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Viscosity solutions for elliptic-parabolic problems

Paola MANNUCCI
Dipartimento di Matematica Pura e Applicata
Università degli Studi di Padova
Via Belzoni 7
35131 Padova, Italy
e-mail: mannucci@math.unipd.it

Juan Luis VAZQUEZ
Departamento de Matemáticas
Universidad Autónoma de Madrid
Campus de Cantoblanco
28049 Madrid, Spain
e-mail: juanluis.vazquez@uam.es

Abstract. We study an elliptic-parabolic problem appearing in the theory of partially saturated flows in the framework of viscosity solutions. This is part of current investigation to understand the theory of viscosity solutions for PDE problems involving free boundaries. We prove that the problem is well posed in the viscosity setting and compare the results with the weak theory. Dirichlet or Neumann boundary conditions are considered.

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1 Introduction

In this paper we are interested in the well-posedness of the following elliptic-parabolic problem:

$$\begin{cases} c(u)_t = \Delta u, & \text{in } \Omega \times (0,T), \\ c(u) = v_0, & \text{on } \Omega \times \{0\}, \\ u = g, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where Ω is an open subdomain of R^N, and the constitutive function $c(\eta) \in C^0(\mathbf{R})$ satisfies $c(\eta) \equiv 0,\ if\ \eta \leq 0$, while it is a C^1 strictly increasing function if $\eta > 0$.

(See figure 1)

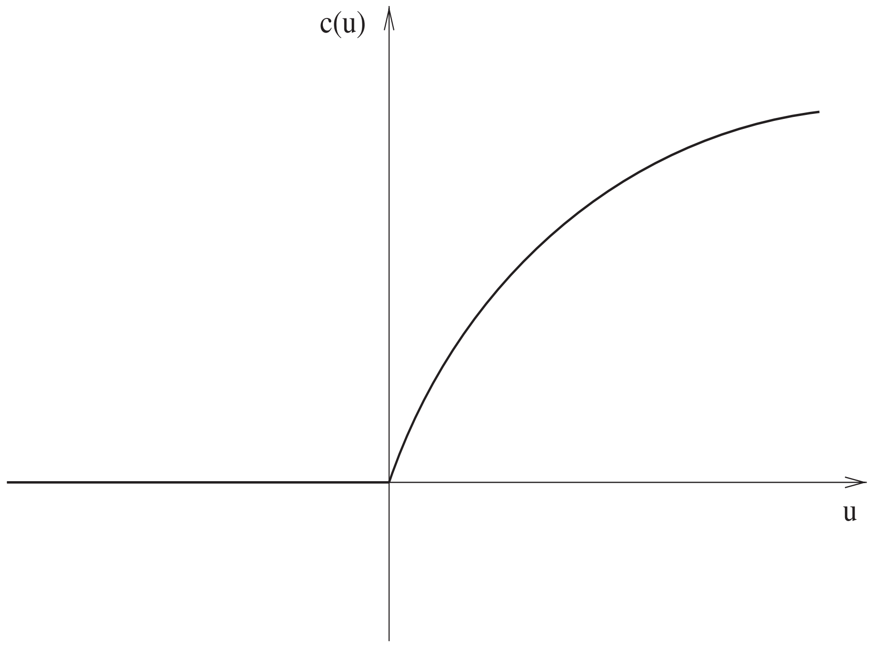

Figure 1

Equation (1.1) is not uniformly parabolic. The level set $\{u = 0\}$ splits the problem in two regions in which the equation is respectively parabolic and elliptic.

This type of equation arises in the theory of partially saturated flows in porous media. In this context, u represents the *pressure* or the *hydrostatic potential* of the fluid, and $c(u)$ the *saturation* or *moisture content* of the medium. In the multidimensional case, the cornerstone is the existence and uniqueness of a weak solution in the paper of Alt - Luckhaus [1]. Further regularity of that solution is obtained by Di Benedetto - Gariepy in [12]. When posed in one space dimension (see [14]), much more is known: a complete analysis of existence, uniqueness and regularity of a weak solution of this problem has been performed by Van Duijn - Peletier and Bertsch - Hulshof in several papers (see [13] and [5] and the references therein).

In this paper we study the problem in the setting of viscosity solutions. Indeed, this work can be seen as part of a general program of defining viscosity solutions and of establishing the well-posedness for parabolic problems involving free boundaries. The standard theory on viscosity solutions, introduced by Crandall, Evans, Lions, [11], can not be used in this case. Some basic references on viscosity solutions are [3], [4], [8]. We consider the one-dimensional case where a general theory of regularity of weak solutions and free boundaries is known.

Our aim is to give a suitable definition of viscosity solution for problem (1.1). This definition allows us to prove that the problem (1.1) is well posed in that setting. In particular, we find a comparison principle between viscosity solutions and weak solutions, thus proving that the viscosity solution in our problem coincides with the weak solution. To do this we use some tools introduced by Caffarelli and Vazquez in [10] where the authors give a suitable definition of a viscosity solution for the porous medium equation:

$$u_t = u\Delta u + |\nabla u|^2.$$

See [6] for an extension of that result. These definitions are inspired in earlier work of Caffarelli on stationary free boundary problems, see [7], continued by him and coauthors for the Stefan Problem, cf. [2]. While for Stefan and porous medium equations the motion of the interface is locally determined by the space slope of the solutions near the point, this is not true in our case, a situation that reminds of the combustion problem treated in [9] and complicated the analysis.

Similar definitions of viscosity solution can be used for a large class of problems involving free boundaries. Where a general theory for weak solutions exists, the aim is to prove that the viscosity solution exists, it is unique and it coincides with the weak solution, thus making for a foundation for the viscosity approach. In more general situations, the viscosity theory could be developed without the help of weak solutions.

2 Problem setting. Classical free boundary solutions

Let the space domain be an open interval in $\mathbf{R}$. We take, for the sake of simplicity, $I = (0, 1)$. We consider the following problem

$$\frac{\partial c(u)}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{in} \quad (0,1) \times (0,T) \equiv: Q_T, \tag{2.1}$$

$$c(u(x,0)) = v_0(x) \qquad \textit{on } (0,1), \tag{2.2}$$

$$u(0,t) = g_0(t) \qquad \textit{on } (0,T), \tag{2.3}$$

$$u(1,t) = g_1(t) \qquad \textit{on } (0,T), \tag{2.4}$$

with the stated assumptions on c. This is the problem with Dirichlet boundary conditions, that we will refer as (PD). We make the following assumptions on the data of this problem. On the boundary data we assume that $g_0(t), g_1(t)$ are continuous functions on $[0, T]$ and

$$g_0(t) > 0, \quad g_1(t) < 0, \ \forall t \in (0, T), \tag{2.5}$$

while the initial data satisfies:

$$v_0(x) \text{ is a nonnegative continuous function} \qquad (2.6)$$
$$v_0(x) > 0 \iff x \in [0, x_0),\ x_0 \in (0, 1).$$

Replacing conditions (2.3) and (2.4) by

$$u_x(0,t) = f_0(t), \quad u_x(1,t) = f_1(t),$$

we obtain the corresponding Neumann problem, (PN), that will be briefly considered in the last section.

As stated in the Introduction, problems consisting of (2.1)–(2.2) with either Dirichlet or Neumann conditions are well studied from the classical point of view. In the one-dimensional case we have quoted the papers of [13], [5] and their references. Under suitable assumptions on the data, the authors extensively study the regularity of the solution and of the free boundary for the Dirichlet and for the Neumann problem. More precisely, they prove that there exists an unique weak solution u of problem (2.1)–(2.4) such that $c(u) \in C(\overline{Q}_T)$, $u_x \in C(Q_T)$, $u_t \in L^{\infty}_{loc}(Q_T)$, where T is a suitable constant determined by the data. It is shown that the solution $u(x,t)$ is smooth and a classical solution of the equation where it does not vanish, but it has limited regularity on the set of points (x,t) where it vanishes.

2.1 Classical free boundary theory

Therefore, it will be convenient to introduce the following definitions that will be useful in constructing the theory of viscosity solutions.

Definition 2.1 We say that a function $u(x,t) \in C^0(Q_T)$ is a **classical free-boundary solution** of equation (2.1) if
(i) the set $\Gamma(u)$ where $u = 0$ is representable as a function $x = s(t)$ such that $s(t)$ is Lipschitz continuous on $[0,T]$.
(ii) u is $C^{2,1}_{x,t}$ smooth in the positivity set $\mathcal{P}(u)$ where it solves $c'(u)u_t = u_{xx}$. In the negativity set $\mathcal{N}(u)$, u solves $u_{xx} = 0$ and $u_t \in L^\infty$.
(iii) u_x is continuous in Q_T.

In the set $\mathcal{P}(u)$, called the *unsaturated region*, equation (2.1) is parabolic; $\mathcal{S}(u) = \{u \leq 0\}$ is the *saturated region*, inside it equation (2.1) becomes elliptic. The set $\Gamma(u)$ is called the *free boundary*. Note that the most important information in (iii) is that u_x is continuous across the free boundary.

Definition 2.2 We say that a function $u(x,t) \in C^0([0,1] \times [0,T))$ is a **classical free-boundary solution** of problem (PD) if it is a classical free boundary solution of the equation in Q_T and moreover,
(iv) $c(u(x,0)) = v_0(x)$,
(v) $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$.

It is immediate from these definitions that in the saturated set $\mathcal{S}(u) = \{u \le 0\}$, we have the representation

$$u(x,t) = \frac{g_1(t)}{1 - s(t)}(x - s(t)), \tag{2.7}$$

while the following conditions hold on $x = s(t)$ if $s(t) < 1$

$$u(s(t),t) = 0, \ t \in (0,T), \tag{2.8}$$

$$\frac{\partial u}{\partial x}(s(t),t) = \frac{g_1(t)}{1 - s(t)}, \ t \in (0,T). \tag{2.9}$$

For a general solution, the *free boundary* $\Gamma(u)$ is defined as the boundary of the positivity set located inside Q_T:

$$\Gamma(u) = \partial\mathcal{P}(u) \cap Q_T.$$

From the results of [5], [13] we know that, under Lipschitz continuous assumptions on the data, a weak solution u of problem (2.1)–(2.4) exists and is unique. Moreover, this solution u is a classical free-boundary solution as in Definition (2.2).

Further regularity of the solution can be proved assuming that the constitutive function $c(\eta)$ satisfies a smoothness condition at $u = 0$, $c'(0) = 0$ (see [5]). Moreover, from the results of [13], a comparison principle for weak solutions of problem (PD) holds in the following sense:

Comparison. *Let u_1 and u_2 be the weak solutions of* (2.1)–(2.4) *corresponding to the initial data v_{01} and v_{02} and boundary data $u_j(i,t) = g_{ij}(t)$, $i = 0,1$, $j = 1,2$. If the data are ordered, i.e., $v_{01}(x) \ge v_{02}(x)$ in $(0,1)$, and $u_1(i,t) \ge u_2(i,t)$ for $i = 0,1$, $0 \le t \le T$, then $c(u_1) \ge c(u_2)$ on Q_T .*

From the Comparison Principle stated above, if $c(u_1) \ge c(u_2)$ we get the conclusion $\mathcal{P}(u_2) \subseteq \mathcal{P}(u_1)$. In $\mathcal{P}(u_2)$ we have $u_1 \ge u_2$. In $\mathcal{N}(u_2)$, the inequality $u_1 \ge u_2$ follows from the linear expression (2.7) of the solution.

The definition of classical free boundary solution can be extended to sub- and super-solutions as follows:

Classical free boundary sub- and super-solutions. For the classical free boundary sub-solution, Definition 2.1 is changed as follows: the equation to be satisfied by u on both $P(u)$ and $N(u)$ is replaced by an inequality $\le$. Point (iii) becomes: u_x may stay continuous over the free boundary or admit jumps upwards, $u_x(s(t)^-,t) \le u_x(s(t)^+,t)$. Note that $u(x,t)$ is still a continuous function in the whole domain, $u \in C^0(Q_T)$.

For the *classical free boundary super-solution*, the inequalities are reversed.

In Section 2.2 we will show some examples of classical free boundary sub and supersolution for problem (2.1)–(2.4).

2.2 Travelling wave examples

Next, we find some examples of classical free boundary solutions and sub- or supersolutions. The most popular one is the family of *travelling waves*: we look for solutions of problem (2.1)–(2.4) of the type

$$u(x,t) = U(x - x_0 - \lambda t) = U(\eta), \quad \lambda \in \mathbf{R}. \tag{2.10}$$

The curve $\{\eta = 0\}$ separates the region $\{\eta < 0\}$ where $U > 0$ from the region $\{\eta > 0\}$ where $U < 0$. The line $x = x_0 + \lambda t$ is the free boundary.

In the region $\{U > 0\}$ the solution U of equation (2.1) satisfies:

$$-\lambda c(U)'(\eta) - U''(\eta) = 0, \; if \; \eta < 0 \tag{2.11}$$

$$U(0) = 0, \tag{2.12}$$

$$U'(0) = -K, \tag{2.13}$$

where $K > 0$. We remark that the constant λ can be negative (then the free boundary will be decreasing in t). Integrating with respect to η, taking into account (2.12) and (2.13):

$$\begin{aligned} & -\lambda c(U(\eta)) - U'(\eta) = K, \quad for \; \eta \le 0, \\ & U(0) = 0. \end{aligned} \tag{2.14}$$

The Cauchy problem (2.14) has a unique solution $U(\eta)$. If we define

$$F(U) :\equiv \int_0^U \frac{dU}{K + \lambda c(U)}, \quad U > 0,$$

then (2.14) is equivalent to

$$F(U) = -\eta.$$

Note that F depends on the speed parameter λ. If $\lambda \ge 0$ function $F(U)$ is invertible for $U > 0$, since

$$F'(U) = \frac{1}{K + \lambda c(U)} > 0.$$

On the other hand, if $\lambda < 0$, we get an orbit $U(\xi)$ starting from $U(0) = 0$, such that

$$0 \le U < c^{-1}(-\frac{K}{\lambda}).$$

In this range $F(U)$ is invertible. Hence, we obtain $U = F^{-1}(x_0 + \lambda t - x)$.

In the particular case of linear c: $c(u) = u$ for $u > 0$, simple calculations give explicit formulas. We obtain that $U(x - x_0 - \lambda t)$ has the following form:

$$U(x - x_0 - \lambda t) = \frac{K}{\lambda}\Bigg(\exp(-\lambda(x - x_0 - \lambda t)) - 1\Bigg), \quad if \; x < x_0 + \lambda t,$$

$$U(x - x_0 - \lambda t) = -K(x - x_0 - \lambda t), \quad if \; x \ge x_0 + \lambda t.$$

We can construct classical free boundary sub- and supersolution of problem (2.1)–(2.4), taking U satisfying equation (2.11)–(2.13) with suitable constraints on x_0 and on K (for example $K \geq |g_1(t)|/(1 - x_0 - \lambda t)$) and replacing the equality in (2.11) respectively by ≤ 0 or ≥ 0.

The only constraint on λ is: $\lambda < \frac{1-x_0}{T}$, if $\lambda > 0$ and $\lambda > -\frac{x_0}{T}$, if $\lambda < 0$.

In a similar way we can write explicit classical free boundary sup- and supersolutions for the Neumann problem.

Remark 2.1 Note that, by adjusting the free parameters K, slope at the free boundary, and λ, wave speed, any classical free boundary solution can be locally approximated in C^1 norm by a travelling wave solution. A corresponding statement can be formulated for super- and subsolutions.

In the following section we give the definition of viscosity solution by means of the definition of classical free boundary super- and sub solutions given above. In the subsequent Sections 4 and 5 we prove the well-posedness in this viscosity setting for Lipschitz continuous data, and finally for merely continuous data.

3 Viscosity solutions

Following the standard approach, we introduce first the definitions of super- and sub-solution. Let $Q_T = (0,1) \times (0,T)$.

Definition 3.1 A function $u(x,t) \in C^0(Q_T)$ is a **viscosity subsolution** of equation (2.1) if:
(i) At every point $P_0 = (x_0, t_0)$ in $\mathcal{P}(u) \cap Q_T$ and for every $\varphi \in C^{2,1}(Q_T)$ that touches u from above at point P_0, the inequality

$$\frac{\partial c(\varphi)}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2} \leq 0 \tag{3.1}$$

holds in P_0.
(ii) At every point $P_0 = (x_0, t_0)$ in $\mathcal{N}(u) \cap Q_T$ and for every function $\varphi \in C^{2,1}(Q_T)$ that touches u from above at point P_0, the inequality

$$-\frac{\partial^2 \varphi}{\partial x^2} \leq 0 \tag{3.2}$$

holds in P_0.
(iii) Let $P_0 = (x_0, t_0) \in \partial\mathcal{P}(u) \cap Q_T$ and let $\mathcal{R}$ be a parabolic neighborhood of P_0, $\mathcal{R} = (x_0 - \delta_1, x_0 + \delta_1) \times (t_0 - \delta_2, t_0)$, with δ_1 and $\delta_2 > 0$ and $\mathcal{R} \subset Q_T$. Then, for every classical free-boundary supersolution U_+, if $u < U_+$ on the parabolic boundary of $\mathcal{R}$ it follows that $u \leq U_+$ on $\mathcal{R}$.

A function $u(x,t) \in C^0([0,1] \times [0,T))$ is a **viscosity subsolution** of problem (2.1)–(2.4) if it is a viscosity solution of equation (2.1) and the initial and boundary

data are well adjusted:
(iv) $c(u(x,0)) \leq v_0(x)$, $\forall x \in (0,1)$; $u(i,t) \leq g_i(t)$, $i = 0,1$, $t \in (0,T)$.

A similar definition applies to **viscosity supersolutions** after changing the sign of the inequalities involved in the definition in the obvious way.

Definition 3.2 A **viscosity solution** of problem (2.1)–(2.4) is a continuous function defined in Q_T which is at the same time a sub- and a super-solution.

Any classical free boundary sub- (resp. supersolution) can be locally approximated by a sub- (resp. supersolution) of the type "travelling wave" found in the previous section (see Remark 2.1). Hence, in the definition of viscosity solution we can replace any classical free boundary sub- (resp. supersolution) by a travelling wave sub- (resp. supersolution). In this way we restrict the class of functions to test the comparison principle in the neighborhoods of the points of $\partial\mathcal{P}(u) \cap Q_T$ to a very special explicit family.

4 Well posedness for Lipschitz continuous data

As a preliminary for our general well-posedness result, we prove that, in the case of Lipschitz continuous data, the viscosity solution exists and coincides with the classical free boundary solution.

Theorem 4.1 *Under the assumptions*
(i) $g_i(t)$*,* $i = 0,1$ *are Lipschitz continuous functions on* $[0,T]$*,*
(ii) $g_0(t) > 0$ *for all* $t \in (0,T)$*, and* $g_1(t) < 0$ *for all* $t \in (0,T)$*,*
(iii) $v_0(x) \geq 0$*, and there exists a Lipschitz continuous function* $u_0 : (0,1) \to \mathbf{R}$ *such that* $v_0 = c(u_0)$*,*
(iv) $u_0(x) > 0$ *iff* $x \in [0,x_0)$*,* $x_0 \in (0,1)$*,*
(iv) $u_0(0) = g_0(0)$*,* $u_0(1) = g_1(1)$*,*
the initial boundary value problem (2.1)–(2.4) *is well posed in the class of viscosity solutions: the viscosity solution coincides with the continuous weak solution of the problem.*

Proof. Let w be the continuous weak solution of the problem (2.1)–(2.4). Under the above assumptions, we know from the results of [5], [13] that w exists, it is unique and is a classical free boundary solution.

Viscosity solution. Existence. We now prove that such a w is a viscosity solution. First we prove that w is a viscosity subsolution. The argument is quite standard. Let $\varphi \in C^{2,1}(Q_T)$ be a test function that touches w from above in $P_0 = (x_0,t_0) \in Q_T$, where $w(P_0) > 0$. In P_0 the function $\varphi - w$ has a minimum zero, then at this point the following is true:

$$\varphi = w,\ \varphi_t \leq w_t,\ \varphi_{xx} \geq w_{xx}.$$

Since w is a classical solution in $\mathcal{P}(w)$, it satisfies $c(w)_t - w_{xx} = 0$ in P_0. Hence,

$$c(\varphi)_t - \varphi_{xx} \le 0, \ \ in \ P_0.$$

If $w(P_0) < 0$, we analogously obtain

$$-\varphi_{xx} \le 0, \ \ in \ P_0.$$

If $w(P_0) = 0$, let I be a parabolic neighbourhood of P_0 and let U_+ be a classical free boundary supersolution such that $w < U_+$ on the parabolic boundary of I. A comparison principle between weak solutions holds ([13]), then we conclude that $w \le U_+$ on I.

Hence, we have proved that w is a viscosity subsolution. In a similar way, we prove that w is a viscosity supersolution. Besides, every classical free boundary solution is a viscosity solution (consistency).

Uniqueness. We want to prove that if u is another viscosity solution to problem (2.1)–(2.4), then $u = w$. We will prove that $u \le w$ and then that $u \ge w$.

A) Proof that $u \le w$. Under the assumptions of the theorem, we know that a Lipschitz function u_0 exists such that $c(u_0) = v_0$. Let us consider w_ϵ the weak solution of

$$\begin{array}{ll} c(w_\epsilon)_t - w_{\epsilon xx} = 0, & in \ Q_T, \\ c(w_\epsilon)(x,0) = c(u_0(x) + \epsilon), & on \ (0,1), \\ w_\epsilon(0,t) = g_0(x,t) + \epsilon, & on \ (0,T), \\ w_\epsilon(1,t) = g_1(x,t) + \epsilon, & in \ (0,T), \end{array} \tag{4.1}$$

The constant $\epsilon > 0$ is taken sufficiently small such that $g_1(x,t) + \epsilon < 0$ and such that $u_0(x) + \epsilon > 0$, for $x \in [0, x_\epsilon)$ and $u_0(x) + \epsilon \le 0$, for $x \in [x_\epsilon, 1]$. We know from [5] that there exists an unique weak solution w_ϵ of problem (4.1). This is a classical free boundary supersolution of problem (2.1)–(2.4).

From the comparison principle between weak solutions ([13]) we know that: if w and w_ϵ are solution respectively of problems (2.1)–(2.4) and (4.1), then $c(w) \le c(w_\epsilon)$. As shown in Section 2.1, this implies that also $w \le w_\epsilon$ in Q_T.

Moreover, we have that w_ϵ is a nondecreasing sequence of ϵ, hence there exists the limit as ϵ tends to 0 and from the uniqueness of the weak solution, this limit is w:

$$\lim_{\epsilon \to 0} w_\epsilon = w.$$

We want to prove that for any $u(x,t)$, viscosity solution of problem (2.1)–(2.4), we have $u(x,t) < w_\epsilon(x,t)$, in Q_T, $\forall \epsilon > 0$, sufficiently small. Suppose for contradiction that there exists a first point $\overline{P} = (\overline{x}, \overline{t}) \in Q_T$ such that $u(\overline{x}, \overline{t}) = w_\epsilon(\overline{x}, \overline{t})$. We examine the different possibilities.

(i) The point $\overline{P} = (\overline{x}, \overline{t})$ lies in the positivity set $\mathcal{P}(u)$, so that $u(\overline{x}, \overline{t}) = w_\epsilon(\overline{x}, \overline{t}) > 0$.

Then, there exists a rectangular parabolic neighborhood B of $\overline{P} = (\overline{x}, \overline{t})$, such that $u(x,t) > 0$ and $w_\epsilon(x,t) > 0$, $\forall (x,t) \in B$ and such that $u < w_\epsilon$ on its parabolic boundary. The function w_ϵ is smooth in its positivity set and it solves equation $c'(w_\epsilon)w_{\epsilon t} - w_{\epsilon xx} = 0$ in the classical sense in B.

Now we can find a classical solution $\tilde{u}$ such that $\tilde{u} = u + \gamma$ on $\partial_p B$, with $\gamma > 0$ such that $\tilde{u} < w_\epsilon$ on $\partial_p B$. From the Strong Maximum Principle between classical solutions we have that $\tilde{u}(\overline{P}) < w_\epsilon(\overline{P})$. We prove that $u \leq \tilde{u}$ in B: we approximate $\tilde{u}$ by $\tilde{u}_\delta$, $\delta > 0$, classical solution in B of

$$c'(\tilde{u}_\delta)\tilde{u}_{\delta t} - \tilde{u}_{\delta xx} = \delta,$$

with boundary data $\tilde{u}_\delta = \tilde{u} + \delta$. If $\tilde{u}_\delta$ touches from above u at a point P_1 of B, from the definition of viscosity subsolution and the smoothness of $\tilde{u}_\delta$, we have $c'(\tilde{u}_\delta)\tilde{u}_{\delta t} - \tilde{u}_{\delta xx} \leq 0$ which is a contradiction. Hence, $u < \tilde{u}_\delta$ in B. Since $\tilde{u}_\delta \searrow \tilde{u}$, in the limit $u \leq \tilde{u}$ in B. Hence, $u \leq \tilde{u}(\overline{P}) < w_\epsilon(\overline{P})$, thus a contradiction.

(ii) $\overline{P} = (\overline{x}, \overline{t}) \in \mathcal{N}(u)$. Then, we have $u(\overline{x}, \overline{t}) = w_\epsilon(\overline{x}, \overline{t}) < 0$. There exists a neighborhood of $\overline{P}$, $B = (x_1, x_2) \times (t_1, \overline{t})$, such that $u < 0$, $w_\epsilon < 0$ in B and $u < w_\epsilon$ on its parabolic boundary. Since w_ϵ is smooth in B, we have that $w_{\epsilon xx}(x,t) = 0$ in B.

Consider now $w_{\epsilon\delta} = w_\epsilon - \delta(x - x_1)^2$, with $\delta > 0$ suitably small such that $w_{\epsilon\delta}(x_2, t) > u(x_2, t)$. If, in a point $P_0 \in \overline{B}$, $w_{\epsilon\delta}$ touches u from above, from the definition of viscosity solution we have that $w_{\epsilon\delta xx} \geq 0$ in P_0, on the other hand $w_{\epsilon\delta xx} = -2\delta < 0$, hence in $\overline{B}$ we have $u < w_{\epsilon\delta} \leq w_\epsilon$, i.e. a contradiction.

(iii) Finally, we examine the interesting case when contact takes place at the free boundary: $\overline{P} = (\overline{x}, \overline{t}) \in \Gamma(u) :\equiv \partial\mathcal{P}(u) \cap Q_T$. Then $(\overline{x}, \overline{t})$ belongs to the free boundary $x = s_\epsilon(t)$, which is the curve where $w_\epsilon = 0$ and $u(\overline{x}, \overline{t}) = w_\epsilon(\overline{x}, \overline{t}) = 0$.

Note that the curve $x = s_\epsilon(t)$ can not touch the axes $x = 0, 1$ since $g_0 + \epsilon > 0$ and $g_1 + \epsilon < 0$, hence $\overline{x} = s_\epsilon(\overline{t}) < 1$.

We take a rectangular parabolic neighborhood B of $\overline{P} = (\overline{x}, \overline{t})$ such that $u < w_\epsilon$ on its parabolic boundary. We may also assume that contact does not take place at other points at time $\overline{t}$, hence there is a positive constant $\gamma = \gamma(\epsilon)$ such that

$$u \leq w_\epsilon - \gamma(\epsilon) \quad \text{on } \partial_p B.$$

We now define $w_{\epsilon\delta}$ as the translation of w_ϵ to the left

$$w_{\epsilon\delta}(x,t) = w_\epsilon(x + \delta, t),$$

where δ is taken small enough so that we still have

$$u(x,t) < w_{\epsilon\delta}(x,t)$$

on the parabolic boundary $\partial_p B$. From the definition of viscosity solution, it follows that $u \leq w_{\epsilon\delta}$ in $\overline{B}$. But this implies that u must be negative in the set of points of B where $x > s_\epsilon(t) - \delta$. In the limit $t = \overline{t}$ we get $u(x, \overline{t}) < 0$ for $x > s_\epsilon(\overline{t}) - \delta$, a contradiction with the assumption that u and w_ϵ touch at $t = \overline{t}$, $x = s_\epsilon(\overline{t})$.

We have proved that $u < w_\epsilon$ in Q_T. Passing to the limit as ϵ tends to zero, this implies that $u \leq w$ in Q_T.

B) To prove that $u \geq w$, we consider the function $u_{-\epsilon}$ obtained by solving the problem with initial boundary conditions $w_{-\epsilon}(x,0) = c(u_0(x) - \epsilon)$, $u_{-\epsilon}(i,t) = g_i(t) - \epsilon$, $i = 0,1$ and, by a similar technique as above, we prove that $w_{-\epsilon} < u$, in Q_T, thus obtaining $u \geq w$ in Q_T. □

5 Well-posedness for general data

The aim of this section is to prove a well-posedness result for the problem with initial and boundary data which are merely continuous.

The viscosity solution u is constructed as the limit of viscosity solutions whose data are Lipschitz continuous functions and whose existence has been proved in the previous section.

Theorem 5.1 *Under assumptions* (2.5), (2.6)*, the initial boundary value problem* (2.1)–(2.4) *is well posed in the class of viscosity solutions and the viscosity solution coincides with the continuous weak solution of the problem.*

Proof. Let us consider the sequences $\{\overline{g}_{0n}(t)\}$, $\{\underline{g}_{0n}(t)\}$, $\{\overline{g}_{1n}(t)\}$, $\{\underline{g}_{1n}(t)\}$, $\{\overline{u}_{0n}(x)\}$, $\{\underline{u}_{0n}(x)\}$ such that:

$$\begin{aligned}
&\overline{g}_{0n}(t),\ \underline{g}_{0n}(t) > 0,\ \overline{g}_{1n}(t),\ \underline{g}_{1n}(t) < 0,\ \forall n \in \mathbf{N},\ \forall t \in (0,T),\\
&\underline{u}_{0n}(x) > 0,\ iff\ x \in [0, \underline{x}_{0n}),\ \underline{x}_{0n} \in (x_0, 1),\\
&\overline{u}_{0n}(x) > 0,\ iff\ x \in [0, \overline{x}_{0n}),\ \underline{x}_{0n} \in (0, x_0),\\
&\overline{g}_{in}, \underline{g}_{in},\ i = 0,1,\ \overline{u}_{0n},\ \underline{u}_{0n} \text{ are Lipschitz continuous functions, } \forall n \in \mathbf{N},\\
&\{\overline{g}_{in}\}, \{\overline{u}_{0n}\} \text{ are decreasing sequences of functions such that}\\
&\overline{g}_{in} \searrow g_0,\ i = 0,1,\ \overline{u}_{0n} \searrow u_0\ in\ C^0,\ as\ n \to +\infty,\\
&\{\underline{g}_{in}\}, \{\underline{u}_{0n}\} \text{ are increasing sequences of functions such that}\\
&\underline{g}_{in} \nearrow g_i,\ i = 0,1,\ \underline{u}_{0n} \nearrow u_0,\ in\ C^0,\ as\ n \to +\infty.
\end{aligned} \tag{5.1}$$

From Theorem 4.1 we know that problem (2.1)–(2.4) with initial and boundary data $\overline{g}_{0n}$, $\overline{g}_{1n}$, $\overline{u}_{0n}$ has a unique viscosity solution $\overline{u}_n$ which coincides with the classical free boundary solution. Analogously, we denote by $\underline{u}_n$ the viscosity solution corresponding to the data $\underline{g}_{0n}(t)$, $\underline{g}_{1n}(t)$, $\underline{u}_{0n}$.

From the comparison principle between classical free boundary solutions we have that $\{\overline{u}_n\}$ and $\{\underline{u}_n\}$ are respectively decreasing and increasing sequences. Moreover $\underline{u}_n \leq \overline{u}_n$, for all $n \in \mathbf{N}$.

From the monotonicity of the sequences, we have that there exist the following two limits:

$$\lim_n \overline{u}_n \equiv: \overline{u}, \quad \lim_n \underline{u}_n \equiv: \underline{u}. \tag{5.2}$$

From the continuity and the monotonicity of $\overline{u}_n$ and $\underline{u}_n$ we have that $\overline{u}$ is an upper semicontinuous function and $\underline{u}$ is a lower semicontinuous function.

CONTINUITY. We will prove that $\overline{u} = \underline{u}$ by means of a continuous dependence result for the functions $\overline{u}_n$, $\underline{u}_n$.

Let $\overline{v}_n$ and $\underline{v}_n$ be such that $c(\overline{u}_n) = \overline{v}_n$ and $c(\underline{u}_n) = \underline{v}_n$. Denote by $v_n := \overline{v}_n - \underline{v}_n \geq 0$, $u_n := \overline{u}_n - \underline{u}_n \geq 0$. The functions v_n, u_n solve the following problem

$$\begin{cases} \dfrac{\partial v_n}{\partial t} - \dfrac{\partial^2 u_n}{\partial x^2} = 0, \ in \ (0,1) \times (0,T), \\ v_n(x,0) = c(\overline{u}_{0n}(x)) - c(\underline{u}_{0n}(x)) =: v_{0n}(x) > 0, \ on \ (0,1), \\ u(0,t) = \overline{g}_{0n}(t) - \underline{g}_{0n}(t) =: g_{0n}(t) > 0, \ on \ (0,T), \\ u(1,t) = \overline{g}_{1n}(t) - \underline{g}_{1n}(t) =: g_{1n}(t) > 0, \ on \ (0,T). \end{cases} \tag{5.3}$$

We now take a test function $\varphi(x) \in C^2([0,1])$ with the following properties:

$$\varphi(0) = \varphi(1) = 0, \ \frac{\partial^2 \varphi}{\partial x^2} = -1. \tag{5.4}$$

Under assumptions (5.4), we have $|\partial\varphi/\partial x| \leq C$. Multiplying equation (5.3) by $\varphi(x)$ and integrating in $(0,1) \times (0,t)$, we obtain

$$\int_0^1 v_n \, \varphi \, dx - \int_0^1 v_{0n} \, \varphi \, dx = \int_0^t u_{nx} \, \varphi /_0^1 dt - \int_0^t u_n \, \varphi_x /_0^1 \, dt \tag{5.5}$$
$$+ \int_0^t \int_0^1 u_n \, \varphi_{xx} \, dx \, dt.$$

Taking into account (5.4), we obtain

$$\int_0^1 v_n(x,t) \, \varphi(x) \, dx + \int_0^t \int_0^1 u_n(x,t) \, dx \, dt \leq \int_0^1 v_{0n}(x)\varphi(x) \, dx \tag{5.6}$$
$$+ C \int_0^t (g_{0n}(t) + g_{1n}(t)) \, dt.$$

If we choose the data such that

$$\int_0^t (g_{0n} + g_{1n}) \, dt \leq \epsilon, \ \int_0^1 v_{0n} \, dx \leq \epsilon,$$

from (5.6), we obtain that

$$\int_0^t \int_0^1 u_n(x,t) \, dx \, dt \leq 2\epsilon,$$

and, from the nonnegativity of u_n, we obtain that $u_n = \overline{u}_n - \underline{u}_n \to 0$ as $n \to +\infty$. Hence $\overline{u} = \underline{u} =: u$ almost everywhere.

Now we prove that $\overline{u} = \underline{u} =: u$ for all $(x,t) \in Q_T$. Where $\overline{u}$ and $\underline{u}$ are both smooth functions, there the equality $\overline{u} = \underline{u}$ holds and also $\overline{v} = c(\overline{u}) \equiv \underline{v} = c(\underline{u})$. If we prove that the corresponding free boundaries coincide, then we can conclude that $\overline{u} \equiv \underline{u} =: u$ for all $(x,t) \in Q_T$.

Let $\{\overline{s}_n(t)\}$, $\{\underline{s}_n(t)\}$ be the sequences of the free boundaries related to $\{\overline{u}_n\}$, $\{\underline{u}_n\}$. Clearly $\{\overline{s}_n(t)\}$ is a decreasing sequence and $\{\underline{s}_n(t)\}$ is increasing. Define

$$\lim_n \overline{s}_n \equiv: \overline{s}, \qquad \lim_n \underline{s}_n \equiv: \underline{s}. \tag{5.7}$$

From the continuity and the monotonicity of $\overline{s}_n$ and $\underline{s}_n$ we have that $\overline{s}$ is an upper semicontinuous function and $\underline{s}$ is a lower semicontinuous function, $\overline{s}_n \geq \underline{s}_n$ for all $n \in \mathbf{N}$.
Suppose that there exists a point t^* such that $\overline{s}_n(t^*) > \underline{s}_n(t^*)$, then integrating on $(0,1)$ for $t = t^*$ we have

$$\int_0^1 \overline{v}(x,t^*)\,dx > \int_0^1 \underline{v}(x,t^*)\,dx,$$

which is a contradiction with the integral inequality (5.6) if we choose n sufficiently large.
Hence $\overline{s}(t) = \underline{s}(t) \equiv: s(t)$ for all $t \in [0,T]$ and $s(t)$ is a continuous function. This implies that $\overline{u} = \underline{u} =: u$ for all $(x,t) \in Q_T$.

Hence, we have proved that the limit function u obtained as lower limit of decreasing functions or as upper limit of increasing functions is continuous.

Let us point out that the continuity of u, $\overline{u}_n$, $\underline{u}_n$ and the monotonicity of the sequences $\{\overline{u}_n\}$, $\{\underline{u}_n\}$ implies that the convergences in (5.2) are locally uniform.

The limit u is a viscosity solution. We prove that the common limit u is a viscosity solution to problem (2.1)–(2.4) with initial and boundary data satisfying assumptions (2.5), (2.6).

We first prove that u is a viscosity subsolution.

1) $P_0 = (x_0, t_0)$ is a point such that $u(P_0) > 0$ or $u(P_0) < 0$:

in this case the procedure is standard, by using the definition of viscosity solution for the functions $\overline{u}_n$ and using that the sequence $\{\overline{u}_n\}$ converges uniformly to u (see, for example, Proposition 2.2 of [3]).

2) $P_0 = (x_0, t_0)$ is a point such that $u(P_0) = 0$: we have to verify that every classical free-boundary supersolution U_+ such that $u < U_+$ on the parabolic boundary of an arbitrary neighbourhood I of P_0, then $u \leq U_+$ on I.

Since $\overline{u}_n$ converges uniformly to u, we choose an n sufficiently large such that $\overline{u}_n < U_+$ on the parabolic boundary of I. Since $\overline{u}_n$ is a viscosity solution we know that $\overline{u}_n \leq U_+$ on I and then $u < \overline{u}_n \leq U_+$.

We proceed analogously to prove that u is a viscosity supersolution.

Uniqueness. Let $\tilde{u}$ be an other viscosity solution, we prove that $\tilde{u} = u$. In Theorem 4.1 we have proved that the viscosity solutions $\overline{u}_n$ and $\underline{u}_n$ coincide with the classical free-boundary solutions corresponding to the same data. By construction we have that $\tilde{u} < \overline{u}_n$, for all $n \in \mathbf{N}$, on the parabolic boundary of Q_T. Hence, from the comparison between viscosity solutions and classical free-boundary solutions (see the proof of Theorem 5.1) we have that $\tilde{u} < \overline{u}_n$, for all $n \in \mathbf{N}$, hence passing to the limit as $n \to +\infty$, $\tilde{u} \leq u$. Analogously $\tilde{u} > \underline{u}_n$ for all $n \in \mathbf{N}$, hence $\tilde{u} \geq \underline{u}$.

Continuous dependence for general data. Passing to the limit as $n \to +\infty$ we have that the equality (5.5) holds true also for the viscosity solution u,

$$\int_0^1 c(u)\,\varphi\,dx - \int_0^t \int_0^1 u\,\varphi_{xx}\,dx\,dt = \int_0^1 v_0\,\varphi\,dx - \int_0^t u\,\varphi_x/_0^1\,dt, \tag{5.8}$$

for all $\varphi(x) \in C^{2,1}$ such that $\varphi(0) = \varphi(1) = 0$. Hence, u is also a weak solution in the sense specified in ([5]), ([13]).

We have also found that the free boundary is a continuous function $x = s(t)$. Moreover, from the uniqueness results of F. Otto ([16]) of the weak solution of problem (2.1)–(2.4), we obtain that the unique viscosity solution u of problem (2.1)–(2.4) with continuous data coincides with the unique weak solution of this problem. □

6 Extensions and comments

Problem in several dimensions. We want to emphasize that Theorem 5.1 can be proved only in the one dimensional case since we strongly use the results on existence, uniqueness, regularity, comparison principle for weak solutions by [13] and [5] which are true only in this case. As shown by the work on the problem of the heat equation with combustion-type boundary conditions, [9], [15], uniqueness of viscosity solutions in these free boundary problems in not easy in space several dimensions.

Neumann and mixed problems. Similar results can be proved for the well posedness of the Neumann problem (PN), obtained by replacing conditions (2.3) and (2.4) by

$$u_x(0,t) = f_0(t), \quad u_x(1,t) = f_1(t). \tag{6.1}$$

In this case the Definition (2.2) of classical free boundary solution is naturally modified in conditions (v). We recall that classical results on the existence and uniqueness of a weak solution with Neumann boundary conditions can be still found in [13] and [5].

For the Neumann problem (2.1), (2.2), (6.1) we can give a similar definition of viscosity subsolution replacing condition iv) in Definition (3.1) by

$$c(u(x,0)) \leq v_0(x),\ u_x(0,t) \geq f_0(t),\ u_x(1,t) \leq f_1(t), \tag{6.2}$$

and in Definition of supersolution by (6.2) with reversed inequalities.

Since the techniques used above are local, we can use the same procedure to prove the existence and uniqueness of a viscosity solution which coincides with the weak solution.

Analogous results can be obtained by considering mixed boundary conditions.

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