NoDEA
Nonlinear differ. equ. appl. 14 (2007) 491—498
1021–9722/07/060491–8
DOI 10.1007/s00030-007-4005-8

Nonlinear Differential Equations
and Applications NoDEA

On the exact limit cycle for some class of planar differential systems

A. BENDJEDDOU
Laboratoire de Mathématiques Appliquées
Faculté des Sciences
Université de Sétif
19 000 Sétif, Algérie

R. CHEURFA
Laboratoire de Mathématiques, Fondamentales et Numériques
Faculté des Sciences
Université de Sétif
19 000 Sétif, Algérie

Abstract. Inspired by a recent paper of Giacomini et al [3], we give the exact expression of the limit cycles for a class of two-dimensional differential sytems. We study also the uniqueness of such limit cycles. An application to Liénard equation and several examples are given at the end.

2000 *Mathematics Subject Classification*: 34C25.
Key words: Liénard equation, polynomial system, periodic solution, limit cycle, uniqueness.

1 Introduction

We consider here a class of autonomous two dimensional dynamical systems of the form

$$\begin{cases} \dot{x} = X(x, y), \\ \dot{y} = Y(x, y), \end{cases} \tag{1}$$

where (X, Y) is a vector field defined in an open subset A of $\mathbb{R}^2$. A dot over a letter denotes derivative with respect to the time variable t. When (X, Y) is a polynomial vector field, this type of dynamical systems appears frequently in several areas of science, such as biology, chemistry, astrophysics, mechanics, electronic, fluid mechanics, ecology, economy, etc (see textbooks [4,7]).

The most interesting solutions of (1) are the limit cycles, firstly defined by Poincaré in a serie of fundamental papers [8,9]. In the (x, y) plan, a limit cycle is an isolated closed path corresponding to a periodic solution, and is of course independent of the initial conditions $x(t_0)$ and $y(t_0)$. All other solutions of (1) approach or recede from the limit cycle either from its interior or from its exterior depending on the initial conditions asymptotically as $t \longrightarrow \infty$. Indoubtfully the most famous problem on limit cycles is the unsolved sixteenth problem of Hilbert [5,6] : If (X, Y) is a polynomial vector field of degree not greater than n, what is the maximum number of limit cycles, and what are their relative positions in the phase plane. In fact, even for the case $n = 2$, the problem remains not fully answered despite great efforts (hundreds of papers were devoted to the subject, see [10]). The determination of the exact expression for a limit cycle corresponding to a given planar system of the type (1), is rarely, if ever, analytically possible. In a recent paper [1], Mostafa A. Abdelkader has given an exact expression for the limit cycle in the phase plane for a relaxation oscillator modeled by the autonomous Liénard differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{2}$$

Also in a serie of papers, Giacomini, Chavarriga et al (see [2,3] and references therein) have introduced two new criteria for studying the nonexistence, existence and uniqueness of limit cycles of (1). We will use mainly their first criteria given by the following theorem :

Theorem 1 *([3]) Let (X, Y) be a C^1 vector field defined in the open subset A of $\mathbb{R}^2$, $(x(t), y(t))$ a periodic solution of (X, Y) of period T, $R : A \longrightarrow \mathbb{R}$ a C^1 map such that $\int_0^T R(x(t), y(t))dt \neq 0$ and $V = V(x, y)$ a C^1- solution of the linear partial differential equation*

$$X\frac{\partial V}{\partial x} + Y\frac{\partial V}{\partial y} = RV, \ (x, y) \in A. \tag{3}$$

Then the closed trajectory $\gamma = \{((x(t), y(t)) \in A : t \in [0, T]\}$ is contained in $\Sigma = \{(x, y) \in A : V(x, y) = 0\}$, and γ is not contained in a period annulus of (X, Y). Moreover, if the vector field (X, Y) and the functions R and V are analytic, then γ is a limit cycle.

We note that Theorem 1 shows that when we know explicitly the trajectories $V(x, y)$=0 of a vector field (X, Y) through equation (3) for a given R, we additionally have information on the periodic solutions of (X, Y), because if γ is a closed trajectory it must satisfy that either γ is contained in $\{(x,y) \in A : V(x,y) = 0\}$, or $\oint_\gamma R = 0$. Without loss of generality, we take $A = \mathbb{R}^2$ and consider the system

$$\begin{cases} \dot{x} = (y - h(x))n(x,y) + k(x)ym(x,y), \\ \dot{y} = (h'(x)y - h(x)h'(x) + k'(x))n(x,y) - (f(x)y + g(x))k(x)m(x,y), \end{cases} \tag{1}$$

where m, n are analytic functions of $\mathbb{R}^2$, h, k are real analytic functions and the real functions f and g are not specified for the moment. Here a prime stands for derivative with respect to the variable x.
According to Theorem 1, the associated partial differential equation is :

$$((y-h)n + kym)\frac{\partial V}{\partial x} + ((h'y - hh' + k')n - (fy+g)km)\frac{\partial V}{\partial y} = RV, \tag{5}$$

2 The main result

Our main result is contained in the following theorem :

Theorem 2 *Let α and β two real constants such that $\alpha < \beta$. Suppose that*
i) h and k are real analytic functions such that $k(x) > 0$ for $x \in]\alpha, \beta[$, $k(\alpha) = k(\beta) = 0$ and $k(x) < 0$ elsewhere and h is such that $h^2(x) - 2k(x) \neq 0$ everywhere on $]\alpha, \beta[$;
ii) m and n are analytic functions on $\mathbb{R}^2$ for which

$$\int_\alpha^\beta \frac{k'h}{k}\left(\frac{m(x,y_1)}{y_1 m(x,y_1) - \sqrt{2k}n(x,y_1)} - \frac{m(x,y_2)}{y_2 m(x,y_2) + \sqrt{2k}n(x,y_2)}\right)dx, \tag{6}$$

where $y_1 = h(x) - \sqrt{2k(x)}$ and $y_2 = h(x) + \sqrt{2k(x)}$, is a convergent nonvanishing integral.
Then the curve $(\gamma) : \left\{(x,y) \ : \frac{1}{2}(y - h(x))^2 - k(x) = 0\right\}$ is the unique limit cycle of the system

$$\begin{cases} \dot{x} = kmy + n(y-h), \\ \dot{y} = m((h'k + \frac{k'h}{2})y + k'(k - \frac{h^2}{2})) + n(h'(y-h) + k'). \end{cases} \tag{7}$$

To state this result, we consider system (4), let us first proof the following lemmas :

Lemma 3 *For $f = -(h' + \frac{k'h}{2k})$ and $g = k'(\frac{h^2}{2k} - 1)$, the C^1-function $V = U = \frac{1}{2}(y - h(x))^2 - k(x)$ is a solution of (5) with $R(x,y) = h(x)k'(x)m(x,y)$.*

Proof. We observe that for the smooth function $V = U = \frac{1}{2}(y-h)^2 - k$, we have $(ykm + (y-h)n)\,U_x + ((h'y - h'h + k')n - (fy+g)km)\,U_y = -2\,(f + h')\,kmU$ $+ (k' + fh + g + hh')\,kmy + \left(hg + h^2 f + h'h^2 - 2kf - 2kh'\right)km.$

If we choose $f = -(h' + \frac{k'h}{2k})$ and $g = k'(\frac{h^2}{2k} - 1)$, the function U satisfy the linear partial differential equation

$$(ykm + (y-h)n)\,U_x + ((h'y - h'h + k')n - (fy+g)km)\,U_y = RU, \tag{8}$$

with $R(x,y) = -2\,(f+h')\,km(x,y) = k'hm(x,y)$, which is a C^1- function as required.

Lemma 4 *The curve* $(\gamma) : \{(x,y) : U(x,y) = 0\}$ *is a smooth Jordan curve.*

Proof. In the (X,Y) plan, the curve $(C) : Y^2 = k(X)$ is a smooth and closed curve, since it is just the reunion of the two arcs $(C_1) : Y = \sqrt{k(X)}$ and $(C_2) : Y = -\sqrt{k(X)}$ and is completely located between α and β. At the points $(\alpha, 0)$ and $(\beta, 0)$, the tangent to (C) is parallel to the Y-axis. The curve (γ) : $\frac{1}{2}(y - h(x))^2 - k(x) = 0$ is the image of the curve (C) by the global diffeomorphism

$$\begin{cases} x = X, \\ y = \sqrt{2}Y + h(X), \end{cases} \tag{9}$$

of $\mathbb{R}^2$, hence it is smooth and closed. At the points $(\alpha, h(\alpha))$ and $(\beta, h(\beta))$, the tangent to (γ) is parallel to the y-axis.

Proof of Theorem 2. From Lemma 1 and Theorem 1, it follows that the curve $(\gamma) : U(x,y) = 0$ defines a periodic solution of (6), and obviously there are no other closed trajectories in the set $\{(x,y) \in \mathbb{R}^2 : U(x,y) = 0\}$.
(γ) is the union of the two arcs defined by : $(\gamma_1) : y_1 = h(x) + \sqrt{2k(x)}$ and $(\gamma_2) : y_2 = h(x) - \sqrt{2k(x)}$.
If T is the period of this solution, we have successfully

$$\begin{aligned}
&\int_0^T R(x(t), y(t))dt \\
&= \int_0^T m(x(t), y(t))k'(x(t))h(x(t))dt \\
&= \oint_\gamma \frac{k'(x)h(x)m(x,y)dx}{k(x)m(x,y)y + n(x,y)(y - h(x))} \\
&= \oint_{\gamma_1} \frac{k'hm(x,y)dx}{km(x,y)y + n(x,y)(y-h)} + \oint_{\gamma_2} \frac{k'hm(x,y)dx}{km(x,y)y + n(x,y)(y-h)} \\
&= \int_\alpha^\beta \frac{k'h}{k} \left(\frac{m(x,y_1)}{y_1 m(x,y_1) - \sqrt{2k}n(x,y_1)} - \frac{m(x,y_2)}{y_2 m(x,y_2) + \sqrt{2k}n(x,y_2)} \right) dx,
\end{aligned}$$

and by condition (ii), $\int_0^T R(x(t), y(t))dt \neq 0$.

According to Theorem 1 again, (γ) is the unique limit cycle of system (6).

Remark 1 The condition ii) of Theorem 2, is easily ensured here, for example for $m(x,y) = k'h(l^2(x,y) + 1)$, were l is an arbitrary analytical function.

3 Applications

3.1 Liénard-type

If we put $m(x,y) = 1$ and $n(x,y) = 1 - k(x)$ in system (7), we have the following analytic system in $\mathbb{R}^2$

$$\begin{cases} \dot{x} = y + (k-1)\,h, \\ \dot{y} = \left(\frac{1}{2}k'h + h'\right)y + k' - \frac{1}{2}k'h^2 - h'h + h'hk. \end{cases} \tag{10}$$

The associated partial differential equation is

$$(y - (1-k)\,h)\frac{\partial U}{\partial x} + \left(\left(\frac{1}{2}k'h + h'\right)y + k' - \frac{1}{2}k'h^2 - h'h + h'hk\right)\frac{\partial U}{\partial y} = k'hU. \tag{11}$$

Under the diffeomorphim

$$\begin{cases} X = x, \\ Y = y + h(x)(k(x) - 1), \end{cases} \tag{12}$$

the curve $U(x,y) = \frac{1}{2}(y-h)^2 - k = 0$ is transformed into the smooth Jordan curve $\tilde{U}(X,Y) = \frac{1}{2}(Y - k(X)h(X))^2 - k(X) = 0$.
The function $\tilde{U}$ satisfy the linear partial differential quation

$$Y\frac{\partial \tilde{U}}{\partial x} + \left(\left(h'k + \frac{3}{2}k'h\right)Y + k' - \frac{1}{2}k'kh^2\right)\frac{\partial \tilde{U}}{\partial y} = k'h\tilde{U}, \tag{13}$$

which means that $\tilde{U}(X,Y) = 0$ defines a periodic solution of the system

$$\begin{cases} \dot{X} = Y, \\ \dot{Y} = (h'k + \frac{3}{2}k'h)Y + k' - \frac{1}{2}k'kh^2. \end{cases} \tag{14}$$

This system is nothing but the Liénard-type oscillator :

$$\ddot{X} - \left(\frac{3}{2}k'h + h'k\right)\dot{X} - k'\left(1 - \frac{1}{2}h^2k\right) = 0. \tag{15}$$

Here the associated R satisfy

$$\begin{aligned} \int_0^T R(x(t),y(t))dt &= \int_0^T k'(x(t))h(x(t))dt \\ &= 2\sqrt{2}\int_\alpha^\beta \frac{k'h\sqrt{k}}{h^2 - 2k}dx. \end{aligned}$$

When the expressions of h and k are given, we can in principle compute this integral in order to ensure the condition ii) of Theorem 2, or study it at the vicinnity of α and β taking into account that $h^2 - 2k \neq 0$ between α and β and making additional hypothesis on the functions h and k. But to avoid any calculations here, let us put $h = k'$, we have

$$\int_0^T R(x(t), y(t))dt = \int_0^T k'^2(x(t))dt > 0,$$

hence the curve $\tilde{U}(x, y) = \frac{1}{2}(y - k(x)k'(x))^2 - k(x) = 0$ defines the unique limit cycle of (15). We have proved the following :

Theorem 5 *If the function k satisfies the assumptions of Theorem 2, the Liénard-type oscillator*

$$\ddot{x} - \left(\frac{3}{2}k'^2 + k''k\right)\dot{x} - k'\left(1 - \frac{1}{2}k'^2 k\right) = 0, \tag{16}$$

admits a unique limit cycle defined in the phase plan by the equation $\frac{1}{2}(y - k(x)k'(x))^2 - k(x) = 0$.

Exemple : For $k(x) = 1 - x^2$ in the above theorem, the oscillator

$$\ddot{x} - 2\left(4x^2 - 1\right)\dot{x} + 4x^5 - 3x^3 + 2x = 0 \tag{17}$$

admits the curve $\frac{1}{2}(y + 2x(1 - x^2))^2 - (1 - x^2) = 0$ as a single limit cycle enclosing the equilibrium point $(0, 0)$.

Remark 2 It is interesting to look to the converse problem :
When the equation (2) admit a unique limit cycle given by an equation of the form $\frac{1}{2}(y - h(x))^2 - k(x) = 0$?
To be more precise, if the functions f and g are subjected to the well known sufficient conditions of existence and uniqueness of the limit cycle, we can show that :

$$k(x) = \frac{1}{h^2(x)} \exp(-2\int_0^x \frac{f(s)}{h(s)}ds), \tag{18}$$

and $h(x)$ verify the very hard nonlinear ordinary differential equation

$$h'' = \frac{h^2(f + h')(g' + h'^2 + fh' + f'h) + (f'h^3 - f^2 - 3h^2h')(g + hh' + fh)}{h^3(f + h' + g + hh' + fh)}. \tag{19}$$

It is clear that this peculiar choice of the equation of the limit cycle causes serious limitations on the prescribed functions f and g and even if (19) is solved, we are not sure that the functions h and k obtained ensure the closedness.

3.2 Polynomial systems

In order to get a cubic system from (7) with a unique limit cycle, we take $m(x,y) = 1$, $n(x,y) = a_0 + a_1 x + a_2 y$, $h = ax$ and $k = b - cx^2$, where a_0, a_1, a_2, a, b and c are real constants with $b > 0$, $c > 0$ and $a \neq 0$. The system (7) reduces to :

$$\begin{cases} \dot{x} = (b - cx^2)y + (a_0 + a_1 x + a_2 y)(y - ax), \\ \dot{y} = (a(b - cx^2) - acx^2)y - 2cx(b - cx^2 - \frac{a^2x^2}{2}) \\ \qquad +(a_0 + a_1 x + a_2 y)(a(y - ax) - 2cx). \end{cases}$$

It is preferable to put the former system into the standard form

$$\begin{cases} \dot{x} = -a_0 ax + (b + a_0)y + (a_1 - a_2 a)xy - a_1 ax^2 + a_2 y^2 - cx^2 y, \\ \dot{y} = -(2bc + a_0 a^2 + 2a_0 c)x + (ab + a_0 a)y + (a_1 a - a_2 a^2 - 2a_2 c)xy \\ -(a_1 a^2 + 2a_1 c)x^2 + a_2 ay^2 - 2acx^2 y + (2c^2 + ca^2)x^3. \end{cases} \tag{20}$$

For this system,

$$\begin{aligned} \int_0^T R(x(t), y(t))dt &= \int_0^T k'(x(t))h(x(t))dt \\ &= -2ac \int_0^T x^2(t))dt \neq 0. \end{aligned}$$

Hence, the system (20) admits as a unique limit cycle the ellipse

$$(y - ax)^2 = 2(c - dx^2),$$

which encloses the equilibrium point $(0,0)$.

4 Conclusion

In this work, we have obtained the explicit expressions of the limit cycles for a wide class of analytical planar systems and shown their uniqueness easily. The techniques used are simple and can be applied to investigate other planar systems. We have recovered the results of [1] for the Liénard-type oscillators, we hope that such systems are enough general to model some dynamical systems arising from other areas of science or at least to built someone (governing the behavior of some electric or electronic circuits for example) with desirable asymptotic behaviour. They can also serve to compare the accuracy of the numerical methods generally used to approximate the shape of the limit cycle.

Acknowledgement

This work is supported by the Algerian Ministry of Higher Education and Scientific Research under project N° B*1901/02/02.

References

[1] M. A. ABDELKADER, Relaxation oscillator with exact limit cycles, *J. of Math. Anal. and Appl.* **218** (1998), 308–312.

[2] J. CHAVARRIGA, H. GIACOMINI and J. GINÉ, On a new type of limit cycles for a planar cubic systems, *Nonlin.Anal.* **36** (1999), 139–149.

[3] H. GIACOMINI, J. LIBRE and M. VIANO, On the nonexistence, existence, and uniqueness of limit cycles, *Nonlinearity* **9** (1996), 501–516.

[4] J. GUCKENHEIMER and P. HOLMES, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, New York (1983).

[5] D. HILBERT, Mathematical problems, *Bull. Amer. Math. Soc.* **8** (1902), 437–479.

[6] Y. S. ILYASHENKO, Finiteness theorems for limit cycles, *Trans. Amer. Math. Soc.* (Providence, RI: AMS, 1991).

[7] L. PERKO, Differential Equations and Dynamical Systems, Springer, Berlin, second edition (1998).

[8] H. POINCARÉ, Mémoires sur les courbes définies par une équation differentielle, I, II, *J.Math Pures Appl.* **7** (1882), 375–422, **8** (1882), 251–296, **1** (1885), 167–244, **2** (1886), 155–217.

[9] H. POINCARÉ, Mémoires sur les courbes définies par les équations differentielles, III, IV ibid. **1** (1885), 167–244, **2** (1886), 155–217.

[10] J. W. REYN, 1994 A bibliography of the qualitative theory of quadratic systems of differential equations in the plan 3rd Report of the faculty of Technical Math. and Infor. 94-02, Delft. Available online at : http://dutita0.twi.tudelft.nl/Staff/Reyn/Biblio).

[11] YE YANQIAN, Theory of Limit Cycles, Translations of Math. Monographs, Vol. 66, *Amer. Math. Soc., Providence*, 1986.

Received 27 January 2004; accepted 8 April 2004