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Multiple solutions of nonlinear elliptic systems

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Abstract. We proved a multiplicity result for a nonlinear elliptic system in $\mathbf{R}^N$. The functional related to the system is strongly indefinite. We investigated the relation between the number of solutions and the topology of the set of the global maxima of the coefficients.

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1 Introduction

The main object of this paper is to investigate the existence of multiple positive solutions of the problem

$$-\epsilon^2\Delta u + u = h_2(x)|v|^{q-1}v, \ -\epsilon^2\Delta v + v = h_1(x)|u|^{p-1}u \ \text{ in } \mathbb{R}^N, \tag{1.1}$$

where the exponents $p, q > 1$ are below the critical hyperbola, that is,

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad N \geq 3. \tag{1.2}$$

Problem (1.1) in bounded domains, even with more general nonlinearities, were studied in [5], [7], [8], [10] and references therein. The problem in the whole space was considered in [9], [11] and [13]. The typical feature of Problem (1.1) is that its related functional is strongly indefinite at zero. Linking type theorems and dual variational methods are used in studying the problem. A multiplicity result was obtained by T. Bartsch and D.G. Figueiredo [2] by min-max theorems. On the other hand, it is well-known that the geometry and topology of the set of critical points of coefficients h_1 and h_2 affect the existence of number of positive solutions for a single equation. See [6], [14] and [15] for recent results. In particular, the number of solutions of a single equation is related to the topology of the set of global minimum points of a ground energy function. See [6]. The purpose of the present paper is to consider this problem for the system (1.1). We will study the relation between the number of positive solutions and the topology of the set of maximum points of coefficients h_1 and h_2. Because the functional associated to Problem (1.1) is strongly indefinite, the arguments used in recent works cannot be pursued straightly. Our idea is to combine the dual variational method with the Nehari technique. Thus we may relate the set of maximum points of the coefficients of its dual variational problem to the number of positive solutions of Problem (1.1). To state our results, let

$$M_i = \{x \in \mathbb{R}^N : h_i(x) = \sup_{x\in\mathbb{R}^N} h_i(x)\},\ i = 1, 2,$$

and $M = M_1 \times M_2$. Let us define

$$h_i^\infty = \lim_{|x|\to\infty} \sup h_i(x),\ i = 1, 2.$$

We assume

(A1) $h_i \in L^\infty(\mathbf{R}^N)$, *is continuous,* $h_i(x) \geq \alpha > 0$ *for all* $x \in \mathbf{R}^N$, *with* $i = 1, 2$.

(A2) $h_i(x) > h_i^\infty, \forall x \in M_i$ *for* $i = 1, 2$.

We note that $(A1)$ and $(A2)$ imply M is bounded. Let us denote $M_\delta = \{x : \mathrm{dist(x, M)} < \delta\}$ and $\mathrm{cat}_B(A)$ the relative category of A in B. We obtain the following multiplicity result.

Theorem 1.1 *Suppose that* h_1, h_2 *satisfy* (A1)–(A2). *Then, for any* $\delta > 0$ *there exists* $\epsilon_0 = \epsilon_0(\delta) > 0$ *such that* (1.1) *possesses at least* $\mathrm{cat}_{M_\delta}(M)$ *positive solutions for* $0 < \epsilon < \epsilon_0$.

To prove the theorem we will use the following results connecting the relative category and the multiplicity of critical points.

Lemma 1.1 (Lemma 2.2, [6]) *Let* H, Ω^+ *and* Ω^- *be closed subsets with* $\Omega^- \subset \Omega^+$, *let* $\beta : H \to \Omega^+$, $\Phi : \Omega^- \to H$ *be two continuous maps such that* $\beta \circ \Phi$ *is homotopically equivalent to the embedding* $j : \Omega^- \to \Omega^+$. *Then* $\mathrm{cat}_H(H) \geq \mathrm{cat}_{\Omega^+}(\Omega^-)$.

Let X be a Banach space, $\phi \in C^2(X, \mathbf{R})$, $V := \{x \in X : \phi(x) = 1\}$, and for all $v \in V$, $\phi'(v) \neq 0$. Denote by $\phi^d = \{x \in V : \phi(x) \leq d\}$.

Lemma 1.2 (Theorem 5.20, [12]) *If $\phi|_V$ is bounded from below and satisfies the $(PS)_c$ condition for any $c \in [\inf_V \phi, d]$, then $\phi|_V$ contains at least $\operatorname{cat}_{\phi^d}(\phi^d)$ critical points of $\phi|_V$.*

In Section 2, using the dual variational formulation of the problem, we define the Nehari manifold Σ_ϵ and study the critical values associated to the least energy solution of (1.1) and the limit system. In Section 3, we will define a homotopy between the set M and a manifold Σ_ϵ. Finally in Section 4 we will prove the (PS) condition and the Theorem 1.1.

2 The Nehari Manifold and the Critical Value

Let $X = L^{1+\frac{1}{p}}(\mathbf{R}^N) \times L^{1+\frac{1}{q}}(\mathbf{R}^N)$ for $p, q > 1$. We define the canonical injection

$$i_p : H^1(\mathbf{R}^N) \to L^{p+1}(\mathbf{R}^N), \;\; i_p^* : L^{\frac{p+1}{p}}(\mathbf{R}^N) \to H^{-1}(\mathbf{R}^N)$$

and the operator

$$T_\epsilon := i_q \circ (-\epsilon^2 \Delta + id)^{-1} \circ i_p^* : L^{1+\frac{1}{p}}(\mathbf{R}^N) \to L^{q+1}(\mathbf{R}^N).$$

Denote $w = (w_1, w_2) \in X$, and define the functional

$$\begin{aligned} J_\epsilon(w) = & \frac{p}{p+1} \int_{\mathbf{R}^N} h_p |w_1|^{1+\frac{1}{p}} dx + \frac{q}{q+1} \int_{\mathbf{R}^N} h_q |w_2|^{1+\frac{1}{q}} dx \\ & -\frac{1}{2} \int_{\mathbf{R}^N} (w_1 T_\epsilon w_2 + w_2 T_\epsilon w_1) dx, \end{aligned}$$

on X, where $h_p = h_1^{-\frac{1}{p}}$ and $h_q = h_2^{-\frac{1}{q}}$. It is standard to verify that J_ϵ is well defined and it is of class C^1 on X. If w is a critical point of J_ϵ, then w satisfies

$$T_\epsilon w_2 = h_p |w_1|^{\frac{1}{p}-1} w_1, \qquad T_\epsilon w_1 = h_q |w_2|^{\frac{1}{q}-1} w_2.$$

Let $u = T_\epsilon w_2$, $v = T_\epsilon w_1$, then (u, v) is a solution of (1.1). We will obtain solutions of (1.1) by finding critical points of J_ϵ which are contained in the Nehari set

$$\Sigma_\epsilon = \{w \in X \backslash \{0\} : \langle J_\epsilon'(w), w \rangle = 0\}.$$

Lemma 2.1 *Σ_ϵ is a smooth manifold.*

Proof. Let $g(w) = \langle J_\epsilon'(w), w \rangle$, for $w \in \Sigma_\epsilon$. Then

$$\begin{aligned} \langle g'(w), w \rangle = & \left(1 + \frac{1}{p}\right) \int_{\mathbf{R}^N} h_p |w_1|^{1+\frac{1}{p}} dx + \left(1 + \frac{1}{q}\right) \int_{\mathbf{R}^N} h_q |w_2|^{1+\frac{1}{q}} dx \\ & -2 \int_{\mathbf{R}^N} (w_1 T_\epsilon w_2 + w_2 T_\epsilon w_1) dx \end{aligned}$$

Using the characterization of Σ_ϵ, we get

$$\langle g'(w), w\rangle = \left(\frac{1}{p}-1\right)\int_{\mathbf{R}^N} h_p|w_1|^{1+\frac{1}{p}}\,dx + \left(\frac{1}{q}-1\right)\int_{\mathbf{R}^N} h_q|w_2|^{1+\frac{1}{q}}\,dx.$$

Because $w \neq 0$ and $p, q > 1$ we conclude that

$$\langle g'(w), w\rangle < 0, \quad \forall\ w \in \Sigma_\epsilon.$$

The assertion follows by the Implicit Function Theorem. □

We will use the solution of the limit system

$$-\Delta u + u = h_2^\infty |v|^{q-1}v, \qquad -\Delta v + v = h_1^\infty |u|^{p-1}u, \quad \text{in } \mathbf{R}^N \tag{2.1}$$

to compare the solutions of (1.1). The functional associated to this system is

$$I_\infty(u,v) = \int_{\mathbf{R}^N} (\nabla u\cdot\nabla v + uv)dx - \frac{1}{p+1}\int_{\mathbf{R}^N} h_1^\infty |u|^{p+1}\,dx - \frac{1}{q+1}\int_{\mathbf{R}^N} h_2^\infty |v|^{q+1}\,dx.$$

Its dual functional on X is

$$J_\infty(w) = \frac{p}{p+1}\int_{\mathbf{R}^N} h_p^\infty |w_1|^{1+\frac{1}{p}}\,dx + \frac{q}{q+1}\int_{\mathbf{R}^N} h_q^\infty |w_2|^{1+\frac{1}{q}}\,dx - \frac{1}{2}\int_{\mathbf{R}^N} (w_1Tw_2 + w_2Tw_1)dx,$$

where $h_p^\infty = (h_1^\infty)^{-\frac{1}{p}}$, $h_q^\infty = (h_2^\infty)^{-\frac{1}{q}}$, $T = i_2 \circ (-\Delta + id)^{-1} \circ i_2^*$. It is proved in [11] under the condition (1.2) that the ground state (U, V) of (2.1) exists and it has exponential decay at infinity. By a ground state solution we mean a solution with the least positive critical value of the functional I_∞. Let $w = (w_1, w_2) = ((-\Delta + id)U, (-\Delta + id)V)$, we have

$$J_\infty(w) = I_\infty(U, V) := c^*. \tag{2.2}$$

The manifold associated to J_∞ is given by

$$\Sigma_\infty = \{w \in X\backslash\{0\} : \langle J'_\infty(w), w\rangle = 0\}.$$

As it was proved in Lemma 2.1, we can see that Σ_∞ is a smooth manifold. Consider the variational problem

$$c^\infty = \inf_{w\in\Sigma_\infty} J_\infty(w). \tag{2.3}$$

Lemma 2.2 *There exists $w \in \Sigma_\infty$ such that $J_\infty(w) = c^\infty$.*

Proof. Let $w_n = (w_n^1, w_n^2)$ be a minimizing sequence. We will show that w_n has a converging subsequence. First, we will prove that $\{w_n\}$ can be replaced by a positive radially symmetric minimizing sequence. In fact, we may assume $\int_{\mathbf{R}^N} w_n^1 T w_n^2 dx \geq 0$. To prove this fact, let us consider the following cases.

(i) If $w_1 \geq 0$ then $Tw_1 \geq 0$. By the Maximum principle, $Tw_2 \leq T|w_2|$, which implies

$$\int_{\mathbf{R}^N} w_1 T w_2 dx \leq \int_{\mathbf{R}^N} w_1 T|w_2| dx = \int_{\mathbf{R}^N} |w_1| T|w_2| dx.$$

By the definition of Σ_∞ we have

$$\int_{\mathbf{R}^N} (h_p^\infty |w_1|^{1+\frac{1}{p}} dx + h_q^\infty |w_2|^{1+\frac{1}{q}}) dx \leq \int_{\mathbf{R}^N} |w_1| T|w_2| dx. \tag{2.4}$$

Thus, there exists $0 < t \leq 1$ such that $(t|w_1|, t|w_2|) \in \Sigma_\infty$. By (2.4)

$$\begin{aligned} J_\infty(t|w_1|, t|w_2|) \leq & \left(\frac{p}{p+1} t^{1+\frac{1}{p}} - \frac{t^2}{2}\right) \int_{\mathbf{R}^N} h_p^\infty |w_1|^{1+\frac{1}{p}} dx \\ & + \left(\frac{q}{q+1} t^{1+\frac{1}{q}} - \frac{t^2}{2}\right) \int_{\mathbf{R}^N} h_q^\infty |w_2|^{1+\frac{1}{q}} dx. \end{aligned}$$

Let $\gamma(t) = c_1(\frac{p}{p+1} t^{1+\frac{1}{p}} - \frac{1}{2}t^2) + c_2(\frac{q}{q+1} t^{1+\frac{1}{q}} - \frac{1}{2}t^2)$, where $c_1, c_2 > 0$, and $t \in [0,1]$.

It is easy to verify that the maximum is attained at $t = 1$. Thus,

$$J_\infty(t|w_1|, t|w_2|) \leq J_\infty(w_1, w_2). \tag{2.5}$$

We have obtained a positive minimizing sequence for the case (i).

(ii) If w_1 changes in sign, we define $\tilde{w}_1 = |w_1|$ and

$$sgn(w_1) T w_2 = T(sgn w_1) w_2 := T\tilde{w}_2.$$

Therefore

$$\int_{\mathbf{R}^N} w_1 T w_2 dx = \int_{\mathbf{R}^N} \tilde{w}_1 T \tilde{w}_2 dx,$$

and $\tilde{w}_1 \geq 0$. Clearly $(\tilde{w}_1, \tilde{w}_2) \in \Sigma_\infty$ and $J_\infty(w_1, w_2) = J_\infty(\tilde{w}_1, \tilde{w}_2)$. We reduce the problem to the case (i).

Next, we replace the positive sequence $\{w_n\}$ by a sequence of radial functions. The operator T is an integral operator with kernel $K(x,y)$ which is the generalized solution of

$$-\Delta u + u = \delta$$

where δ is the Dirac distribution. This equation can be solved by Fourier transform and we know that the fundamental solution is radially symmetric. Thus,

$$\int_{\mathbf{R}^N} w_1(x) T w_2(x) dx = \int_{\mathbf{R}^{2N}} w_1(x) K(x,y) w_2(y) dx dy.$$

Let us denote by u^* the Steiner symmetrization of the function u. By the convolution inequality (see [3]) we get

$$\int_{\mathbf{R}^{2N}} w_1(x)K(x,y)w_2(y)dxdy \le \int_{\mathbf{R}^{2N}} w_1^*(x)K^*(x,y)w_2^*(y)dxdy$$

$$= \int_{\mathbf{R}^{2N}} w_1^*(x)K(x,y)w_2^*(y)dxdy = \int_{\mathbf{R}^N} w_1^*(x)Tw_2^*(x)dx.$$

It is well known that if $p > 0$ and $q > 0$,

$$\int_{\mathbf{R}^N} (h_p^\infty w_1^{1+\frac{1}{p}} + h_q^\infty w_2^{1+\frac{1}{q}})dx = \int_{\mathbf{R}^N} (h_p^\infty (w_1^*)^{1+\frac{1}{p}} + h_q^\infty (w_2^*)^{1+\frac{1}{q}})dx,$$

then there exists $0 < t^* \le 1$ such that $(t^* w_1^*, t^* w_2^*) \in \Sigma_\infty$ and, as in (2.5), we have

$$J_\infty(w_1, w_2) \ge J_\infty(w_1^*, w_2^*). \tag{2.6}$$

Thus, the infimum of J_∞ in Σ_ϵ is attained among the positive radial solutions.

Finally, we will show that c^∞ is attained at some $w \in \Sigma_\infty$. Denote by X_r the subspace of radial functions in X. From (2.6) we know that

$$c^\infty = \inf_{w \in \Sigma_r} J_\infty(w), \text{ where } \Sigma_r = X_r \cap \Sigma_\infty.$$

By the Ekeland's variational principle, (see Theorem 2.4, [12]) we can replace the minimizing sequence $\{w_n\}$ by a (PS) sequence of J_∞, that is, $\{w_n\} \subset X_r$,

$$J'_\infty(w_n) \to 0, \quad J_\infty(w_n) \to c^\infty, \text{ as } n \to \infty. \tag{2.7}$$

Let $u_n = |w_n^2|^{\frac{1}{p}-1} w_n^2$, $v_n = |w_n^1|^{\frac{1}{q}-1} w_n^1$. From (2.7) we know that $\|w_n\|_X$ is uniformly bounded. Clearly, (u_n, v_n) is radial. Using Theorem 4.1 in [9], $\{(u_n, v_n)\}$ has a convergent subsequence in $L^{p+1}(\mathbf{R}^N) \times L^{q+1}(\mathbf{R}^N)$. Then, $\{w_n\}$ has a convergent subsequence in X_r. Thus c^∞ is attained at some $w \ge 0$, which is a solution of $J'_\infty(w) = 0$. □

Next, we will prove the equivalence between (2.2) and (2.3).

Corollary 2.1

$$c^\infty = c^*.$$

Proof. Because c^* is the critical value corresponding to the minimal energy solution, it is clear that $c^\infty \ge c^*$. To prove the reverse inequality, we note from Lemma 2.2 that c^∞ is attained by some $\overline{w}$. By definition of c^∞, $J_\infty(\overline{w}) \le J_\infty(w) = c^*$, where w denotes the ground state of (2.1). □

To relate the solutions of (1.1) to the set M, we study the system

$$-\Delta u + u = h_2(\xi_2)|v|^{q-1}v, \qquad -\Delta v + v = h_1(\xi_1)|u|^{p-1}u \quad \text{in } \mathbf{R}^N, \tag{2.8}$$

where $\xi = (\xi_1, \xi_2) \in M$. This problem has a ground state solution with exponential decay at infinity. The dual functional

$$\begin{aligned} J_M(w) = \frac{p}{p+1}\int_{\mathbf{R}^N} h_p(\xi_1)|w_1|^{1+\frac{1}{p}}dx + \frac{q}{q+1}\int_{\mathbf{R}^N} h_q(\xi_2)|w_2|^{1+\frac{1}{q}}dx \\ -\frac{1}{2}\int_{\mathbf{R}^N}(w_1Tw_2 + w_2Tw_1)dx. \end{aligned}$$

associated to (2.8) is C^1 on X. Let $\Sigma_M = \{w \in X\backslash\{0\} : \langle J'_M(w), w\rangle = 0\}$, we define

$$c_M = \inf_{\Sigma_M} J_M(w)$$

and

$$c_\epsilon = \inf_{\Sigma_\epsilon} J_\epsilon(w).$$

Using the ground state of (2.8), we will construct a test function which approximates asymptotically this ground state and we will use it to estimate the critical value c_ϵ with $\epsilon^N c_M$.

Let $\delta > 0$ be fixed and η be a smooth non-increasing function defined on $[0, +\infty)$, such that

$$\eta(t) = \begin{cases} 1 & 0 \le t \le \frac{\delta}{2}, \\ 0 & t > \delta, \end{cases}$$

with $0 \le \eta(t) \le 1$, $|\eta'(t)| \le c$. Let (U, V) be a ground state of (2.8) related to $\xi = (\xi_1, \xi_2) \in M$. Let us define the functions

$$(\varphi_{\epsilon,\xi_1}(x), \psi_{\epsilon,\xi_2}(x)) := \left(\eta(|x-\xi_1|)U\left(\frac{x-\xi_1}{\epsilon}\right), \eta(|x-\xi_2|)V\left(\frac{x-\xi_2}{\epsilon}\right)\right).$$

Lemma 2.3 *There exists $t_\epsilon > 0$ such that*

$$(w^1_{\epsilon,\xi_1}, w^2_{\epsilon,\xi_2}) = (t_\epsilon h_1 \varphi^p_{\epsilon,\xi_1}, t_\epsilon h_2 \psi^q_{\epsilon,\xi_2}) \in \Sigma_\epsilon$$

for $\epsilon > 0$ small. Thus, $\Sigma_\epsilon \neq \emptyset$.

Proof. We may assume $\int_{\mathbf{R}^N} w_1 T_\epsilon w_2 dx > 0$. The critical points of $\gamma_\epsilon(t) := J_\epsilon(tw)$ satisfy

$$\begin{aligned} 0 = \gamma'_\epsilon(t) = t^{\frac{1}{p}}\int_{\mathbf{R}^N} h_p|w_1|^{1+\frac{1}{p}}dx + t^{\frac{1}{q}}\int_{\mathbf{R}^N} h_q|w_2|^{1+\frac{1}{q}}dx \\ -t\int_{\mathbf{R}^N}(w_1T_\epsilon w_2 + w_2T_\epsilon w_1)dx. \end{aligned}$$

Since T_ϵ is self-adjoint, there holds $\int_{\mathbf{R}^N} w_1 T_\epsilon w_2 dx = \int_{\mathbf{R}^N} w_2 T_\epsilon w_1 dx$. Thus,

$$t^{\frac{1}{p}-1}\int_{\mathbf{R}^N} h_p|w_1|^{1+\frac{1}{p}}dx + t^{\frac{1}{q}-1}\int_{\mathbf{R}^N} h_q|w_2|^{1+\frac{1}{q}}dx = 2\int_{\mathbf{R}^N} w_1 T_\epsilon w_2 dx > 0.$$

This implies that there exists a unique t_ϵ such that $t_\epsilon w \in \Sigma_\epsilon$. In particular,

$$(w_\epsilon^1, w_\epsilon^2) := (t_\epsilon h_1 \varphi^p_{\epsilon,\xi_1}, t_\epsilon h_2 \psi^q_{\epsilon,\xi_2}) \in \Sigma_\epsilon.$$

□

We turn to study the asymptotic estimate of c_ϵ in terms of c_M.

Lemma 2.4

$$\int_{\mathbf{R}^N} h_1\varphi^p_{\epsilon,\xi_1} T_\epsilon h_2 \psi^q_{\epsilon,\xi_2} dx = \int_{\mathbf{R}^N} h_1 \varphi^{p+1}_{\epsilon,\xi_1} dx + o(1),$$

$$\int_{\mathbf{R}^N} h_2\psi^q_{\epsilon,\xi_2} T_\epsilon h_1 \varphi^p_{\epsilon,\xi_1} dx = \int_{\mathbf{R}^N} h_2 \psi^{q+1}_{\epsilon,\xi_2} dx + o(1)$$

as $\epsilon \to 0$.

Proof. By the definition of the functions φ_{ϵ,ξ_1} and ψ_{ϵ,ξ_2} and the exponential decay at infinity of (U, V) we get

$$\int_{\mathbf{R}^N} h_1(\xi_1 + \epsilon y)(\eta(\epsilon|y|)U)^{p+1} dy = \int_{\mathbf{R}^N} h_1(\xi_1) U^{p+1} dy + o(1), \tag{2.9}$$

$$\int_{\mathbf{R}^N} h_2(\xi_2 + \epsilon y)(\eta(\epsilon|y|)V)^{q+1} dy = \int_{\mathbf{R}^N} h_2(\xi_2) V^{q+1} dy + o(1) \tag{2.10}$$

as $\epsilon \to 0$. Consider the system

$$T_\epsilon h_2 \psi^q_{\epsilon,\xi_2} = \varphi_{\epsilon,\xi_1} + \eta_\epsilon, \quad \text{in } B_\delta, \tag{2.11}$$

$$T_\epsilon h_1 \varphi^p_{\epsilon,\xi_1} = \psi_{\epsilon,\xi_2} + \xi_\epsilon, \quad \text{in } B_\delta.$$

Solving for η_ϵ we have

$$(-\epsilon^2\Delta + id)\eta_\epsilon = h_2\psi^q_{\epsilon,\xi_2} - (-\epsilon^2\Delta + id)\varphi_{\epsilon,\xi_1}.$$

Changing variables to $y = \frac{x-\xi_2}{\epsilon}$ we get

$$(-\Delta + id)\tilde{\eta}_\epsilon = h_2(\epsilon y + \xi_2)\tilde{\psi}^q_{\epsilon,0} - (-\Delta + id)\tilde{\varphi}_{\epsilon,\xi_1+\xi_2}, \text{ in } B\left(\xi_2, \frac{\delta}{\epsilon}\right),$$

where $\tilde{\eta}_\epsilon(y) = \eta_\epsilon(\epsilon y + \xi_2)$, $\tilde{\psi}_\epsilon(y) = \psi_\epsilon(\epsilon y + \xi_2)$. From the Newtonian potential estimates and interpolation theorem we have

$$\|\tilde{\eta}_\epsilon\|_{W^{2,2}(B_{\frac{\delta}{\epsilon}})} \le C\|h_2(\epsilon y + \xi_2)\tilde{\psi}^q_{\epsilon,0} - (-\Delta + id)\tilde{\varphi}_{\epsilon,\xi_1+\xi_2}\|_{L^2(B_{\frac{\delta}{\epsilon}})}.$$

But by the exponential decay of the ground states, the continuity of h_2, and the definition of $(\varphi_{\epsilon,\xi_1}, \psi_{\epsilon,\xi_2})$ we have that

$$\|\tilde{\eta}_\epsilon\|_{W^{2,2}(B_{\frac{\delta}{\epsilon}})} \leq o(1) + \|h_2(\xi_2)\tilde{\psi}^q_{\epsilon,0} - (-\Delta + id)\tilde{\varphi}_{\epsilon,0}\|_{L^2(B_{\frac{\delta}{\epsilon}} \setminus B_{\frac{\delta}{2\epsilon}})} = o(1)$$

as $\epsilon \to 0$. The same estimate is obtained for $\tilde{\xi}_\epsilon$. From the Sobolev embeddings and the change of variables, we obtain

$$\|\eta_\epsilon\|_{L^p(\mathbf{R}^N)} = \|\xi_\epsilon\|_{L^p(\mathbf{R}^N)} = o(1),$$

as $\epsilon \to 0$ and for $1 \leq p \leq \frac{2N}{N-4}$. Thus, using (2.11)

$$\int_{\mathbf{R}^N} h_1 \varphi^p_{\epsilon,\xi_1} T_\epsilon h_2 \psi^q_{\epsilon,\xi_2} dx = \int_{\mathbf{R}^N} h_1 \varphi^{p+1}_{\epsilon,\xi_1} dx + o(1).$$

Similar estimates can be obtained for the term $\int_{\mathbf{R}^N} h_2 \psi^q_{\epsilon,\xi_2} T_\epsilon h_1 \varphi^p_{\epsilon,\xi_1} dx$. Thus, we conclude the proof. □

Finally, we obtain the approximation lemma.

Lemma 2.5

$$J_\epsilon(w^1_{\epsilon,\xi_1}, w^2_{\epsilon,\xi_2}) = \epsilon^N \{c_M + o(1)\}$$

as $\epsilon \to 0$.

Proof. There holds

$$\begin{aligned} J_\epsilon(w^1_{\epsilon,\xi_1}, w^2_{\epsilon,\xi_2}) = {} & \frac{p}{p+1} t_\epsilon^{1+\frac{1}{p}} \int_{\mathbf{R}^N} h_1 \varphi^{p+1}_{\epsilon,\xi_1} dx + \frac{q}{q+1} t_\epsilon^{1+\frac{1}{q}} \int_{\mathbf{R}^N} h_2 \psi^{q+1}_{\epsilon,\xi_2} dx \\ & - \frac{1}{2} t_\epsilon^2 \int_{\mathbf{R}^N} (h_1 \varphi^p_{\epsilon,\xi_1} T_\epsilon h_2 \psi^q_{\epsilon,\xi_2} + h_2 \psi^q_{\epsilon,\xi_2} T_\epsilon h_1 \varphi^p_{\epsilon,\xi_1}) dx. \end{aligned}$$

By the estimates in Lemma 2.4, we obtain that the last term simplifies to

$$\left(\frac{p}{p+1} t_\epsilon^{1+\frac{1}{p}} - \frac{1}{2} t_\epsilon^2\right) \int_{\mathbf{R}^N} h_1 \varphi^{p+1}_{\epsilon,\xi_1} dx + \left(\frac{q}{q+1} t_\epsilon^{1+\frac{1}{q}} - \frac{1}{2} t_\epsilon^2\right) \int_{\mathbf{R}^N} h_2 \psi^{q+1}_{\epsilon,\xi_2} dx + o(1)$$

for ϵ small. Thus, from (2.9), (2.10), and the definition of $(\varphi_{\epsilon,\xi_1}, \psi_{\epsilon,\xi_2})$ we obtain

$$\begin{aligned} J_\epsilon(w^1_{\epsilon,\xi_1}, w^2_{\epsilon,\xi_2}) \; = \; & \epsilon^N \Bigg(\left(\frac{p}{p+1} t_\epsilon^{1+\frac{1}{p}} - \frac{1}{2} t_\epsilon^2\right) \int_{\mathbf{R}^N} h_1(\xi_1) U^{p+1} dy \\ & + \left(\frac{q}{q+1} t_\epsilon^{1+\frac{1}{q}} - \frac{1}{2} t_\epsilon^2\right) \int_{\mathbf{R}^N} h_2(\xi_2) V^{q+1} dy + o(1) \Bigg). \quad (2.12) \end{aligned}$$

Since $(w^1_{\epsilon,\xi_1}, w^2_{\epsilon,\xi_2}) \in \Sigma_\epsilon$, we have

$$(t_\epsilon^{\frac{1}{p}-1} - 1)\int_{\mathbf{R}^N} h_1(\xi_1)U^{p+1}dy + (t_\epsilon^{\frac{1}{q}-1} - 1)\int_{\mathbf{R}^N} h_2(\xi_2)V^{q+1}dy = o(1). \tag{2.13}$$

Then, if $\epsilon \to 0$, $t_\epsilon \to 1$. From (2.12) we obtain

$$J_\epsilon(w^1_{\epsilon,\xi_1}, w^2_{\epsilon,\xi_2}) = \epsilon^N\left(\left(\frac{p}{p+1} - \frac{1}{2}\right)\int_{\mathbf{R}^N} h_1(\xi_1)U^{p+1}dy \right.$$
$$\left. + \left(\frac{q}{q+1} - \frac{1}{2}\right)\int_{\mathbf{R}^N} h_2(\xi_2)V^{q+1}dy + o(1)\right) = \epsilon^N(c_M + o(1)).$$

□

3 Homotopy

We will devote this section to the construction of the homotopy needed for Lemma 1.1.

We are going to construct a mapping Φ_ϵ from M to Σ_ϵ and a mapping β from Σ_ϵ to M_δ such that $\beta \circ \Phi_\epsilon$ is homotopic to the inclusion $j : M \to M_\delta$.

Let $w_{\epsilon,\xi} = (w^1_{\epsilon,\xi_1}, w^2_{\epsilon,\xi_2})$ be as in Lemma 2.3. We define the mapping $\Phi_\epsilon : M \to \Sigma_\epsilon$ by

$$\Phi_\epsilon(\xi) = (w^1_{\epsilon,\xi_1}, w^2_{\epsilon,\xi_2}).$$

Since the set M is bounded, there exists $\rho > 0$ such that $M_\delta \subset B_\rho \times B_\rho$. Let $\chi : \mathbf{R}^N \to \mathbf{R}^N$ be a function given by

$$\chi(x) = \begin{cases} x & \text{if } |x| \le \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| > \rho. \end{cases}$$

We define $\beta : \Sigma_\epsilon \to \mathbf{R}^N \times \mathbf{R}^N$ by

$$\beta(w) = (\beta_1(w_1), \beta_2(w_2)) = \left(\frac{\int_{\mathbf{R}^N} \chi|w_1|^{1+\frac{1}{p}}dx}{\int_{\mathbf{R}^N} |w_1|^{1+\frac{1}{p}}dx}, \frac{\int_{\mathbf{R}^N} \chi|w_2|^{1+\frac{1}{q}}dx}{\int_{\mathbf{R}^N} |w_2|^{1+\frac{1}{q}}dx}\right).$$

This corresponds to a local center of mass of (w_1, w_2). We have the following asymptotic relation.

Lemma 3.1 *For each point $(\xi_1, \xi_2) \in M$, there holds $\beta(\Phi_\epsilon(\xi)) = (\xi_1, \xi_2) + o(1)$ as $\epsilon \to 0$.*

Proof. Note that

$$\beta_1(w^1_{\epsilon,\xi_1}) = \frac{\int_{\mathbf{R}^N} \chi t_\epsilon^{1+\frac{1}{p}} h_1\varphi^{p+1}_{\epsilon,\xi_1}dx}{\int_{\mathbf{R}^N} t_\epsilon^{1+\frac{1}{p}} h_1\varphi^{p+1}_{\epsilon,\xi_1}dx}.$$

Changing variables, we get

$$\begin{aligned}\beta_1(w^1_{\epsilon,\xi_1}) &= \frac{\int_{\mathbf{R}^N} \chi(\epsilon x+\xi_1)h_1(\epsilon x+\xi_1)|\eta(\epsilon|x|)U(x)|^{p+1}dx}{\int_{\mathbf{R}^N} h_1(\epsilon x+\xi_1)|\eta(\epsilon|x|)U(x)|^{p+1}dx}\\ &= \xi_1 + \frac{\int_{\mathbf{R}^N} (\chi(\epsilon x+\xi_1)-\xi_1)h_1(\epsilon x+\xi_1)|\eta(\epsilon|x|)U(x)|^{p+1}dx}{\int_{\mathbf{R}^N} h_1(\epsilon x+\xi_1)|\eta(\epsilon|x|)U(x)|^{p+1}dx}.\end{aligned}$$

Using the continuity of χ we obtain, for ϵ small,

$$\beta_1(w^1_{\epsilon,\xi_1}) = \xi_1 + o(1).$$

Similarly, we get $\beta_2(w^2_{\epsilon,\xi_2}) = \xi_2 + o(1)$. □

Let $h(\epsilon)$ be a positive function tending to 0 as $\epsilon \to 0$ slowly. Because of the asymptotic behavior of $J_\epsilon(w)$, we can define

$$\tilde{\Sigma}_\epsilon = \{w \in \Sigma_\epsilon : J_\epsilon(w) \le \epsilon^N(c_M + h(\epsilon))\},$$

which, by Lemma 2.3, is not empty for $\epsilon > 0$ small.

We claim that $\beta(w)$ restricted to $\tilde{\Sigma}_\epsilon$ approaches to a point in M_δ. We will first prove the following lemma.

Lemma 3.2 *Let $\{w_n\}$ be a minimizing sequence of c_M. Then*

(i) *there exists $\{\overline{w}_n\} \subset\subset \Sigma_M$ such that*

$$J_M(\overline{w}_n) \to c_M, \quad J'_M(\overline{w}_n) \to 0,$$

and

$$\|w_n - \overline{w}_n\|_X \to 0$$

as $n \to \infty$.

(ii) *Furthermore, there exists $\{\xi_n\} \subset \mathbf{R}^N$ such that if we define $\tilde{w}_n(\cdot) = \overline{w}_n(\cdot + \xi_n)$, then $\{\tilde{w}_n\}$ is precompact.*

Proof. (i) It is a direct consequence of the Ekeland's variational principle, (see [12]).

(ii) We will use the Concentration Compactness given in Lemma 1.21, [12]. Let us show that $\{\overline{w}_n\}$ is tight. Because $J_M(\overline{w}_n) \to c_M$ and $\overline{w}_n \in \Sigma_M$,

$$\frac{p-1}{2p+2}\int_{\mathbf{R}^N} h_p(\xi_1)|\overline{w}^n_1|^{1+\frac{1}{p}}dx + \frac{q-1}{2q+2}\int_{\mathbf{R}^N} h_q(\xi_2)|\overline{w}^n_2|^{1+\frac{1}{q}}dx \le c_M + 1$$

for n large. Let $z_n = (T\overline{w}^2_n, T\overline{w}^1_n)$. Thus, $\|z_n\|_{X^*} \le c$. This implies that $z_n \in E$, where

$$E := (W^{2,1+\frac{1}{q}}(\mathbf{R}^N) \cap W^{1,1+\frac{1}{q}}(\mathbf{R}^N)) \times (W^{2,1+\frac{1}{p}}(\mathbf{R}^N) \cap W^{1,1+\frac{1}{p}}(\mathbf{R}^N)).$$

Since the sequence is bounded, we may assume that $z_n \rightharpoonup z$ in E.

Using $J'_M(\overline{w}_n) \to 0$, we have that

$$u_n = h_p(\xi_1)|\overline{w}_n^1|^{\frac{1}{p}-1}\overline{w}_n^1 + o(1), \quad v_n = h_q(\xi_2)|\overline{w}_n^2|^{\frac{1}{q}-1}\overline{w}_n^2 + o(1)$$

as $n \to \infty$, which implies

$$\overline{w}_n^1 = h_1(\xi_1)|u_n|^{p-1}u_n + o(1), \quad \overline{w}_n^2 = h_2(\xi_2)|v_n|^{q-1}v_n + o(1) \tag{3.1}$$

as $n \to \infty$. Therefore,

$$\begin{aligned} c_M + o(1) = J_M(\overline{w}_n) &= \frac{p}{p+1}\int_{\mathbf{R}^N} h_p(\xi_1)|\overline{w}_n^1|^{1+\frac{1}{p}}dx \\ &+ \frac{q}{q+1}\int_{\mathbf{R}^N} h_q(\xi_2)|\overline{w}_n^2|^{1+\frac{1}{q}}dx - \frac{1}{2}\int_{\mathbf{R}^N}(\overline{w}_n^1 T\overline{w}_n^2 + \overline{w}_n^2 T\overline{w}_n^1)dx \\ &= \frac{p-1}{2p+2}\int_{\mathbf{R}^N} h_1(\xi_1)|u_n|^{p+1}dx + \frac{q-1}{2q+2}\int_{\mathbf{R}^N} h_2(\xi_2)|v_n|^{q+1}dx + o(1). \end{aligned}$$

We claim that vanishing does not happen for $\{z_n\}$. By contradiction, suppose that for some $r > 0$

$$\sup_{y\in\mathbf{R}^N}\int_{B(y,r)}|u_n|^{p+1}dx \to 0,$$

and

$$\sup_{y\in\mathbf{R}^N}\int_{B(y,r)}|v_n|^{q+1}dx \to 0.$$

Because of the Sobolev's embedding $H^s(\mathbf{R}^N) \hookrightarrow L^\gamma_{\mathrm{loc}}(\mathbf{R}^N)$, for $2 \le \gamma \le \frac{2N}{N-2s}$ and $s > 0$, we have that $u_n \to 0$ in $L^\gamma_{\mathrm{loc}}(\mathbf{R^N})$ for $2 < \gamma < \frac{2N}{N-2s}$ and $v_n \to 0$ in $L^\mu_{\mathrm{loc}}(\mathbf{R^N})$ for $2 < \mu < \frac{2N}{N-2t}$, with $s + t = 2$. For any $p + 1 < \alpha < \frac{2N}{N-2s}$, the Hölder's inequality and the Sobolev's embedding imply that

$$\begin{aligned} \|u\|_{L^\alpha(B(y,r))} &\le \|u\|^{1-\lambda}_{L^{p+1}(B(y,r))}\|u\|^\lambda_{L^{\frac{2N}{N-2s}}(B(y,r))} \\ &\le c\|u\|^{1-\lambda}_{L^{p+1}(B(y,r))}\|u\|^\lambda_{H^s(B(y,r))}, \end{aligned}$$

where $\lambda = \frac{\alpha-p-1}{\frac{2N}{N-2s}-p-1}\frac{\frac{2N}{N-2s}}{\alpha}$. Choosing $\lambda = \frac{2}{\alpha}$, we obtain

$$\int_{B(y,r)}|u|^\alpha dx \le c^\alpha\|u\|^{\alpha-2}_{L^{p+1}(B(y,r))}\|u\|^2_{H^s(B(y,r))}.$$

Covering $\mathbf{R}^N$ with balls of radius r in such a way that each point of $\mathbf{R}^N$ is contained at most in $m + 1$ balls, we find that

$$\int_{\mathbf{R}^N}|u|^\alpha dx \le (m+1)c^\alpha \sup_{y\in\mathbf{R}^N}\left(\int_{B(y,r)}|u|^{p+1}\right)^{\frac{\alpha-2}{p+1}}\|u\|^2_{H^s(\mathbf{R}^N)}.$$

Then $u_n \to 0$ in $L^\alpha(\mathbf{R}^N)$ for $2 < \alpha < \frac{2N}{N-2s}$. In the same way $v_n \to 0$ in $L^\mu(\mathbf{R}^N)$ for $2 < \mu < \frac{2N}{N-2t}$. Consequently

$$J_M(\overline{w}_n) = \left(\frac{p}{p+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_1(\xi_1)|u_n|^{p+1} dx + \left(\frac{q}{q+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_2(\xi_2)|v_n|^{q+1} dx \to 0$$

as $n \to \infty$, contradicting to the fact that $J_M(\overline{w}_n) \to c_M > 0$. Therefore vanishing does not happen. Thus, there exists $\sigma > 0$ such that

$$\lim_n \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |u_n|^\gamma dx \geq 2\sigma > 0 \quad \text{or} \quad \lim_n \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |v_n|^\mu dx \geq 2\sigma > 0,$$

where $2 < \gamma < \frac{2N}{N-2s}$ and $2 < \mu < \frac{2N}{N-2t}$. Then, there exists a sequence $\{\xi_n\} \subset \mathbf{R}^N$ such that

$$\int_{B(\xi_n,1)} |u_n|^\gamma dx \geq \sigma > 0 \quad \text{or} \quad \int_{B(\xi_n,1)} |v_n|^\mu dx \geq \sigma > 0.$$

Let $\hat{u}_n(x) = u_n(x + \xi_n)$, $\hat{v}_n(x) = v_n(x + \xi_n)$. Then, $\hat{u}_n \rightharpoonup \hat{u}$ in $H^s(\mathbf{R}^N)$, $\hat{v}_n \rightharpoonup \hat{v}$ in $H^t(\mathbf{R}^N)$. By the Sobolev embedding, $\hat{u}_n \to \hat{u}$, $\hat{v}_n \to \hat{v}$ in $L^\gamma_{\text{loc}}(\mathbf{R}^N)$ and $L^\mu_{\text{loc}}(\mathbf{R}^N)$ respectively. In particular, $\hat{u}_n \to \hat{u}$, $\hat{v}_n \to \hat{v}$ a.e in $\mathbf{R}^N$. Then, $(\hat{u}, \hat{v}) \neq 0$ and by (3.1) it solves

$$-\Delta u + u = h_1(\xi_1)|v|^{q-1}v, \quad -\Delta v + v = h_2(\xi_2)|u|^{p-1}u, \tag{3.2}$$

in a weak sense. We know that this is a classical solution by the regularity theory.

Let $\hat{w}_n = ((-\Delta + id)\hat{u}_n, (-\Delta + id)\hat{v}_n)$ and $\hat{w} = ((-\Delta + id)\hat{u}, (-\Delta + id)\hat{v})$. Using the fact that $(\hat{u}, \hat{v})$ is a solution of (3.2), we find that $\hat{w} \in \Sigma_M$. Note that from (i)

$$\|\hat{w}_n - \overline{w}_n(\cdot + \xi_n)\| \to 0 \text{ and } J'_M(\hat{w}_n) \to 0 \text{ as } n \to \infty.$$

We will show that $\{\hat{w}_n\}$ is precompact. By the Brézis-Lieb Lemma (see Lemma 1.32 in [12]), we have

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} |\hat{w}_n^1|^{1+\frac{1}{p}} dx = \lim_{n\to\infty} \left(\int_{\mathbf{R}^N} |\hat{w}_n^1 - \hat{w}_1|^{1+\frac{1}{p}} dx + \int_{\mathbf{R}^N} |\hat{w}_1|^{1+\frac{1}{p}} dx \right),$$

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} |\hat{w}_n^2|^{1+\frac{1}{q}} dx = \lim_{n\to\infty} \left(\int_{\mathbf{R}^N} |\hat{w}_n^2 - \hat{w}_2|^{1+\frac{1}{q}} dx + \int_{\mathbf{R}^N} |\hat{w}_2|^{1+\frac{1}{q}} dx \right).$$

Hence,

$$c_M = J_M(\hat{w}_n) + o(1) = J_M(\hat{w}_n - \hat{w}) + J_M(\hat{w}) + o(1)$$

as $n \to \infty$. Since $\hat{w} \in \Sigma_M$, we have $J_M(\hat{w}) \geq c_M$. If $\hat{w}_n - \hat{w} \to w_0 \neq 0$ then $J(w_0) > 0$, it yields

$$c_M \geq J_M(\hat{w}_n - \hat{w}) + c_M + o(1) \geq c_M + J_M(w_0) + o(1) > c_M$$

which is a contradiction. We conclude that

$$\hat{w}_n \to \hat{w} \text{ in } L^{1+\frac{1}{p}}(\mathbf{R}^N) \times L^{1+\frac{1}{q}}(\mathbf{R}^N).$$

□

To finish the construction of the homotopy, let us prove that $\beta \circ \Phi_\epsilon$ approaches to the injection j as $\epsilon \to 0$.

Lemma 3.3

$$\lim_{\epsilon \to 0} \sup_{w \in \tilde{\Sigma}_\epsilon} \inf_{\xi \in M_\delta} |\beta(w) - \xi| = 0.$$

Proof. Let $\epsilon_n \to 0$. For any n there exists $w_n \in \tilde{\Sigma}_{\epsilon_n}$ such that

$$\inf_{\xi \in M_\delta} |\beta(w_n) - \xi| = \sup_{w \in \tilde{\Sigma}_{\epsilon_n}} \inf_{\xi \in M_\delta} |\beta(w) - \xi| + o(1).$$

Then, it is sufficient to find points $\xi_n \in M_\delta$ such that

$$\lim_{n \to \infty} |\beta(w_n) - \xi_n| = 0. \tag{3.3}$$

Defining $\tilde{\Sigma}(\epsilon_n) := \{w : \int_{\mathbf{R}^N} (h_p(\epsilon_n x)|w_1|^{1+\frac{1}{p}} + h_q(\epsilon_n x)|w_2|^{1+\frac{1}{q}})dx = \int_{\mathbf{R}^N} (w_1 T w_2 + w_2 T w_1)dx\}$ and denoting $\tilde{w}_n(x) = w_n(\epsilon_n x) \in \tilde{\Sigma}(\epsilon_n)$, we get

$$\int_{\mathbf{R}^N} (h_p(\epsilon_n x)|\tilde{w}_n^1|^{1+\frac{1}{p}} + h_q(\epsilon_n x)|\tilde{w}_n^2|^{1+\frac{1}{q}})dx = \int_{\mathbf{R}^N} (\tilde{w}_n^1 T \tilde{w}_n^2 + \tilde{w}_n^2 T \tilde{w}_n^1)dx$$
$$\geq \int_{\mathbf{R}^N} (h_p^M |\tilde{w}_n^1|^{1+\frac{1}{p}} + h_q^M |\tilde{w}_n^2|^{1+\frac{1}{q}})dx.$$

Therefore, there exists $0 < t_n \leq 1$ such that $t_n \tilde{w}_n \in \Sigma_M$. From the definition of $\tilde{\Sigma}_{\epsilon_n}$, we obtain

$$h(\epsilon_n) + c_M \geq J_{\tilde{\Sigma}_{\epsilon_n}}(\tilde{w}_n) \geq J_{\tilde{\Sigma}(\epsilon_n)}(t_n \tilde{w}_n) \geq J_M(t_n \tilde{w}_n) \geq c_M > 0. \tag{3.4}$$

Thus, (3.4) implies that $\{t_n \tilde{w}_n\}$ is a minimizing sequence of c_M. We also have that $t_n \to 1$ as $n \to \infty$.

By Lemma 3.2 there exists $\{\overline{w}_n\}$ such that $\|\overline{w}_n - t_n \tilde{w}_n\| \to 0$ as $n \to \infty$. Thus, $|\beta(\overline{w}_n) - \beta(t_n \tilde{w}_n)| \to 0$ as $n \to \infty$ and there exists a sequence $\{\xi_n\} \subset \mathbf{R}^N$ such that $\{\overline{w}_n(\cdot + \xi_n)\}$ is precompact.

Let $\hat{w}_n(x) = \overline{w}_n(x+\xi_n)$. Then, $\hat{w}_n$ is bounded in X. We may assume that $\hat{w}_n \to \hat{w}$ in X. Clearly $\hat{w}_n \in \Sigma_M$ and $\hat{w} \in \Sigma_M$. We claim that the sequence $\{\epsilon_n \xi_n\}$ is bounded.

In fact, if it is not bounded, we would have

$$\begin{aligned} h(\epsilon_n) + c_M \geq & \left(\frac{p}{p+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_p(\epsilon_n(x+\xi_n))|\hat{w}_n^1|^{1+\frac{1}{p}} dx \\ & + \left(\frac{q}{q+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_q(\epsilon_n(x+\xi_n))|\hat{w}_n^2|^{1+\frac{1}{q}} dx + o(1). \end{aligned}$$

Thus, by Fatou's lemma

$$\begin{aligned} c_M \geq & \liminf_{n\to\infty} \left(\left(\frac{p}{p+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_p(\epsilon_n(x+\xi_n))|\hat{w}_n^1|^{1+\frac{1}{p}} dx \right. \\ & \left. + \left(\frac{q}{q+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_q(\epsilon_n(x+\xi_n))|\hat{w}_n^2|^{1+\frac{1}{q}} dx \right) \\ \geq & \left(\frac{p}{p+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} \liminf_{n\to\infty} h_p(\epsilon_n(x+\xi_n))|\hat{w}_n^1|^{1+\frac{1}{p}} dx \\ & + \left(\frac{q}{q+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} \liminf_{n\to\infty} h_q(\epsilon_n(x+\xi_n))|\hat{w}_n^2|^{1+\frac{1}{q}} dx \\ \geq & \left(\frac{p}{p+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_p^\infty |\hat{w}_1|^{1+\frac{1}{p}} dx + \left(\frac{q}{q+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_q^\infty |\hat{w}_2|^{1+\frac{1}{q}} dx. \end{aligned}$$

Also note that

$$\begin{aligned} & \int_{\mathbf{R}^N} h_p(\epsilon_n(x+\xi_n))|\hat{w}_n^1|^{1+\frac{1}{p}} dx + \int_{\mathbf{R}^N} h_q(\epsilon_n(x+\xi_n))|\hat{w}_n^2|^{1+\frac{1}{q}} dx \\ & \qquad \geq \int_{\mathbf{R}^N} h_p^\infty |\hat{w}_1|^{1+\frac{1}{p}} dx + \int_{\mathbf{R}^N} h_q^\infty |\hat{w}_2|^{1+\frac{1}{q}} dx. \end{aligned} \tag{3.5}$$

Since $\hat{w}_n \in \Sigma_M$, using (3.5) we obtain the existence of t, $0 < t \leq 1$, such that $t\hat{w} \in \Sigma_\infty$. Hence, we have

$$\begin{aligned} c_M \geq & \left(\frac{p}{p+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_p^\infty |\hat{w}_1|^{1+\frac{1}{p}} dx + \left(\frac{q}{q+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_q^\infty |\hat{w}_2|^{1+\frac{1}{q}} dx \\ \geq & J_\infty(t\hat{w}) \geq c^\infty \end{aligned}$$

which is a contradiction. Thus $\{\epsilon_n \xi_n\}$ is bounded. Suppose that $\epsilon_n \xi_n \to \overline{\xi} = \{\overline{\xi_1}, \overline{\xi_2}\}$. Since $\hat{w} \in \Sigma_M$, there holds

$$\begin{aligned} c_M \geq & \left(\frac{p}{p+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_p(\overline{\xi_1}) |\hat{w}_1|^{1+\frac{1}{p}} dx + \left(\frac{q}{q+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_q(\overline{\xi_2}) |\hat{w}_2|^{1+\frac{1}{q}} dx \\ \geq & \left(\frac{p}{p+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_p^M |\hat{w}_1|^{1+\frac{1}{p}} dx + \left(\frac{q}{q+1} - \frac{1}{2}\right) \int_{\mathbf{R}^N} h_q^M |\hat{w}_2|^{1+\frac{1}{q}} dx, \\ = & J_M(\hat{w}) \geq c_M, \end{aligned}$$

where $h_p^M = h_p(\xi_1)$, $h_q^M = h_q(\xi_2)$ for any $(\xi_1, \xi_2) \in M$, then $\xi \in M$.

Next, we prove (3.3). Using the definition of β and w_n, we get

$$\begin{aligned}\beta(w_n) &= \left(\frac{\int_{\mathbf{R}^N} \chi(x)|w_n^1|^{1+\frac{1}{p}}dx}{\int_{\mathbf{R}^N} |w_n^1|^{1+\frac{1}{p}}dx}, \frac{\int_{\mathbf{R}^N} \chi(x)|w_n^2|^{1+\frac{1}{p}}dx}{\int_{\mathbf{R}^N} |w_n^2|^{1+\frac{1}{p}}dx} \right)\\ &= \left(\frac{\int_{\mathbf{R}^N} \chi(\epsilon_n(x+\xi_1^n))|\hat{w}_n^1(x)|^{1+\frac{1}{p}}dx}{\int_{\mathbf{R}^N} |\hat{w}_n^1|^{1+\frac{1}{p}}dx}, \frac{\int_{\mathbf{R}^N} \chi(\epsilon_n(x+\xi_2^n))|\hat{w}_n^2|^{1+\frac{1}{p}}dx}{\int_{\mathbf{R}^N} |\hat{w}_n^2|^{1+\frac{1}{p}}dx} \right)\\ &\to (\xi_1,\xi_2) \in M, \quad \text{as } n \to \infty,\end{aligned}$$

since $\hat{w}_n$ converges strongly in X. □

4 (PS) Condition and Proof of the Theorem

In this last section, we will prove Theorem 1.1. First, we need the (PS) condition.

Lemma 4.1 *J_ϵ satisfies the (PS) condition in $\{w \in \Sigma_\epsilon : J_\epsilon(w) < \epsilon^N c^\infty\}$.*

Proof. It is clear that $\{w \in \Sigma_\epsilon : J_\epsilon(w) < \epsilon^N c^\infty\}$ is not empty because $c^\infty > c_M$ and by the results obtained in Lemma 2.5. Let $\{w_n\}$ be a (PS)-sequence at level c for $0 < c < \epsilon^N c^\infty$. Then,

$$J_\epsilon(w_n) = c + \delta_n, \qquad J'_\epsilon|_{\Sigma_\epsilon}(w_n) = \delta_n$$

with $\delta_n \to 0$ as $n \to \infty$. By the Ekeland's variational principle, we may assume

$$J_\epsilon(w_n) = c + \delta_n, \qquad J'_\epsilon(w_n) = \delta_n, \qquad \delta_n \to 0$$

as $n \to \infty$. By Lemma 3.2, we need only to rule out dichotomy, that is, we need to show that for any $\delta > 0$ there exists $R > 0$ such that

$$\int_{\{|x|\geq R\}} |w_n^1|^{1+\frac{1}{p}}dx < \delta, \quad \int_{\{|x|\geq R\}} |w_n^2|^{1+\frac{1}{q}}dx < \delta, \tag{4.1}$$

which implies in a standard way that $\{w_n\}$ converges strongly in X.

Now, let us prove (4.1). By contradiction, suppose that there exist a subsequence $\{w_k\}$ of $\{w_n\}$ and $\alpha_0 > 0$ such that

$$\int_{\{|x|\geq k\}} |w_k^1|^{1+\frac{1}{p}}\, dx \geq \alpha_0, \qquad \forall\ k. \tag{4.2}$$

From $|\langle J'_\epsilon(w_n), v\rangle| \leq \epsilon\|v\|$, we have for $v = ((1-\chi_R)w_n^1, 0)$, where χ_R is the characteristic function for B_R, that

$$\begin{aligned}\int_{\{|x|\geq k\}} |w_n^1|^{1+\frac{1}{p}}\, dx &\leq \epsilon\|w_n^1\|_{1+\frac{1}{p}} + \int_{\{|x|\geq k\}} w_n^2 T_\epsilon w_n^1\, dx\\ &\leq \|w_n^1\|_{1+\frac{1}{p}}(\epsilon + c\|w_n^2\|_{1+\frac{1}{q}}).\end{aligned}$$

Thus,

$$\|w_n^1\|^{\frac{1}{p}}_{L^{1+\frac{1}{p}}(\{|x|\geq k\})} \leq \epsilon + c\|w_n^2\|_{L^{1+\frac{1}{q}}(\{|x|\geq k\})}.$$

Using (4.2) and the last inequality we obtain

$$\alpha_0^{\frac{1}{p}} \leq \epsilon + c\left(\int_{\{|x|\geq k\}} |w_k^2|^{1+\frac{1}{q}}\,dx\right)^{\frac{q}{q+1}}.$$

For ϵ small and k large, there holds

$$0 < \left(\frac{\alpha_0^{\frac{1}{p}} - \epsilon}{c}\right)^{\frac{(q+1)}{q}} =: \alpha_1 \leq \int_{\{|x|\geq k\}} |w_k^2|^{1+\frac{1}{q}}\,dx, \quad \forall k. \tag{4.3}$$

In particular, for $\eta > 0$ small there exists $r(\eta) > 0$ such that

$$\int_{\{r\leq|x|\leq r+1\}} |w_k^1|^{1+\frac{1}{p}}\,dx < \eta, \quad \int_{\{r\leq|x|\leq r+1\}} |w_k^2|^{1+\frac{1}{q}}\,dx < \eta, \tag{4.4}$$

for a subsequence still indexed by k. In fact, if it is not so, for any $m > r$ there is an index k_m such that

$$\int_{\{m\leq|x|\leq m+1\}} |w_k^1|^{1+\frac{1}{p}}\,dx \geq \eta$$

for all $k > k_m$. Thus,

$$\int_{\mathbf{R}^N} |w_k^1|^{1+\frac{1}{p}}\,dx \geq \int_{\{r\leq|x|\leq m\}} |w_k^1|^{1+\frac{1}{p}}\,dx \geq (m-r)\eta \to \infty$$

if $m \to \infty$. This contradicts the fact that $\|w_k^1\| \leq c$. The same result is obtained if $\int_{\{m\leq|x|\leq m+1\}} |w_k^2|^{1+\frac{1}{q}}\,dx \geq \eta$. Thus, we have achieved (4.4).

Let us prove that this contradicts to our assumption on the energy. For this purpose we write

$$w_k^i = v_k^i + z_k^i, \quad i = 1, 2,$$

where $v_k^i = \rho w_k^i$, $z_k^i = (1-\rho)w_k^i$, $i = 1, 2$ and $\rho : \mathbf{R}^N \to [0, 1]$ is a cut-off function such that

$$\rho(x) = \begin{cases} 1 & \text{if } |x| \leq r, \\ 0 & \text{if } |x| \geq r+1, \end{cases}$$

and $|\nabla\rho| \leq c$ in $\mathbf{R}^N$. Thus,

$$|\langle J'_\epsilon(w_k), v_k\rangle - \langle J'_\epsilon(v_k), v_k\rangle| \leq O(\eta).$$

and

$$|\langle J'_\epsilon(w_k), z_k\rangle - \langle J'_\epsilon(z_k), z_k\rangle| \leq O(\eta).$$

Therefore, we can split the derivative as

$$\langle J_\epsilon'(v_k), v_k\rangle = O(\eta) + \delta_k = \langle J_\epsilon'(z_k), z_k\rangle. \tag{4.5}$$

We also have

$$J_\epsilon(w_k) \geq J_\epsilon(v_k) + J_\epsilon(z_k) + O(\eta).$$

But

$$J_\epsilon(v_k) \geq c(\|v_k^1\|_{1+\frac{1}{p}} + \|v_k^2\|_{1+\frac{1}{q}}) + O(\eta) + \delta_k > \delta_k,$$

for k large. Thus,

$$J_\epsilon(w_k) \geq J_\epsilon(z_k) + O(\eta) + \delta_k.$$

From (4.2) and (4.3), we have

$$\int_{\{|x|\geq R\}} h_p|z_k^1|^{1+\frac{1}{p}}dx + \int_{\{|x|\geq R\}} h_q|z_k^2|^{1+\frac{1}{q}}dx \geq \alpha_0 + \alpha_1 + O(\eta).$$

Let t_k be such that $t_k z_k \in \Sigma_\epsilon$. From (4.5), we have

$$t_k = 1 + O(\eta) + \delta_k,$$

in particular

$$J_\epsilon(t_k z_k) = J_\epsilon(z_k) + O(\eta) + \delta_k.$$

Define $\tilde{z}_k = t_k z_k(\epsilon x)$. Let $\tilde{t}_k$ be such that $\tilde{t}_k \tilde{z}_k$ belongs to the set

$$\Sigma_\eta := \left\{ w : \int_{\mathbf{R}^N} ((h_p^\infty - \eta)|w_1|^{1+\frac{1}{p}} + (h_q^\infty - \eta)|w_2|^{1+\frac{1}{q}})dx \right.$$
$$\left. = \int_{\mathbf{R}^N} (w_1 T w_2 + w_2 T w_1)dx \right\}.$$

Then,

$$\begin{aligned}
&\epsilon^N \left(\int_{\mathbf{R}^N} (h_p^\infty - \eta)|\tilde{z}_k^1|^{1+\frac{1}{p}}dx + \int_{\mathbf{R}^N} (h_q^\infty - \eta)|\tilde{z}_k^2|^{1+\frac{1}{q}}dx \right) \\
&= \int_{\mathbf{R}^N} (h_p^\infty - \eta)|t_k z_k^1|^{1+\frac{1}{p}}dx + \int_{\mathbf{R}^N} (h_q^\infty - \eta)|t_k z_k^2|^{1+\frac{1}{q}}dx \\
&\leq \int_{\mathbf{R}^N} h_p|t_k z_k^1|^{1+\frac{1}{p}}dx + \int_{\mathbf{R}^N} h_q|t_k z_k^2|^{1+\frac{1}{q}}dx \\
&= \int_{\mathbf{R}^N} (t_k z_k^1)T_\epsilon(t_k z_k^2)dx + \int_{\mathbf{R}^N} (t_k z_k^2)T_\epsilon(t_k z_k^1)dx \\
&= \epsilon^N \left(\int_{\mathbf{R}^N} (\tilde{z}_k^1)T(\tilde{z}_k^2)dx + \int_{\mathbf{R}^N} (\tilde{z}_k^2)T(\tilde{z}_k^1)dx \right) dx.
\end{aligned}$$

We conclude that $\tilde{t}_k \leq 1$. Finally, taking η small and $R(\eta)$ large enough such that $h_p^\infty - \eta < h_p^\infty < h_p(x)$ and $h_q^\infty - \eta < h_q^\infty < h_q(x)$ if $|x| > R(\eta)$, then one has

$$\begin{aligned}\epsilon^N c_\eta \leq J_\eta := \frac{p}{p+1}\int_{\mathbf{R}^N}(h_p^\infty - \eta)|\tilde{t}_k\tilde{z}_k^1|^{1+\frac{1}{p}}dx + \frac{q}{q+1}\int_{\mathbf{R}^N}(h_q^\infty - \eta)|\tilde{t}_k\tilde{z}_k^2|^{1+\frac{1}{q}}dx \\ - \int_{\mathbf{R}^N}(\tilde{t}_k\tilde{z}_k^1)T(\tilde{t}_k\tilde{z}_k^2)dx - \int_{\mathbf{R}^N}(\tilde{t}_k\tilde{z}_k^2)T(\tilde{t}_k\tilde{z}_k^1)dx \\ \leq J_\infty(\tilde{t}_k\tilde{z}_k) \leq J_\infty(\tilde{z}_k) \leq J_\epsilon(t_k z_k) = J_\epsilon(z_k) + O(\eta) + \delta_k,\end{aligned}$$

where $c_\eta = \inf_{\Sigma_\eta} J_\eta$. We obtain the estimate

$$\epsilon^N c_\eta \leq c + O(\eta) + \delta_k. \tag{4.6}$$

From the characterization of the critical value, for η small, we can define a test function based on the ground state for c^∞ and prove that $c^\infty - O(\eta) > c_\eta$. If $\lim_{\eta\to 0} c_\eta = \hat{c} < c^\infty$, for a ground state z_η of c_η, we have

$$J_\eta(z_\eta) \leq J_\infty(z_\eta) - O(\eta) < c^\infty.$$

For $\eta_k = \frac{1}{k}$, $\{z_{\eta_k}\}$ is a bounded positive radial sequence. By the Sobolev inclusions, $z_{\eta_k} \to \hat{z}$. Taking limits in the last inequality we obtain that $c^\infty \leq J_\infty(\hat{z}) \leq c^\infty$, which is a contradiction. Thus,

$$\lim_{\eta\to 0} c_\eta = c^\infty.$$

Letting $k \to +\infty$ and using (4.6) we conclude that

$$\epsilon^N c^\infty \leq c,$$

which contradicts the assumption on the level set. □

Proof of Theorem 1.1. Choose $h(\epsilon) > 0$ such that $h(\epsilon) \to 0$ as $\epsilon \to 0$ and $\epsilon^N\{c_M + h(\epsilon)\}$ is not a critical value of J_ϵ. Let $\tilde{\Sigma}_\epsilon$ be as above with $h(\epsilon) < \epsilon^N(c^\infty - c_M)$. Lemma 4.1 shows that J_ϵ satisfies (PS) in $\tilde{\Sigma}_\epsilon$. By Lemma 3.3,

$$\sup_{w\in\tilde{\Sigma}_\epsilon}\inf_{\xi\in M_\delta}|\beta(w) - \xi| \leq \frac{\delta}{2}, \qquad \forall\epsilon < \epsilon_0$$

for some $\epsilon_0 > 0$. By Lemma 2.5, $J_\epsilon(\Phi_\epsilon(\xi)) \leq \epsilon^N(c_M + h(\epsilon)) < \epsilon^N c^\infty$, for $\xi \in M$, $0 < \epsilon < \epsilon_0$. Thus $\Phi_\epsilon(M) \subset \tilde{\Sigma}_\epsilon$. Then, we may assume that

$$\operatorname{dist}(\beta(w), M_\delta) < \frac{\delta}{2},\ \forall\epsilon < \epsilon_0,\ w \in \tilde{\Sigma}_\epsilon.$$

Thus, $\beta(\tilde{\Sigma}_\epsilon) \subset M_\delta$.

The map $\beta \circ \Phi_\epsilon$ is homotopic to the inclusion $j : M \to M_\delta$. Set $\tilde{\Sigma}_\epsilon^+ = \tilde{\Sigma}_\epsilon \cap \{w \in \Sigma_\epsilon : w \geq 0\}$. If w changes in sign, we have that $w^+ = \max\{w, 0\} \neq 0$ and $w^- = \min\{w, 0\} \neq 0$. Using the maximum principle we can prove that $\int_{\mathbf{R}^N} (w_1^+ T_\epsilon w_2^- + w_1^- T_\epsilon w_2^+) dx \leq 0$ Thus, for some positive numbers t^+ and t^-, we have that $t^+ w^+$ and $t^- w^-$ belong to $\tilde{\Sigma}_\epsilon$. We obtain

$$
\begin{aligned}
\epsilon^N (c_M + h(\epsilon)) &\geq J_\epsilon(w) \\
&\geq J_\epsilon(t^+ w^+) + J_\epsilon(t^- |w^-|) - t^+ t^- \int_{\mathbf{R}^N} (w_1^+ T_\epsilon w_2^- + w_1^- T_\epsilon w_2^+) dx \\
&\geq J_\epsilon(t^+ w^+) + J_\epsilon(t^- |w^-|) \\
&\geq J_M(t^+ \tilde{w}^+) + J_M(t^- |\tilde{w}^-|) \geq 2\epsilon^N c_M,
\end{aligned}
$$

where $\tilde{w}(x) = w(\epsilon x)$. This is a contradiction for ϵ small. Thus, the solutions obtained does not change in sign. If the solution w satisfies $w \leq 0$, then $\hat{w} = -w$ is also a positive solution. Therefore, in any case we may obtain positive solutions. By Lemma 1.1,

$$
\mathrm{cat}_{\tilde{\Sigma}_\epsilon}(\tilde{\Sigma}_\epsilon) = \mathrm{cat}_{\tilde{\Sigma}_\epsilon}(\tilde{\Sigma}_\epsilon^+) \geq \mathrm{cat}_{M_\delta}(M).
$$

We deduce by Theorem 1.2 that J_ϵ has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions.

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