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The Cauchy problem for the 1-D Dirac–Klein–Gordon equation

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Abstract. The Cauchy problem for the Dirac−Klein−Gordon equation are discussed in one space dimension. Time local and global existence for solutions with rough data, especially the solutions for Klein−Gordon equation in the critical and super critical Sobolev norm of [4] are considered. The solutions with general propagation speeds are dealt with.

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1 Introduction

We study the Cauchy problem for the following Dirac–Klein–Gordon equation in one space dimension:

$$\begin{cases} \mathcal{D}\mathbf{u} + m\mathbf{u} = w\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}_0, \\ \Box w + Mw = \mathbf{u}^\dagger\mathbf{u}, \quad w(0) = w_0,\ \partial_t w(0) = w_1. \end{cases} \tag{1.1}$$

where $m, M \geq 0$ are masses, $\mathbf{u}$ is a function from $\mathbb{R}^2$ to $\mathbb{C}^2$ such that

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix}, \tag{1.2}$$

and w is a function from $\mathbb{R}^2$ to $\mathbb{R}$ such that $w = w(t,x)$. The Dirac operator $\mathcal{D}$ is defined by $\mathcal{D} = i\gamma_0\partial_t + i\gamma_1\partial_x$ with the Dirac matrices, see also section 6,

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

The d'Alembert operator stands for $\Box = \Box_c = \partial_t^2 - c^2\partial_x^2$ with the propagation speed c. It is well known that $\mathcal{D}^2 = -\Box_1 I$ where I is a unit matrix. We write down the right hand side of equations in full,

$$w\mathbf{u} = \begin{pmatrix} wu \\ wv \end{pmatrix}, \quad \mathbf{u}^\dagger\mathbf{u} = (\overline{u}, \overline{v})\gamma_0 \begin{pmatrix} u \\ v \end{pmatrix} = 2\Re(\overline{u}v) \quad \text{for} \quad \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.4}$$

where $\overline{u}$ is the complex conjugate of u. The initial data $\mathbf{u}_0$ and w_0, w_1 are functions from $\mathbb{R}$ to $\mathbb{C}^2$ and from $\mathbb{R}$ to $\mathbb{R}$ respectively such that

$$\mathbf{u}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} \quad \text{and} \quad w_0 = w_0(x), \ w_1 = w_1(x). \tag{1.5}$$

We study the existence of solutions $(\mathbf{u}, w)$ for this problem. Especially we are interested in the smaller order of Sobolev space where solutions being. There are available results. In '73, Chadam [2] proved the time global existence of $(\mathbf{u}, w) \in H^1 \otimes H^1$. In '00, Bournaveas [1] proved the time global existence of $(\mathbf{u}, w) \in L^2 \otimes H^1$. In '02, Fang [3] gave the parallel proof with [1] and later in '04 [4] he proved the time local existence of $(\mathbf{u}, w) \in H^{-1/4+\varepsilon} \otimes H^{1/2+\delta}$ with $0 < \varepsilon \leq 1/4, \ 0 < \delta \leq 2\varepsilon$, and the time global existence of $(\mathbf{u}, w) \in L^2 \otimes H^{1/2+\delta}$. So the regularities of solution $\mathbf{u}$ have been successively reduced. The appropriate null form estimates on $\mathbf{u}^\dagger\mathbf{u}$ were derived and utilized effectively for the solutions with low regularity. However, the orders of Sobolev space for w are greater than $1/2$. The case $w \in H^{1/2}$ is called critical in [4]. Indeed the Sobolev embedding theorem in one dimension $H^s \hookrightarrow L^\infty, \ s > 1/2$ plays an important role for the estimates on $w\mathbf{u}$ in available papers [1], [2], [3], [4]. We remark that the embedding $H^{1/2} \hookrightarrow L^\infty$ fails.

In this paper we deal with the solutions in critical setting $(\mathbf{u}, w) \in H^{-1/4+\varepsilon} \otimes H^{1/2}$. Moreover we consider the super critical solution $w \in H^s, \ s < 1/2$ in exchange for the additional regularity of solution $\mathbf{u}$. For a proof, we employ the arguments in the series of papers by Fang and Fang–Grillakis [4, 5, 6] for the study on the same problem to ours in space dimensions 1,2 and 3 respectively with some improvements. See also [12].

We investigate the solutions of wave equations in Fourier space for variables (τ, ξ). We take a notice of that the Fourier transform of solutions for wave equations with a propagation speed c are described mainly by variables for the light lines $\tau - c\xi$ and $\tau + c\xi$, and Dirac equation consists of the system of wave equations with a propagation speed $c = 1$. In this paper we deal with the Klein–Gordon equation with the general propagation speed $c > 0$, against [4]. We make use of the rooms for the arguments on the solution w for Klein–Gordon equation which come from the enough smoothing effect in comparison to Dirac equation. It is sufficient for the proof to derive the bilinear estimates which work with these variables $\tau - c\xi$ and $\tau + c\xi$ for only $c = 1$.

Here we introduce the notation. We denote the Fourier transform with respect to x and t variables by $\mathcal{F}_x$ and $\mathcal{F}_t$ respectively or simply taking a hat$\widehat{\ }$, and use a tilde $\widetilde{f} = \mathcal{F}_t\mathcal{F}_x f$. We use $\langle x\rangle = 1 + |x|$. For $s \in \mathbb{R}$, norm of Sobolev space H^s is defined by

$$\|f\|_{H^s_x} = \|\langle\xi\rangle^s \widehat{f}\|_{L^2_\xi}. \tag{1.6}$$

The mixed norm for $L^2_\tau L^2_\xi$ is defined by

$$\|f\|_{L^2_\tau L^2_\xi} = \left(\int_{\mathbb{R}^2} |f(\tau,\xi)|^2 d\xi d\tau\right)^{1/2}. \tag{1.7}$$

The following cut-off function ψ is provided and be often used through this paper,

$$\mathcal{C}_0^\infty \ni \psi(t) = \begin{cases} 1, & |t| \le 1, \\ 0, & |t| \ge 2. \end{cases} \tag{1.8}$$

We denote the rescaled function by $\psi_T(t) := \psi(t/T)$.

We state our main theorem.

Theorem 1.1 *Let* $m, M \ge 0$. *Let* $c > 0$. *Let* $0 \le 2a < s \le 1/2$ *or* $0 \le a < 1/4,\ 1/2 < s \le 1 - 2a$. *Then for any* $\mathbf{u}_0 \in H^{-a},\ w_0 \in H^s,\ w_1 \in H^{s-1}$, *there exists* $T > 0$ *and a unique solution* $(\mathbf{u}, w)$ *for* (1.1) *in the time interval* $[-T, T]$ *satisfying*

$$\mathbf{u} \in C(-T, T; H^{-a}), \quad w \in C(-T, T; H^s) \cap C^1(-T, T; H^{s-1}). \tag{1.9}$$

Moreover when $0 = a < s < 1/2$, *the solution* $(\mathbf{u}, w)$ *can be extended to global time,*

$$\mathbf{u} \in C(\mathbb{R}; L^2), \quad w \in C(\mathbb{R}; H^s) \cap C^1(\mathbb{R}; H^{s-1}). \tag{1.10}$$

Remark 1.2 Time local solutions $(\mathbf{u}, w) \in H^{-a} \otimes H^s$ with the case $0 \le a < 1/4,\ 1/2 < s \le 1 - 2a$ were obtained by Fang [4] for $c = 1$. We remark that when even the critical case $s = 1/2$, we can take any $a < 1/4$ in this theorem. In regard to a time global solution, we impose the lower regularities s than previous works too.

The rest of this paper is organized as follows. In section 2, we gather the elemental estimates which play a role for the calculation in the other sections. In section 3, we give the estimates on the solutions to the reduced-wave equations which constitute to the Dirac equation, and wave equation respectively. In section 4, we derive the bilinear estimates for the Sobolev space with negative index. In section 5, we prove the existence results by applying the standard contraction mapping principle. In section 6, we discuss about the Dirac equation with another sets of Dirac matrices to (1.3).

2 Preliminaries

We often use the following elementary estimates, see [7], [11] and so on.

Lemma 2.1 *Let $a, b \in \mathbb{R}$. Let p, q satisfy $p + q > 1$. Then the following integral converges.*

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x-a\rangle^p \langle x-b\rangle^q}. \tag{2.1}$$

Set $\kappa = \min(p, q, p+q-1)$. The following estimates hold,

(i) *In the case* $\max(p, q) = 1$, *for any* $\varepsilon > 0$

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x-a\rangle^p \langle x-b\rangle^q} \lesssim \frac{1}{\langle a-b\rangle^{\kappa-\varepsilon}}. \tag{2.2}$$

(ii) *In the case* $\max(p, q) \neq 1$,

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x-a\rangle^p \langle x-b\rangle^q} \lesssim \frac{1}{\langle a-b\rangle^{\kappa}}. \tag{2.3}$$

(iii) *For any p, q,*

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x-a\rangle^p \langle x-b\rangle^q} \gtrsim \frac{1}{\langle a-b\rangle^{\kappa}}. \tag{2.4}$$

The following estimates are also easy.

Lemma 2.2 *Let $c > 0$. There exists C such that*

$$\frac{\langle \tau+\xi\rangle\langle \tau-\xi\rangle}{\langle \xi\rangle\langle \tau+c\xi\rangle\langle \tau-c\xi\rangle} < C \quad \textit{for any } \tau,\ \xi. \tag{2.5}$$

Proof. In the case $c > 1$. When $0 < \xi < \tau$, we estimate

$$|\tau+\xi| = \tau+\xi < \tau+c\xi = |\tau+c\xi|, \tag{2.6}$$
$$|\tau-\xi| = \tau-c\xi+(c-1)\xi \le |\tau-c\xi|+(c-1)|\xi|. \tag{2.7}$$

When $0 < \tau < \xi$, we estimate

$$|\tau+\xi| = \tau+\xi < \tau+c\xi = |\tau+c\xi|, \tag{2.8}$$
$$|\tau-\xi| = \xi-\tau < \xi = |\xi|. \tag{2.9}$$

We finish the proof for the case $\tau, \xi > 0$. The other cases follow by taking $\tau' = -\tau$ or $\xi' = -\xi$.

In the case $c < 1$. When $0 < \xi < \tau$, we estimate

$$|\tau + \xi| = \tau + \xi < \tau/c + \xi = |\tau + c\xi|/c, \tag{2.10}$$

$$|\tau - \xi| = \tau - \xi < \tau - c\xi = |\tau - c\xi|. \tag{2.11}$$

When $0 < \tau < \xi$, we estimate

$$|\tau + \xi| = \tau + \xi < \tau/c + \xi = |\tau + c\xi|/c, \tag{2.12}$$

$$|\tau - \xi| = \xi - \tau < \xi = |\xi|. \tag{2.13}$$

We finish the proof for the case $\tau, \xi > 0$. The other cases follow by taking $\tau' = -\tau$ or $\xi' = -\xi$. □

3 Linear estimates

In this section we give the linear estimates for the solutions to the following inhomogeneous reduced-wave equations and inhomogeneous wave equation respectively:

$$\partial_t u + \partial_x u = F, \quad u(0,x) = u_0(x), \tag{3.1}$$

$$\partial_t v - \partial_x v = G, \quad v(0,x) = v_0(x), \tag{3.2}$$

$$\partial_t^2 w - c^2 \partial_x^2 w = H, \quad w(0,x) = w_0(x),\ \partial_t w(0,x) = w_1(x). \tag{3.3}$$

We prove the following estimates.

Proposition 3.1 *Let* $a + b < 1/2$, $\max\{-a, 1-b\} \neq 1/2$ *and* $\max\{b-1, a+b-1/2\} \leq a$. *Let u and v be solutions to* (3.1) *and* (3.2) *respectively. Then the following estimates hold respectively,*

$$\left\| \frac{\langle \tau + \xi \rangle^{1-b}}{\langle \tau - \xi \rangle^{a}} \widehat{\psi_T} * \widetilde{u} \right\|_{L^2_\tau L^2_\xi} \lesssim \|u_0\|_{H^{-a}} + \left\| \frac{\widetilde{F}}{\langle \tau + \xi \rangle^{b} \langle \tau - \xi \rangle^{a}} \right\|_{L^2_\tau L^2_\xi}, \tag{3.4}$$

$$\left\| \frac{\langle \tau - \xi \rangle^{1-b}}{\langle \tau + \xi \rangle^{a}} \widehat{\psi_T} * \widetilde{v} \right\|_{L^2_\tau L^2_\xi} \lesssim \|v_0\|_{H^{-a}} + \left\| \frac{\widetilde{G}}{\langle \tau + \xi \rangle^{a} \langle \tau - \xi \rangle^{b}} \right\|_{L^2_\tau L^2_\xi}. \tag{3.5}$$

Remark 3.2 For instance, when $a + b < 1/2$ and $a \geq 0$, estimates (3.4) and (3.5) hold.

Proposition 3.3 *Let* $a < 1/4$, $b \in \mathbb{R}$. *Let w be solution to* (3.3). *Then the following estimate holds,*

$$\left\| \frac{\langle \tau + c\xi \rangle^{1-a} \langle \tau - c\xi \rangle^{1-a}}{\langle \xi \rangle^{b}} \widehat{\psi_T} * \widetilde{w} \right\|_{L^2_\tau L^2_\xi}$$

$$\lesssim \|w_0\|_{H^{1-a-b}} + \|w_1\|_{H^{-a-b}} + \left\| \frac{\widetilde{H}}{\langle \xi \rangle^{b} \langle \tau + c\xi \rangle^{a} \langle \tau - c\xi \rangle^{a}} \right\|_{L^2_\tau L^2_\xi}. \tag{3.6}$$

Proof of Proposition 3.1.

We firstly prove (3.5). We have two expressions for the Fourier transform of the solution $\widetilde{v}$ to (3.2), see [12].

$$\widetilde{v}(\tau,\xi)=\delta(\tau-\xi)\widehat{v_0}(\xi)-i\times\begin{cases}\sum_{k=1}^{\infty}(k!)^{-1}\widehat{(it)}^k *_\tau \delta(\tau-\xi)\int(\sigma-\xi)^{k-1}\widetilde{G}(\sigma,\xi)d\sigma\\ \quad\quad\text{or}\\ (\tau-\xi)^{-1}\widetilde{G}(\tau,\xi)-\delta(\tau-\xi)\int(\sigma-\xi)^{-1}\widetilde{G}(\sigma,\xi)d\sigma,\end{cases} \tag{3.7}$$

We split a inhomogeneous term $G =: G_1 + G_2$ with regard to the support in Fourier space, neighborhood of $\tau=\xi$ and otherwise.

$$\widetilde{G_1}(\tau,\xi)=\psi(\tau-\xi)\widetilde{G}(\tau,\xi),\qquad \widetilde{G_2}(\tau,\xi)=(1-\psi(\tau-\xi))\widetilde{G}(\tau,\xi). \tag{3.8}$$

As in [12], we try to estimate for the following terms,

$$\widetilde{v}(\tau,\xi)=\delta(\tau-\xi)\widehat{v_0}(\xi)-i\sum_{k=1}^{\infty}\frac{1}{k!}\widehat{(it)}^k *_\tau \delta(\tau-\xi)\int(\sigma-\xi)^{k-1}\widetilde{G_1}(\sigma,\xi)d\sigma$$
$$-i\frac{\widetilde{G_2}(\tau,\xi)}{\tau-\xi}+i\delta(\tau-\xi)\int\frac{\widetilde{G_2}(\sigma,\xi)}{\sigma-\xi}d\sigma. \tag{3.9}$$

We estimate for the terms with Delta's function. For any function $f(\xi)$ and $k\ge 0$ we have

$$\left\|\frac{\widehat{\psi_T}*\widehat{(it)}^k*\delta(\tau-\xi)f(\xi)}{\langle\tau-\xi\rangle^{b-1}\langle\tau+\xi\rangle^a}\right\|_{L^2_\tau L^2_\xi}=\left\|\left\|\frac{\widehat{\psi_T}*\widehat{(it)}^k*\delta(\tau-\xi)}{\langle\tau-\xi\rangle^{b-1}\langle\tau+\xi\rangle^a}\right\|_{L^2_\tau}f(\xi)\right\|_{L^2_\xi} \tag{3.10}$$

We set $h_k(\tau):=\widehat{\psi_T}*\widehat{(it)}^k$ and concentrate on L^2_τ norm to calculate

$$\left\|\frac{h_k(\tau)*\delta(\tau-\xi)}{\langle\tau-\xi\rangle^{b-1}\langle\tau+\xi\rangle^a}\right\|_{L^2_\tau}=\left\|\frac{\langle\tau\rangle^{1+a-b}h_k(\tau)}{\langle\tau\rangle^a\langle\tau+2\xi\rangle^a}\right\|_{L^2_\tau} \tag{3.11}$$

$$\lesssim\frac{1}{\langle\xi\rangle^a}\|\langle\tau\rangle^{1+a-b}h_k(\tau)\|_{L^2_\tau}\lesssim\frac{(k+1)^{1+a-b}}{\langle\xi\rangle^a}. \tag{3.12}$$

Consequently we have the boundedness by $\|\langle\xi\rangle^{-a}f\|_{L^2_\xi}$. For the second term in R.H.S. of (3.9),

$$\left\|\frac{1}{\langle\xi\rangle^a}\int(\sigma-\xi)^{k-1}\widetilde{G}_1(\sigma,\xi)d\sigma\right\|_{L^2_\xi} \tag{3.13}$$

$$\lesssim 2^{k+b}\left\|\frac{1}{\langle\xi\rangle^a}\left(\int_{|\sigma-\xi|<2}\langle\sigma+\xi\rangle^{2a}d\sigma\right)^{1/2}\right\|_{L^\infty_\xi}\left\|\frac{\widetilde{G}_1(\tau,\xi)}{\langle\tau+\xi\rangle^a\langle\tau-\xi\rangle^b}\right\|_{L^2_\tau L^2_\xi}, \tag{3.14}$$

here

$$\frac{1}{\langle\xi\rangle^{2a}}\int_{|\sigma-\xi|<2}\langle\sigma+\xi\rangle^{2a}d\sigma \lesssim \int_{|\sigma-\xi|<2}\langle\sigma-\xi\rangle^{2a}d\sigma \le C. \tag{3.15}$$

Similarly for the forth term,

$$\left\|\frac{1}{\langle\xi\rangle^{a}}\int\frac{\widetilde{G}_2(\sigma,\xi)}{\sigma-\xi}d\sigma\right\|_{L^2_\xi} \tag{3.16}$$

$$\lesssim \left\|\frac{1}{\langle\xi\rangle^{a}}\left(\int\left|\frac{\langle\sigma+\xi\rangle^{a}}{\langle\sigma-\xi\rangle^{1-b}}\right|^2 d\sigma\right)^{1/2}\right\|_{L^\infty_\xi}\left\|\frac{\widetilde{G}_2(\tau,\xi)}{\langle\tau+\xi\rangle^{a}\langle\tau-\xi\rangle^{b}}\right\|_{L^2_\tau L^2_\xi}, \tag{3.17}$$

here we obtain from assumptions on a, b and Lemma 2.1,

$$\frac{1}{\langle\xi\rangle^{2a}}\int\left|\frac{\langle\sigma+\xi\rangle^{a}}{\langle\sigma-\xi\rangle^{1-b}}\right|^2 d\sigma \le C. \tag{3.18}$$

We commute $\langle\tau-\xi\rangle^{\alpha}$ and $\widehat{\psi_T} *_\tau H(\tau,\xi)$ for any function $H(\tau,\xi)$. When $\alpha \ge 0$ we estimate pointwisely, from $\langle\tau-\xi\rangle \le \langle\tau-\sigma\rangle + \langle\sigma-\xi\rangle$

$$|\langle\tau-\xi\rangle^{\alpha}\widehat{\psi_T} * H(\tau,\xi)| \le \int(\langle\tau-\sigma\rangle^{\alpha} + \langle\sigma-\xi\rangle^{\alpha})|\widehat{\psi_T}(\tau-\sigma)H(\sigma,\xi)|d\sigma \tag{3.19}$$

$$\le |\langle\tau\rangle^{\alpha}\widehat{\psi_T}| * |H(\tau,\xi)| + |\widehat{\psi_T}| * \langle\tau-\xi\rangle^{\alpha}|H(\tau,\xi)|. \tag{3.20}$$

When $\alpha \le 0$, from $\langle\sigma-\xi\rangle \le \langle\sigma-\tau\rangle + \langle\tau-\xi\rangle$

$$|\langle\tau-\xi\rangle^{\alpha}\widehat{\psi_T} * H(\tau,\xi)| \le \langle\tau-\xi\rangle^{\alpha}\int(\langle\tau-\sigma\rangle^{-\alpha} + \langle\tau-\xi\rangle^{-\alpha})\frac{|\widehat{\psi_T}(\tau-\sigma)H(\sigma,\xi)|}{\langle\sigma-\xi\rangle^{-\alpha}}d\sigma \tag{3.21}$$

$$= \langle\tau-\xi\rangle^{\alpha}\left(|\langle\tau\rangle^{-\alpha}\widehat{\psi_T}| * \frac{|H(\tau,\xi)|}{\langle\tau-\xi\rangle^{-\alpha}}\right) + |\widehat{\psi_T}| * \frac{|H(\tau,\xi)|}{\langle\tau-\xi\rangle^{-\alpha}}. \tag{3.22}$$

From these pointwise estimates, we have for $\alpha \in \mathbb{R}$,

$$\|\langle\tau-\xi\rangle^{\alpha}\widehat{\psi_T} * H(\tau,\xi)\|_{L^2_\tau L^2_\xi} \lesssim \|\langle\tau-\xi\rangle^{\alpha}H(\tau,\xi)\|_{L^2_\tau L^2_\xi}. \tag{3.23}$$

We also obtain from above calculations twice, for $\alpha, \beta \in \mathbb{R}$,

$$\|\langle\tau-\xi\rangle^{\alpha}\langle\tau+\xi\rangle^{\beta}\widehat{\psi_T} * H(\tau,\xi)\|_{L^2_\tau L^2_\xi} \lesssim \|\langle\tau-\xi\rangle^{\alpha}\langle\tau+\xi\rangle^{\beta}H(\tau,\xi)\|_{L^2_\tau L^2_\xi}. \tag{3.24}$$

We then estimate the third term in R.H.S of (3.9) easily,

$$\left\| \frac{\langle \tau - \xi \rangle^{1-b}}{\langle \tau + \xi \rangle^{a}} \widehat{\psi_T} * \frac{\widetilde{G_2}(\tau,\xi)}{\tau - \xi} \right\|_{L^2_\tau L^2_\xi} \lesssim \left\| \frac{\widetilde{G}_2}{\langle \tau + \xi \rangle^{a} \langle \tau - \xi \rangle^{b}} \right\|_{L^2_\tau L^2_\xi}. \tag{3.25}$$

We complete the proof for (3.5). The proof for (3.4) is similar. □

Proof of Proposition 3.3.

We see easily it is sufficient to prove the case $b = 0$ in (3.6). The solution for wave equation consists of the solutions for reduced-wave equation. The Fourier transform with respect to x of the solution $\widehat{w}$ for (3.3) is given by

$$\widehat{w}(t,\xi) = \frac{e^{ic\xi t} + e^{-ic\xi t}}{2} \widehat{w_0}(\xi) + \frac{e^{ic\xi t} - e^{-ic\xi t}}{2ic\xi} \widehat{w_1}(\xi) \tag{3.26}$$

$$+ \int_0^t \frac{e^{ic\xi(t-t')} - e^{-ic\xi(t-t')}}{2ic\xi} \widehat{H}(t',\xi)dt' \tag{3.27}$$

So the estimate (3.6) for the region $|\xi| > 1$ is derived similarly with the above arguments. From here, we restrict the region $|\xi| < 1$ in $L^2_\tau L^2_\xi$. For the second term,

$$\frac{e^{ic\xi t} - e^{-ic\xi t}}{2ic\xi} \widehat{w_1}(\xi) = e^{-ic\xi t} \sum_{k=1}^{\infty} \frac{t^k (2ic\xi)^{k-1}}{k!} \widehat{w_1}(\xi). \tag{3.28}$$

We take Fourier transform with respect to t and estimate similarly to (3.10),

$$\|\langle \tau + c\xi \rangle^{1-a} \langle \tau - c\xi \rangle^{1-a} \widehat{\psi_T} * \delta(\tau + c\xi) * \widehat{t^k}\ \xi^{k-1} \widehat{w_1}(\xi)\|_{L^2_\tau L^2_\xi} \tag{3.29}$$

$$\lesssim \|\langle \xi \rangle^{1-a} \xi^{k-1} \widehat{w_1}(\xi)\|_{L^2_\xi(|\xi|<1)} \lesssim \|w_1\|_{H^{-a}}. \tag{3.30}$$

For the third term, we split again the inhomogeneous term $H =: H_1 + H_2$ as follows,

$$\widetilde{H_1}(\tau,\xi) = \psi(|\tau| - |\xi|)\widetilde{H}(\tau,\xi), \qquad \widetilde{H_2}(\tau,\xi) = (1 - \psi(|\tau| - |\xi|))\widetilde{H}(\tau,\xi). \tag{3.31}$$

We estimate on H_1,

$$\int_0^t \frac{e^{ic\xi(t-t')} - e^{-ic\xi(t-t')}}{2ic\xi} \widehat{H}_1(t',\xi)dt' \tag{3.32}$$

$$= \frac{1}{2c\xi} \left(\int \frac{e^{ic\xi t} - e^{i\sigma t}}{\sigma - c\xi} \widetilde{H_1}(\sigma,\xi)d\sigma - \int \frac{e^{-ic\xi t} - e^{i\sigma t}}{\sigma + c\xi} \widetilde{H_1}(\sigma,\xi)d\sigma \right) \tag{3.33}$$

$$= \sum_{k=1}^{\infty} \frac{(-it)^k}{k!} \int e^{i\sigma t} \frac{(\sigma - c\xi)^{k-1} - (\sigma + c\xi)^{k-1}}{2c\xi} \widetilde{H_1}(\sigma,\xi)d\sigma \tag{3.34}$$

We take Fourier transform with respect to t and estimate,

$$\left\| \langle \tau + c\xi \rangle^{1-a} \langle \tau - c\xi \rangle^{1-a} \widehat{\psi_T} * \widehat{t^k} * \frac{(\tau - c\xi)^{k-1} - (\tau + c\xi)^{k-1}}{2c\xi} \; \widetilde{H_1}(\tau, \xi) \right\|_{L^2_\tau L^2_\xi} \tag{3.35}$$

$$\lesssim \left\| \langle \tau + c\xi \rangle^{1-a} \langle \tau - c\xi \rangle^{1-a} \frac{(\tau - c\xi)^{k-1} - (\tau + c\xi)^{k-1}}{2c\xi} \; \widetilde{H_1}(\tau, \xi) \right\|_{L^2_\tau L^2_\xi} \tag{3.36}$$

$$\lesssim \left\| \frac{\widetilde{H}}{\langle \tau + c\xi \rangle^a \langle \tau - c\xi \rangle^a} \right\|_{L^2_\tau L^2_\xi}, \tag{3.37}$$

at the last inequality, we use $|\tau|, |\xi| \lesssim 1$.

On the other hand, we estimate on H_2,

$$\int_0^t \frac{e^{ic\xi(t-t')} - e^{-ic\xi(t-t')}}{2ic\xi} \; \widehat{H}_2(t', \xi) dt' \tag{3.38}$$

$$= - \int e^{i\sigma t} \; \frac{\widetilde{H_2}(\sigma, \xi)}{(\sigma - c\xi)(\sigma + c\xi)} \; d\sigma + \frac{e^{ic\xi t}}{2c\xi} \int \frac{\widetilde{H_2}(\sigma, \xi)}{\sigma - c\xi} \; d\sigma$$
$$- \frac{e^{-ic\xi t}}{2c\xi} \int \frac{\widetilde{H_2}(\sigma, \xi)}{\sigma + c\xi} \; d\sigma \tag{3.39}$$

$$= - \int e^{i\sigma t} \; \frac{\widetilde{H_2}(\sigma, \xi)}{(\sigma - c\xi)(\sigma + c\xi)} \; d\sigma + e^{ic\xi t} \int \frac{\widetilde{H_2}(\sigma, \xi)}{(\sigma + c\xi)(\sigma - c\xi)} d\sigma \tag{3.40}$$

$$+ \frac{e^{ic\xi t} - e^{-ic\xi t}}{2c\xi} \int \frac{\widetilde{H_2}(\sigma, \xi)}{(\sigma + c\xi)} d\sigma. \tag{3.41}$$

We take Fourier transform with respect to t,

$$- \frac{\widetilde{H_2}(\tau, \xi)}{(\tau - c\xi)(\tau + c\xi)} + \delta(\tau - c\xi) \int \frac{\widetilde{H_2}(\sigma, \xi)}{(\sigma - c\xi)(\sigma + c\xi)} \; d\sigma \tag{3.42}$$

$$+ \sum_{k=1}^{\infty} \delta(\tau + c\xi) * \widehat{(it)^k} \; \frac{(2c\xi)^{k-1}}{k!} \int \frac{\widetilde{H_2}(\sigma, \xi)}{\sigma + c\xi} \; d\sigma. \tag{3.43}$$

The estimates for the first term is obvious. We estimate the second and third terms,

$$\left\| \langle \xi \rangle^{1-a} \int \frac{\widetilde{H_2}(\sigma, \xi)}{(\sigma - c\xi)(\sigma + c\xi)} \; d\sigma \right\|_{L^2_\xi(|\xi|<1)} \tag{3.44}$$

$$\lesssim \left\| \langle \xi \rangle^{1-a} \int \frac{1}{\langle \sigma - c\xi \rangle^{1-a} \langle \sigma + c\xi \rangle^{1-a}} d\sigma \right\|_{L^\infty_\xi(|\xi|<1)} \left\| \frac{\widetilde{H}}{\langle \tau + c\xi \rangle^a \langle \tau - c\xi \rangle^a} \right\|_{L^2_\tau L^2_\xi}, \tag{3.45}$$

$$\left\| \langle\xi\rangle^{1-a}\xi^{k-1}\int \frac{\widetilde{H_2}(\sigma,\xi)}{\sigma+c\xi}\,d\sigma \right\|_{L^2_\xi(|\xi|<1)} \tag{3.46}$$

$$\lesssim \left\| \frac{1}{\langle\xi\rangle^a}\int \frac{\widetilde{H_2}(\sigma,\xi)}{\sigma+c\xi}\,d\sigma \right\|_{L^2_\xi(|\xi|<1)} \lesssim \left\| \frac{\widetilde{H}}{\langle\tau+c\xi\rangle^a\langle\tau-c\xi\rangle^a} \right\|_{L^2_\tau L^2_\xi} \tag{3.47}$$

We complete the proof for (3.6). □

4 Bilinear estimates

Here we give the bilinear estimates for the inhomogeneous terms of Dirac–Klein–Gordon equation. The following proposition is essential in this section. The special case $\beta > 1/2, \beta \geq \alpha = \gamma \geq -\beta$ is given in [8].

Proposition 4.1 *Let $\alpha+\beta > 1/2$. Let $\gamma <$ (resp. $\leq$) $\min\{\alpha,\beta,\alpha+\beta-1/2\}$ when $\max\{\alpha,\beta\} = 1/2$ (resp. $\neq 1/2$). Then the following estimate holds,*

$$\|fg\|_{H^{-\alpha}} \lesssim \|f\|_{H^\beta}\|g\|_{H^{-\gamma}}. \tag{4.1}$$

Proof. We employ the duality argument. We prove for any $\varphi \in L^2_\xi$,

$$\int \langle\xi\rangle^{-\alpha}\widehat{fg}\;\varphi\;d\xi \lesssim \|\langle\xi\rangle^\beta \widehat{f}\|_{L^2}\|\langle\xi\rangle^{-\gamma}\widehat{g}\|_{L^2}\|\varphi\|_{L^2}. \tag{4.2}$$

We set $u = \langle\xi\rangle^\beta\widehat{f}$ and $v = \langle\xi\rangle^{-\gamma}\widehat{g}$.

$$\int \langle\xi\rangle^{-\alpha}\widehat{fg}\;\varphi\;d\xi = \int \frac{\langle\eta\rangle^\gamma u(\xi-\eta)v(\eta)\varphi(\xi)}{\langle\xi\rangle^\alpha\langle\xi-\eta\rangle^\beta}d\eta d\xi \tag{4.3}$$

$$\leq \|v\|_{L^2_\eta}\left(\int \langle\eta\rangle^{2\gamma}\left|\int \frac{u(\xi-\eta)\varphi(\xi)}{\langle\xi\rangle^\alpha\langle\xi-\eta\rangle^\beta}\,d\xi\right|^2\,d\eta\right)^{1/2} \tag{4.4}$$

$$\leq \|v\|_{L^2_\eta}\left\{\int \langle\eta\rangle^{2\gamma}\left(\int |u(\xi-\eta)\varphi(\xi)|^2\,d\xi\right)\left(\int \frac{1}{\langle\xi\rangle^{2\alpha}\langle\xi-\eta\rangle^{2\beta}}\,d\xi\right)d\eta\right\}^{1/2} \tag{4.5}$$

$$\leq \|v\|_{L^2_\eta}\|u\|_{L^2_\eta}\|\varphi\|_{L^2_\xi}\left(\sup_\eta \langle\eta\rangle^{2\gamma}\int \frac{1}{\langle\xi\rangle^{2\alpha}\langle\xi-\eta\rangle^{2\beta}}\,d\xi\right)^{1/2}, \tag{4.6}$$

here we have the boundedness from Lemma 2.1,

$$\int \frac{1}{\langle\xi\rangle^{2\alpha}\langle\xi-\eta\rangle^{2\beta}}\,d\xi \lesssim \frac{1}{\langle\eta\rangle^{2\gamma}}. \tag{4.7}$$

We obtain (4.2). □

As a corollary, we obtain the following.

Corollary 4.2 *For $j = 1, 2$, let $\alpha_j + \beta_j > 1/2$ and $\gamma_j <$ (resp. $\leq$) $\min\{\alpha_j, \beta_j, \alpha_j + \beta_j - 1/2\}$ when $\max\{\alpha_j, \beta_j\} = 1/2$ (resp. $\neq 1/2$). Then the following estimates hold,*

$$\left\| \frac{\widetilde{uv}}{\langle \xi + \tau \rangle^{\alpha_1} \langle \xi - \tau \rangle^{\alpha_2}} \right\|_{L^2_\tau L^2_\xi} \lesssim \| \langle \xi + \tau \rangle^{\beta_1} \langle \xi - \tau \rangle^{\beta_2} \widetilde{u} \|_{L^2_\tau L^2_\xi} \left\| \frac{\widetilde{v}}{\langle \xi + \tau \rangle^{\gamma_1} \langle \xi - \tau \rangle^{\gamma_2}} \right\|_{L^2_\tau L^2_\xi}, \tag{4.8}$$

$$\left\| \frac{\widetilde{uv}}{\langle \xi + \tau \rangle^{\alpha_1} \langle \xi - \tau \rangle^{\alpha_2}} \right\|_{L^2_\tau L^2_\xi} \lesssim \left\| \frac{\langle \xi + \tau \rangle^{\beta_1}}{\langle \xi - \tau \rangle^{\gamma_2}} \widetilde{u} \right\|_{L^2_\tau L^2_\xi} \left\| \frac{\langle \xi - \tau \rangle^{\beta_2}}{\langle \xi + \tau \rangle^{\gamma_1}} \widetilde{v} \right\|_{L^2_\tau L^2_\xi}. \tag{4.9}$$

Proof. We prove (4.9) only. We set the following operator P,

$$(PF)(\tau, \xi) = F\left(\tfrac{\tau - \xi}{2}, \tfrac{\tau + \xi}{2}\right) \quad \text{for} \quad F = F(\tau, \xi) \tag{4.10}$$

and which implies

$$(PF)(\xi + \tau, \xi - \tau) = F(\tau, \xi). \tag{4.11}$$

We remark the following property of P,

$$\|PF\|_{L^2_\tau L^2_\xi} \sim \|F\|_{L^2_\tau L^2_\xi}. \tag{4.12}$$

We calculate

$$\left\| \frac{\widetilde{uv}}{\langle \xi + \tau \rangle^{\alpha_1} \langle \xi - \tau \rangle^{\alpha_2}} \right\|_{L^2_\tau L^2_\xi} = \left\| \frac{(P\widetilde{u})(\xi + \tau, \xi - \tau) * (P\widetilde{v})(\xi + \tau, \xi - \tau)}{\langle \xi + \tau \rangle^{\alpha_1} \langle \xi - \tau \rangle^{\alpha_2}} \right\|_{L^2_\tau L^2_\xi}. \tag{4.13}$$

By changing the variables $\tau + \xi \to \tau,\ \tau - \xi \to \xi$, we continue

$$\sim \left\| \frac{(P\widetilde{u})(\tau, \xi) * (P\widetilde{v})(\tau, \xi)}{\langle \tau \rangle^{\alpha_1} \langle \xi \rangle^{\alpha_2}} \right\|_{L^2_\tau L^2_\xi} \tag{4.14}$$

We then apply Proposition 4.1 to both the variables τ and ξ, and continue

$$\lesssim \left\| \frac{\langle \tau \rangle^{\beta_1}}{\langle \xi \rangle^{\gamma_2}} (P\widetilde{u})(\tau, \xi) \right\|_{L^2_\tau L^2_\xi} \left\| \frac{\langle \xi \rangle^{\beta_2}}{\langle \tau \rangle^{\gamma_1}} (P\widetilde{v})(\tau, \xi) \right\|_{L^2_\tau L^2_\xi} \tag{4.15}$$

$$\sim \left\| \frac{\langle \xi + \tau \rangle^{\beta_1}}{\langle \xi - \tau \rangle^{\gamma_2}} \widetilde{u}(\tau, \xi) \right\|_{L^2_\tau L^2_\xi} \left\| \frac{\langle \xi - \tau \rangle^{\beta_2}}{\langle \xi + \tau \rangle^{\gamma_1}} \widetilde{v}(\tau, \xi) \right\|_{L^2_\tau L^2_\xi}. \tag{4.16}$$

□

5 Time local and global existence

Now we are ready to prove Theorem 1.1. We restrict time interval $[-T, T]$ and consider the following system,

$$\begin{cases} \mathcal{D}\mathbf{u} = -m\psi_T\mathbf{u} + \psi_T^2 w\mathbf{u} \\ \Box w = -M\psi_T w + \psi_T^2 \mathbf{u}^\dagger\mathbf{u} \end{cases} \tag{5.1}$$

We apply the standard contraction mapping principle to obtain a solution for the rescaled equations.

Proof of Theorem 1.1. Firstly we investigate the equation by a scaling argument. We consider the following system for $\lambda > 0$

$$\begin{cases} \mathcal{D}\mathbf{u} = -m\lambda\mathbf{u} + w\mathbf{u} \\ \Box w = -M\lambda^2 w + \mathbf{u}^\dagger\mathbf{u} \end{cases} \tag{5.2}$$

Then $(\mathbf{u}, w)$ is a solution for (5.2) with $\lambda = 1$ if and only if the following $(\mathbf{u}_\lambda, w_\lambda)$ is a solution for (5.2),

$$\mathbf{u}_\lambda(t, x) = \lambda^{3/2}\mathbf{u}(\lambda t, \lambda x), \tag{5.3}$$

$$w_\lambda(t, x) = \lambda w(\lambda t, \lambda x). \tag{5.4}$$

We calculate the initial data $\mathbf{u}_{0\lambda}(x) = \mathbf{u}_\lambda(0, x) = \mathbf{u}_0(\lambda x)$ and $w_{0\lambda}(x) = w_\lambda(0, x) = w_0(\lambda x)$ where $\mathbf{u}_0$ and w_0 are in (1.5). We have

$$\|\mathbf{u}_{0\lambda}\|_{H^{-a}} \leq \lambda^{1-a}\|\mathbf{u}_0\|_{H^{-a}}, \quad \|w_{0\lambda}\|_{H^s} \leq \lambda^{1/2+s}\|w_0\|_{H^s}. \tag{5.5}$$

Therefore we can reduce the size of initial data and mass terms for small λ with $a < 1, s > -1/2$. In the following, we seek for the solution $(\mathbf{u}_\lambda, w_\lambda)$ of (5.2), and for simplicity, we abbreviate $(\mathbf{u}_\lambda, w_\lambda) = (\mathbf{u}, w)$.

As in [4], we consider the iteration map $\Phi(\mathbf{u}^k, w^k) = (\mathbf{u}^{k+1}, w^{k+1})$. We show that Φ is a contraction map on the space,

$$\Theta_R = \{(\mathbf{u}, w) = (u, v, w);\ \|(u, v, w)\|_\Theta \leq R\} \tag{5.6}$$

where

$$\|(u, v, w)\|_\Theta = \left\| \frac{\langle \tau + \xi \rangle^{1-b}}{\langle \tau - \xi \rangle^{a}}\, \widetilde{u} \right\|_{L_\tau^2 L_\xi^2} + \left\| \frac{\langle \tau - \xi \rangle^{1-b}}{\langle \tau + \xi \rangle^{a}}\, \widetilde{v} \right\|_{L_\tau^2 L_\xi^2} \tag{5.7}$$

$$+ \left\| \frac{\langle \tau + c\xi \rangle^{1-a}\langle \tau - c\xi \rangle^{1-a}}{\langle \xi \rangle^{1-a-s}}\, \widetilde{w} \right\|_{L_\tau^2 L_\xi^2}. \tag{5.8}$$

We restrict the case $0 \le 2a < s \le 1/2$. We set b such that

$$a + b < 1/2, \quad 0 \le a < b + s - 1/2. \tag{5.9}$$

We firstly give the estimate for the norm of u. From Proposition 3.1,

$$\left\| \frac{\langle \tau+\xi \rangle^{1-b}}{\langle \tau-\xi \rangle^{a}} \ \widehat{\psi_T} * \widetilde{u} \right\|_{L^2_\tau L^2_\xi} \lesssim \|u_0\|_{H^{-a}} + \left\| \frac{m\lambda \widetilde{\psi_T v} - \widetilde{\psi_T^2 w v}}{\langle \tau+\xi \rangle^{b} \langle \tau-\xi \rangle^{a}} \right\|_{L^2_\tau L^2_\xi}. \tag{5.10}$$

Obviously we obtain from $a < b$,

$$\left\| \frac{\widetilde{\psi_T v}}{\langle \tau+\xi \rangle^{b} \langle \tau-\xi \rangle^{a}} \right\|_{L^2_\tau L^2_\xi} \lesssim \left\| \frac{\langle \tau-\xi \rangle^{1-b}}{\langle \tau+\xi \rangle^{a}} \ \widehat{\psi_T} * \widetilde{v} \right\|_{L^2_\tau L^2_\xi}. \tag{5.11}$$

We remark $b + s > 1/2$ and $1 - b > 1/2$ to apply Corollary 4.2,

$$\left\| \frac{\widetilde{\psi_T^2 w v}}{\langle \tau+\xi \rangle^{b} \langle \tau-\xi \rangle^{a}} \right\|_{L^2_\tau L^2_\xi} \lesssim \|\langle \tau+\xi \rangle^{s} \widehat{\psi_T} * \widetilde{w}\|_{L^2_\tau L^2_\xi} \left\| \frac{\langle \tau-\xi \rangle^{1-b}}{\langle \tau+\xi \rangle^{a}} \ \widehat{\psi_T} * \widetilde{v} \right\|_{L^2_\tau L^2_\xi}. \tag{5.12}$$

We calculate the weight of norm for $\widetilde{w}$ with $a + s < 1$,

$$\langle \tau+\xi \rangle^{s} \le \langle \tau+c\xi \rangle^{s} \langle \tau-c\xi \rangle^{s} \le \frac{\langle \tau+c\xi \rangle^{1-a} \langle \tau-c\xi \rangle^{1-a}}{\langle \xi \rangle^{1-a-s}}. \tag{5.13}$$

The estimates for v is similar to above, that is

$$\left\| \frac{\langle \tau-\xi \rangle^{1-b}}{\langle \tau+\xi \rangle^{a}} \ \widehat{\psi_T} * \widetilde{v} \right\|_{L^2_\tau L^2_\xi} \lesssim \|v_0\|_{H^{-a}} + \left\| \frac{m\lambda \widetilde{\psi_T u} - \widetilde{\psi_T^2 w u}}{\langle \tau-\xi \rangle^{b} \langle \tau+\xi \rangle^{a}} \right\|_{L^2_\tau L^2_\xi}. \tag{5.14}$$

The estimates for the inhomogeneous terms are given by

$$\left\| \frac{\widetilde{\psi_T u}}{\langle \tau-\xi \rangle^{b} \langle \tau+\xi \rangle^{a}} \right\|_{L^2_\tau L^2_\xi} \lesssim \left\| \frac{\langle \tau+\xi \rangle^{1-b}}{\langle \tau-\xi \rangle^{a}} \ \widehat{\psi_T} * \widetilde{u} \right\|_{L^2_\tau L^2_\xi}, \tag{5.15}$$

$$\left\| \frac{\widetilde{\psi_T^2 w u}}{\langle \tau-\xi \rangle^{b} \langle \tau+\xi \rangle^{a}} \right\|_{L^2_\tau L^2_\xi} \lesssim \|\langle \tau-\xi \rangle^{s} \widehat{\psi_T} * \widetilde{w}\|_{L^2_\tau L^2_\xi} \left\| \frac{\langle \tau+\xi \rangle^{1-b}}{\langle \tau-\xi \rangle^{a}} \ \widehat{\psi_T} * \widetilde{u} \right\|_{L^2_\tau L^2_\xi}. \tag{5.16}$$

For w, we have from Proposition 3.3,

$$\left\| \frac{\langle \tau + c\xi \rangle^{1-a} \langle \tau - c\xi \rangle^{1-a}}{\langle \xi \rangle^{1-a-s}} \widehat{\psi_T} * \widetilde{w} \right\|_{L^2_\tau L^2_\xi}$$

$$\lesssim \|w_0\|_{H^s} + \|w_1\|_{H^{s-1}} + \left\| \frac{M\lambda^2 \widetilde{\psi_T w} - \widetilde{\psi_T^2 \mathbf{u}^\dagger \mathbf{u}}}{\langle \xi \rangle^{1-a-s} \langle \tau + c\xi \rangle^a \langle \tau - c\xi \rangle^a} \right\|_{L^2_\tau L^2_\xi}. \quad (5.17)$$

We have

$$\left\| \frac{\widetilde{\psi_T w}}{\langle \xi \rangle^{1-a-s} \langle \tau + c\xi \rangle^a \langle \tau - c\xi \rangle^a} \right\|_{L^2_\tau L^2_\xi} \lesssim \left\| \frac{\langle \tau + c\xi \rangle^{1-a} \langle \tau - c\xi \rangle^{1-a} \widehat{\psi_T} * \widetilde{w}}{\langle \xi \rangle^{1-a-s}} \right\|_{L^2_\tau L^2_\xi}. \quad (5.18)$$

We remark $a \leq 1 - a - s$ to apply Lemma 2.2,

$$\left\| \frac{\widetilde{\psi_T^2 \mathbf{u}^\dagger \mathbf{u}}}{\langle \xi \rangle^{1-a-s} \langle \tau + c\xi \rangle^a \langle \tau - c\xi \rangle^a} \right\|_{L^2_\tau L^2_\xi} \lesssim \left\| \frac{\widetilde{\psi_T^2 \mathbf{u}^\dagger \mathbf{u}}}{\langle \tau + \xi \rangle^a \langle \tau - \xi \rangle^a} \right\|_{L^2_\tau L^2_\xi}. \quad (5.19)$$

We apply Corollary 4.2 with $1 - b > 1/2$ again

$$\left\| \frac{\widetilde{\psi_T^2 \overline{u} v}}{\langle \tau + \xi \rangle^a \langle \tau - \xi \rangle^a} \right\|_{L^2_\tau L^2_\xi} \lesssim \left\| \frac{\langle \tau + \xi \rangle^{1-b}}{\langle \tau - \xi \rangle^a} \widehat{\psi_T} * \widetilde{u} \right\|_{L^2_\tau L^2_\xi} \left\| \frac{\langle \tau - \xi \rangle^{1-b}}{\langle \tau + \xi \rangle^a} \widehat{\psi_T} * \widetilde{v} \right\|_{L^2_\tau L^2_\xi}. \quad (5.20)$$

Consequently we obtain

$$\|\Phi(\psi_T u, \psi_T v, \psi_T w)\|_\Theta \lesssim \|u_0\|_{H^{-a}} + \|v_0\|_{H^{-a}} + \|w_0\|_{H^s} + \|w_1\|_{H^{s-1}} \quad (5.21)$$

$$+ (\lambda + \lambda^2) \|(\psi_T u, \psi_T v, \psi_T w)\|_\Theta + \|(\psi_T u, \psi_T v, \psi_T w)\|_\Theta^2. \quad (5.22)$$

We obtain for sufficiently small λ which also means sufficiently small initial data, and appropriate R,

$$\|\Phi(\psi_T u, \psi_T v, \psi_T w)\|_\Theta \leq R. \quad (5.23)$$

The estimate for the difference of each solutions is similar, see for reference [4]

$$\|\Phi(\psi_T(u^1 - u^2), \psi_T(v^1 - v^2), \psi_T(w^1 - w^2))\|_\Theta \quad (5.24)$$

$$\leq C(\lambda + \lambda^2 + R) \|(\psi_T(u^1 - u^2), \psi_T(v^1 - v^2), \psi_T(w^1 - w^2))\|_\Theta. \quad (5.25)$$

We take $C(\lambda + \lambda^2 + R) < 1$ to conclude the proof for the time local existence. The continuous properties of solutions $\mathbf{u}$ and w with respect to t in H^{-a} and H^s

norms respectively follow from the embeddings,

$$\begin{aligned}\left\|\frac{\widehat{u}(t,\xi)}{\langle\xi\rangle^a}\right\|_{L_t^\infty L_\xi^2} &\lesssim \left\|\frac{\widetilde{u}(\tau,\xi)}{\langle\xi\rangle^a}\right\|_{L_\xi^2 L_\tau^1} \\ &\lesssim \left\|\left\|\frac{\langle\tau-\xi\rangle^a}{\langle\tau+\xi\rangle^{1-b}\langle\xi\rangle^a}\right\|_{L_\tau^2}\left\|\frac{\langle\tau+\xi\rangle^{1-b}\widetilde{u}(\tau,\xi)}{\langle\tau-\xi\rangle^a}\right\|_{L_\tau^2}\right\|_{L_\xi^2} \\ &\lesssim \left\|\frac{\langle\tau+\xi\rangle^{1-b}}{\langle\tau-\xi\rangle^a}\widetilde{u}(\tau,\xi)\right\|_{L_\tau^2 L_\xi^2},\end{aligned} \tag{5.26}$$

and from similar calculation,

$$\left\|\frac{\widehat{v}(t,\xi)}{\langle\xi\rangle^a}\right\|_{L_t^\infty L_\xi^2} \lesssim \left\|\frac{\langle\tau-\xi\rangle^{1-b}}{\langle\tau+\xi\rangle^a}\widetilde{v}(\tau,\xi)\right\|_{L_\tau^2 L_\xi^2}, \tag{5.27}$$

$$\|\langle\xi\rangle^s\widehat{w}(t,\xi)\|_{L_t^\infty L_\xi^2} \lesssim \left\|\frac{\langle\tau+c\xi\rangle^{1-a}\langle\tau-c\xi\rangle^{1-a}}{\langle\xi\rangle^{1-a-s}}\widetilde{w}(\tau,\xi)\right\|_{L_\tau^2 L_\xi^2}. \tag{5.28}$$

We now turn to the proof for the global existence in the case $a = 0$ and $0 < s < 1/2$. We give the a priori estimate. For $\mathbf{u} = {}^t(u, v)$, we use the conservation of charge, see [2, 3, 4]

$$\|\mathbf{u}(t)\|_{L_x^2}^2 = \|u(t)\|_{L_x^2}^2 + \|v(t)\|_{L_x^2}^2 = \|u_0\|_{L_x^2}^2 + \|v_0\|_{L_x^2}^2 = \|\mathbf{u}_0\|_{L_x^2}^2 \tag{5.29}$$

for any t. On the other hand, for the bound of w, we use the energy estimate for the inhomogeneous Klein–Gordon equation,

$$\|w(t)\|_{H^s} \lesssim \|w_0\|_{H^s} + \|w_1\|_{H^{s-1}} + \int_0^t \|\mathbf{u}^\dagger\mathbf{u}(t')\|_{H^{s-1}} dt'. \tag{5.30}$$

We apply Proposition 4.1 with $1 - s > 1/2$ for the inhomogeneous term,

$$\|\overline{u}v(t')\|_{H^{s-1}} \lesssim \|u(t')\|_{L_x^2}\|v(t')\|_{L_x^2} \le C. \tag{5.31}$$

Therefore we obtain a priori estimate and derive the global existence. □

6 Discussions

In this section, we explain that our choice of Dirac matrices (1.3) is not the special case to apply the arguments in this paper. We rewrite the Dirac operator $\mathcal{D}$ and the inhomogeneous term $\mathbf{u}^\dagger\mathbf{u}$ for the Klein–Gordon equation as follows,

$$\mathcal{D} = i\alpha_j\partial_t + \alpha_k\partial_x, \qquad \mathbf{u}^\dagger\mathbf{u} = {}^t\overline{\mathbf{u}}\alpha_j\mathbf{u} \quad \text{for} \quad j \neq k \tag{6.1}$$

where $\alpha_j,\ j = 1, 2, 3$ are the Pauli matrices

$$\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.2}$$

We could deal with any pairs of $(\alpha_j, \alpha_k), j \neq k$ to compose $\mathcal{D}$ and $\dagger$. The choice of (1.3) corresponds to the case $(j, k) = (1, 2)$. We consider the case $(j, k) = (3, 2)$, and the other cases are similar. We calculate in this case,

$$0 = \mathcal{D}\mathbf{u} + (m - w)\mathbf{u} = \begin{pmatrix} iu_t - iv_x + (m - w)u \\ -iv_t + iu_x + (m - w)v \end{pmatrix}, \tag{6.3}$$

$$\mathbf{u}^\dagger \mathbf{u} = |u|^2 - |v|^2 \tag{6.4}$$

for $\mathbf{u} = {}^t(u, v)$. We set $f = u + v,\ g = u - v$ to obtain

$$0 = \begin{pmatrix} if_t - if_x + (m - w)g \\ ig_t + ig_x + (m - w)f \end{pmatrix}, \tag{6.5}$$

$$\mathbf{u}^\dagger \mathbf{u} = \Re(\overline{f} g) \tag{6.6}$$

which turns to the case $(j, k) = (1, 2)$.

References

[1] N. BOURNAVEAS, A new proof of global existence for the Dirac Klein-Gordon equations in one space dimension, *J. Funct. Anal.* **173**(1) (2000), 203–213.

[2] CHADAM, JOHN M., Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac equations in one space dimension, *J. Functional Analysis* **13** (1973), 173–184. MR 51 #4881

[3] YUNG-FU FANG, A direct proof of global existence for the Dirac-Klein-Gordon equations in one space dimension, *Taiwanese J. Math.* **8** (2004), 33–41.

[4] YUNG-FU FANG, On the Dirac-Klein-Gordon equation in one space dimension, *Differential Integral Equations* **17** (2004), 1321–1346.

[5] YUNG-FU FANG and MANOUSSOS G. GRILLAKIS, On the Dirac-Klein-Gordon system in $2+1$ dimensions, preprint.

[6] YUNG-FU FANG and MANOUSSOS G. GRILLAKIS, On the Dirac-Klein-Gordon equations in three space dimensions, preprint.

[7] J. GINIBRE, Y. TSUTSUMI and G. VELO, On the Cauchy problem for the Zakharov system, *J. Funct. Anal.* **151**(2) (1997), 384–436.

[8] KEEL, MARKUS and TAO, TERENCE, Local and global well-posedness of wave maps on $\mathbf{R}^{1+1}$ for rough data, *Internat. Math. Res. Notices* **21** (1998), 1117–1156.

[9] S. KLAINERMAN and M. MACHEDON, Space-time estimates for null forms and the local existence theorem, *Comm. Pure Appl. Math.* **46**(9) (1993), 1221–1268.

[10] S. KLAINERMAN and M. MACHEDON, Smoothing estimates for null forms and applications, *Duke Math. J.* **81**(1) (1995), 99–133 (1996), A celebration of John F. Nash, Jr.

[11] S. MACHIHARA, One dimensional Dirac equation with quadratic nonlinearities, *Discrete Contin. Dyn. Syst.* **13**(2) (2005), 277–290.

[12] S. MACHIHARA, Dirac equation with certain nonlinearities in one space dimension, Commun. Contemp. Math. **9**(3) (2007), 421–435.

[13] MICHAEL REED, Abstract non-linear wave equations, Springer-Verlag, Berlin, 1976, *Lecture Notes in Mathematics*, Vol. 507.

[14] WALTER E. THIRRING, A soluble relativistic field theory, *Ann. Physics* **3** (1958), 91–112. MR 19,1016a

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