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A Zubov's method for stochastic differential equations

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Abstract. We consider a stochastic differential equation with an asymptotically stable equilibrium point. We show that the domain of attraction of the equilibrium, i.e. the set of points which are attracted with positive probability to it, can be characterized by the solution of a suitable partial differential equation.

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1 Introduction

The study of the domain of attraction of an equilibrium point of a deterministic system has been one of the central topics in stability theory. A general approach to this problem was proposed by V.I.Zubov (see [11], [21]). The Zubov's method gives a characterization of the domain of attraction by means of a Lyapounov function solving an appropriate first order partial differential equation. This equation involves the coefficients of the system and a function to be chosen in such a way to guarantee the existence of a smooth solution.

During the last years the Zubov's method has been widely applied both in the study of theoretical properties of the domain of attraction (see [11], [13]) and in its numerical approximation (see [14] and reference therein).

In [5] the Zubov's method has been considered in the framework of the Crandall-Lions viscosity solution theory [7]. The viscosity solution approach simplifies the application of the Zubov's method and allows the extension to more general classes of deterministic systems.

Aim of this paper is to extend the Zubov's method to stochastic systems. Lyapunov's theory for stochastic systems relies on two different concepts: stability in probability and moment stability (see [1], [9], [10], [15]).

In this paper we are concerned with stability in probability. We consider a stochastic differential equation with a stable point x_0 and we require that x_0 is locally almost surely exponential stable (see [1], [10], [18]). We call *domain of attraction* of x_0 the set of points $x \in \mathbb{R}^N$ for which the trajectories of the stochastic equation starting from x are attracted to x_0 with positive probability.

Following the Zubov's idea, we show that the domain of attraction of the stochastic system coincides with the set $\{x \in \mathbb{R}^N : v(x) < 1\}$, where v is a functional of the trajectories of the system. The function v is a robust Lyapounov function for the stochastic system and it can be characterized as the unique solution of a second order partial differential equation related to the stochastic system. Since this equation is not uniformly elliptic, we consider solutions in viscosity sense ([7], [8]).

We also consider the case of perturbed stochastic systems, where we assume that the perturbation does not affect the stability of the equilibrium point. In this case we characterize the set of points which are attracted with positive probability to the equilibrium for at least one perturbation.

Characterizations of stability and viability properties of stochastic systems via weak solutions to partial differential equations can be found in [2], [3], [4], [12], [19]. A peculiar feature of the Zubov's method respect to other methods is to give an explicit representation formula for the Lyapounov function representing the domain of attraction. Moreover the characterization of Zubov function as unique solution of a partial differential equation allows the construction of numerical methods for the approximation of the domain of attraction (see [6] for the deterministic case). In a future work we hope to extend this technique to the characterization of the set of points which are attracted with probability one and to the study of moment stability.

The paper is organized as follows.

In Section 2 we introduce the basic concepts and we prove several characterizations of the domain of attraction. In Section 3 we study some properties of the domain of attraction, of its complementary and of the function v. In Section 4 we show that the function v can be characterized as the unique viscosity solution of a second order partial differential equation. Section 5 is devoted to the study of a regularity property of v and of a regularization procedure that can be useful in view of the numerical approximation. In Section 6 we sketch the extension of the previous results to perturbed systems.

2 Characterizations of the domain of attraction

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a fixed probability space with a right continuous increasing filtration and consider the autonomous stochastic differential equation

$$\begin{cases} dX(t) &= b(X(t))\,dt + \sigma(X(t))\,dW(t) \\ X(0) &= x \in \mathbb{R}^N \end{cases} \tag{2.1}$$

where $W(t)$ is an M-dimensional Wiener process adapted to $\mathcal{F}_t$ and $b : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $\sigma : \mathbb{R}^N \longrightarrow \mathbb{R}^{N\times M}$ satisfy

$$|b(x)| + |\sigma(x)| \le M \qquad \text{for any } x \in \mathbb{R}^N, \tag{2.2}$$

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \le L|x-y| \qquad \text{for any } x,\, y \in \mathbb{R}^N. \tag{2.3}$$

Under the previous assumptions there exists a unique continuous process which is a strong solution to (2.1) for any $t > 0$ (see [1], [9]). We denote the solution of (2.1) by $X(t,x)$ or by $X(t)$ when this will be not ambiguous.

We assume that the origin is an equilibrium point for (2.1), i.e.

$$b(0) = 0, \quad \sigma(0) = 0$$

and that it is *locally almost surely exponentially stable* (see [1], [10], [18]), i.e. there exist two positive constants r and λ and a finite r.v. β such that for any $x \in B(0,r)$

$$|X(t,x)| \le \beta e^{-\lambda t} \quad \text{almost surely for any } t > 0. \tag{2.4}$$

We set

$$\mathbb{C} = \{x \in \mathbb{R}^N :\ \mathbb{P}[\lim_{t\to+\infty} |X(t,x)| = 0] > 0\},$$
$$\mathcal{E} = \{x \in \mathbb{R}^N :\ \mathbb{P}[\lim_{t\to+\infty} |X(t,x)| = 0] = 0\}.$$

$\mathbb{C}$ is the set of points x for which the probability that some trajectories $X(t,x)$ is attracted to the origin is positive and it will be called the *domain of attraction* of the equilibrium point. It is immediate that $B(0,r) \subset \mathbb{C}$. Note that if $\sigma(x) \equiv 0$, i.e. the system is deterministic, $\mathbb{C}$ coincides with the domain of attraction of 0 as defined for example in [11]. $\mathcal{E}$, the set of points which have 0-probability of being attracted from the equilibrium position, will be called the *escape set*.

We start giving a characterization of $\mathbb{C}$ and $\mathcal{E}$ in terms of a stopping time of the process $X(t)$.

Proposition 2.1 *Set $t(x) = \inf\{t > 0 : \; X(t,x) \in B(0,r)\}$. Then*

$$\begin{aligned} \mathbb{C} &= \{x \in \mathbb{R}^N : \; \mathbb{P}[t(x) < +\infty] > 0\}, \\ \mathcal{E} &= \{x \in \mathbb{R}^N : \; \mathbb{P}[t(x) < +\infty] = 0\}. \end{aligned}$$

Proof. If ω is such that $\lim_{t\to+\infty} |X(t, x_0, \omega)| = 0$, clearly $t(x_0, \omega) < +\infty$. Therefore if $x_0 \in \mathbb{C}$, we have $\mathbb{P}[t(x_0) < +\infty] > 0$.

Assume now that $\mathbb{P}[t(x_0) < +\infty] > 0$. Since $|X(t(x_0), x_0)| = r$ and (2.4), we have

$$\begin{aligned} &\mathbb{P}[\{t(x_0) < +\infty\} \cap \{\lim_{t\to+\infty} |X(t,x_0)| = 0\}] \\ &\quad = \mathbb{E}[\chi_{\{t(x_0)<+\infty\}}\; \mathbb{E}[\chi_{\{\lim_{t\to+\infty} |X(t,x_0)|=0\}} | \mathcal{F}_{t(x_0)}]] \\ &\quad = \mathbb{E}[\chi_{\{t(x_0)<+\infty\}} \mathbb{E}_{X(t(x_0),x_0)}[\chi_{\{\lim_{t\to+\infty} |X(t)|=0\}}]] = \mathbb{P}\{t(x_0) < +\infty\}. \end{aligned}$$

It follows that $\mathbb{P}[\lim_{t\to+\infty} |X(t, x_0)| = 0] > 0$, hence $x_0 \in \mathbb{C}$. □

A second characterization of the set $\mathcal{E}$ and $\mathbb{C}$ involves the expectation of the stopping time $t(x)$.

Proposition 2.2

$$\mathbb{C} = \{x \in \mathbb{R}^N : \; \mathbb{E}[e^{-t(x)}] > 0\} \tag{2.5}$$

$$\mathcal{E} = \{x \in \mathbb{R}^N : \; \mathbb{E}[e^{-t(x)}] = 0\}. \tag{2.6}$$

Proof. Note that $0 \le \mathbb{E}[e^{-t(x)}] \le 1$ and

$$\mathbb{E}[e^{-t(x)}] = 0 \text{ if and only if } t(x) = +\infty \text{ a.s.} \tag{2.7}$$

Hence we immediately get (2.5) and (2.6) from (2.7) and Proposition 2.1. □

We now introduce a function whose level sets characterize the domain of attraction and the escape set. Let $v : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\begin{aligned} v(x) &= \mathbb{E}\left\{ \int_0^{+\infty} g(X(t,x)) e^{-\int_0^t g(X(s,x))ds} dt \right\} \\ &= 1 - \mathbb{E}[e^{-\int_0^{+\infty} g(X(t,x))dt}] \end{aligned} \tag{2.8}$$

where we assume that $g(0) = 0$, $g(x) > 0$ for $|x| > 0$ and

$$\begin{aligned} &|g(x)| \le M_g && \text{for any } x \in \mathbb{R}^N, \\ &|g(x) - g(y)| \le L_g |x - y| && \text{for any } x,\, y \in \mathbb{R}^N, \\ &g(x) \ge g_0 > 0 && \text{for any } x \in \mathbb{R}^N \setminus B(0,r). \end{aligned} \tag{2.9}$$

It is easy to see that $0 \le v(x) \le 1$ for any $x \in \mathbb{R}^N$, $v(0) = 0$ and $v(x) > 0$ for $|x| > 0$. Moreover, v satisfies the probabilistic formula

$$v(x) = \mathbb{E}\left\{ \int_0^{\tau\wedge T} g(X(t,x))e^{-\int_0^t g(X(s,x))ds}dt + e^{-\int_0^{\tau\wedge T} g(X(t,x))dt}\, v(X(\tau\wedge T, x))\right\} \qquad (2.10)$$

for any stopping time τ adapted to the filtration of the process and for any $T > 0$ (see for example [8], [16]).

The following theorem gives the characterization of the sets $\mathcal{E}$ and $\mathbb{C}$ by means of the 1-level set of v.

Theorem 2.3

$$\mathbb{C} = \{x \in \mathbb{R}^N : \ v(x) < 1\} \qquad (2.11)$$
$$\mathcal{E} = \{x \in \mathbb{R}^N : \ v(x) = 1\}. \qquad (2.12)$$

Proof. If $x \in \mathcal{E}$, from Prop. 2.2, $\mathbb{E}[e^{-t(x)}] = 0$, hence $\mathbb{E}[e^{-g_0 t(x)}] = 0$. Since $g(X(t,x)) \ge g_0$ a.s. if $t < t(x)$, we get

$$\begin{aligned} v(x) &\ge 1 - \mathbb{E}[e^{-\int_0^{t(x)} g(X(t,x))dt}] \\ &\ge 1 - \mathbb{E}[e^{-g_0 t(x)}] = 1. \end{aligned}$$

If $x \in \mathbb{C}$, then $\mathbb{P}[t(x) < +\infty] > 0$. Take T and K sufficiently large in such a way $\mathbb{P}[\Delta] := \mathbb{P}\left(\{t(x) \le T\} \cap \{\beta \le K\}\right) \ge \eta$ for some $\eta \in (0,1)$. For $t > T$, we have

$$\begin{aligned} \mathbb{E}[\mathbb{E}[|X(t,x)|\big|\Delta]\ \chi_\Delta] &= \mathbb{E}[\mathbb{E}[|X(t-t(x), X(t(x),x))||\Delta]\chi_\Delta] \\ &\le \ Ke^{-\lambda(t-T)}. \end{aligned}$$

Then

$$\begin{aligned} v(x) &\le 1 - \mathbb{E}[\mathbb{E}[e^{-\int_0^T g(X(t,x))dt + \int_T^{+\infty} g(X(t,x))dt}|\Delta]\chi_\Delta] \\ &\le 1 - e^{-(M_g T + L_g K/\lambda)} < 1. \end{aligned}$$

□

3 Properties of the domain of attraction

In this section we will study some properties of the sets $\mathcal{E}$ and $\mathbb{C}$ and of the function v.

Proposition 3.1 i) *The set $\mathbb{C}$ is open, connected and contains $B(0,r)$ as a proper subset.*

ii) *The set $\mathcal{E}$ is closed and invariant for the process, i.e. if $x \in \mathcal{E}$ then $X(t,x) \in \mathcal{E}$ a.s. for any $t > 0$.*

Proof. Recall that for any x, $y \in \mathbb{R}^N$ and $\varepsilon > 0$

$$\lim_{|x-y|\to 0} \mathbb{P}\big[\sup_{t\in[0,T]} |X(t,x) - X(t,y)| > \varepsilon\big] = 0. \tag{3.1}$$

We first prove that $B(0,r)$ is a proper subset of $\mathbb{C}$. Fix $x_0 \in B(0,r)$ and take K be such that $\mathbb{P}[\Delta] := \mathbb{P}[\beta \le K] \ge \eta$ for some $\eta \in (0,1)$. From (2.4), there exists $T > 0$ such that for $\omega \in \Delta$ and $t \ge T$

$$|X(t,x_0)| \le r/2. \tag{3.2}$$

From (3.1), select δ such that for any $t \le T$ and for any $x \in B(x_0,\delta)$,

$$P[A_x] := P[|X(t,x) - X(t,x_0)| > r/2] \le \eta/2.$$

If $\omega \in B := A_x^c \cap \Delta$,

$$|X(T,x)| \le |X(T,x) - X(T,x_0)| + |X(T,x_0)| \le r$$

and therefore from (2.4)

$$\begin{aligned}
&\mathbb{P}[\{\lim_{t\to+\infty} |X(t,x)| = 0\} \cap B] \\
&\quad = \mathbb{E}[\mathbb{E}[\chi_{\{\lim_{t\to+\infty}|X(t,x)|=0\}}\big|B]\ \chi_B] \\
&\quad = \mathbb{E}\big[\mathbb{E}_{X(T,x)}\big[\chi_{\{\lim_{t\to+\infty}|X(t)|=0\}}\big]\ \chi_B\big] = \mathbb{P}[B].
\end{aligned}$$

Moreover

$$\mathbb{P}[B] = 1 - \mathbb{P}[A_x \cup \Delta^c] \ge 1 - (\mathbb{P}[A_x] + \mathbb{P}[\Delta^c]) \ge \frac{\eta}{2}.$$

It follows that $\mathbb{P}[\{\lim_{t\to+\infty} |X(t,x)| = 0\}]$ is positive for any $x \in B(x_0,\delta)$ and therefore $B(x_0,\delta) \subset \mathbb{C}$. The statement then follows by a compactness argument.

To prove that $\mathbb{C}$ is open, we can repeat a similar argument. In fact, if $x_0 \in \mathbb{C}$, then $\mathbb{P}[\lim_{t\to+\infty} |X(t,x_0)| = 0] > 0$ and therefore we can find Δ and $T > 0$ such that $\mathbb{P}[\Delta] > 0$ and if $\omega \in \Delta$ then

$$|X(t,x_0)| \le r/2 \qquad \text{for any } t \ge T.$$

From this point we can follow the same argument after (3.2).

The set $\mathbb{C}$ is connected since for any $x \in \mathbb{C}$ there exists a continuous path joining x to the origin.

$\mathcal{E}$ is closed since it is the complementary of an open set. To prove that $\mathcal{E}$ is invariant, we make use of its characterization by means of the function v. Assume by contradiction that there exists $x_0 \in \mathcal{E}$ and $t_0 > 0$ for which

$\mathbb{P}[X(t_0,x_0) \in \mathbb{C}] > 0$. Since $\mathbb{C} \setminus \{0\} = \cup_{n=1}^{\infty}\{x \in \mathbb{R}^N : 1/(n+1) \le v(x) < 1/n\}$, there exists n_0 such that $\mathbb{P}[v(X(t_0,x_0)) \le 1/n_0] > 0$. Set $\Gamma = \{x \in \mathbb{R}^N : v(x) \le 1/n_0\}$. Then, from (2.10)

$$\begin{aligned}
v(x_0) &= \mathbb{E}\Bigg\{\int_0^{t_0} g(X(t))e^{-\int_0^t g(X(s))ds}dt + e^{-\int_0^{t_0} g(X(t))dt}v(X(t_0))\chi_{\Gamma^c} \\
&\qquad + e^{-\int_0^{t_0} g(X(t))dt}v(X(t_0))\chi_{\Gamma}\Bigg\} \\
&\le \mathbb{E}\Bigg\{\int_0^{t_0} g(X(t))e^{-\int_0^t g(X(s))ds}dt +^{-\int_0^{t_0} g(X(t))dt}\,\chi_{\Gamma^c} \\
&\qquad + \frac{1}{n_0}e^{-\int_0^{t_0} g(X(t))dt}\chi_{\Gamma}\Bigg\} \\
&= 1 - \left(1 - \frac{1}{n_0}\right)\mathbb{E}\{e^{-\int_0^{t_0} g(X(t))dt}\chi_{\Gamma}\} < 1
\end{aligned}$$

and therefore a contradiction since $v(x_0) = 1$. □

Remark 3.2 A worthwhile consequence of the invariance of the set $\mathcal{E}$ is that the diffusion has to degenerate on its boundary. In fact the invariance of $\mathcal{E}$ is equivalent (see [3], [12]) to the following condition

$$\frac{1}{2}tr(\sigma(x)\sigma^t(x)X) + b(x)p \ge 0, \qquad \text{if } x \in \partial\mathcal{E} \text{ and } (p,X) \in \mathbb{N}^2_{\mathcal{E}}(x)$$

where $\mathbb{N}^2_{\mathcal{E}}(x) = \{(p,X) \in \mathbb{R}^N \times S^N : \mathcal{E} \ni y \to x, p(y-x) + \frac{1}{2}(y-x)X(y-x) \ge o(|x-y|^2)\}$ is the second order normal cone to $\mathcal{E}$ at x.

If $\mathcal{E}$ is smooth at $x \in \partial\mathcal{E}$, the previous condition reduces to

$$\begin{aligned}
&\sigma(x)\overrightarrow{n}(x) = 0, \\
&\frac{1}{2}S(P(\sigma(x)) \ge b(x)\overrightarrow{n}(x)
\end{aligned}$$

(where $\overrightarrow{n}(x)$ is the exterior normal, P the orthogonal projection on the tangent space, S the second order fundamental form of $\partial\mathcal{E}$ at x), i.e. the diffusion is tangential to $\partial\mathcal{E}$.

The next step is to prove the continuity of the function v in $\mathbb{R}^N$. For this, we need a preliminary lemma.

Lemma 3.3 *If $x_n \in \mathbb{C}$ and $x_n \to x_0 \in \partial\mathbb{C}$, then*

$$\mathbb{E}[e^{-t(x_n)}] \to 0 \qquad \textit{for } n \to +\infty. \tag{3.3}$$

Proof. If we show that $t(x_n) \to +\infty$ a.s., then we immediately get the assert from the dominated convergence's theorem.

Assume by contradiction that $\mathbb{P}[\lim_{n\to\infty} t(x_n) = +\infty] < 1$. Hence there exists $T_0 > 0$ such that $\lim_{m\to+\infty} \mathbb{P}[\cap_{n>m}\{t(x_n) > T_0\}] < 1$ and therefore there exists $\varepsilon > 0$ such that for any m, we can find $n_m > m$ for which $\mathbb{P}[t(x_{n_m}) \le T_0] \ge \varepsilon$.

Let K be such that $\mathbb{P}[\Delta] := \mathbb{P}[\{t(x_{n_m}) \le T_0\} \cap \{\beta \le K\}] \ge \varepsilon/2$ and δ be such that if $|x - x_0| < \delta$, then

$$P[A_x] := P\big[\sup_{[0,T]} |X(t,x) - X(t,x_0)| > r/2\big] \le \varepsilon/4.$$

From (2.4), we can find $T_1 > T_0$ such that if $\omega \in \Delta$ then $|X(t, x_{n_m})| \le r/2$ for $t \ge T_1$.

Taking m sufficiently large in such a way that $|x_{n_m} - x_0| < \delta$, for $\omega \in A^c_{x_{n_m}} \cap \Delta$ we get

$$|X(T_1, x_0)| \le |X(T_1, x_{n_m}) - X(T_1, x_0)| + |X(T_1, x_{n_m})| \le r.$$

Hence $|X(t, x_0)| \longrightarrow 0$ for $t \longrightarrow +\infty$ and

$$\mathbb{P}[A^c_{x_{n_m}} \cap \Delta] \ge 1 - (\mathbb{P}[A_{x_{n_m}}] + \mathbb{P}[\Delta^c]) \ge \frac{\varepsilon}{4}.$$

It follows that $\mathbb{P}[\lim_{t\to+\infty} |X(t, x_0)| = 0] > 0$ and therefore $x_0 \in \mathbb{C}$. This gives a contradiction since $x_0 \in \partial\mathbb{C}$ and $\mathbb{C}$ is open. □

Proposition 3.4 *v is continuous in $\mathbb{R}^N$.*

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Proof. The function v is continuous in the interior of $\mathcal{E}$ since it is constant. We are going to show that

i) $v(x_n) \to 1$ if $x_n \in \mathbb{C}$ and $x_n \to x_0 \in \partial\mathbb{C}$ for $n \to +\infty$;

ii) v is continuous in $B(0, r)$ where r is given as in (2.4).

If we prove i) and ii) then the continuity of v in $\mathbb{C}\setminus B(0, r/2)$ and therefore in all $\mathbb{R}^N$ can be proved as in Theorem II.2 in [16, part I], since $g(x) > 0$ in $\mathbb{R}^N \setminus B(0, r/2)$ and i), ii).

To prove i), from Lemma 3.3 we have $\mathbb{E}[e^{-g_0 t(x_n)}] \to 0$ for $n \to +\infty$ and therefore

$$\begin{aligned} v(x_n) &\ge 1 - \mathbb{E}[e^{-\int_0^{t(x_n)} g(X(t,x))dt}] \\ &\ge 1 - \mathbb{E}[e^{-g_0 t(x_n)}] \to 1 \qquad \text{for } n \to +\infty. \end{aligned}$$

Now fix $x_0 \in B(0, r)$ and $\varepsilon > 0$. Let K be such that $\mathbb{P}[\Delta] := P[\{\beta \le K\}] \ge 1-\varepsilon/8$. Take T such that $L_g K e^{-\lambda T}/\lambda < \varepsilon/4$ and δ sufficiently small in such a way that $\mathbb{E}|X(t, x_0) - X(t, x)| \le \varepsilon/4L_g T$ if $x \in B(0, r) \cap B(x_0, \delta)$ and $t \le T$.

For any $x \in B(0, r)$, we have

$$\mathbb{E}\left[\int_T^\infty |X(t,x)|dt\, \chi_\Delta\right] \le \mathbb{E}\left[\int_0^\infty |X(t+T,x)|dt\, \chi_\Delta\right] \le Ke^{-\lambda T}/\lambda.$$

Hence if $x \in B(0,r) \cap B(x_0, \delta)$, we get

$$\begin{aligned} |v(x) - v(x_0)| &\le \mathbb{E}|e^{-\int_0^{+\infty} g(X(t,x))dt} - e^{-\int_0^{+\infty} g(X(t,x_0))dt}| \\ &\le 2\mathbb{P}(\Delta^c) + \mathbb{E}\Bigg[L_g\Bigg(\int_0^T |X(t,x_0) - X(t,x)|dt + \int_T^\infty |X(t,x)|dt \\ &\qquad + \int_T^\infty |X(t,x_0)|dt\Bigg)\chi_\Delta\Bigg] \le \varepsilon. \end{aligned}$$

□

Theorem 3.5 *The function v is a Lyapunov function for the stochastic system* (2.1)*, i.e.* (*see* [*1*], [*10*])

$$\mathbb{E}[v(X(t,x_0)) - v(x_0)] < 0$$

for any $x_0 \in \mathbb{C} \setminus \{0\}$ and any $t > 0$.

Proof. v is continuous and from (2.10) it follows that

$$\mathbb{E}[v(X(t,x_0)) - v(x_0)] \le \mathbb{E}\left[(1 - e^{-\int_0^t g(X(s))ds})(v(X(t,x_0)) - 1)\right] < 0.$$

□

In Section 5 we will show that v can be made globally Lipschitz with an appropriate choice of g.

4 A characterization of v

In this section we will show that the function v can be characterized by a second order partial differential equation. Since this equation is not uniformly elliptic, in general it does not admit smooth solutions. Therefore we will use the concept of viscosity solution introduced for the second order case in [16] (for more details about this theory we refer to [7], [8]).

Definition 4.1 Given an open subset $\mathcal{O}$ of $\mathbb{R}^N$ and a continuous function $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R}$, we say that a lower semicontinuous function $w : \mathcal{O} \to \mathbb{R}$ (resp. an upper semicontinuous function $u : \mathcal{O} \to \mathbb{R}$) is a viscosity supersolution (resp. subsolution) of the equation

$$F(x, u, Du, D^2u) = 0 \qquad x \in \mathcal{O} \tag{4.1}$$

if for all $\phi \in C^2(\mathcal{O})$ and $x \in \mathrm{argmin}_{\mathcal{O}}(w-\phi)$ (resp., $x \in \mathrm{argmax}_{\mathcal{O}}(u-\phi)$) we have

$$F(x, w(x), D\phi(x), D^2\phi(x)) \geq 0$$
$$\big(\text{resp., } F(x, u(x), D\phi(x), D^2\phi(x)) \leq 0\big).$$

A continuous function $u : \mathcal{O} \to \mathbb{R}$ is said to be a viscosity solution of (4.1) in $\mathcal{O}$ if u is a viscosity supersolution and a viscosity subsolution of (4.1) in $\mathcal{O}$.

Set $a(x) = \sigma(x)\sigma^t(x)$. As consequence of the probabilistic formula (2.10) we get the following result (for the proof see f.e. [8], [16]).

Proposition 4.2 *The function v is a viscosity solution in $\mathbb{R}^N$ of*

$$-\frac{1}{2}Tr(a(x)D^2v(x)) - b(x)Dv(x) - (1-v(x))g(x) = 0. \tag{4.2}$$

Since $g(x) = 0$ for $x = 0$ we cannot apply standard comparison theorems in viscosity solution theory, which require the coefficient of zero-th order term v to be strictly positive. Instead we will make use of lower and upper optimality principles satisfied by sub and supersolutions (see [16], [17]).

Proposition 4.3 *Let w, u be a viscosity supersolution, respectively a viscosity subsolution, of* (4.2) *in an open set $\mathcal{O} \subset \mathbb{R}^N$ not containing* 0. *Set $\mathcal{O}_\delta = \{x \in \mathcal{O} : d(x, \partial\mathcal{O}) > \delta,\ |x| < 1/\delta\}$. Then for any $x \in\ \mathcal{O}_\delta$ and for any $T > 0$*

$$\begin{aligned} w(x)\ \geq \mathbb{E}\Bigg\{ &\int_0^{T\wedge\tau_\delta} g(X(t,x))e^{-\int_0^t g(X(s,x))ds}dt \\ &+e^{-\int_0^{T\wedge\tau_\delta} g(X(t,x))dt}w(X(T\wedge\tau_\delta, x))\Bigg\} \end{aligned}$$

and respectively

$$\begin{aligned} u(x)\ \leq \mathbb{E}\Bigg\{ &\int_0^{T\wedge\tau_\delta} g(X(t,x))e^{-\int_0^t g(X(s,x))ds}dt \\ &+e^{-\int_0^{T\wedge\tau_\delta} g(X(t,x))dt}u(X(T\wedge\tau_\delta, x))\Bigg\} \end{aligned}$$

where τ_δ is the first exit time of the process $X(t)$ from $\mathcal{O}_\delta$.

The proof of previous proposition can be found in [16]. It is done for continuous viscosity sub and supersolutions but it can be easily adapted to the semicontinuous case.

We can now prove a comparison principle for equation (4.2).

Theorem 4.4 *Let u be a bounded subsolution of* (4.2) *on $\mathbb{R}^N$ such that $u(0) \le 0$, w a bounded supersolution of* (4.2) *on $\mathbb{R}^N$ such that $w(0) \ge 0$. Then $u \le w$ in $\mathbb{R}^N$.*

Proof. Set $\mathcal{O}_\delta = \{x \in \mathbb{R}^N : \ \delta < |x| < 1/\delta\}$, let τ_δ be the first exit time of $X(t)$ from $\mathcal{O}_\delta$ and $\varepsilon_\delta > 0$ be such that $g(x) \ge \varepsilon_\delta$ for $x \in \mathcal{O}_\delta$.

Consider first $x \in B(0,r)$ and set $\Delta_n := \{\beta \le n\}$. Applying the sub and superoptimality principles in Ω_δ with $\delta < |x|$, we have

$$\begin{aligned} u(x) - w(x) \ &\le \mathbb{E}[e^{-\int_0^{\tau_\delta} g(X(t,x))dt}(u(X(\tau_\delta, x)) \\ &-w(X(\tau_\delta,x)))\chi_{\{\tau_\delta \le T\}}] + 2Me^{-\varepsilon_\delta T} \end{aligned} \tag{4.3}$$

for any $T > 0$ where M is a bound for u and w in $\mathbb{R}^N$.

If $\omega \in \Delta_n$, then for any δ sufficiently small we have

$$|X(\tau_\delta, x)|\chi_{\{\tau_\delta \le T\}} = \delta$$

a.s. for any $T > 0$. Sending $T \to +\infty$ in (4.3), we get

$$u(x) - w(x) \le \sup_{|y|=\delta}\{u(y) - w(y)\}\mathbb{P}(\Delta_n) + 2M\mathbb{P}(\Delta_n^c)$$

for any δ sufficiently small. Since β is a finite r.v., $\mathbb{P}(\Delta_n^c) \longrightarrow 0$ for $n \longrightarrow +\infty$. Therefore sending $\delta \longrightarrow 0$ and then $n \longrightarrow +\infty$ in the previous inequality and recalling that $u - w$ is u.s.c. and $u(0) - w(0) \le 0$, we finally get $u(x) \le w(x)$ for $x \in B(0,r)$.

Since $u(x) \le w(x)$ on $|x| = r$ and $g(x) \ge g_0 > 0$ on $\mathbb{R}^N \setminus B(0,r)$, applying a standard comparison theorem (see for example [7], [8]), we get $u \le w$ in $\mathbb{R}^N$ and therefore the statement. □

Corollary 4.5 *Let u be a bounded continuous solution of* (4.2) *on $\mathbb{R}^N$ with $u(0) = 0$. Then $u = v$, for v defined in* (2.8)*, and $\mathbb{C} = \{x \in \mathbb{R}^N : u(x) < 1\}$.*

5 Regularity and regularization

We start giving a regularity result for v. Recall that from Ito's formula, we have

$$\mathbb{E}[|X(t,x) - X(t,y)|^2 e^{-2L_0\theta}] \le |x-y|^2 \tag{5.1}$$

for any x, $y \in \mathbb{R}^N$ and for any finite stopping time θ, where L_0 is a constant depending on the constant L in (2.3).

Proposition 5.1 *Besides assumptions* (2.2)–(2.3) *and* (2.9), *we also assume that*

i) β *is bounded, i.e. there exists* $K > 0$ *such that*

$$\beta(\omega) \le K \quad a.s. \tag{5.2}$$

where β *is as in* (2.4)*;*

ii) *there exists* $\delta > 0$ *such that*

$$|g(x) - g(y)| \le C \max\{|x| \ , |y|\}^q |x - y| \tag{5.3}$$

for any x, $y \in B(0, \delta)$, *where* $q > L_0/\lambda$ *with* L_0 *as in* (5.1) *and* λ *as in* (2.4).

Then v *is Lipschitz continuous in* $\mathbb{R}^N$*.*

Proof. We first prove that v is Lipschitz continuous in $B(0, r)$.

From (5.2), for any $x \in B(0, r)$

$$|X(t, x)| \le K e^{-\lambda t} \quad \text{a.s. for any } t > 0. \tag{5.4}$$

and therefore there exists $T > 0$ such that $X(t, x) \in B(0, r)$ a.s. for any $t > T$. If x, $y \in B(0, r)$, then

$$\begin{aligned} |v(x) - v(y)| &\le \mathbb{E}|e^{-\int_0^{+\infty} g(X(t,x))dt} - e^{-\int_0^{+\infty} g(X(t,y))dt}| \\ &\le L_g \mathbb{E}\left[\int_0^T |X(t, x) - X(t, y)|dt\right] \\ &\quad + C\mathbb{E}\left[\int_T^{+\infty} \max\{|X(t, x)| \ , |X(t, y)|\}^q |X(t, x) - X(t, y)|dt\right]. \end{aligned}$$

Now applying (5.1), (5.4) in the previous inequality we get

$$\begin{aligned} |v(x) - v(y)| \le\ & L_g|x - y| \textstyle\int_0^T e^{L_0 T} dt + CK|x - y| \int_T^\infty e^{(L_0-\lambda q)t} dt \\ :=\ & L_{B(0,r)}|x - y|. \end{aligned}$$

Now we prove the Lipschitz continuity in all $\mathbb{R}^N$. Define

$$\begin{aligned} \tau(x) &= \inf\{t > 0 : X(t, x) \in B(0, r/2)\}, \\ g_1 &= \inf_{\mathbb{R}^N \setminus B(0,r/2)} g(x) > 0. \end{aligned}$$

and for x, $y \in \mathbb{R}^N$ and $T > 0$ set $\theta = \tau(x) \wedge \tau(y)$ and $B_T = \{\omega \in \Omega : \ |X(\theta \wedge T, x) - X(\theta \wedge T, y)| > r/2\}$. We have

$$\begin{aligned} P[B_T] &\le \frac{2}{r}\mathbb{E}|X(\theta \wedge T, x) - X(\theta \wedge T, y)| \\ &\le \frac{2}{r}\mathbb{E}[|X(\theta \wedge T, x) - X(\theta \wedge T, y)|^2 e^{-2L_0\theta\wedge T}]^{1/2}\mathbb{E}[e^{2L_0\theta\wedge T}]^{1/2} \\ &\le \frac{2}{r}|x - y|e^{L_0 T}. \end{aligned}$$

Applying (2.10), we get

$$\begin{aligned}
|v(x)-v(y)| &\le \mathbb{E}|e^{-\int_0^{\theta\wedge T} g(X(t,x))dt} - e^{-\int_0^{\theta\wedge T} g(X(t,y))dt}| \\
&\quad + \mathbb{E}|e^{-\int_0^{\theta\wedge T} g(X(t,x))dt} v(X(\theta\wedge T,x)) - e^{-\int_0^{\theta\wedge T} g(X(t,y))dt} v(X(\theta\wedge T,y))| \\
&\le 2L_g\mathbb{E}\Bigg\{\int_0^{\theta\wedge T} |X(t,x)-X(t,y)|e^{-g_1 t}dt + 2e^{-g_1\theta\wedge T}\chi_{B_T} \\
&\quad + L_{B(0,r)}e^{-g_1\theta\wedge T}|X(\theta\wedge T,x)-X(\theta\wedge T,y)|\chi_{B_T^c}\Bigg\} \\
&\le 2L_g|x-y|\int_0^T e^{(L_0-g_1)t}dt + \frac{4}{r}|x-y|e^{L_0 T} \\
&\quad + L_{B(0,r)}|x-y|\min\{e^{(L_0-g_1)T},1\} := L(T)|x-y|
\end{aligned}$$

and we can choose T in such a way that the $L(T)$ is minimal. □

Especially in view of its numerical approximation, it is useful to consider a regularization of (4.2). In fact, as explained in [6] for the deterministic case (but the same drawback can happen also in the stochastic case), since $g(0)=0$ the discrete problem obtained by a direct approximation of equation (4.2) can display instability phenomena and also fails to converge to the correct solution.

Let us define

$$v_\varepsilon(x) = \mathbb{E}\left\{\int_0^{+\infty} g(X(t,x))e^{-\int_0^t g_\varepsilon(X(s,x))ds}dt\right\}$$

where $g_\varepsilon(x) = g(x)\vee\varepsilon$.

Theorem 5.2 *The following statements hold for any $\varepsilon \le g_0$ with g_0 given as in* (2.9)

i) $0\le v_\varepsilon\le 1$, $v_\varepsilon(0)=0$, $v_\varepsilon(x)>0$ *for* $x\in\mathbb{R}^N\setminus\{0\}$ *and* $v_\varepsilon\le v$ *for any* $x\in\mathbb{R}^N$. *Moreover* v_ε *is continuous in* $\mathbb{R}^N$.

ii) $\mathbb{C}=\{x\in\mathbb{R}^N:\ v_\varepsilon(x)<1\}$ *and* $\mathcal{E}=\{x\in\mathbb{R}^N:\ v_\varepsilon(x)=1\}$.

iii) v_ε *converges to* v *uniformly on* $\mathbb{R}^N$.

iv) v_ε *is the unique bounded viscosity solution in* $\mathbb{R}^N$ *of*

$$-\frac{1}{2}\mathrm{Tr}(a(x)D^2v(x)) - b(x)Dv(x) - g_\varepsilon(x) + v(x)g_\varepsilon(x) = 0. \tag{5.5}$$

Proof. The first part of i) follows immediately from the definition of v_ε, while the continuity of v_ε can be proved as in Proposition 3.4.

Since $v_\varepsilon\le v$, we have $\{x\in\mathbb{R}^N:\ v_\varepsilon(x)=1\}\subset\mathcal{E}$. Moreover if $x\in\mathcal{E}$ then $t(x)=+\infty$ a.s. and therefore $|X(t,x)|>r$ a.s. for any $t>0$. It follows that $g_\varepsilon(X(t,x)) = g(X(t,x))$ a.s. for any $t > 0$ and therefore $v_\varepsilon(x) = v(x)$. This implies $v_\varepsilon(x) = 1$ for $x \in \mathcal{E}$ and therefore ii).

Since v is continuous and $v(x) > 0$ for $|x| \neq 0$, for any $\eta > 0$ there exists $\varepsilon > 0$ such that

$$\{x \in \mathbb{R}^N : \ g_\varepsilon(x) \geq g(x)\} \subset \{x \in \mathbb{R}^N : \ v(x) \leq \eta\}.$$

Set $\Gamma_\eta = \{x \in \mathbb{R}^N : v(x) \leq \eta\}$ and define the stopping time $\tau_\eta = \inf\{t > 0 : X(t,x) \in \Gamma_\eta\}$. If $x \notin \Gamma_\eta$, we have for any $t > 0$

$$\begin{aligned} v(x) - v_\varepsilon(x) &= \mathbb{E}\{e^{-\int_0^{\tau_\eta \wedge t} g(X(t,x))dt}(v(X(\tau_\eta \wedge t, x) - v_\varepsilon(X(\tau_\eta \wedge t, x))\} \\ &\leq \mathbb{E}\{e^{-\int_0^{\tau_\eta} g(X(t,x))dt} v(X(\tau_\eta, x))\chi_{\{\tau_\eta \leq t\}} + 2e^{-t\varepsilon}\chi_{\{\tau_\eta > t\}}\} \end{aligned}$$

Sending $t \to +\infty$, we get $v(x) - v_\varepsilon(x) \leq \eta$ if $x \notin \Gamma_\eta$, while if $x \in \Gamma_\eta$ we have immediately $v(x) - v_\varepsilon(x) \leq 2\eta$. Since η is arbitrary, we get the assert.

Finally iv) follows from a standard comparison theorem (see for example [7]) since $g_\varepsilon(x) \geq \varepsilon > 0$ for any $x \in \mathbb{R}^N$. □

Note that, even if the characterization of $\mathbb{C}$ is preserved by the approximating problem (5.5), respect to the Zubov's method which always gives a Lyapounov function for the stochastic system (see Theorem 3.5), v_ε does not in general enjoy of this property.

6 Domain of attraction for controlled diffusions

In this section we will consider the case of a controlled stochastic differential equation. The control can be interpreted either as an additional perturbation to the system or a way of the controller to contrast the stochastic perturbation. Since many of the proof are similar to the uncontrolled case, we will only sketch them.

Consider the controlled stochastic differential equation

$$\begin{cases} dX(t) &= b(X(t), \alpha(t))\, dt + \sigma(X(t), \alpha(t))\, dW(t) \\ X(0) &= x \end{cases} \tag{6.1}$$

where $\alpha(t)$, the control applied to the system, is a progressively measurable process having values in a compact metric space A. We denote with $\mathcal{A}$ the set of the control laws $\alpha(t)$.

The functions $b : \mathbb{R}^N \times A \longrightarrow \mathbb{R}^N$, $\sigma : \mathbb{R}^N \times A \longrightarrow \mathbb{R}^{N \times M}$ are continuous and satisfy assumptions (2.2), (2.3) uniformly with respect to $a \in A$. We assume that

$$b(0, a) = 0, \quad \sigma(0, a) = 0 \qquad \text{for any } a \in A,$$

i.e. 0 is an equilibrium point of $X(t,\alpha)$ for any $\alpha(t) \in \mathcal{A}$. Moreover there exist r and λ positive and a finite r.v. $\beta(\omega)$ such that for any $x \in B(0,r)$ and $\alpha \in \mathcal{A}$

$$|X(t,x,\alpha)| \leq \beta e^{-\lambda t} \quad \text{a.s. for any } t > 0.$$

We set

$$\mathbb{C} = \{x \in \mathbb{R}^N : \text{there exists } \alpha \in \mathcal{A} \text{ s.t. } \mathbb{P}[\lim_{t\to+\infty} |X(t,x,\alpha)| = 0] > 0\}$$
$$\mathcal{E} = \{x \in \mathbb{R}^N : \text{ for any } \alpha \in \mathcal{A},\, \mathbb{P}[\lim_{t\to+\infty} |X(t,x,\alpha)| = 0] = 0\}.$$

Similarly to the uncontrolled case, we define a function $v : \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$\begin{aligned} v(x) &= \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left\{\int_0^{+\infty} g(X(t),\alpha(t))e^{-\int_0^t g(X(s),\alpha(s))ds}dt\right\} \\ &= \inf_{\alpha\in\mathcal{A}} \{1 - \mathbb{E}[e^{-\int_0^{+\infty} g(X(t),\alpha(t))dt}]\} \end{aligned} \tag{6.2}$$

where $g : \mathbb{R}^N \times A \longrightarrow \mathbb{R}$ is continuous and satisfy the same assumptions of the uncontrolled case, uniformly in $a \in A$ (in particular $\inf_{(\mathbb{R}^N\setminus B(0,r))\times A} g(x,a) \geq g_0 > 0$). Note that now (2.10) becomes the dynamic programming principle

$$\begin{aligned} v(x) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\Bigg\{ &\int_0^{\tau\wedge T} g(X(t),\alpha(t))e^{-\int_0^t g(X(s),\alpha(s))ds}dt \\ &+ e^{-\int_0^{\tau\wedge T} g(X(t),\alpha(t))dt} v(X(\tau\wedge T))\Bigg\}. \end{aligned}$$

Theorem 6.1

$$\mathbb{C} = \{x \in \mathbb{R}^N : \ v(x) < 1\}$$
$$\mathcal{E} = \{x \in \mathbb{R}^N : \ v(x) = 1\}.$$

Proof. Set $t(x,\alpha) = \inf\{t > 0 : \ X(t,x,\alpha) \in B(0,r)\}$. Repeating the same argument of Propositions 2.1 and 2.2, it is easy to see that the following characterizations hold

$$\begin{aligned} \mathbb{C} &= \{x \in \mathbb{R}^N : \text{ there exists } \alpha \in \mathcal{A} \text{ s.t. } \quad \mathbb{P}[t(x,\alpha) < +\infty] > 0\} \\ &= \{x \in \mathbb{R}^N : \text{ there exists } \alpha \in \mathcal{A} \text{ s.t. } \quad \mathbb{E}[e^{-t(x,\alpha)}] > 0\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{E} &= \{x \in \mathbb{R}^N : \text{ for any } \alpha \in \mathcal{A} \ \ \mathbb{P}[t(x,\alpha) < +\infty] = 0\} \\ &= \{x \in \mathbb{R}^N : \text{ for any } \alpha \in \mathcal{A} \ \ \mathbb{E}[e^{-t(x,\alpha)}] = 0\} \end{aligned}$$

If $x \in \mathcal{E}$, for any control $\alpha(t)$ we have $\mathbb{E}[e^{-t(x,\alpha)}] = 0$. Then

$$1 - \mathbb{E}[e^{-\int_0^{t(x,\alpha)} g(X(t),\alpha(t))dt}] \geq 1 - \mathbb{E}[e^{-g_0 t(x,\alpha)}] = 1.$$

and therefore $v(x) = \inf_{\alpha \in \mathcal{A}}\{1 - \mathbb{E}[e^{-\int_0^{+\infty} g(X(t),\alpha(t))dt}]\} = 1$.

If $x \in \mathbb{C}$, then there exists $\overline{\alpha}$ such that $\mathbb{P}[t(x,\overline{\alpha}) < +\infty] > 0$. As in Theorem 2.3, we get

$$v(x) \leq 1 - \mathbb{E}[e^{-\int_0^T g(X(t),\overline{\alpha}(t))dt + \int_T^{+\infty} g(X(t),\overline{\alpha}(t))dt}] < 1.$$

□

The analogous of Propositions 3.1 and 3.4 can be proved for the perturbed system (6.1). Note that in this case the set $\mathcal{E}$ is invariant for the process $X(t, x, \alpha)$ for any $\alpha \in \mathcal{A}$.

Moreover v is a viscosity solution of the Hamilton-Jacobi-Bellman equation

$$\sup_{a \in A}\left\{-\frac{1}{2}Tr\left(\Sigma(x,a)D^2 v(x)\right) - b(x)Dv(x) - (1 - v(x))g(x)\right\} = 0, \qquad (6.3)$$

for $x \in \mathbb{R}^N$, where $\Sigma(x, a) = \sigma(x, a)\sigma^t(x, a)$ for any $a \in A$ (see [8], [16]). Applying the controlled version of the optimality principles of Prop 4.3, we can prove the analogous of Theorem 4.4 and consequently we get the following characterization of $\mathcal{C}$ and v.

Theorem 6.2 *Let u be a bounded continuous nonnegative solution of* (6.3) *with $u(0) = 0$. Then $u = v$, for v defined in* (6.2)*, and $\mathcal{E} = \{x \in \mathbb{R}^N : u(x) = 1\}$.*

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