x_3}u^a(0) = \xi^b(0') \end{array} \right\}, \tag{2.4}$$

where $C^\infty_a([0,1];\mathbb{R}^3)=\{u^a \in C^\infty([0,1];\mathbb{R}^3) : u^a(1)=0,\ D^i u^a(1) = 0\ \forall i \in \mathbb{N}\}$. We point out that assumption (1.10) gives a meaning to the junction conditions in the definition (2.4). The following density result will be exploited in the sequel (see Proposition 4.1 in [17]).

**Proposition 2.1** *Let $V^p$ and $V$ be as in* (2.3) *and* (2.4)*, respectively. Then $V$ is dense in $V^p$.*

In order to obtain the representation result we need to recall some well known notions of convex analysis, as well as some more general properties, essentially concerning $\mathcal{A}$-quasiconvexity. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a function, the convex envelope of $g$ is the function

$$g^{**} = \sup\{h \leq g : h \text{ convex } \}. \tag{2.5}$$

Consider a collection of linear operators $A^{(i)} \in Lin(\mathbb{R}^d, \mathbb{R}^l)$, $i = 1, \ldots, N$, and define

$$\mathcal{A}v := \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i}, \quad v : \mathbb{R}^N \to \mathbb{R}^d,$$

$$\mathcal{A}(w) := \sum_{i=1}^{N} A^{(i)} w_i \in Lin(\mathbb{R}^d, \mathbb{R}^l), w \in \mathbb{R}^N,$$

where $Lin(X, Y)$ is the vector space of linear mappings from the vector space $X$ into the vector space $Y$. Furthermore assume that $\mathcal{A}$ satisfies the *constant rank* property, i.e. there exists $r \in \mathbb{N}$ such that

$$\text{rank}\mathbb{A}(w) = r \text{ for all } w \in S^{N-1}$$

(where $S^{N-1}$ is the unit sphere in $\mathbb{R}^N$). Here and in the sequel $Q$ will denote the unit cube in $\mathbb{R}^d$: i.e. $Q := ]0, 1[^d$.

**Definition 2.2** A Borel function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-quasiconvex if

$$f(v) \leq \int_Q f(v + w(x))dx \ \text{ for every } w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \text{Ker}\mathcal{A}, \int_Q w(y)dy = 0.$$

Recall the notion of $\mathcal{A}$-quasiconvexification, which extends to the $\mathcal{A}$-free setting (i.e. test functions in Ker$\mathcal{A}$ ) the notion of 'quasiconvexity'.

**Definition 2.3** Given a Borel function $f : \mathbb{R}^d \to \mathbb{R}$, the $\mathcal{A}-$quasiconvexification of $f$ at $v \in \mathbb{R}^d$ is given by

$$Q_\mathcal{A} f(v) := \inf \left\{ \int_Q f(v + w(x))dx : w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \text{Ker}\mathcal{A}, \int_Q w(y)dy = 0 \right\}.$$

i) It is worthwhile to observe that the convexification in (2.5) can be obtained as a particular case of $\mathcal{A}$-quasiconvexity, i.e. when the operator $\mathcal{A}$ is 0, i.e. with no constraints on the admissible fields.

Let $\Omega$ be connected and simply connected.

ii) Consider a matrix-valued function $v : \Omega \subset \mathbb{R}^N \to \mathbb{M}^{m\times N}$ and define the operator $\mathcal{A}$ as the curl operator, i.e.

$$\mathrm{curl} v = 0, i.e. \frac{\partial v_{jk}}{\partial x_i} - \frac{\partial v_{ji}}{\partial x_k} = 0, \quad 1 \le j \le m, 1 \le i,k \le N,$$

then

$$\mathrm{Ker}\mathcal{A}(w) = \{v \in \mathbb{M}^{m\times N} : \mathcal{A}(x)v = 0\} = \{v \in \mathbb{M}^{m\times N} : v = a \otimes w, \text{ for some } a \in \mathbb{R}^m\}.$$

In this case the well known notion of quasiconvexity is recovered and the formula stated in Definition 2.2 becomes

$$f(v) \le \int_Q f(v + \nabla w(x))dx \text{ for every } w \in C_0^\infty(Q;\mathbb{R}^m) \tag{2.6}$$

analogously the $\mathcal{A}$-quasiconvexification in Definition 2.3 recovers the usual quasiconvexification, or equivalently the quasiconvex envelope,

$$\mathcal{Q}f(v) := \inf\left\{\int_Q f(v + \nabla w(x))dx : w \in C_0^\infty(Q;\mathbb{R}^m)\right\}, \tag{2.7}$$

with $Q = ]0,1[^N$.

Let $E$ be a nonempty set and $Q_2$ be the cube $]0,1[^2$ and $C^\infty_{\mathrm{per}}(Q_2;E)$ the set of smooth functions defined and periodic on $Q_2$ with values in $E$. In the sequel it will be made a large use of the subsequent differential operator $\mathcal{A}^2 := (\mathcal{A}^2_2, \mathcal{A}^2_1)$ given by

$$\mathcal{A}^2 : v \equiv (h,\xi) \in Sym(\mathbb{R}^{2\times 2}) \times \mathbb{M}^{3\times 2} \to (\mathcal{A}^2_2 h, \mathcal{A}^2_1 \xi) \tag{2.8}$$

where

$$\mathcal{A}^2_2 h = \left(\frac{\partial h^i_{j1}}{\partial x_2} - \frac{\partial h^i_{j2}}{\partial x_1}\right)_{i=1,2,3, j=1,2} \quad \text{and } \mathcal{A}^2_1 \xi = \left(\frac{\partial \xi^i_1}{\partial x_2} - \frac{\partial \xi^i_2}{\partial x_1}\right)_{i=1,2,3}.$$

**Remark 2.4** It is easily verified that

$$\left\{h \in C^\infty(Q_2; Sym(\mathbb{R}^{2\times 2})) : \mathcal{A}^2_2 h = 0, \int_{Q_2} h dx = 0\right\} = \left\{D^2_{x'} u : u \in C^\infty_{\mathrm{per}}(Q_2, \mathbb{R}^3)\right\}. \tag{2.9}$$

In fact, for every $i = 1,2,3$, if $\mathcal{A}_2^2 h^i = 0$ then $h^i_{jk} = \frac{\partial w^i_j}{\partial x_k}$ for some functions $w^i_j \in C^\infty_{\rm per}(Q_2, \mathbb{M}^{3\times 2})$ with average zero. Note that $w^i_j$ is periodic since $h^i$ is periodic and $\int_Q h^i dx = 0$. Then, by the symmetry of $h^i_{jk}$ with respect to $i$ and $j$, it results curl$w^i = 0$ and we conclude that $h^i_{jk} = \frac{\partial^2 u^i}{\partial x_k \partial x_j}$ for some $u^i \in C^\infty_{\rm per}(\mathbb{R}^2;\mathbb{R})$. The operator is a constant rank operator. Indeed, for every $w \in S^1$, we have

$$\begin{aligned}\ker\mathbb{A}_2^2(w) &= \left\{X \in Sym(\mathbb{R}^{2\times 2}) : w_i X^l_{jk} - w_j X^l_{ik} = 0, i,j = 1,2, k = 1,2, l = 1,2,3\right\} \\ &= \left\{b \otimes w \otimes w, b \in \mathbb{R}^3\right\}, \end{aligned} \tag{2.10}$$

so dim $\mathrm{Ker}\mathbb{A}_2^2(w) = 3$. Also

$$\left\{\xi \in C^\infty(Q_2, M^{3\times 2}) : \mathcal{A}_1^2 \xi = 0, \int_{Q_2} \xi dx = 0\right\} = \left\{D_\alpha \varphi : \varphi \in C^\infty_{\rm per}(Q_2, \mathbb{R}^3)\right\}, \tag{2.11}$$

and for every $w \in S^1$ it results $\mathrm{Ker}\mathbb{A}_1^2(w) = \left\{V \in M^{3\times 2} : \mathbb{A}_2(w) V^l = 0, l = 1,2,3\right\}$ $= \left\{w_i V^l_j - w_j V^l_i = 0, l = 1,2,3, i,j = 1,2\right\} = \left\{a \otimes w, a \in \mathbb{R}^3\right\}$ and dim $\mathrm{Ker}\mathbb{A}_1^2(w)$ $= 3$. It follows immediately that $\mathcal{A}^2$ is a constant rank operator, and for every $w \in S^1$, $\mathrm{Ker}\mathbb{A}^2(w) = \left\{(X,V) \in Sym(\mathbb{R}^2) \times \mathbb{M}^{3\times 2} : (X,V) = (b \otimes w^{\otimes 2}, a \otimes w), b \in \mathbb{R}^3, a \in \mathbb{R}^3\right\}$, where $w^{\otimes 2}$ stands for $w \otimes w$. For every $v \in Sym(\mathbb{R}^{2\times 2}) \times \mathbb{M}^{3\times 2}$, with $v = (h,\xi)$, for every Borel function $f : Sym(\mathbb{R}^{2\times 2}) \times \mathbb{M}^{3\times 2} \to \mathbb{R}$, we have

$$\begin{aligned} Q_{\mathcal{A}^2} f(v) \;=\; \inf \Bigg\{ & \int_Q f(v + w(x)) dx \;: w \in C^\infty_{per}(Q_2; Sym(\mathbb{R}^{2\times 2}) \times \mathbb{R}^{3\times 2}) \cap \mathrm{Ker}\mathcal{A}^2, \\ & \int_{Q_2} w dx = 0, \Bigg\}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} Q_{\mathcal{A}^2} f((h,\xi)) \;=\; \inf \Big\{ & \textstyle\int_{Q_2} f((h + D^2_{x'} u, \xi + D_{x'}\varphi)) dx : \varphi \in C^\infty_0(Q_2;\mathbb{R}^3), \\ & u \in C^\infty_0(Q_2, \mathbb{R}^3) \Big\}. \end{aligned} \tag{2.12}$$

We end this section by recalling the main features of $\Gamma$ convergence, see [9] for more details.

Let $(X,\tau)$ be a topological space. Given a sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$

$$\begin{aligned} &\Gamma - \liminf_{n\to\infty} F_n(x) := \inf\{\liminf_{n\to\infty} F_n(x_n) : x_n \to x \text{ in } \tau\}, \\ &\Gamma - \limsup_{n\to\infty} F_n(x) := \inf\{\limsup_{n\to\infty} F_n(x_n) : x_n \to x \text{ in } \tau\} \end{aligned} \tag{2.13}$$

which are called, respectively, the $\Gamma$ lower limit and the $\Gamma$-upper limit at the point $x$.

**Definition 2.5** We say that a sequence $F_n : X \to [-\infty; +\infty]$ $\Gamma$- converges to a functional $F : X \to [-\infty, +\infty]$ at a point $x \in X$, and we write

$$\Gamma - \lim_{n\to\infty} F_n(x) = F(x)$$

if

$$\Gamma - \liminf_{n\to\infty} F_n(x) = \Gamma - \limsup_{n\to\infty} F_n(x) = F(x).$$

We say that a sequence of functionals $\Gamma$-converges on a set if it $\Gamma$-converges at every point of the set.

It can be easily verified that a sequence $F_n : X \to [-\infty; +\infty]$ $\Gamma$-converges to a functional $F : X \to [-\infty; +\infty]$ if the following two conditions hold:

i) for every sequence $x_n \in X$ converging to $x$ in $\tau$ one has

$$\liminf_{n\to\infty} F_n(x_n) \geq F(x);$$

ii) there exists a sequence $\overline{x_n} \in X$ converging to $x$ in $\tau$ and

$$\lim_{n\to\infty} F_n(\overline{x_n}) = F(x).$$

The next definition and result enlighten the behavior of the 'almost'- minimizers of a sequence $\{F_n\}$ with respect to the the minimizers of its $\Gamma$-limit $F$.

**Definition 2.6** The sequence $F_n$ is said to be equi-coercive if for any real number $M$ there exists a $\tau$-compact and a $\tau$-closed subset $K_M$ of $X$ such that

$$\{x : F_m(x) \leq M\} \subseteq K_M \text{ for every } n.$$

**Proposition 2.7** *Let us assume that* $\Gamma\text{-}\lim_{n\to\infty} F_n = F$ *on* $X$ *and that the sequence* $F_n$ *be equi-coercive. Then it results that*

*i)* $F$ *is* $\tau$*-lower semicontinuous;*

*ii)* $F$ *is* $\tau$*-coercive;*

*iii) if* $x_n \in X$ *satisfy* $\liminf_{n\to 0} F_n(x_n) = \liminf_{n\to\infty} \inf F_n$ *(e.g. if* $x_n$ *minimizes* $F_n$*) then*

   *a) if* $x_n \to x$ *in* $\tau$ *then* $x$ *is a minimizer of* $F$ *on* $X$ *and* $\lim_{n\to\infty} F_n(x_n) = F(x)$;

   *b) there is a sequence* $\{x_{n_k}\} \subset X$ *and a minimizer* $x$ *of* $F$ *on* $X$ *such that* $x_{n_k} \to x$ *in* $\tau$.

# 3 Compactness and Lower Semicontinuity

In order to perform our asymptotic analysis we will also make use of the following compactness result whose proof can be found in Section 6 of [17], observing that the proof only uses (2.2) and the fact that the sequences are energy bounded ones.

**Proposition 3.1** *Let $W$ be a function satisfying* (2.1) *and* (2.2)*, and let,* $\{(\overline{u}_n^a, \overline{u}_n^b)\}$ *an energy bounded sequence, in the sense that* $(\overline{u}_n^a, \overline{u}_n^b) \in \mathcal{U}_n$ *for every* $n$ *and* $\limsup_n \left(K_n^a(\overline{u}_n^a) + \frac{h_n}{r_n^2} K_n^b(\overline{u}_n^b)\right) \leq C$*. Let* $V^p$ *be as in* (2.3)*. Assume that* (1.2) *and* (1.6) *hold. Then, there exist an increasing sequence of positive integer numbers* $\{n_i\}_{i\in\mathbb{N}}$ *and* $((\overline{u}^a, \overline{u}^b), (\overline{\xi}^a, \overline{\xi}^b)) \in V^p$*, depending possibly on the selected subsequence* $\{n_i\}_{i\in\mathbb{N}}$*, such that*

$$\begin{cases} \overline{u}_{n_i}^a \rightharpoonup \overline{u}^a \text{ weakly in } W^{2,p}(\Omega^a;\mathbb{R}^3), \\ \\ \overline{u}_{n_i}^b \rightharpoonup \overline{u}^b \text{ weakly in } W^{2,p}(\Omega^b;\mathbb{R}^3), \end{cases} \tag{3.1}$$

$$\begin{cases} \dfrac{1}{r_{n_i}} D_{x'} \overline{u}_{n_i}^a \rightharpoonup \overline{\xi}^a \text{ weakly in } W^{1,p}(\Omega^a;\mathbb{R}^{3\times 2}), \\ \\ \dfrac{1}{h_{n_i}} D_{x_3} \overline{u}_{n_i}^b \rightharpoonup \overline{\xi}^b \text{ weakly in } W^{1,p}(\Omega^b;\mathbb{R}^3). \end{cases} \tag{3.2}$$

Recalling that for every $H \in Sym(\mathbb{R}^{3\times 3})$, the decomposition, after the rescaling, is the following

$$H = \begin{pmatrix} h & \xi^T \\ \xi & c \end{pmatrix}$$

with $H \in Sym(\mathbb{R}^{2\times 2})$, $\xi \in M^{3\times 2}$ and $c \in \mathbb{R}^3$ in the sequel we exploit the functions

$$\hat{W}(c,\xi) = \inf_{h\in Sym(\mathbb{R}^{2\times 2})} W \begin{pmatrix} h & \xi^T \\ \xi & c \end{pmatrix}, \tag{3.3}$$

and

$$W_0(h,\xi) = \inf_{c\in\mathbb{R}^3} W \begin{pmatrix} h & \xi^T \\ \xi & c \end{pmatrix} \tag{3.4}$$

**Remark 3.2** We observe that under the assumptions made on $W$ (see 2.2), arguing as in Proposition 1 in [24], it follows that $\hat{W}$ and $W_0$ are continuous and satisfy a growth condition of order $p$.

**Lemma 3.3** *Let $W$ be the energy density in* (2.1), *satisfying assumptions* (2.2) *and let $\hat{W}$ and $W_0$ the functions in* (3.3) *and* (3.4) *respectively. For every $((u^a, u^b), (\xi^a, \xi^b)) \in V^p$ define the functionals*

$$J_0^p(u^a, u^b, \xi^a, \xi^b) = \inf \left\{ \begin{array}{l} \liminf_{n\to\infty} \left[ \begin{array}{l} \int_0^1 \hat{W}(D^2_{x_3} u^a_n, D_{x_3}\xi^a_n) dx_3 \\ +q \int_\omega W_0(D^2_{x'} u^b_n, D_{x'}\xi^b_n) dx' : \end{array} \right] \\ ((u^a_n, u^b_n), (\xi^a_n, \xi^b_n)) \in V^p, u^a_n \rightharpoonup u^a \textit{ in } W^{2,p}(]0,1[;\mathbb{R}^3), \\ \xi^a_n \rightharpoonup \xi^a \textit{ in } W^{1,p}(]0,1[;\mathbb{R}^{3\times 2}), u^b_n \rightharpoonup u^b \textit{ in} \\ W^{2,p}(\omega;\mathbb{R}^3), \xi^b_n \rightharpoonup \xi^b \textit{ in } W^{1,p}(\omega;\mathbb{R}^3) \end{array} \right\}, \tag{3.5}$$

*then it results*

$$J_0^p(u^a, u^b, \xi^a, \xi^b) = \int_0^1 \hat{W}^{**}(D^2_{x^3} u^a, D_{x_3}\xi^a) dx_3 + q \int_\omega Q_{\mathcal{A}^2} W_0(D^2_{x'} u^b, D_{x'}\xi^b) dx' \tag{3.6}$$

*for every $((u^a, u^b), (\xi^a, \xi^b)) \in V^p$.*

*Proof.* Standard relaxation arguments, (see [8], [4] among the others) ensure that

$$\inf \left\{ \begin{array}{l} \liminf_{n\to\infty} \int_0^1 \hat{W}(D^2_{x_3} u^a_n, D_{x_3}\xi^a_n) dx_3 : u^a_n \rightharpoonup u^a \text{ in } W^{2,p}(]0,1[;\mathbb{R}^3), \\ \xi^a_n \rightharpoonup \xi^a \text{ in } W^{1,p}(]0,1[;\mathbb{R}^3) \end{array} \right\} \\ = \int_0^1 \hat{W}^{**}(D^2_{x_3} u^a, D_{x_3}\xi^a) dx_3 \tag{3.7}$$

for every $(u^a, \xi^a) \in ((c^a + W^{2,p}_a(]0,1[;\mathbb{R}^3)) \times (d^a + W^{1,p}(]0,1[;\mathbb{R}^{3\times 2})))$. Analogously Theorem 5.1 in [18] (see also [4]) guarantees that

$$\inf \left\{ \begin{array}{l} \liminf_{n\to\infty} \int_\omega W_0(D^2_{x'} u^b_n, D_{x'}\xi^b_n) dx' : u^b_n \rightharpoonup u^b \text{ in } W^{2,p}(\omega;\mathbb{R}^3), \\ \xi^b_n \rightharpoonup \xi^b \text{ in } W^{1,p}(\omega;\mathbb{R}^3) \end{array} \right\} \\ = \int_\omega Q_{\mathcal{A}^2} W_0(D^2_{x'} u^b, D_{x'}\xi^b) dx' \tag{3.8}$$

for every $(u^b, \xi^b) \in (f^b + W^{2,p}_0(\omega;\mathbb{R}^3)) \times (g^b + W^{1,p}_0(\omega;\mathbb{R}^3))$. Again classical arguments with smooth cut-off functions allow to prove (3.7) and (3.8) in the case when the boundary data are prescribed, i.e.

$$\inf \left\{ \begin{array}{l} \liminf_{n\to\infty} \int_0^1 \hat{W}(D^2_{x_3} u^a_n, D_{x_3}\xi^a_n) dx_3 : u^a_n \rightharpoonup u^a \text{ in } W^{2,p}(]0,1[;\mathbb{R}^3), \\ u^a_n(1) = u^a(1), Du^a_n(1) = Du^a(1) \end{array} \right. \\ \left. \xi^a_n \rightharpoonup \xi^a \text{ in } W^{1,p}(]0,1[;\mathbb{R}^{3\times 2}), \xi^a_n(1) = \xi^a(1) \right\} = \int_0^1 \hat{W}^{**}(D^2_{x_3} u^a, D_{x_3}\xi^a) dx_3 \tag{3.9}$$

for every $(u^a, \xi^a) \in ((c^a + W_a^{2,p}(]0,1[;\mathbb{R}^3)) \times (d^a + W_a^{1,p}(]0,1[;\mathbb{R}^3))$, and

$$\begin{aligned}
&\inf\left\{\liminf_{n\to\infty}\int_\omega W_0(D^2_{x'}u^b_n, D_{x'}\xi^b_n)dx' : u^b_n \rightharpoonup u^b \text{ in } W^{2,p}(\omega), u^b_n = u^b,\right.\\
&\qquad\left. D_{x'}u^b_n = D_{x'}u^b \text{ on } \partial\omega \xi^b_n \rightharpoonup \xi^b \text{ in } W^{1,p}(\omega;\mathbb{R}^3), \xi^b_n = \xi^b \text{ on } \partial\omega\right\} \qquad (3.10)\\
&\quad = \int_\omega Q_{\mathcal{A}^2}W_0(D^2_{x'}u^b, D_{x'}\xi^b)dx'
\end{aligned}$$

for every $(u^b, \xi^b) \in (f^b + W_0^{2,p}(\omega;\mathbb{R}^3)) \times (g^b + W_0^{1,p}(\omega;\mathbb{R}^3))$.

Formulas (3.9) and (3.10) entail the following result

$$\begin{aligned}
J'^p_0(u^a,u^b,\xi^a,\xi^b) = \inf\left\{\begin{array}{l}
\liminf\limits_{n\to\infty}\left[\begin{array}{l}\int_0^1 \hat{W}(D^2_{x_3}u^a_n, D_{x_3}\xi^a_n)dx_3\\ \quad +q\int_\omega W_0(D^2_{x'}u^b_n, D_{x'}\xi^b_n)dx'\end{array}\right] :\\
u^a_n \rightharpoonup u^a \text{ in } W^{2,p}(]0,1[;\mathbb{R}^3), u^a_n, u^a \in c^a\\
\quad +W_a^{2,p}(]0,1[;\mathbb{R}^3)\\
\xi^a_n \rightharpoonup \xi^a \text{ in } W^{1,p}(]0,1[;\mathbb{R}^{3\times 2}), \xi^a_n, \xi^a \in d^a\\
\quad +W_a^{1,p}(]0,1[;\mathbb{R}^{3\times 2}),\\
u^b_n \rightharpoonup u^b \text{ in } W^{2,p}(\omega;\mathbb{R}^3), u^b_n, u^b \in f^b\\
\quad +W_0^{2,p}(\omega;\mathbb{R}^3)\\
\xi^b_n \rightharpoonup \xi^b \text{ in } W^{1,p}(\omega;\mathbb{R}^3), \xi^b_n, \xi^b \in g^b + W_0^{1,p}(\omega;\mathbb{R}^3)
\end{array}\right\}\\
= \int_0^1 \hat{W}^{**}(D^2_{x^3}u^a, D_{x_3}\xi^a)dx_3 + q\int_\omega Q_{\mathcal{A}^2}W_0(D^2_{x'}u^b, D_{x'}\xi^b)dx' \qquad (3.11)
\end{aligned}$$

for every $((u^a,u^b),(\xi^a,\xi^b)) \in V^p$. Since, obviously $J'^p_0(u^a,u^b,\xi^a,\xi^b) \le J^p_0(u^a,u^b,\xi^a,\xi^b)$ for every $(u^a,u^b,\xi^a,\xi^b) \in V^p$, in order to conclude the proof it will be enough to prove the converse inequality. To this end, we should consider the two cases. First assume that $1 < p \le 2$, then the only difference in the definition of $J'^p_0$ and $J^p_0$ consists of requiring that the test functions $u^a_n$ and $u^b_n$ in the latter functional have to coincide at the origin, i.e. $u^a_n(0) = u^b_n(0')$. The proof will be concluded if we show that $J'^p_0$ admits on $V^p$ the same representation as

$$J''^p_0(u^a,u^b,\xi^a,\xi^b) = \inf\left\{\begin{array}{l}
\liminf\limits_{n\to\infty}\left[\begin{array}{l}\displaystyle\int_0^1 \hat{W}(D^2_{x_3}u^a_n, D_{x_3}\xi^a_n)dx_3\\ \displaystyle\quad +q\int_\omega W_0(D^2_{x'}u^b_n, D_{x'}\xi^b_n)dx'\end{array}\right] :\\
u^a_n \rightharpoonup u^a \text{ in } W^{2,p}(]0,1[;\mathbb{R}^3), u^a_n, u^a \in c^a\\
\quad +W_a^{2,p}(]0,1[;\mathbb{R}^3)\\
\xi^a_n \rightharpoonup \xi^a \text{ in } W^{1,p}(]0,1[;\mathbb{R}^{3\times 2}), \xi^a_n, \xi^a \in d^a\\
\quad +W_a^{1,p}(]0,1[;\mathbb{R}^{3\times 2}),\\
u^b_n \rightharpoonup u^b \text{ in } W^{2,p}(\omega;\mathbb{R}^3), u^b_n, u^b \in f^b + W_0^{2,p}(\omega;\mathbb{R}^3)\\
\xi^b_n \rightharpoonup \xi^b \text{ in } W^{1,p}(\omega;\mathbb{R}^3), \xi^b_n, \xi^b \in g^b\\
\quad +W_0^{1,p}(\omega;\mathbb{R}^3), u^a_n(0) = u^a(0), u^b_n(0') = u^b(0')
\end{array}\right\},$$

since obviously the following chain of inequalities holds on $V^p$, $(1 < p \leq 2)$

$$J'^p_0 \leq J^p_0 \leq J''^p_0.$$

For every $u^a \in W^{2,p}(]0,1[;\mathbb{R}^3)$, $\xi^a \in W^{1,p}(]0,1[;\mathbb{R}^{3\times 2})$ and $u^b \in W^{2,p}(\omega;\mathbb{R}^3)$, $\xi^b \in W^{1,p}(\omega;\mathbb{R}^3)$ define

$$\begin{array}{l}\mathcal{G}_p(u^a,\xi^a;A) := \int_A (1+|D\xi^a|^p + |D^2u^a|^p)dx_3,\\ \mathcal{H}_p(u^b,\xi^b;B) := \int_B (1+|D\xi^b|^p + |D^2u^b|^p)dx_1dx_2,\end{array}$$

where $A$ is any open subset of $]0,1[$ and $B$ is any open subset of $\omega$. Thus, given $\varepsilon > 0$, take a sequence $((u^a_n,u^b_n),(\xi^a_n,\xi^b_n)) \in (c^a + W^{2,p}_a(]0,1[;\mathbb{R}^3), f^b + W^{2,p}_0(\omega;\mathbb{R}^3)) \times (d^a + W^{1,p}_a(]0,1[;\mathbb{R}^{3\times 2}), g^b + W^{1,p}_0(\omega;\mathbb{R}^3))$ such that $u^a_n \rightharpoonup u^a$ in $W^{2,p}(]0,1[;\mathbb{R}^3)$, $u^b_n \rightharpoonup u^b$ in $W^{2,p}(\omega;\mathbb{R}^3)$, $\xi^a_n \rightharpoonup \xi^a$ in $W^{1,p}(]0,1[;\mathbb{R}^{3\times 2})$ and $\xi^b_n \rightharpoonup \xi^b$ in $W^{1,p}(\omega;\mathbb{R}^3)$ and

$$\begin{array}{l}\liminf\limits_{n\to\infty}\left[\int_0^1 \hat{W}(D^2_{x_3}u^a_n, D_{x_3}\xi^a_n)dx_3 + q\int_\omega W_0(D^2_{x'}u^b_n, D_{x'}\xi^b_n)dx'\right]\\ \quad \leq J'^p_0(u^a,u^b,\xi^a,\xi^b) + \varepsilon.\end{array}$$

One can extract a subsequence $\{n_k\}$ such that

$$\liminf_n \left[\int_0^1 \hat{W}(D^2_{x_3}u^a_n, D_{x_3}\xi^a_n)dx_3 + q\int_\omega W_0(D^2_{x'}u^b_n, D_{x'}\xi^b_n)dx'\right]$$

$$= \lim_{k\to\infty}\left[\int_0^1 \hat{W}(D^2_{x_3}u^a_{n_k}, D_{x_3}\xi^a_{n_k})dx_3 + q\int_\omega W_0(D^2_{x'}u^b_{n_k}, D_{x'}\xi^b_{n_k})dx'\right],$$

and the sequences of measures $\nu_k := \mathcal{G}_p(u^a,\xi^a;\cdot) + \mathcal{G}_p(u^a_{n_k},\xi^a_{n_k};\cdot)$ and $\mu_k := \mathcal{H}_p(u^b,\xi^b;\cdot) + \mathcal{H}_p(u^b_{n_k},\xi^b_{n_k};\cdot)$ converge weakly $*$ to some Radon measures $\nu$ and $\mu$ respectively.

Denoting for every $t > 0$ by $B_t$ a spherical neighborhood of $0'$ in $\omega$, centered at $0'$ with radius $t$ and by $A_t := ]0,t[$, we fix $\eta > 0$ and for every $0 < \delta < \eta$ we define the subsets $L_\delta := B_{\eta+\delta} \setminus \overline{B_{\eta-2\delta}}$. Analogously on the $x_3$ edge, we may consider the set $A_\eta := ]0,\eta[$ and $L'_\delta := A_{\eta+\delta} \setminus \overline{A_{\eta-2\delta}} = ]\eta-2\delta,\eta+\delta[$. Consider two smooth cut-off functions $\varphi_\delta \in C^\infty_0(A_{\eta+\delta};[0,1])$ and $\psi_\delta \in C^\infty_0(B_{\eta+\delta};[0,1])$ such that $\varphi_\delta = 1$ on $B_\eta$ and $\psi_\delta = 1$ on $A_\eta$. Since the thickness of the strips $L_\delta$ and $L'_\delta$ is of order $\delta$ the following upper bounds hold $\|D_{x'}\varphi_\delta\|_{L^\infty(A_{\eta+\delta})} \leq \frac{C}{\delta}$, $\|D^2_{x'}\varphi_\delta\|_{L^\infty(A_{\eta+\delta})} \leq \frac{C}{\delta^2}$ and $\|D_{x_3}\psi_\delta\|_{L^\infty(B_{\eta+\delta})} \leq \frac{C}{\delta}$, $\|D^2_{x_3}\psi_\delta\|_{L^\infty(B_{\eta+\delta})} \leq \frac{C}{\delta^2}$.

Define

$$\begin{array}{l}w^a_n := u^a\psi_\delta + (1-\psi_\delta)u^a_n\\ w^b_n := u^b\varphi_\delta + (1-\varphi_\delta)u^b_n.\end{array}$$

Clearly, $w^a_n \rightharpoonup u^a$ in $W^{2,p}(]0,1[;\mathbb{R}^3)$ and $w^a_n(0) = u^a(0)$, and analogously $w^b_n \rightharpoonup u^b$ in $W^{2,p}(\omega;\mathbb{R}^3)$ and $w^b_n(0') = u^b(0')$. Then, since we are dealing with integral functionals and by virtue of Remark 3.2

$$\begin{aligned}
&\int_0^1 \hat{W}(D^2_{x_3} w^a_{n_k}, D_{x_3}\xi^a_{n_k})dx_3 + q\int_\omega W_0(D^2_{x'} w^b_{n_k}, D_{x'}\xi^b_{n_k})dx' \\
&\quad\le \int_0^\eta \hat{W}(D^2_{x_3} u^a, D_{x_3}\xi^a_{n_k})dx_3 + q\int_{B_\eta} W_0(D^2_{x'} u^b, D_{x'}\xi^b_{n_k})dx' \\
&\quad+ \int_{\eta+\delta}^1 \hat{W}(D^2_{x_3} u^a_{n_k}, D_{x_3}\xi^a_{n_k})dx_3 + q\int_{\omega\setminus\overline{B_{\eta+\delta}}} W_0(D^2_{x'} u^b_{n_k}, D_{x'}\xi^b_{n_k})dx' \\
&\quad+ \int_{\eta-2\delta}^{\eta+\delta} \hat{W}(D^2_{x_3} w^a_{n_k}, D_{x_3}\xi^a_{n_k})dx_3 + q\int_{B_{\eta+\delta}\setminus\overline{B_{\eta-2\delta}}} W_0(D^2_{x'} w^b_{n_k}, D_{x'}\xi^b_{n_k})dx' \\
&\quad\le \int_0^1 \hat{W}(D^2_{x_3} u^a_{n_k}, D_{x_3}\xi^a_{n_k})dx_3 + q\int_\omega W_0(D^2_{x'} u^b_{n_k}, D_{x'}\xi^b_{n_k})dx' \\
&\quad+C\int_0^\eta [|D^2_{x_3} u^a|^p + |D_{x_3}\xi^a_{n_k}|^p]dx_3 + Cq\int_{B_\eta} [|D^2_{x'} u^b|^p + |D_{x'}\xi^b_{n_k}|^p]dx' \\
&\quad+C\left(\mathcal{G}_p(u^a,\xi^a;L'_\delta) + \mathcal{G}_p(u^a_{n_k},\xi^a_{n_k};\mathcal{L}'_\delta) + \mathcal{H}_p(u^b,\xi^b;L_\delta) + \mathcal{H}_p(u^b_{n_k},\xi^b_{n_k};L_\delta)\right) \\
&\quad+C\Bigg(\int_{L'_\delta}\left[\tfrac{1}{\delta^p}|D_{x_3}u^a_{n_k} - D_{x_3}u^a|^p + \frac{1}{\delta^{2p}}|u^a_{n_k}-u^a|^p\right]dx_3 \\
&\quad+q\int_{L_\delta}\left[\frac{1}{\delta^p}|D_{x'}u^b_{n_k} - D_{x'}u^b|^p + \frac{1}{\delta^{2p}}|u^b_{n_k}-u^b|^p\right]dx'\Bigg).
\end{aligned}$$

Then, passing to the limit as $k \to +\infty$, one gets

$$\begin{aligned}
&\liminf_{k\to\infty}\left[\int_0^1 \hat{W}(D^2_{x_3} w^a_{n_k}, D_{x_3}\xi^a_{n_k})dx_3 + q\int_\omega W_0(D^2_{x'} w^b_{n_k}, D_{x'}\xi^b_{n_k})dx'\right] \\
&\quad\le {J'}^p_0(u^a,u^b,\xi^a,\xi^b) + \varepsilon + C\left(\nu(A_\eta) + \mu(B_\eta) + \mu(\overline{L^\delta}) + \nu(\overline{L'_\delta})\right),
\end{aligned}$$

Letting $\delta$ go to zero one obtains

$${J''}^p_0(u^a,u^b,\xi^a,\xi^b) \le {J'}^p_0(u^a,u^b,\xi^a,\xi^b) + \varepsilon + C(\nu(A_\eta) + \mu_(B_\eta) + \nu(\partial A_\eta) + \mu(\partial B_\eta)).$$

hence, by choosing a subsequence $\{\eta_k\}$ such that $\eta_k \to 0^+$ and $\nu(\partial A_{\eta_k}) = \mu$ $(\partial B_{\eta_k}) = 0$, the thesis follows in the case $1 < p \le 2$, by letting first $k \to +\infty$ and then $\varepsilon \to 0^+$. The case $p > 2$ can be proved in analogy with the previous step, just considering four cut-off functions $\psi_\delta$, $\psi^1_\delta$ defined in $A_{\eta+\delta}$, and $\varphi_\delta$, $\varphi^1_\delta$ defined in $B_{\eta+\delta}$, in order to ensure the three junction conditions defining $V^p$. So far, one construct four sequences

$$\begin{aligned}
w^a_n &:= u^a\psi_\eta + (1-\psi_\eta)u^a_n \\
w^b_n &:= u^b\varphi_\eta + (1-\varphi_\eta)u^b_n \\
\tilde{\xi^a_n} &:= \xi^a\psi^1_\eta + (1-\psi^1_\eta)\xi^a_n \\
\tilde{\xi^b_n} &:= \xi^b\varphi^1_\eta + (1-\varphi^1_\eta)\xi^b_n,
\end{aligned}$$

that guarantee $\xi^a_n(0) = \xi^a(0) = D_{x'}u^b(0') = D_{x'}u^b_n(0')$ and $\xi^b_n(0') = \xi^b(0') = D_{x_3}$ $u^a(0) = D_{x_3}u^a_n(0)$ and finally one passes to the limit as in the case $1 < p \le 2$, exploiting the growth condition and the continuity of $\hat{W}$ and $W_0$ established in Remark 3.2. □

Next we recall a technical lemma (see [9])

**Lemma 3.4** *Let $X \hookrightarrow Y$ be two Banach spaces such that $X$ is reflexive and compactly embedded in $Y$. Consider a functional $G : X \to \mathbb{R}$ such that for all $v \in X$, $G(v) \geq g(\|v\|_X)$ where $g$ is such that $g(t) \to +\infty$ as $t \to +\infty$. Let $G_* : Y \to \mathbb{R}$ be defined by $G_*(v) = G(v)$ if $v \in X$, $G_*(v) = +\infty$ otherwise. Let $\Gamma - G$ denote the sequential lower semi-continuous envelope of $G$ for the weak topology of $X$ and let $\Gamma - G_*$ denote the lower semi-continuous envelope of $G_*$ for the strong topology of $Y$. Then $\Gamma - G_* = (\Gamma - G)_*$.*

**Remark 3.5** From the very definition (3.5), the coercivity assumption (2.2) on $W$, Remark 3.2, Lemma 3.3 and Lemma 3.4, it follows the sequential lower semi-continuity of the functional

$$\int_0^1 \hat{W}^{**}(D^2_{x_3} u^a, D_{x_3}\xi^a)dx_3 + q\int_\omega Q_{\mathcal{A}^2} W_0(D^2_{x'} u^b, D_{x'}\xi^b)dx',$$

extended to $+\infty$ in $(W^{1,p}(]0,1[;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)\times L^p(]0,1[;\mathbb{R}^{3\times 2})\times L^p(\omega;\mathbb{R}^3)) \setminus V^p$, with respect to the weak topology in $V^p$.

# 4 Γ-convergence

In order to deal with the classical arguments of Γ-convergence (see [9]) it is convenient, also in view of the coercivity assumption (2.2), to extend the energies to a metric space and define for every $(u^a, u^b) \in W^{1,p}(\Omega^a;\mathbb{R}^3)\times W^{1,p}(\Omega^b;\mathbb{R}^3)$

$$\left(K^a_n(\cdot) + \frac{h_n}{r_n^2}K^b_n(\cdot)\right)_* (u^a, u^b) := \begin{cases} K^a_n(u^a) + \frac{h_n}{r_n^2}K^b_n(u^b) & \text{if } (u^a, u^b) \in \mathcal{U}_n, \\ +\infty & \text{otherwise.}\end{cases} \tag{4.1}$$

Next the Γ-limit can be computed for the sequence $(K^a_n + \frac{h_n}{r_n^2}K^b_n)_*$ with respect to the strong topology of $W^{1,p}$ for what concerns the convergence of $(u^a_n, u^b_n) \to (u^a, u^b)$ and the strong topology of $L^p$ for what concerns the convergence of $\left(\frac{1}{r_n}D_{x'}u^a_n, \frac{1}{h_n}D_{x_3}u^b_n\right) \to (\xi^a, \xi^b)$. The final result will be achieved through the two next propositions, according to Definition 2.5. We also emphasize that we are not in presence of the ‘classical’ Γ-convergence because of the rescalings, but all the results stated at the end of section 2 can be easily adapted to our context.

**Proposition 4.1** *Let $W$ be a function satisfying* (2.1), *and* (2.2). *For every $n \in \mathbb{N}$, let $K^a_n$, $K^b_n$ and $\mathcal{U}_n$ be as in* (1.7), (1.8), *and* (1.9), *respectively. Let $V$ be as in* (2.4). *Assume that* (1.2) *and* (1.6) *hold.Then, for every $(u^a, u^b, \xi^a, \xi^b) \in V$*

$$\begin{aligned}&\inf\Big\{\limsup_n \left(K^a_n(\cdot) + \tfrac{h_n}{r_n^2}K^b_n(\cdot)\right)_* (u^a_n, u^b_n), (u^a_n, u^b_n) \in \mathcal{U}_n :\\ &(u^a_n, u^b_n) \to (u^a, u^b) \text{ in } W^{1,p}, \left(\tfrac{1}{r_n}D_{x'}u^a_n, \tfrac{1}{h_n}D_{x_3}u^b_n\right) \to (\xi^a, \xi^b) \text{ in } L^p\Big\} \leq\\ &\int_{-1}^1 \hat{W}^{**}(D^2_{x_3}u^a, D_{x_3}\xi^a)dx_3 + q\int_\omega Q_{\mathcal{A}^2}W_0(D^2_{x'}v^b, D_{x'}\xi^b)dx'.\end{aligned} \tag{4.2}$$

*Proof.* The proof is a by now rather standard argument (see [24], [13] and [17]), apart from the fact that it involves functionals which are formally coupled. We present some details for the reader's convenience. The role of the recovery sequence is played by the following sequences (4.3) and (4.4). Fix $(u^a, u^b, \xi^a, \xi^b) \in V$ and $(z^a, z^b) \in W_0^{2,p}(]0,1[;\mathbb{R}^{3\times 4}) \times W_0^{2,p}(\omega;\mathbb{R}^3)$, and for every $n \in \mathbb{N}$, set

$$u_n^a(x', x_3) =$$

$$\begin{cases} u^a(x_3) + r_n \xi^a(x_3) \cdot x' + r_n^2 x'^T \cdot z^a(x_3) \cdot x', \\ \qquad \text{for a.e. } x = (x', x_3) \in \omega \times ]\varepsilon_n, 1[; \\ \Big[ -\frac{2}{\varepsilon_n^3} u^a(\varepsilon_n) - 2\frac{r_n}{\varepsilon_n^3} \xi^a(\varepsilon_n) \cdot x' - 2\frac{r_n^2}{\varepsilon_n^3} z^a(\varepsilon_n) + \frac{2}{\varepsilon_n^3} u^b(r_n x') + \\ \frac{1}{\varepsilon_n^2} D_{x_3} u^a(\varepsilon_n) + \frac{r_n}{\varepsilon_n^2} D_{x_3} \xi^a(\varepsilon_n) \cdot x' + \frac{r_n^2}{\varepsilon_n^2} x'^T \cdot D_{x_3} z^a(\varepsilon_n) \cdot x' + \frac{1}{\varepsilon_n^2} \xi^b(r_n x') \Big] \\ x_3^3 + \Big[ \frac{3}{\varepsilon_n^2} u^a(\varepsilon_n) + 3\frac{r_n}{\varepsilon_n^2} \xi^a(\varepsilon_n) \cdot x' + 3\frac{r_n^2}{\varepsilon_n^2} x'^T z^a(\varepsilon_n) \cdot x' - \frac{3}{\varepsilon_n^2} u^b(r_n x') - \\ \frac{1}{\varepsilon_n} D_{x_3} u^a(\varepsilon_n) - \frac{r_n}{\varepsilon_n} D_{x_3} \xi^a(\varepsilon_n) \cdot x' - \frac{r_n^2}{\varepsilon_n} x'^T D_{x_3} z^a(\varepsilon_n) x' - \frac{2}{\varepsilon_n} \xi^b(r_n x') \Big] \\ x_3^2 + \left[ \xi^b(r_n x') \right] x_3 + u^b(r_n x'), \quad \text{for a.e. } x = (x', x_3) \in \omega \times ]0, \varepsilon_n[; \end{cases} \tag{4.3}$$

and

$$u_n^b(x', x_3) = u^b(x') + h_n \xi^b(x') x_3 + h_n^2 x_3^2 z^b(x'), \quad \text{for a.e. } x = (x', x_3) \in \Omega^b; \tag{4.4}$$

where $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset ]0,1[$ is suitably chosen, for example one can choose $\varepsilon_n = r_n^2$.

Clearly $(u_n^a, u_n^b) \in \mathcal{U}_n$, $(u_n^a, u_n^b) \to (u^a, u^b)$ strongly in $W^{2,p}(\Omega^a;\mathbb{R}^3) \times W^{2,p}(\Omega^b;\mathbb{R}^3)$, and $\left( \frac{1}{r_n} D_{x'} u_n^a, \frac{1}{h_n} D_{x_3} u_n^b \right) \to (\xi^a, \xi^b)$ strongly in $W^{1,p}(\Omega^a;\mathbb{R}^{3\times 2}) \times W^{1,p}(\Omega^b;\mathbb{R}^3)$. Arguing as in Proposition 5.1 in [17] where the convexity on the density $W$ has not been exploited and with the only difference in the choice of $z^a$ and $z^b$, which now do not depend on all the variables, it can be proved that

$$\begin{aligned} K_n^b(u_n^b) &= \int_{\Omega^b} W \left( D_{x'}^2 u_n^b, \frac{1}{h_n} D_{x',x_3}^2 u_n^b, \frac{1}{h_n^2} D_{x_3}^2 u_n^b \right) dx \\ &= \int_{\Omega^b} W \begin{pmatrix} \begin{matrix} D_{x'}^2 u^b + h_n D_{x'}^2 \xi^b(x') x_3 \\ + h_n^2 x_3^2 D_{x'}^2 z^b(x') \end{matrix} & (D_{x'} \xi^b(x') + 2h_n x_3 D_{x'} z^b(x'))^T \\ (D_{x'} \xi^b(x') + 2h_n x_3 D_{x'} z^b(x')) & 2z^b(x') \end{pmatrix} dx \\ &\to \int_{\Omega^b} W \begin{pmatrix} D_{x'}^2 u^b & (D_{x'} \xi^b(x'))^T \\ (D_{x'} \xi^b(x')) & 2z^b(x') \end{pmatrix} dx \end{aligned} \tag{4.5}$$

as $n \to +\infty$. By managing the first term in (4.2) as in Proposition 5.1 in [17], observe that

$$K_n^a(u_n^a)$$

$$= \int_\omega \int_{\varepsilon_n}^1 W \begin{pmatrix} 2z^a(x_3) & (D_{x_3}\xi^a + 2r_n D_{x_3} z^a(x_3)\cdot x')^T \\ D_{x_3}\xi^a + 2r_n D_{x_3} z^a(x_3)\cdot x' & D^2_{x_3}u^a + r_n D^2_{x_3}\xi^a \cdot x' + r_n^2 {x'}^T D^2_{x_3} z^a \cdot x' \end{pmatrix} dx$$

$$+ \int_\omega \int_0^{\varepsilon_n} W \begin{pmatrix} \frac{1}{r_n^2} D^2_{x'} u_n^a & \left(\frac{1}{r_n} D^2_{x',x_3} u_n^a\right)^T \\ \frac{1}{r_n} D^2_{x',x_3} u_n^a & D^2_{x_3} u_n^a \end{pmatrix} dx$$

still converges to

$$\int_{\Omega^a} W \begin{pmatrix} 2z^a(x_3) & (D_{x_3}\xi^a)^T \\ D_{x_3}\xi^a & D^2_{x_3}u^a \end{pmatrix} dx. \tag{4.6}$$

Since (4.5) and (4.6) are true for all $z^b \in W_0^{2,p}(\omega;\mathbb{R}^3)$, and $z^a \in W_0^{2,p}(]0,1[;\mathbb{R}^{3\times 4})$ respectively, from the properties of $\Gamma$ convergence, it results that

$$\inf\left\{\liminf_n \left(K_n^a(u_n^a) + \frac{h_n}{r_n^2} K_n^b(u_n^b)\right), (u_n^a,u_n^b) \in \mathcal{U}_n : (u_n^a,u_n^b) \to (u^a,u^b) \text{ in } W^{1,p}, \left(\tfrac{1}{r_3} D_{x'} u_n^a, \tfrac{1}{h_n} D_{x_3} u_n^b\right) \to (\xi^a,\xi^b) \text{ in } L^p\right\}$$
$$\leq \inf_{z^a \in W_0^{2,p}(]0,1[;\mathbb{R}^{3\times 4}), z^b \in W_0^{2,p}(\omega;\mathbb{R}^3)} \left\{ \int_{\Omega^a} W \begin{pmatrix} 2z^a & (D_{x_3}\xi^a)^T \\ (D_{x_3}\xi^a) & D^2_{x_3}u^a \end{pmatrix} dx + q\int_{\Omega^b} W \begin{pmatrix} D^2_{x'}u^b & (D_{x'}\xi^b)^T \\ (D_{x'}\xi^b) & 2z^b \end{pmatrix} dx \right\}. \tag{4.7}$$

On the other hand, from the density of $W_0^{2,p}(\omega;\mathbb{R}^3)$ in $L^p(\omega;\mathbb{R}^3)$, and of $W_0^{2,p}(]0, 1[;\mathbb{R}^{3\times 4})$ in $L^p(]0,1[;\mathbb{R}^{3\times 4})$ the dominated convergence theorem

$$\inf_{z^a \in W_0^{2,p}(]0,1[;\mathbb{R}^{3\times 4}), z^b \in W_0^{2,p}(\omega;\mathbb{R}^3)} \left\{ \int_{\Omega^a} W \begin{pmatrix} 2z^a & (D_{x_3}\xi^a)^T \\ (D_{x_3}\xi^a) & D^2_{x_3}u^a \end{pmatrix} dx + q\int_{\Omega^b} W \begin{pmatrix} D^2_{x'}u^b & (D_{x'}\xi^b)^T \\ (D_{x'}\xi^b) & 2z^b \end{pmatrix} dx \right\}$$
$$= \inf_{z^a \in L^p(]0,1[;\mathbb{R}^{3\times 4}), z^b \in L^p(\omega;\mathbb{R}^3)} \left\{ \int_{\Omega^a} W \begin{pmatrix} 2z^a & (D_{x_3}\xi^a)^T \\ (D_{x_3}\xi^a) & D^2_{x_3}u^a \end{pmatrix} dx + q\int_{\Omega^b} W \begin{pmatrix} D^2_{x'}u^b & (D_{x'}\xi^b)^T \\ (D_{x'}\xi^b) & 2z^b \end{pmatrix} dx \right\}. \tag{4.8}$$

Since $z^a$ and $z^b$ are independent from each other, one can re-write (4.8) as

$$\inf_{z^a\in L^p(]0,1[;\mathbb{R}^{3\times 4})}\left\{\int_{\Omega^a} W\begin{pmatrix} 2z^a & (D_{x_3}\xi^a)^T \\ (D_{x_3}\xi^a) & D^2_{x_3}u^a\end{pmatrix} dx\right\} + q\inf_{z^b\in L^p(\omega;\mathbb{R}^3)}\left\{\int_{\Omega^b} W\begin{pmatrix} D^2_{x'}u^b & (D_{x'}\xi^b)^T \\ (D_{x'}\xi^b) & 2z^b\end{pmatrix} dx\right\}$$

Define two auxiliary functions $g\ :]0,1[\times\mathbb{R}^3 \to \mathbb{R}$ by $g(x_3,z^a) := W\begin{pmatrix} 2z^a & (D_{x_3}\xi^a(x_3))^T \\ (D_{x_3}\xi^a(x_3)) & D^2_{x_3}u^a\end{pmatrix}$ and $f\ :\ \omega\times\mathbb{R}^3 \to \mathbb{R}$ by $f(x',z^b) = W\begin{pmatrix} D^2_{x'}u^b(x') & (D_{x'}\xi^b(x'))^T \\ (D_{x'}\xi^b(x')) & 2z^b\end{pmatrix}$. It is easily seen that they are Carathéodory functions. By (3.3), (3.4) and applying the measurable selection lemma, cf. [10] and [24], it can be deduced the existence of two measurable functions $z^a_0$ and $z^b_0$, such that

$$\hat{W}(D^2_{x_3}u^a(x_3), D_{x_3}\xi^a(x_3)) = W\begin{pmatrix} 2z^a_0(x_3) & (D_{x_3}\xi^a(x_3))^T \\ (D_{x_3}\xi^a(x_3)) & D^2_{x_3}u^a(x_3)\end{pmatrix} \text{ for a.e. } x_3\in]0,1[,$$

and

$$W_0(D^2_{x'}u^b(x'), D_{x'}\xi^b(x')) = W\begin{pmatrix} D^2_{x'}u^b(x') & (D_{x'}\xi^b(x'))^T \\ (D_{x'}\xi^b(x')) & 2z^b_0(x')\end{pmatrix} \text{ for a.e. } x'\in\omega.$$

Due to the coerciveness assumption (2.2), $z^a_0 \in L^p(]0,1[;\mathbb{R}^{3\times 4})$ and $z^b_0 \in L^p(\omega;\mathbb{R}^3)$, and thus

$$\begin{aligned}&\inf_{z^a\in W^{2,p}_0(]0,1[;\mathbb{R}^{3\times 4}),z^b\in W^{2,p}_0(\omega;\mathbb{R}^3)}\left\{\int_{\Omega^a} W\begin{pmatrix} 2z^a & (D_{x_3}\xi^a)^T \\ (D_{x_3}\xi^a) & D^2_{x_3}u^a\end{pmatrix} dx\right.\\ &\left.+q\int_{\Omega^b} W\begin{pmatrix} D^2_{x'}u^b & (D_{x'}\xi^b)^T \\ (D_{x'}\xi^b) & 2z^b\end{pmatrix} dx\right\} \le \int_0^1 \hat{W}(D^2_{x_3}u^a, D_{x_3}\xi^a)dx_3 \\ &+q\int_\omega W_0(D^2_{x'}u^b, D_{x'}\xi^b)dx'.\end{aligned} \tag{4.9}$$

(4.7), (4.9), Lemma 3.3, Remark 3.5 and the sequential lower semicontinuity of 'Γ upper limit' entail that

$$\begin{aligned}&\inf\left\{\liminf_n \left(\left(K^a_n(\cdot)+\frac{h_n}{r_n^2}K^b_n(\cdot)\right)_* (u^a_n,u^b_n)\right) : (u^a_n,u^b_n)\in\mathcal{U}_n, (u^a_n,u^b_n)\to(u^a,u^b) \text{ in } W^{1,p},\right.\\ &\left.\left(\frac{1}{r_n}D_{x'}u^a_n, \frac{1}{h_n}D_{x_3}u^b_n\right) \to (\xi^a,\xi^b) \text{ in } L^p\right\}\\ &\le \int_0^1 \hat{W}^{**}(D^2_{x_3}u^a, D_{x_3}\xi^a)dx_3 + q\int_\omega Q_{\mathcal{A}^2}W_0(D^2_{x'}u^b, D_{x'}\xi^b)dx'.\end{aligned} \tag{4.10}$$

□

The next result deals with the lower bound for our $\Gamma$ limit.

**Proposition 4.2** *Let $W$ be a function satisfying* (2.1), *and* (2.2). *For every $n \in \mathbb{N}$, let $K_n^a$, $K_n^b$ and $\mathcal{U}_n$ be as in* (1.7), (1.8), *and* (1.9), *respectively. Let $V^p$ be as in* (2.3). *Assume that* (1.2) *and* (1.6) *hold. For every $(u^a, u^b, \xi^a, \xi^b) \in V^p$ it results that*

$$\inf \left\{ \liminf_n \left( (K_n^a(\cdot) + \frac{h_n}{r_n^2} K_n^b(\cdot))_*(u_n^a, u_n^b) \right) : (u_n^a, u_n^b) \in \mathcal{U}_n, \right.$$
$$\left. (u_n^a, u_n^b) \to (u^a, u^b) \text{ in } W^{1,p}, \left( \frac{1}{r_n} D_{x'} u_n^a, \frac{1}{h_n} D_{x_3} u_n^b \right) \to (\xi^a, \xi^b) \text{ in } L^p \right\}$$
$$\geq \int_0^1 \hat{W}^{**}(D_{x_3}^2 u^a, D_{x_3}\xi^a) dx_3 + q \int_\omega Q_{\mathcal{A}^2} W_0(D_{x'}^2 u^b, D_{x'}\xi^b) dx'. \tag{4.11}$$

*Proof.* The arguments used in the proof are very close to those used in [24], but here there is the presence of suitable $\mathcal{A}$-quasiconvex functions and of coupled functionals. Take $(u^a, u^b, \xi^a, \xi^b) \in V^p$. Considered the sequence $\{(u^a(x_3) + r_n x' \cdot \xi^a(x_3), u^b(x') + h_n x_3 \xi^b(x'))\}_n$, it results that $K_n^a(u^a + r_n x' \cdot \xi^a) + \frac{h_n}{r_n^2} K_n^b(u^b + h_n x_3 \xi^b)$ is bounded uniformly in $n$. Consequently $\inf\{\liminf_n \left( (K_n^a(\cdot) + \frac{h_n}{r_n^2} K_n^b(\cdot))_* \right)\}(u^a, u^b, \xi^a, \xi^b)$ is also bounded. By the very definition of $\Gamma$ -lower limit there exists a sequence $\{u_n^a, u_n^b\} \in \mathcal{U}_n$ such that $(u_n^a, u_n^b) \to (u^a, u^b)$ in $W^{1,p}$ and $\left( \frac{1}{r_n} D_{x'} u_n^a, \frac{1}{h_n} D_{x_3} u_n^b \right) \to (\xi^a, \xi^b)$ in $L^p$. Clearly Proposition 3.1 provides the weak convergence of $(u_n^a, u_n^b)$ to $(u^a, u^b)$ in $W^{2,p}(\Omega^a; \mathbb{R}^3) \times W^{2,p}(\Omega^b; \mathbb{R}^3)$ and of $\left( \frac{1}{r_n} D_{x'} u_n^a, \frac{1}{h_n} D_{x_3} u_n^b \right)$ to $(\xi^a, \xi^b)$ in $W^{1,p}(\Omega^a; \mathbb{R}^{3\times 4}) \times W^{1,p}(\Omega^b; \mathbb{R}^3)$. Moreover it results that

$$\int_{\Omega^a} W \begin{pmatrix} \frac{1}{r_n^2} D_{x'}^2 u_n^a & \left( \frac{1}{r_n} D_{x_3,x'}^2 u_n^a \right)^T \\ \left( \frac{1}{r_n} D_{x_3,x'}^2 u_n^a \right) & D_{x_3}^2 u_n^a \end{pmatrix} dx \geq \int_{\Omega^a} \hat{W} \left( \frac{1}{r_n} D_{x_3,x'}^2 u_n^a, D_{x_3}^2 u_n^a \right) dx$$
$$\geq \int_{\Omega^a} \hat{W}^{**} \left( \frac{1}{r_n} D_{x_3,x'}^2 u_n^a, D_{x_3}^2 u_n^a \right) dx.$$
$$\int_{\Omega^b} W \begin{pmatrix} \frac{1}{h_n^2} D_{x_3}^2 u_n^b & \left( \frac{1}{h_n} D_{x_3,x'}^2 u_n^b \right)^T \\ \left( \frac{1}{h_n} D_{x_3,x'}^2 u_n^b \right) & D_{x'}^2 u_n^b \end{pmatrix} dx \geq \int_{\Omega^b} W_0 \left( \frac{1}{h_n} D_{x_3,x'}^2 u_n^b, D_{x'}^2 u_n^b \right) dx$$
$$\geq \int_{\Omega^b} Q_{\mathcal{A}^2} W_0(D_{x'}^2 u_n^b, \frac{1}{h_n} D_{x',x_3}^2 u_n^b) dx.$$

To obtain the last inequalities one can observe that for fixed $x'$:

$$\int_0^1 \hat{W} \left( \frac{1}{r_n} D_{x_3,x'}^2 u_n^a, D_{x_3}^2 u_n^a \right) dx_3 \geq \int_0^1 \hat{W}^{**} \left( \frac{1}{r_n} D_{x_3,x'}^2 u_n^a, D_{x_3}^2 u_n^a \right) dx_3,$$

and for fixed $x_3$:

$$\int_\omega W_0\left(D^2_{x'}u^b_n, \frac{1}{h_n}D^2_{x_3,x'}u^b_n\right)dx' \geq \int_\omega Q_{\mathcal{A}^2}W_0(D^2_{x'}u^b_n, \frac{1}{h_n}D^2_{x',x_3}u^b_n)dx'.$$

We can observe that $Q_{\mathcal{A}^2}W_0 : Sym(\mathbb{R}^{2\times 2})\times M^{3\times 2} \to [0,+\infty[$ is $\mathcal{A}^2-$quasiconvex and continuous (see [27]).

We can define a new function $Z : Sym(\mathbb{R}^{3\times 3})\times M^{3\times 3} \to \mathbb{R}$ as

$$Z(H|B) = Q_{\mathcal{A}^2}W_0(h|\xi), \tag{4.12}$$

where $\xi_{ij} = B_{ij},\ \ i = 1,2,3,\ \ j = 1,2$ and $h$ is such that $h^i_{jk} := H^i_{jk}$, $i = 1,2,3$ and $j,k = 1,2$. The function $Z$ is $\mathcal{A}-$ quasiconvex in the sense of the differential operator $\mathcal{A}^3$ defined by the system of PDE's:

$$\mathcal{A}^3V := (\mathcal{A}^3_2H, \mathcal{A}^3_1B) \tag{4.13}$$

where

$$\mathcal{A}^3_2H := \left(\frac{\partial}{\partial x_i}H^l_{jk} - \frac{\partial}{\partial x_j}H^l_{ik}\right)_{i,j,k,l=1,\dots,3},$$

where $Q_3 = ]0,1[^3$ and $H \in C^\infty_{\mathrm{per}}(Q_3, Sym(\mathbb{R}^3))$. Furthermore

$$\mathcal{A}^3_1B = \left(\frac{\partial B^i_j}{\partial x_k} - \frac{\partial B^i_k}{\partial x_j}\right)_{i,j,k=1,2,3}.$$

Arguing as in Remark 2.4, it can be shown that $\mathcal{A}^3$ satisfies the constant rank property and that

$$\begin{aligned}&\mathrm{Ker}\mathbb{A}^3(w)\\ &\quad= \left\{(X,V)\in Sym(\mathbb{R}^3)\times M^{3\times 3}:(X,V)=(b\otimes w^{\otimes 2}, a\otimes w),\ b\in\mathbb{R}^3, a\in\mathbb{R}^3\right\},\end{aligned} \tag{4.14}$$

where $w^{\otimes 2}$ stands for $w\otimes w$ (see also [18] for details). To prove the $\mathcal{A}^3$-quasiconvexity of $Z$, we consider a test function $\omega \in C^\infty_{per}(\mathbb{R}^3, Sym(\mathbb{R}^{3\times 3})\times M^{3\times 3})$, $\ \omega = (\omega_H, \omega_B)$ which satisfies the constraint $\mathcal{A}^3\omega = 0$ and such that $\int_{Q_3}\omega(y)dy = 0$. It is easy to see that for every $x_3$ fixed we have $\omega(\cdot, x_3) \in C^\infty_{per}(\mathbb{R}^2, Sym(\mathbb{R}^{2\times 2})\times M^{3\times 2})$, as $\omega(\cdot,x_3) = (\omega_h(\cdot,x_3), \omega_\xi(\cdot,x_3))$, in the sense that $\mathcal{A}^2_2\omega_h = 0$, $\mathcal{A}^2_1\omega_\xi = 0$.

We also have $\int_{Q_2}\omega dx' = 0$, which follows from (4.13), $\int_{Q_3}\omega dx = 0$ and the periodicity of $\omega$. Indeed, given $(H,B) \in Sym(\mathbb{R}^{3\times 3})\times M^{3\times 3}$, from (4.12) and Definition 2.2 it follows

$$\begin{aligned}\int_Q Z((H|B)+\omega(x))dx &= \int_0^1\int_{Q_2} Q_{\mathcal{A}^2}W_0((h|b)+(\omega_h|\omega_b))dx'dx_3\\ &\geq \int_0^1\int_{Q_2} Q_{\mathcal{A}^2}W_0(h,b)dx'dx_3 = \int_Q Z(H|B)dx.\end{aligned}$$

It can be easily verified that the function $Z$ is bounded from below and also satisfies the same growth condition as $W$, hence Theorem 3.7 in [11] ensures that $G_2 : W^{2,p}(\Omega^b;\mathbb{R}^3)\times W^{1,p}(\Omega^b;\mathbb{R}^3) \to \mathbb{R}$, defined as $G_2(u^b,\xi^b) := \int_{\Omega^b} Q_{\mathcal{A}^2}W_0\left(D^2_{x'}u^b, D_{x'}\xi^b\right) dx$ is lower semicontinuous. An analogous (even easier) argument, ensures the same property for $G_1 : W^{2,p}(\Omega^a;\mathbb{R}^3)\times W^{1,p}(\Omega^a;\mathbb{R}^{3\times 2}) \to \mathbb{R}$, defined as $G_1(u^a, \xi^a) := \int_{\Omega^a} \hat{W}^{**}\left(D^2_{x_3}u^a, D_{x_3}\xi^a\right) dx$. Hence one can write

$$\begin{aligned}
&\liminf_n \left(\int_{\Omega^a} W\begin{pmatrix} \frac{1}{r_n^2}D^2_{x'}u_n^a & \left(\frac{1}{r_n}D^2_{x_3,x'}u_n^a\right)^T \\ \left(\frac{1}{r_n}D^2_{x_3,x'}u_n^a\right) & D^2_{x_3}u_n^a\end{pmatrix} dx\right.\\
&\left.+\frac{h_n}{r_n^2}\int_{\Omega^b} W\begin{pmatrix} \frac{1}{h_n^2}D^2_{x_3}u_n^b & \left(\frac{1}{h_n}D^2_{x_3,x'}u_n^b\right)^T \\ \left(\frac{1}{h_n}D^2_{x_3,x'}u_n^b\right) & D^2_{x'}u_n^b\end{pmatrix} dx\right)\\
&\geq \liminf_n \left(G_1\left(u_n^a, \frac{1}{r_n}D_{x'}u_n^a\right) + \frac{h_n}{r_n^2}G_2\left(u_n^b, \frac{1}{h_n}D_{x_3}u_n^b\right)\right)\\
&\geq \liminf_n G_1\left(u_n^a, \frac{1}{r_n}D_{x'}u_n^a\right) + \liminf_n \frac{h_n}{r_n^2}G_2\left(u_n^b, \frac{1}{h_n}D_{x_3}u_n^b\right)\\
&\geq G_1(u^a,\xi^a) + qG_2(u^b,\xi^b)\\
&= \int_0^1 \hat{W}^{**}(D^2u^a, D\xi^a)dx_3 + q\int_\omega Q_{\mathcal{A}^2}W_0(D^2u^b, d\xi^b)dx',
\end{aligned}$$

and this concludes the proof. □

**Theorem 4.3** *Let $W$ be a function satisfying* (2.1), *and* (2.2). *For every $n \in \mathbb{N}$, let $K_n^a$, $K_n^b$ and $\mathcal{U}_n$ be as in* (1.7), (1.8), *and* (1.9), *respectively. Let $V$ be as in* (2.4). *Assume that* (1.2) *and* (1.6) *hold.*

$$\begin{aligned}
&\inf\left\{\liminf_n \left((K_n^a(\cdot) + \frac{h_n}{r_n^2}K_n^b(\cdot))_*(u_n^a,u_n^b)\right) : (u_n^a,u_n^b)\in\mathcal{U}_n, (u_n^a,u_n^b)\right.\\
&\quad\left.\to (u^a,u^b) \text{ in } W^{1,p}, \left(\frac{1}{r_n}D_{x'}u_n^a, \frac{1}{h_n}D_{x_3}u_n^b\right) \to (\xi^a,\xi^b) \text{ in } L^p\right\}\\
&\quad= \int_0^1 \hat{W}^{**}(D^2_{x_3}u^a, D_{x_3}\xi^a)dx_3 + q\int_\omega Q_{\mathcal{A}^2}W_0(D^2_{x'}u^b, D_{x'}\xi^b)dx'.
\end{aligned} \tag{4.15}$$

*for every $(u^a,u^b,\xi^a,\xi^b)\in V$. Furthermore the $\Gamma$ lower limit is a $\Gamma$ limit.*

*Proof.* The theorem is a direct consequence of Propositions 4.1 and 4.2 and of Definition 2.5. □

**Theorem 4.4** *Let $W$ be a function satisfying* (2.1), *and* (2.2). *For every $n \in \mathbb{N}$, let $K_n^a$, $K_n^b$ and $\mathcal{U}_n$ be as in* (1.7), (1.8), *and* (1.9), *respectively. Assume that* (1.2) *and* (1.6) *hold.*

$$\begin{aligned}&\inf\left\{\liminf_n\left(\left(K_n^a(\cdot)+\frac{h_n}{r_n^2}K_n^b(\cdot)\right)_*(u_n^a,u_n^b)\right):(u_n^a,u_n^b)\in\mathcal{U}_n,(u_n^a,u_n^b)\right.\\&\quad\left.\to(u^a,u^b)\ \text{in}\ W^{1,p},\left(\frac{1}{r_n}D_{x'}u_n^a,\frac{1}{h_n}D_{x_3}u_n^b\right)\to(\xi^a,\xi^b)\ \text{in}\ L^p\right\}\\&\quad=\int_0^1\hat{W}^{**}(D_{x_3}^2u^a,D_{x_3}\xi^a)dx_3+q\int_\omega Q_{\mathcal{A}^2}W_0(D_{x'}^2u^b,D_{x'}\xi^b)dx'\end{aligned}\tag{4.16}$$

*for every* $(u^a,u^b,\xi^a,\xi^b)\in V^p$. *Furthermore the* $\Gamma$ *lower limit is a* $\Gamma$ *limit.*

*Proof.* The statement will be achieved by double inequality. Take $(u^a,u^b,\xi^a,\xi^b)\in V^p$, then one inequality follows from Proposition 4.2. It remains to prove the converse one. By virtue of Proposition 2.1, the growth condition (2.2) inherited by $Q_{\mathcal{A}^2}W_0$ and $\hat{W}^{**}$, and standard relaxation arguments, allow us to say that smooth recovery sequence can be used in Lemma 3.3, thus getting

$$\begin{aligned}&\int_0^1\hat{W}^{**}(D_{x_3}^2u^a,D_{x_3}\xi^a)dx_3+q\int_\omega Q_{\mathcal{A}^2}W_0(D_{x'}^2u^b,D_{x'}\xi^b)dx'\\&=\inf\left\{\begin{array}{l}\liminf\limits_{n\to\infty}\left[\int_0^1\hat{W}(D_{x_3}^2u_n^a,D_{x_3}\xi_n^a)dx_3+q\int_\omega W_0(D_{x'}^2u_n^b,D_{x'}\xi_n^b)dx'\right]:\\((u_n^a,u_n^b),(\xi_n^a,\xi_n^b))\in V,u_n^a\rightharpoonup u^a\ \text{in}\ W^{2,p}(]0,1[;\mathbb{R}^3),\\\xi_n^a\rightharpoonup\xi^a\ \text{in}\ W^{1,p}(]0,1[;\mathbb{R}^{3\times4}),u_n^b\rightharpoonup u^b\ \text{in}\ W^{2,p}(\omega;\mathbb{R}^3),\\\xi_n^b\rightharpoonup\xi^b\ \text{in}\ W^{1,p}(\omega;\mathbb{R}^3)\end{array}\right\}.\end{aligned}\tag{4.17}$$

Since (4.17) holds, for any $(u^a,u^b,\xi^a,\xi^b)\in V^p$ one can construct a diagonal sequence still denoted by $\{(u_n^a,u_n^b)\}_n\subset\mathcal{U}_n$ such that $(u_n^a,u_n^b)\to(u^a,u^b)$ in $W^{1,p}$, $\left(\frac{1}{r_n}u_n^a,\frac{1}{h_n}u_n^b\right)\to(\xi^a,\xi^b)$ in $L^p$ and

$$\begin{aligned}\lim_n\left(K_n^a(\cdot)+\frac{h_n}{r_n^2}K_n^b(\cdot)\right)_*(u_n^a,u_n^b)&=\int_0^1\hat{W}^{**}(D_{x_3}^2u^a,D_{x_3}\xi^a)dx_3\\&\quad+q\int_\omega Q_{\mathcal{A}^2}W_0(D_{x'}^2u^b,D_{x'}\xi^b)dx'\end{aligned}$$

and this concludes the proof. □

# 5 The Case $s=1$

In this section we generalize the result presented in [13] (for a quasilinear Neumann problem) to the non-convex case. We just sketch the main steps needed to get the main result, since most of the theorems are already established in [13] (in the Neumann context, yet with a straightforward generalisation to the Dirichlet framework) and other results can be easily obtained by adapting to the gradient context the propositions and lemmata proven in the previous sections for the case of non simple grade two materials. The thin multi-domain, that we consider here, is the multi-structure $\Omega_n$ as in figure 1, with the characteristic lengths $h_n$ and $r_n$ satisfying assumption (1.2) and (1.6). Let $p > 1$, and assume that the energy density

$$W : \mathbb{M}^{3\times 3} \to \mathbb{R} \tag{5.1}$$

is a continuous function such that there exist three constants $\alpha \in \mathbb{R}$ and $\beta, C > 0$ for which

$$\alpha + \frac{1}{C}|F|^p \leq W(F) \leq \beta + C|F|^p \tag{5.2}$$

for every $F \in \mathbb{M}^{3\times 3}$. The model can be described through an energy of the kind

$$\int_{\Omega_n} W(DU_n)dx, \tag{5.3}$$

where the admissible fields $U_n \in W^{1,p}(\Omega_n;\mathbb{R}^3)$ verify the boundary condition as in section 2, namely we require that $U_n$ coincide with $c^a$ on the top of $\Omega_n^a$ with $f^b$ on the lateral boundary of $\Omega_n^b$, for $f^b \in W^{1,p}(\omega;\mathbb{R}^3)$. By adopting the same re-scaling as in (1.4) we obtain the following energy in $\Omega = \Omega^a \cup \Omega^b = \omega\times]-1,1[$

$$E_n(u_n^a, u_n^b) := \int_{\Omega^a} W\left(\frac{1}{r_n} D_{x'} u_n^a, D_{x_3} u_n^a\right) dx + \frac{h_n}{r_n^2}\int_{\Omega^b} W\left(D_{x'} u_n^b, \frac{1}{h_n} D_{x_3} u_n^b\right) dx, \tag{5.4}$$

with the following junction condition

$$u_n^a(x', 0) = u_n^b(r_n x', 0) \quad \text{a.e. in } \omega. \tag{5.5}$$

We describe the limit energy as $n \to +\infty$, when the volumes of $\Omega_n^a$ and $\Omega_n^b$ tend to zero with the same rate, i.e. (1.2) and (1.6) hold. Define

$$\mathcal{U'}_n = \left\{ \begin{array}{l} (u^a, u^b) \in \left(c^a + W_a^{1,p}(\Omega^a;\mathbb{R}^3)\right) \times \left(f^b + W_b^{1,p}(\Omega^b;\mathbb{R}^3)\right) : \\ u^a(x', 0) = u^b(r_n x', 0), \quad x' \text{ a.e. in } \omega \end{array} \right\} \tag{5.6}$$

with $r_n$, $h_n$, $c^a$, and $f^b$ as defined before, $W_a^{1,p}(\Omega^a;\mathbb{R}^3)$ the closure, in $W^{1,p}$, of $\{u^a \in C^\infty(\overline{\Omega^a};\mathbb{R}^3) : u^a = 0$ in a neighbourhood of $\omega\times\{1\}\}$ and $W_b^{1,p}(\Omega^b;\mathbb{R}^3)$ the closure in $W^{1,p}$, of $\{u^b \in C^\infty(\overline{\Omega^b};\mathbb{R}^3) : u^b = 0$ in a neighbourhood of $\partial\omega\times]-1,0[\}$. Furthermore we can still assume that (1.10) holds. A compactness result analogous to Proposition 3.1 still holds, (cf. Proposition 2.1 in [13] where the dead loads played the role of prescribed boundary data) by replacing the boundary conditions of $\mathcal{U}_n$, with those of $\mathcal{U}'_n$ and the convergences in Proposition 3.1 with the weak convergence in $W^{1,p}(\Omega^a;\mathbb{R}^3)\times W^{1,p}(\Omega^b;\mathbb{R}^3)$. Actually the arguments used there can be adapted to this framework by using standard cut-off functions. This fact leads to consider the following problem:

$$\inf\left\{\liminf_{n\to\infty} E_n(u_n^a,u_n^b) : (u_n^a,u_n^b)\in\mathcal{U}_n',(u_n^a,u_n^b)\to(u^a,u^b) \text{ in } L^p\right\}, \tag{5.7}$$

where $E_n(u_n^a,u_n^b)$ has been extended to $+\infty$ in $(L^p(\Omega^a;\mathbb{R}^3)\times L^p(\Omega^b;\mathbb{R}^3))\backslash\mathcal{U}_n'$. The limit functions $u^a$ and $u^b$, do not depend on $x'$ and $x_3$ respectively, and junction conditions, in this case, as emphasized in [13], are the following:

i) the limit problem is uncoupled if $1 < p \leq 2$,

ii) coupled by the junction condition: $u^a(0) = u^b(0')$ if $p > 2$.

The following theorem can be proved

**Theorem 5.1** *Let $W$ be the energy density in* (5.1)*, satisfying* (5.2)*. For every $n \in \mathbb{N}$, let $E_n$ be the functional introduced in* (5.4) *and $\mathcal{U}_n'$ be as in* (5.6)*. Assume that* (1.2) *and* (1.6) *hold. Then*

$$\begin{aligned}&\inf\left\{\liminf_{n\to\infty} E_n(u_n^a,u_n^b) : (u_n^a,u_n^b)\in\mathcal{U}_n',(u_n^a,u_n^b)\to(u^a,u^b) \textit{ in } L^p\right\} =\\ &\int_0^1 \tilde{W}^{**}(D_{x_3}u^a) + \int_\omega \mathcal{Q}W_1(D_{x'}u^b)dx'\end{aligned} \tag{5.8}$$

*for every $(u^a,u^b)\in W^{1,p}(]0,1[;\mathbb{R}^3)\times W^{1,p}(\omega;\mathbb{R}^3)$ with $u^a(1)=c^a$, $u^b(x')=f^b(x')$ on $\partial\omega$ and $u^a(0)=u^b(0')$ if $p>2$.*

We recall that $\tilde{W}:\mathbb{R}^3\to\mathbb{R}$ is the function defined by

$$\tilde{W}(f) := \inf_{g,h\in\mathbb{R}^3} W(f|g|h) \tag{5.9}$$

while $W_1:\mathbb{R}^{3\times 2}\to\mathbb{R}$ is the function defined by

$$W_1(F') := \inf_{I\in\mathbb{R}^3} W(F'|I) \tag{5.10}$$

The analogous of Remark 3.2 still holds. Furthermore the energy densities appearing in (5.8) are respectively the convexification of $\tilde{W}$ according to definition (2.5) and the quasiconvexification of $W_1$ defined in (2.7).

*Sketch of the proof.* The structure of the proof (again relying on $\Gamma$ convergence arguments) is analogous to the case $s = 2$. Namely (5.8) follows by double inequality. By the same token as above, we need the following analogue of Lemma 3.3, dealing with relaxation of 'coupled' functionals, if $p > 2$, whose proof is omitted since entirely similar to the mentioned Lemma:

**Lemma 5.2** *Let $W$ be the energy density in* (5.1)*, satisfying* (5.2)*, let $\tilde{W}$ and $W_1$ be the functions in* (5.9) *and* (5.10)*. Let $p > 2$, for every $(u^a, u^b) \in W^{1,p}(]0,1[;\mathbb{R}^3) \times W^{1,p}(\omega;\mathbb{R}^3)$ with $u^a(1) = c^a$, $u^b(x') = f^b(x')$ on $\partial\omega$, define the functionals*

$$I_0(u^a, u^b) = \inf \left\{ \begin{array}{l} \liminf\limits_{n\to\infty} \left[ \int_0^1 \tilde{W}(Dx_3 u_n^a) dx_3 + q \int_\omega W_1(Dx' u_n^b) dx' \right] : \\ (u_n^a, u_n^b) \in W^{1,p}(]0,1[;\mathbb{R}^3) \times W^{1,p}(\omega;\mathbb{R}^3), \\ u_n^a(1) = c^a, u_n^b(x') = f^b(x') \text{ on } \partial\omega, \\ u_n^a \rightharpoonup u^a \text{ in } W^{1,p}(]0,1[;\mathbb{R}^3), \\ u_n^b \rightharpoonup u^b \text{ in } W^{1,p}(\omega;\mathbb{R}^3), u_n^a(0) = u_n^b(0') \end{array} \right\}, \tag{5.11}$$

*then*

$$I_0(u^a, u^b) = \int_0^1 \tilde{W}^{**}(D_{x_3} u^a) dx_3 + q \int_\omega \mathcal{Q}W_1(D_{x'} u^b) dx'$$

*for every $((u^a, u^b) \in W^{1,p}(]0,1[;\mathbb{R}^3) \times W^{1,p}(\omega;\mathbb{R}^3)$ with $u^a(1) = c^a$, $u^b(x') = f^b(x')$ on $\partial\omega$.*

If $1 < p \leq 2$ there is no need of proving such results because the limit problem is uncoupled, the relaxed functionals have separate behavior and no junction enters in the proof of the main theorem. The proof of the lower bound in entirely similar to that of Proposition 4.2. The main difference consists of replacing the $\mathcal{A}^2$ quasiconvexification of $W_0$ with the quasiconvexification of $W_1$ and in applying the well known semicontinuity result, established in [2], instead of Theorem 3.7 in [11]. The upper bound is first obtained in a dense subset of the limit space, in analogy with [13]. More precisely a density result similar to Proposition 3.1 in [13] still holds (using the proof of that result and cut-off functions) by considering a class of functions which also verifies the boundary conditions of the limit ones. Next one can 'mimic' the proof of Proposition 4.1 to get the $\Gamma$ lim sup inequality, by adopting the following 'recovery sequence':

$$u_n^a(x) := \begin{cases} (r_n x' \cdot z^a(\varepsilon_n) + u^a(\varepsilon_n)) \frac{x_3}{\varepsilon_n} + v^b(r_n x') \frac{\varepsilon_n - x_n}{\varepsilon_n} & \text{if } x = (x', x_3) \in \omega \times ]0, \varepsilon_n[, \\ r_n x' \cdot z^a(x_3) + u^a(x_3) & \text{if } x = (x', x_3) \in ]\varepsilon_n, 1[ \end{cases}$$

and

$$u_n^b(x) := h_n x_3 z^b(x') + u^b(x') \text{ if } x \in \Omega^b,$$

where $\{\varepsilon_n\}$ is any sequence converging to 0 as $\{r_n\}$, the functions $u^a$ and $u^b$ belong to the dense subspace, $z^a \in W_0^{1,p}(]0,1[;\mathbb{R}^{3\times 2})$ and $z^b \in W_0^{1,p}(\omega;\mathbb{R}^3)$. The final step consists of applying the same diagonal argument as in Theorem 4.4. □

# 6 Conclusions

- It is worthwhile to mention the fact that by virtue of the coercivity assumptions (2.2) and (5.2) made on the energy density both in the case $s = 2$ and

$s = 1$, the families of functionals $\{K_n^a(\cdot) + \frac{h_n}{r_n^2} K_n^b\}$ and $\{E_n\}$ are equicoercive according to what stated in Definition 2.6, convergence of almost minimizers of these families in the classes $\mathcal{U}_n$ and $\mathcal{U}'_n$ respectively, to the minimizers of $\int_0^1 \hat{W}^{**}(D^2 u^a, D\xi^a)dx_3 + q\int_\omega \mathcal{Q}_{\mathcal{A}^2} W_0(D^2 u^b, D\xi^b)dx'$ and of $\int_0^1 \tilde{W}^{**}(Du^a)dx_3 + q\int_\omega \mathcal{Q}W_1(Du^b)dx'$ respectively, follows, in the spirit of Theorem 2.7.

- The argument used in the proof of Proposition 4.1 for the construction of the 'recovery sequences', i.e. the choice of $z^a$ and $z^b$ independent of $x'$ and $x_3$ respectively and the definitions of $W_0$, $\hat{W}$ and $W_1$, $\tilde{W}$ in (3.4), (3.3) and (5.10), (5.9) can be regarded as a way of obtaining a slight improvement to the representation results obtained in Theorem 1.1 in [17] and Theorem 1.1 of [13] also in the convex case, in the sense that one could first take the infimum on the last variables $z$, thus obtaining energy densities still convex but explicitly depending only on $(u^a, u^b, \xi^a, \xi^b)$ if $s = 2$ ($(u^a, u^b)$ for $s = 1$).

- As we already observed this paper is not aimed by investigating junction conditions but just by showing what is the asymptotic behavior of hyperelastic or grade two materials filled multi-structures as in figure 1, when the energy density is not convex. Because of the physical relevance of this question, both for describing thin hyperelastic structures ($s = 1$) and for non simple materials thin multi-domains ($s = 2$), we just considered the case $N = 3$. The mathematical problems can nonetheless be managed also for $N > 3$, as they have been developed in [13] and [17] in the convex context. Clearly the junction possibilities in the limit could be different from the cases presented here, since they explicitly depend on the relations between the Sobolev exponent $p$ and the space dimension $N$. Actually when $s = 1$ there is no difference in the proof of Theorem 5.1 (since the only junction conditions are analogous to $i$) and $ii$)), namely the limit problem is

  i) uncoupled if $1 < p \leq N - 1$,

  ii) coupled by the junction condition: $u^a(0) = u^b(0')$ if $p > N - 1$.

  The proof differs just for the fact that the limit functions $(u^a, u^b)$ are defined in $\mathbb{R}\times\mathbb{R}^{N-1}$ respectively, rather than $\mathbb{R}\times\mathbb{R}^2$ and being $\mathbb{R}^N$ rather than $\mathbb{R}^3$.

  A new facet shows up for $s = 2$. Apart from the obvious variations in domain and codomain of functions, in this case the junction conditions can be summarized as follows: the limit problem is uncoupled if $1 < p \leq \frac{N-1}{2}$, "partially" coupled if $\frac{N-1}{2} < p \leq N - 1$, and coupled if $N - 1 < p$. On the other hand also the transmission conditions in $\mathcal{U}_n$ are different since the traces on the junction surface at $x_3 = 0$ if $1 < p \leq \frac{N-1}{2}$ are meaningless. As in the case $s = 1$ both the compactness theorem and Theorem 4.4 carry over to the $N$- dimensional case. The same can be said also for the previous results and lemmata. The proofs are identical to case $N = 3$ when the limit problem is partially coupled and coupled. The situation is slightly different albeit simpler when $1 < p \leq \frac{N-1}{2}$, since in this case there is no need for a result such as Lemma 3.3, because the limit functionals are really independent and the relaxation procedure works separately.

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