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Uniqueness of solutions for some nonlinear Dirichlet problems

ALESSIO PORRETTA
Dipartimento di Matematica
Università di Roma Tor Vergata
Via della Ricerca Scientifica, 1
00133 Roma, Italy
e-mail: porretta@mat.uniroma2.it

Abstract. We consider here a class of nonlinear Dirichlet problems, in a bounded domain Ω, of the form

$$\begin{cases} -\mathrm{div}(a(x,u)\nabla u) + \mathrm{div}(\Phi(u)) = f \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

investigating the problem of uniqueness of solutions. The functions $\Phi(s)$ and $s \mapsto a(x,s)$ satisfy rather general assumptions of locally Lipschitz continuity (with possibly exponential growth) and the datum f is in $L^1(\Omega)$. Uniqueness of solutions is proved both for coercive $a(x,s)$ and for the case of $a(x,s)$ degenerating for s large.

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1 Introduction

In this work we are concerned with the problem of uniqueness of solutions for a class of elliptic equations. Precisely, we consider the following Dirichlet problem in a bounded open set $\Omega \subset \mathbf{R}^N$ ($N \geq 1$):

$$\begin{cases} -\mathrm{div}(a(x,u)\nabla u) + \mathrm{div}(\Phi(u)) = f \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We let the second order operator $A(u) = -\mathrm{div}(a(x,u)\nabla u)$ satisfy suitable coercivity conditions, and we assume that Φ belongs to $C(\mathbf{R}, \mathbf{R}^N)$ and f belongs to $L^1(\Omega)$. Unless $a(x,s)$ is constant with respect to s, A is a very simple example of nonmonotone operator, so that, even if $\Phi = 0$ and f is smooth, uniqueness can not be obtained through standard monotonicity arguments.

In fact, for smooth data f, N. Trudinger proved in [18] the uniqueness of smooth solutions of (1.1) if $a(x,s)$ and $\Phi(s)$ are Lipschitz continuous with respect to s. In case f belongs to $H^{-1}(\Omega)$, uniqueness in the class of weak solutions in $H^1_0(\Omega)$ was proved in [1] if $\Phi = 0$, and then in [7], where Φ is still assumed to be Lipschitz continuous. Recent further generalizations on the growth conditions on $a(x,s)$ as a function of s can be found in [4], [15]. Other uniqueness results for equations as in (1.1) can be found in [10], [11], [12].

These results are always in the context of finite energy solutions, that is for solutions u such that $a(x,u)|\nabla u|^2$ belongs to $L^1(\Omega)$. With respect to the datum f, this means, roughly speaking, that f can be taken in $L^m(\Omega)$, with $m \geq \frac{2N}{N+2}$. In this paper we obtain two main extensions of the results previously mentioned. In fact, we allow both the datum f to belong to $L^1(\Omega)$ and the functions $s \mapsto a(x,s)$ and $\Phi(s)$ to satisfy fairly general assumptions of **locally** Lipschitz continuity, so that our results are new even in the case that f belongs to $H^{-1}(\Omega)$.

Let us recall that assuming f to be in $L^1(\Omega)$ implies that the solutions of (1.1) may have infinite energy and the weak formulation is not appropriate for the uniqueness problem. In fact, even if $\Phi = 0$ and $a(x,s)$ does not depend on s, it has been shown (see [16], [17]) that the distributional formulation is not strong enough to give uniqueness in case of L^1 data. For this reason, in [3] it was introduced the notion of entropy solution (which we will recall later), and the authors proved that, for a nonlinear monotone operator, which only depends on ∇u, the Dirichlet problem in L^1 is well-posed in the formulation of entropy solutions. Following this approach, here we prove uniqueness of entropy solutions of (1.1), extending then the results proved in [3] (see also [8], [13]) to a class of nonmonotone operators.

On the other hand, since we do not assume that Φ is globally Lipschitz continuous (hence Φ does not need to grow linearly at infinity), the term $\Phi(u)$ may not belong to $L^1(\Omega)$ even if f belongs to $H^{-1}(\Omega)$. Thus in this general context the distributional formulation can not even be applied, in fact existence results for (1.1) have been proved in the framework of so-called renormalized solutions or of entropy solutions both if f is in $H^{-1}(\Omega)$ (see [6]) and if f is in $L^1(\Omega)$ (see [5], [14]). Here we prove the uniqueness of the solutions found in these papers, in presence of L^1 data and only assuming that Φ is locally Lipschitz continuous and has at most an exponential growth at infinity.

As an example, we prove that there exists a unique entropy solution of the problem

$$\begin{cases} -\mathrm{div}((1 + a(x)e^{|u|})\nabla u) + \mathrm{div}(e^{\Lambda u}) = f \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Lambda \in \mathbf{R}^N$, $a(x)$ is a nonnegative bounded function (for instance, $a = \chi_E$, the characteristic function of a set $E \subset \Omega$) and f is in $L^1(\Omega)$.

To obtain our results, we take great advantage from the unifying framework of so-called entropy solutions introduced in [3], which allow to handle the cases where the dependence on u of the operator is of possibly different types. Actually, the only regularity asked on entropy solutions is that u is almost everywhere finite and the truncations $T_k(u) = \max(-k, \min(u,k))$ belong to $H_0^1(\Omega)$ for every $k > 0$. Of course, the notion of entropy solution coincides with that of weak solution for solutions in $H_0^1(\Omega)$. This definition also needs a new functional setting introduced in [3].

Definition 1.1 We define $\mathcal{T}_0^{1,2}(\Omega)$ as the set of measurable functions $u : \Omega \to \mathbf{R}$ almost everywhere finite and such that $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$.

With very easy examples it can be checked that $\mathcal{T}_0^{1,2}(\Omega)$ is not a vector space. However, if u is in $\mathcal{T}_0^{1,2}(\Omega)$ and φ is in $H_0^1(\Omega) \cap L^\infty(\Omega)$ then $u + \varphi$ belongs to $\mathcal{T}_0^{1,2}(\Omega)$. It is also proved in [3], Lemma 2.1, that it is possible to extend the notion of gradient to this class of functions, by setting

$$\nabla u(x) = \nabla T_k(u)(x) \qquad \text{a.e. in } \{x : |u(x)| < k\} \quad \forall k > 0. \tag{1.2}$$

In fact, it is proved that there is only one function (up to almost everywhere equivalence) ∇u satisfying (1.2) for every $k > 0$ and which we will call the gradient of u (it coincides with the gradient defined as a distribution if $u \in W_0^{1,1}(\Omega)$). Henceforth, the gradient appearing in the equation will always be the gradient as defined above.

Definition 1.2 A function u in $\mathcal{T}_0^{1,2}(\Omega)$ is an entropy solution of (1.1) if it satisfies

$$\int_\Omega a(x,u)\nabla u \nabla T_k(u-\varphi)\,dx - \int_\Omega \Phi(u)\nabla T_k(u-\varphi)\,dx = \int_\Omega f\,T_k(u-\varphi)\,dx, \\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \forall k > 0. \tag{1.3}$$

Remark 1.3 Every term in (1.3) has a meaning since $T_k(u-\varphi)$ is bounded (hence the right hand side is well defined) and since $\nabla T_k(u-\varphi) = 0$ if $|u| > M = k + \|\varphi\|_{L^\infty(\Omega)}$, so that $a(x,u)\nabla u \nabla T_k(u-\varphi) = a(x, T_M(u))\nabla T_M(u)\nabla T_k(T_M(u)-\varphi)$ and $\Phi(u)\nabla T_k(u-\varphi) = \Phi(T_M(u))\nabla T_k(T_M(u)-\varphi)$ and both these expressions are in $L^1(\Omega)$ due to $u \in \mathcal{T}_0^{1,2}(\Omega)$.

Note also that in the original formulation of entropy solution given in [3] the equality sign in (1.3) is replaced by an inequality and would read as follows in our context:

$$\int_\Omega a(x,u)\nabla u \nabla T_k(u-\varphi)\,dx - \int_\Omega \Phi(u)\nabla T_k(u-\varphi)\,dx \le \int_\Omega f\,T_k(u-\varphi)\,dx. \tag{1.4}$$

On the other hand, using similar arguments to those of Lemma 2.1 below (in particular, as to prove (2.5), choosing $\varphi = T_h(u) + T_k(u-\psi)$ in (1.4) with $\psi \in$

$H_0^1(\Omega) \cap L^\infty(\Omega)$, $h > k + \|\psi\|_{L^\infty(\Omega)}$, writing $T_k(u - T_h(u) - T_k(u-\psi)) = T_{2k}(u - T_h(u)) - T_k(u - \psi)$ and using estimate (2.2)), one can prove that (1.4) implies (1.3), so that the inequality and the equality sign in the entropy equation are equivalent (see also [9]).

The main result we obtain on (1.1) is the following one.

Theorem 1.4 *Let $N \geq 1$, and let f belong to $L^1(\Omega)$. Assume that $a(x, s)$ is a Carathéodory function, $\Phi \in C(\mathbf{R}, \mathbf{R}^N)$ and that there exist constants α, $\mu > 0$ such that $a(x, s)$ and $\Phi(s)$ satisfy:*

$$\begin{aligned} &\alpha \leq a(x,s) \leq \mu e^{\mu|s|}, \qquad \forall\, s \in \mathbf{R}, \quad \text{a.e.} \quad x \in \Omega, \\ &|a(x,s) - a(x,t)| \leq \mu(e^{\mu|s|} + e^{\mu|t|})|s-t|, \quad \forall\, s\,,\, t \in \mathbf{R}, \quad \text{a.e.} \quad x \in \Omega, \\ &|\Phi(s) - \Phi(t)| \leq \mu(e^{\mu|s|} + e^{\mu|t|})|s-t|\,, \quad \forall\, s\,,\, t \in \mathbf{R}, \quad \text{a.e.} \quad x \in \Omega. \end{aligned} \tag{1.5}$$

Then there exists a unique entropy solution of (1.1).

Note that a large number of examples fit the assumptions of the previous theorem, including the classical case that $a(x, s)$ is bounded and Lipschitz continuous with respect to s and $\Phi(s)$ is Lipschitz continuous as well. Rapidly oscillating functions like $a(x, s) = (1 + a(x)e^{|s|} \sin^2(e^{|s|}))$ are also allowed as well as nonlinear local perturbations of the Laplacian, if $a(x, s) = 1 + a(x)|s|^m + b(x)e^s$ for nonnegative bounded functions $a(x)$ and $b(x)$.

Since the proof of Theorem 1.4 will be achieved after several steps, let us explain first the idea of our work. In order to handle the difficulty of having infinite energy solutions, we review the method used in [7] for finite energy solutions, showing how it can be adapted to possibly different problems which also lead to consider the case of $s \mapsto a(x, s)$ only locally Lipschitz continuous. Roughly speaking, this method consists in using that the difference of two solutions u_1, u_2 of (1.1) satisfies the elliptic equation

$$\begin{cases} -\text{div}(a(x, u_1)\nabla(u_1 - u_2)) = \text{div}(\Phi(u_2) - \Phi(u_1) \\ \quad + [a(x, u_1) - a(x, u_2)]\nabla u_2) \text{ in } \Omega, \\ u_1 - u_2 = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.6}$$

Then $u_1 - u_2$ is proved to be zero by using the Lipschitz continuity of $a(x, s)$ and $\Phi(s)$ and the fact that ∇u_2 lies in $L^2(\Omega)$. This method fails to work for singular data since the right hand side of (1.6) lacks of regularity and in particular ∇u_2 is no longer in $L^2(\Omega)$. We first remark that this obstacle can be overcome at least for those Lipschitz functions whose derivative rapidly goes to zero at infinity, for which we can recover the fact that $\frac{a(x,u_1)-a(x,u_2)}{u_1-u_2}\nabla u_2$ and $\Phi(u_1) - \Phi(u_2)$ belong to $L^2(\Omega)$. Then we show that problem (1.1) can always be reduced to this case by performing a suitable change of unknown. Indeed, setting $u = \varphi(v)$, with φ an unbounded increasing function on $\mathbf{R}$ such that $\varphi(0) = 0$, (1.1) can be transformed into the following problem:

$$\begin{cases} -\mathrm{div}(\tilde{a}(x,v)\nabla v) + \mathrm{div}(\tilde{\Phi}(v)) = f \text{ in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\tilde{a}(x,s) = a(x,\varphi(s))\varphi'(s)$, $\tilde{\Phi}(s) = \Phi(\varphi(s))$, and the function φ can be chosen in such a way that the modulus of Lipschitz continuity of $s \mapsto \tilde{a}(x,s)$ and $\tilde{\Phi}(s)$ rapidly goes to zero at infinity. On the other hand, it can be easily seen that a similar change of unknown affects the coercivity properties of the equation, so that problem (1.7) may be no longer uniformly elliptic. The main novelty in the technique we use is an extension of the method of [7] which allows to work under possibly degenerate coercivity conditions and only using a locally Lipschitz continuity of $a(x,s)$ and $\Phi(s)$. The main results we obtain on (1.1) are in fact derived from those we can prove on (1.7) where φ is a logarithmic function.

In order to perform this program, the paper is planned in the following way. Some basic properties of entropy solutions will be recalled in Section 2, where we also prove the equivalence of problems (1.1) and (1.7) in terms of entropy solutions. Next we will prove a general uniqueness result which can be obtained through the method developed in [7] both for a coercive and for a non coercive operator. The proof of Theorem 1.4 will then follow as an easy corollary from the study of (1.7) and is left to Section 3, together with an extension to more general operators satisfying a possibly degenerate ellipticity condition.

Finally, let us stress that the quasilinear structure of the operator A in (1.1) is not necessary but it has been chosen to allow a simpler presentation (see Remark 3.4), and that the same method used to get uniqueness also provides a comparison principle with respect to possibly different data f_1, f_2 (see Corollary 3.3).

2 The basic uniqueness argument

Let us first set our assumptions in a more general framework. Precisely, let $a(x,s)$ be a Carathéodory function (that is $a(\cdot,s)$ is measurable for every s in $\mathbf{R}$ and $a(x,\cdot)$ is continuous for almost every x in Ω) and let Φ belong to $W^{1,\infty}_{\mathrm{loc}}(\mathbf{R},\mathbf{R}^N)$, the space of locally Lipschitz functions from $\mathbf{R}$ into $\mathbf{R}^N$. There is no loss of generality in assuming, henceforth, the normalization condition

$$\Phi(0) = 0.$$

Setting $\mathbf{R}^+ = (0,+\infty)$ and $C(\mathbf{R},\mathbf{R}^+)$ the space of continuous functions from $\mathbf{R}$ into $\mathbf{R}^+$, we assume that for every $s \in \mathbf{R}$ and almost every x in Ω:

$$a(x,s) \geq \alpha(s) > 0 \qquad \alpha(s) \in C(\mathbf{R},\mathbf{R}^+). \tag{2.1}$$

The first step in the proof of our uniqueness results is the following lemma which collects some fundamental properties of entropy solutions.

Lemma 2.1 *Assume that* (2.1) *holds true. Let* u *be an entropy solution of* (1.1) *and let* f *belong to* $L^1(\Omega)$. *Then we have:*

$$\lim_{h\to+\infty}\int_{\{h\le|u|\le h+k\}} a(x,u)|\nabla u|^2\,dx = 0 \quad \forall\, k>0, \tag{2.2}$$

$$\int_\Omega g(u)|\nabla u|^2 < +\infty,$$
$$\textit{for any } g\in C(\mathbf{R},\mathbf{R}^+) \textit{ such that } \int_{-\infty}^{+\infty}\frac{g(s)}{\alpha(s)}ds < +\infty, \tag{2.3}$$

$$\int_\Omega a(x,u)\nabla u\nabla T_k(\psi(u)-\varphi)dx - \int_\Omega \Phi(u)\nabla T_k(\psi(u)-\varphi)dx$$
$$= \int_\Omega f\,T_k(\psi(u)-\varphi)dx,$$
$$\forall\varphi\in H^1_0(\Omega)\cap L^\infty(\Omega)\,,\forall k>0, \tag{2.4}$$

for any $\psi\in C^1(\mathbf{R})$ *such that* $\psi(0)=0, \lim_{s\to\pm\infty}\psi(s)=\pm\infty$ *and* $\psi'(s)>0\,\forall s\in\mathbf{R}$.

$$\int_\Omega a(x,u)\nabla u\nabla(\psi(u)\varphi) = \int_\Omega \nabla\varphi\left(\int_0^u \Phi'(s)\psi(s)\,ds\right)dx + \int_\Omega f\,\psi(u)\varphi, \tag{2.5}$$

for any $\psi\in W^{1,\infty}(\mathbf{R})$ *with compact support and for any* $\varphi\in H^1_0(\Omega)\cap L^\infty(\Omega)$.

Proof. To prove (2.2), we take $\varphi=T_h(u)$ in (1.3). Since we have:

$$\int_\Omega \Phi(u)\nabla T_k(u-T_h(u))\,dx = \int_\Omega \Phi(T_{h+k}(u))\nabla T_{h+k}(u)\,\chi_{\{h\le|u|<h+k\}}\,dx$$
$$= \int_\Omega \operatorname{div}\left[\int_0^{T_{h+k}(u)}\Phi(s)\,\chi_{\{h\le|s|<h+k\}}\,ds\right]dx,$$

from the divergence theorem in Sobolev spaces and since $T_{h+k}(u)\in H^1_0(\Omega)$ we conclude that

$$\int_\Omega \Phi(u)\nabla T_k(u-T_h(u))\,dx = 0.$$

Then we get:

$$\int_{\{h\le|u|\le h+k\}} a(x,u)|\nabla u|^2\,dx \le k\int_{\{|u|\ge h\}}|f|,$$

and (2.2) follows from Lebesgue's theorem since f is in $L^1(\Omega)$ and u is almost everywhere finite.

To prove (2.3), we choose $T_h(u)-\psi_h(u)$ as test function in (1.3), where now $\psi_h(u)=\int_0^{T_h(u)}\frac{g(t)}{\alpha(t)}\,dt$. Then we have:

$$\int_\Omega a(x,u)\nabla u\nabla T_k(u-T_h(u)+\psi_h(u))\,dx$$

$$-\int_\Omega \Phi(u)\nabla T_k(u - T_h(u) + \psi_h(u))\,dx \qquad (2.6)$$
$$= \int_\Omega f\,T_k(u - T_h(u) + \psi_h(u))\,dx.$$

Now, setting $M = h + k + \int_{-\infty}^{+\infty} \frac{g(t)}{\alpha(t)}\,dt$ and $\Gamma(s) = s - T_h(s) + \psi_h(s)$, we have that

$$\begin{aligned}\int_\Omega &\Phi(u)\nabla T_k(u - T_h(u) + \psi_h(u))dx \\ &= \int_\Omega \Phi(T_M(u))\nabla T_M(u)\Gamma'(T_M(u))\chi_{\{|\Gamma(u)|\le k\}}dx \\ &= \int_\Omega \operatorname{div}\left[\int_0^{T_M(u)} \Phi(s)\Gamma'(s)\,\chi_{\{|\Gamma(s)|\le k\}}\,ds\right] dx = 0,\end{aligned}$$

so that, choosing $k = \int_{-\infty}^{+\infty} \frac{g(t)}{\alpha(t)}\,dt$ we obtain from (2.6):

$$\int_{\{|u|\le h\}} a(x,u)|\nabla u|^2\,\frac{g(u)}{\alpha(u)}\,dx \le k\|f\|_{L^1(\Omega)},$$

which yields, using (2.1),

$$\int_{\{|u|\le h\}} g(u)|\nabla u|^2\,dx \le C.$$

Letting h tend to infinity, we can easily conclude using Fatou's lemma.

We deal now with (2.4). Let $\psi \in C^1(\mathbf{R})$ be such that $\psi(0) = 0$, $\lim_{s\to\pm\infty}\psi(s) = \pm\infty$ and $\psi'(s) > 0\ \forall\, s \in \mathbf{R}$. First observe that the function $\tilde{u} = \psi(u)$ is almost everywhere finite and its truncations belong to $H_0^1(\Omega)$. This is easy to establish, in fact $\tilde{u}$ is almost everywhere finite since such is u and ψ is unbounded. Moreover we have, for every $k > 0$, that $T_k(\tilde{u}) = T_k(\psi(u)) = T_k(\psi(T_M(u)))$, with $M = \max(\psi^{-1}(k), -\psi^{-1}(-k))$. Since $T_M(u)$ is in $H_0^1(\Omega)\cap L^\infty(\Omega)$ and ψ is $C^1(\mathbf{R})$ and $\psi(0) = 0$, we deduce that $T_k(\tilde{u})$ belongs to $H_0^1(\Omega)$ for every $k > 0$.

Thus $\psi(u)$ belongs to $\mathcal{T}_0^{1,2}(\Omega)$, and moreover note that by definition of $\nabla\tilde{u}$ given by (1.2) we have:

$$\begin{aligned}\nabla\tilde{u}(x)\chi_{\{|\tilde{u}(x)|<k\}} &= \nabla T_k(\tilde{u}(x))\chi_{\{|\tilde{u}(x)|<k\}} \\ &= (\psi)'(T_M(u))\nabla T_M(u)\,\chi_{\{|\psi(T_M(u(x)))|<k\}}\,\chi_{\{|\tilde{u}(x)|<k\}},\end{aligned}$$

so that by definition of $\tilde{u}$ and ∇u we get:

$$\nabla\tilde{u}(x)\chi_{\{|\tilde{u}(x)|<k\}} = \psi'(u(x))\nabla u\chi_{\{|\tilde{u}(x)|<k\}} \qquad \forall k > 0,$$

which yields that (recall that ∇u and $\nabla\tilde{u}$ are unique up to almost everywhere equivalence)

$$\nabla\tilde{u} = \psi'(u)\nabla u \qquad \text{a.e. in } \Omega,$$

which proves that the well-known chain rule for gradients also holds for these "new" gradients. Since we have proved that $\psi(u) \in \mathcal{T}_0^{1,2}(\Omega)$, we can take $T_h(u) - T_M(\psi(u)) + \varphi$ as test function in (1.3), with $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $M = k + \|\varphi\|_{L^\infty(\Omega)}$, to get

$$\int_\Omega a(x,u)\nabla u \nabla T_k(u - T_h(u) + T_M(\psi(u)) - \varphi)\,dx$$
$$- \int_\Omega \Phi(u)\nabla T_k(u - T_h(u) + T_M(\psi(u)) - \varphi)\,dx$$
$$= \int_\Omega f\,T_k(u - T_h(u) + T_M(\psi(u)) - \varphi)\,dx.$$

Note that for $h > \max(-\psi^{-1}(-M), \psi^{-1}(M))$, we have that in the set $\{|u| > h\}$ either $u > h$ so that $\psi(u) > M = k + \|\varphi\|_{L^\infty(\Omega)}$, hence $u - T_h(u) + T_M(\psi(u)) - \varphi = u - h + k + \|\varphi\|_{L^\infty(\Omega)} - \varphi > k$; or $u < -h$, so that $\psi(u) < -M$ and $u - T_h(u) + T_M(\psi(u)) - \varphi = u + h - k - \|\varphi\|_{L^\infty(\Omega)} - \varphi < -k$. In both cases we can deduce that $\nabla T_k(u - T_h(u) + T_M(\psi(u)) - \varphi) = 0$ in the set $\{|u| > h\}$ provided h is large enough. Then we have:

$$\int_{\{|u|\le h\}} a(x,u)\nabla u \nabla T_k(T_M(\psi(u)) - \varphi)\,dx$$
$$- \int_{\{|u|\le h\}} \Phi(u)\nabla T_k(T_M(\psi(u)) - \varphi)\,dx$$
$$= \int_\Omega f\,T_k(u - T_h(u) + T_M(\psi(u)) - \varphi)\,dx.$$

Since $T_k(T_M(\psi(u)) - \varphi) = T_k(\psi(u) - \varphi)$, letting h tend to infinity and using Lebesgue's theorem we get (2.4).

Note that we also proved that the function $\tilde{u} = \psi(u)$ satisfies the entropy formulation

$$\int_\Omega \tilde{a}(x,\tilde{u})\nabla \tilde{u} \nabla T_k(\tilde{u} - \varphi)\,dx - \int_\Omega \tilde{\Phi}(\tilde{u})\nabla T_k(\tilde{u} - \varphi)\,dx$$
$$= \int_\Omega f\,T_k(\tilde{u} - \varphi)\,dx, \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \forall k > 0,$$

where

$$\tilde{a}(x,s) = \frac{a(x,\psi^{-1}(s))}{\psi'(\psi^{-1}(s))}$$

and $\tilde{\Phi}(s) = \Phi(\psi^{-1}(s))$.

Finally, we are left with the proof of (2.5). Let then $\psi \in W^{1,\infty}(\mathbf{R})$ have compact support, namely assume that $\psi(s) \equiv 0$ if $|s| \ge L$.

Choosing $k = \|\varphi\|_{L^\infty(\Omega)} \|\psi\|_{L^\infty(\mathbf{R})}$ we have from (1.3):

$$\int_\Omega a(x,u)\nabla u \nabla T_k(u - T_h(u) + \psi(u)\varphi)\,dx \\ - \int_\Omega \Phi(u)\nabla T_k(u - T_h(u) + \psi(u)\varphi)\,dx = \int_\Omega f\,T_k(u - T_h(u) + \psi(u)\varphi)\,dx.$$

Note that for every $h > L$ since $k = \|\varphi\|_{L^\infty(\Omega)} \|\psi\|_{L^\infty(\mathbf{R})}$ we have $T_k(u - T_h(u) + \psi(u)\varphi) = T_k(u - T_h(u)) + \psi(u)\varphi$, so that we get

$$\int_{\{h \le |u| \le h+k\}} a(x,u)|\nabla u|^2\,dx + \int_\Omega a(x,u)\nabla u \nabla(\psi(u)\varphi)\,dx \\ - \int_\Omega \Phi(u)\nabla T_k(u - T_h(u))\,dx - \int_\Omega \Phi(u)\nabla(\psi(u)\varphi)\,dx \\ = \int_\Omega f\,T_k(u - T_h(u))\,dx + \int_\Omega f\,\psi(u)\varphi\,dx.$$

Since

$$\int_\Omega \Phi(u)\nabla T_k(u - T_h(u))\,dx = \int_\Omega \operatorname{div}\left[\int_0^{T_{h+k}(u)} \Phi(s)\chi_{\{h \le |s| \le h+k\}}\,ds\right]dx = 0,$$

it follows that:

$$\int_{\{h \le |u| \le h+k\}} a(x,u)|\nabla u|^2\,dx + \int_\Omega a(x,u)\nabla u \nabla(\psi(u)\varphi)\,dx \\ = \int_\Omega \Phi(u)\nabla(\psi(u)\varphi)\,dx + \int_\Omega f\,T_k(u - T_h(u))\,dx + \int_\Omega f\,\psi(u)\varphi\,dx.$$

Letting h tend to infinity we deduce from Lebesgue's theorem and (2.2):

$$\int_\Omega a(x,u)\nabla u \nabla(\psi(u)\varphi)\,dx = \int_\Omega \Phi(u)\nabla(\psi(u)\varphi)\,dx + \int_\Omega f\,\psi(u)\varphi\,dx. \tag{2.7}$$

On the other hand, since $\operatorname{Supp}(\psi(s)) \subset [-L, L]$ and $\varphi \in H_0^1(\Omega)$ we have:

$$\int_\Omega \Phi(u)\nabla(\psi(u)\varphi)\,dx = \int_\Omega \operatorname{div}\left(\int_0^u \Phi(s)\psi'(s)\,ds\right)\varphi\,dx \\ + \int_\Omega \Phi(u)\psi(u)\nabla\varphi\,dx = \int_\Omega \left[\Phi(u)\psi(u) - \left(\int_0^u \Phi(s)\psi'(s)\,ds\right)\right]\nabla\varphi\,dx.$$

Since $\Phi(s) \in W^{1,\infty}_{\text{loc}}(\mathbf{R})$ and $\psi(s) \in W^{1,\infty}(\mathbf{R})$ we have

$$\Phi(t)\psi(t) - \int_0^t \Phi(s)\psi'(s)\,ds = \int_0^t \Phi'(s)\,\psi(s)\,ds \qquad \forall\, t \in \mathbf{R},$$

so that we deduce

$$\int_\Omega \Phi(u)\nabla(\psi(u)\varphi)\,dx = \int_\Omega \left(\int_0^u \Phi'(t)\,\psi(t)\,dt\right)\nabla\varphi\,dx,$$

and then (2.5) follows from (2.7). □

As a consequence of (2.4) of Lemma 2.1 we obtain the following important result.

Corollary 2.2 *Let f belong to $L^1(\Omega)$, and let $\psi \in C^1(\mathbf{R})$ be such that $\psi(0) = 0$, $\lim_{s\to\pm\infty} \psi(s) = \pm\infty$ and $\psi'(s) > 0$ for any $s \in \mathbf{R}$. Then u is an entropy solution of* (1.1) *if and only if $\tilde u = \psi(u)$ is an entropy solution of*

$$\begin{cases} -\mathrm{div}(\tilde a(x,\tilde u)\nabla\tilde u) + \mathrm{div}(\tilde\Phi(\tilde u)) = f \;\; in\ \Omega, \\ \tilde u = 0 \quad on\ \partial\Omega, \end{cases} \tag{2.8}$$

where $\tilde a(x,s) = \frac{a(x,\psi^{-1}(s))}{\psi'(\psi^{-1}(s))}$ and $\tilde\Phi(s) = \Phi(\psi^{-1}(s))$.

Our next purpose is to see how suitable Lipschitz type assumptions on $\Phi(s)$ and $s \mapsto a(x,s)$ may yield a uniqueness result for solutions of (1.1). To this aim, we assume henceforth that there exist δ, $\gamma > 0$ and functions L, $L_1 \in C(\mathbf{R},\mathbf{R}^+)$ such that:

$$|a(x,s) - a(x,t)| \le [L(s) + L(t)]\,|s-t| \qquad \forall s\,,\ t\ : |s-t| \le \delta, \tag{2.9}$$

$$|\Phi(s) - \Phi(t)| \le [L_1(s) + L_1(t)]\,|s-t| \qquad \forall s\,,\ t\ : |s-t| \le \delta, \tag{2.10}$$

$$\gamma^{-1} \le \frac{L(s+\theta)}{L(s)} \le \gamma \qquad \forall s \in \mathbf{R}\,, \quad \forall\theta \in [0,1]. \tag{2.11}$$

$$\gamma^{-1} \le \frac{L_1(s+\theta)}{L_1(s)} \le \gamma \qquad \forall s \in \mathbf{R}\,, \quad \forall\theta \in [0,1]. \tag{2.12}$$

Clearly, the functions $L(s)$, $L_1(s)$ are meant to play the role of the modulus of locally Lipschitz continuity of the functions $s \mapsto a(x,s)$ and $\Phi(s)$ respectively. In fact, (2.9)–(2.10) are equivalent to assuming that $s \mapsto a(x,s)$ and $\Phi(s)$ belong to $W^{1,\infty}_{\mathrm{loc}}(\mathbf{R})$ and that their weak derivatives are bounded (uniformly on x in Ω) by two continuous functions L, L_1 satisfying (2.11)–(2.12). In particular, if $\Phi'(s)$ denotes the weak derivative of Φ in $W^{1,\infty}_{\mathrm{loc}}(\mathbf{R})$, (2.10) implies that $|\Phi'(s)| \le 2L_1(s)$ for every s. Note also that the function $L(s)$ contains some informations on the growth assumptions from above on $a(x,s)$ (take $t = 0$ in (2.9)). However, we do not assume that $L(s)$ is nondecreasing, since for our argument it will be essential to admit Lipschitz functions with rapidly decreasing derivatives, for instance, $a(x,s) = b(x) + \log(1 + |s|)$, with $\alpha \le b(x) \le \beta$ (in this case, we can take $\delta = \frac{1}{2}$ and $L(s) = \min(1, \frac{1}{|s|})$). Further examples of $a(x,s)$, $\Phi(s)$ satisfying (2.9)–(2.12) are given by $a(x,s) = 1 + a(x)|s|^m + b(x)\sin^2(|s|)\,|s|^q$ with m, $q \ge 1$ (here $L(s) \sim c\,|s|^{\max(q,m-1)}$), or similarly $a(x,s) = 1 + a(x)|s|^m + b(x)e^s$, where

$a(x)$ and $b(x)$ are nonnegative bounded functions, and $\Phi(s) = e^{|s|}\sin^2(e^{|s|}))$ (here $L_1(s) = ce^{2|s|}$).

Finally, note that assumptions (2.11) and (2.12) imply that

$$\frac{1}{\gamma} \le \frac{L(s)}{L(t)} \le \gamma\ , \qquad \frac{1}{\gamma} \le \frac{L_1(s)}{L_1(t)} \le \gamma \qquad \forall s\, ,\, t \in \mathbf{R}\ :\ |s-t| \le 1, \tag{2.13}$$

and moreover that L as well as L_1 have at most an exponential growth at infinity; indeed, writing $s = [s] + s - [s]$, where $[s]$ is the highest integer smaller or equal to s, we have $L(s) \le \gamma L([s]) \le \gamma^{[s]+1}L(0) \le L(0)\gamma^{s+1}$.

The conclusion of Corollary 2.2 says that, in order to obtain uniqueness for problem (1.1), we can decide to replace it with problem (2.8) with a suitable change of variable $\tilde{u} = \psi(u)$. Of course, performing this change of unknown both the coercivity function $\alpha(s)$ and the Lipschitz modulus functions $L(s)$ and $L_1(s)$ of problem (1.1) are changed into different functions $\tilde{\alpha}$, $\tilde{L}$ and $\tilde{L}_1(s)$ of problem (2.8). It may happen, for instance, that the new problem (2.8) is no longer uniformly elliptic, so that, in order to keep a higher number of possibilities to transform our problem (1.1) into (2.8), it is essential for us to obtain uniqueness results even in the case that $\alpha(s)$ goes to zero as s tends to infinity. For instance, the change of unknown $\tilde{u} = (e^{|u|}-1)\frac{u}{|u|}$ gets $\alpha(s)$ into $\tilde{\alpha}(s) = \frac{\alpha(\frac{s}{|s|}\log(1+|s|))}{1+|s|}$, so that if $\alpha(s) \ge \alpha_0 > 0$ we find $\tilde{\alpha}(s) \ge \frac{\alpha_0}{1+|s|}$, which means that $\tilde{u}$ is solution of a non uniformly elliptic problem. The aim of the next propositions is to refine, in this general context, the uniqueness argument presented in [7] for weak solutions.

Proposition 2.3 *Assume* (2.1), (2.9)–(2.12) *and let* u, v *be two entropy solutions of* (1.1) *such that* $\frac{L(u)^2}{\alpha(u)}|\nabla u|^2$, $\frac{L(v)^2}{\alpha(v)}|\nabla v|^2$, $\frac{L_1(u)^2}{\alpha(u)}$ *and* $\frac{L_1(v)^2}{\alpha(v)}$ *belong to* $L^1(\Omega)$. *Then we have:*

$$\lim_{\varepsilon\to 0}\ \frac{1}{\varepsilon^2}\int_\Omega a(x,u)|\nabla T_\varepsilon(u-v)|^2\,dx = 0.$$

Proof. Let us introduce the function:

$$\psi_h(s) = \begin{cases} 1 & \text{if } 0 \le s \le h, \\ h+1-s & \text{if } h \le s \le h+1, \\ 0 & \text{if } s > h+1, \\ h(-s) & \text{if } s \le 0. \end{cases}$$

Note that $\psi_h \in W^{1,\infty}(\mathbf{R})$, $\mathrm{Supp}(\psi_h) \subset [-h-1, h+1]$, $0 \le \psi_h \le 1$ and ψ_h tends to 1 as h tends to infinity. We take $\psi = \psi_h$ and $\varphi = T_\varepsilon(T_{h+1}(u) - v)$ in (2.5) written for u and for v, with $\varepsilon \le \min(\delta, 1)$ (δ appears in (2.9)–(2.10)). Subtracting the two equations we get:

$$\int_\Omega a(x,u)\psi_h(u)\,\nabla u \nabla T_\varepsilon(T_{h+1}(u)-v)dx$$
$$+\int_\Omega a(x,u)|\nabla u|^2\psi_h'(u)T_\varepsilon(T_{h+1}(u)-v)dx$$

$$
\begin{aligned}
&+\int_\Omega a(x,v)\psi_h(v)\nabla v\nabla T_\varepsilon(v-T_{h+1}(u))dx\\
&+\int_\Omega a(x,v)|\nabla v|^2\psi_h'(v)T_\varepsilon(v-T_{h+1}(u))dx\\
=&\int_\Omega\left(\int_v^u\Phi'(t)\,\psi_h(t)dt\right)\,\nabla T_\varepsilon(T_{h+1}(u)-v)dx\\
&+\int_\Omega f(\psi_h(v)-\psi_h(u))T_\varepsilon(v-T_{h+1}(u))dx.
\end{aligned}
$$

which yields, by definition of ψ_h,

$$
\begin{aligned}
&\int_\Omega a(x,u)\psi_h(u)\,|\nabla T_\varepsilon(u-v)|^2\,dx\\
&\leq\int_\Omega[a(x,u)\psi_h(u)-a(x,v)\psi_h(v)]\,\nabla v\nabla T_\varepsilon(v-T_{h+1}(u))\,dx\\
&\quad+\varepsilon\int_{\{h\leq|u|\leq h+1\}}a(x,u)|\nabla u|^2\,dx+\varepsilon\int_{\{h\leq|v|\leq h+1\}}a(x,v)|\nabla v|^2\,dx \qquad (2.14)\\
&\quad+\int_\Omega\left(\int_v^{T_{h+1}(u)}\Phi'(t)\,\psi(t)\,dt\right)\,\nabla T_\varepsilon(T_{h+1}(u)-v)\,dx\\
&\quad+\int_\Omega f\,(\psi_h(v)-\psi_h(u))T_\varepsilon(v-T_{h+1}(u))\,dx.
\end{aligned}
$$

We will henceforth denote by $\omega^\varepsilon(h)$ those terms which converge to zero as h tends to infinity for every fixed $\varepsilon>0$ and by $\omega(\varepsilon)$ those which do not depend on h and converge to zero as ε tends to zero. Moreover we denote with C possibly different constants which do not depend on ε or on h.

Clearly, by Lebesgue's theorem we have that for every $\varepsilon>0$

$$
\lim_{h\to\infty}\int_\Omega f\,(\psi_h(v)-\psi_h(u))T_\varepsilon(v-T_{h+1}(u))\,dx=0,
$$

so that from (2.14) we obtain, using also (2.2),

$$
\begin{aligned}
&\int_\Omega a(x,u)\psi_h(u)\,|\nabla T_\varepsilon(u-v)|^2\,dx\\
&\leq\int_\Omega[a(x,u)\psi_h(u)-a(x,v)\psi_h(v)]\,\nabla v\nabla T_\varepsilon(v-T_{h+1}(u))\,dx \qquad (2.15)\\
&\quad+\int_\Omega\left(\int_v^{T_{h+1}(u)}\Phi'(t)\,\psi(t)\,dt\right)\,\nabla T_\varepsilon(T_{h+1}(u)-v)\,dx+\omega^\varepsilon(h),
\end{aligned}
$$

and then

$$
\begin{aligned}
&\int_\Omega a(x,u)\psi_h(u)|\nabla T_\varepsilon(u-v)|^2\,dx\\
&\leq\int_\Omega\psi_h(u)\,|a(x,u)-a(x,v)|\,|\nabla v|\,|\nabla T_\varepsilon(v-T_{h+1}(u))|\,dx
\end{aligned}
$$

$$+\int_\Omega a(x,v)|\psi_h(u)-\psi_h(v)|\,|\nabla v|\,|\nabla T_\varepsilon(v-T_{h+1}(u))|\,dx \qquad (2.16)$$
$$+\int_\Omega\left(\int_v^{T_{h+1}(u)}\Phi'(t)\,\psi(t)\,dt\right)\nabla T_\varepsilon(T_{h+1}(u)-v)\,dx+\omega^\varepsilon(h).$$

Using (2.9) and (2.1), and since $\psi_h(u)\equiv 0$ if $|u|>h+1$, we have:

$$\begin{aligned}&\int_\Omega \psi_h(u)\,|a(x,u)-a(x,v)|\,|\nabla v|\,|\nabla T_\varepsilon(v-T_{h+1}(u))|\,dx\\&\le\varepsilon\int_{\{0<|u-v|\le\varepsilon\}}|\nabla v|\,|\nabla T_\varepsilon(v-u)|\,\psi_h(u)\,[L(u)+L(v)]\,dx\\&\le\varepsilon\int_{\{0<|u-v|\le\varepsilon\}}|\nabla v|\;a(x,v)^{\frac12}\,|\nabla T_\varepsilon(v-u)|\,\psi_h(u)\,\frac{L(u)+L(v)}{\alpha(v)^{\frac12}}\,dx.\end{aligned}$$

Note that thanks to assumption (2.9) and to (2.13) we have that if $|u-v|\le\varepsilon$ then $a(x,v)\le a(x,u)+\varepsilon[L(u)+L(v)]$ and also $\frac{1}{\gamma}\le\frac{L(v)}{L(u)}\le\gamma$ almost everywhere, so that we obtain:

$$\begin{aligned}&\int_\Omega \psi_h(u)\,|a(x,u)-a(x,v)|\,|\nabla v|\,|\nabla T_\varepsilon(v-T_{h+1}(u))|\,dx\\&\le C\varepsilon\int_{\{0<|u-v|\le\varepsilon\}}|\nabla v|\,a(x,u)^{\frac12}\,|\nabla T_\varepsilon(v-u)|\,\psi_h(u)\,\frac{L(v)}{\alpha(v)^{\frac12}}\,dx\\&\quad+C\varepsilon\int_{\{L(u)\ge\frac{a(x,u)}{\varepsilon}\,,0<|u-v|\le\varepsilon\}}|\nabla v|\,(\varepsilon\,L(u))^{\frac12}\,\psi_h(u)\,|\nabla T_\varepsilon(v-u)|\,\frac{L(v)}{\alpha(v)^{\frac12}}\,dx.\end{aligned}$$

By Young's inequality we finally get:

$$\begin{aligned}&\int_\Omega \psi_h(u)\,|a(x,u)-a(x,v)|\,|\nabla v|\,|\nabla T_\varepsilon(v-T_{h+1}(u))|\,dx\\&\le\frac12\int_\Omega a(x,u)\psi_h(u)|\nabla T_\varepsilon(u-v)|^2\,dx \qquad (2.17)\\&\quad+\varepsilon^2\,C\int_{\{0<|u-v|\le\varepsilon\}}|\nabla v|^2\,\frac{L(v)^2}{\alpha(v)}\,dx\\&\quad+\varepsilon\,C\int_{\{L(u)\ge\frac{a(x,u)}{\varepsilon}\,,|u-v|\le\varepsilon\}}L(u)|\nabla T_\varepsilon(v-u)|^2\,dx.\end{aligned}$$

Let us consider last integral in (2.17). Using (2.13) we have:

$$\begin{aligned}&\int_{\{L(u)\ge\frac{a(x,u)}{\varepsilon},|u-v|\le\varepsilon\}}L(u)|\nabla T_\varepsilon(u-v)|^2\,dx\\&\le C\int_{\{L(u)\ge\frac{a(x,u)}{\varepsilon},\,|u-v|\le\varepsilon\}}(L(u)|\nabla u|^2+L(v)|\nabla v|^2)\,dx,\end{aligned}$$

and since in the set $\{L(u) \geq \frac{a(x,u)}{\varepsilon}, |u-v| \leq \varepsilon\}$ we have that $L(v) \geq \frac{1}{\gamma} L(u) \geq \frac{1}{\gamma} \frac{a(x,u)}{\varepsilon} \geq [\frac{a(x,v)}{\gamma\varepsilon} - 2L(v)]$ we deduce, using also (2.1),

$$\begin{aligned}
\varepsilon \int_{\{L(u)\geq \frac{a(x,u)}{\varepsilon}, |u-v|\leq\varepsilon\}} L(u)|\nabla T_\varepsilon(v-u)|^2\, dx &\leq C\varepsilon \int_{\{L(u)\geq \frac{a(x,u)}{\varepsilon}\}} L(u)|\nabla u|^2\, dx \\
&+ C\varepsilon \int_{\{L(v)\geq C\frac{a(x,v)}{\varepsilon}\}} L(v)|\nabla v|^2\, dx \\
&\leq \varepsilon^2 C \int_{\{L(u)\geq \frac{a(x,u)}{\varepsilon}\}} \frac{L(u)^2}{\alpha(u)}|\nabla u|^2\, dx + \varepsilon^2 C \int_{\{L(v)\geq C\frac{a(x,v)}{\varepsilon}\}} \frac{L(v)^2}{\alpha(v)}|\nabla v|^2\, dx.
\end{aligned}$$

Therefore (2.17) becomes:

$$\begin{aligned}
\int_\Omega \psi_h(u)\, |a(x,u) - a(x,v)|\, |\nabla v|\, |\nabla T_\varepsilon(v - T_{h+1}(u))|\, dx \\
\leq \frac{1}{2}\int_\Omega a(x,u)\psi_h(u)|\nabla T_\varepsilon(u-v)|^2\, dx \\
+ \varepsilon^2 C \int_{\{0<|u-v|\leq\varepsilon\}} |\nabla v|^2 \frac{L(v)^2}{\alpha(v)}\, dx \\
+ \varepsilon^2\, C \int_{\{L(u)\geq \frac{a(x,u)}{\varepsilon}\}} \frac{L(u)^2}{\alpha(u)}|\nabla u|^2\, dx \\
+ \varepsilon^2\, C \int_{\{L(v)\geq C\frac{a(x,v)}{\varepsilon}\}} \frac{L(v)^2}{\alpha(v)}|\nabla v|^2\, dx.
\end{aligned} \tag{2.18}$$

Since $\frac{L(u)^2}{\alpha(u)}|\nabla u|^2$ and $\frac{L(v)^2}{\alpha(v)}|\nabla v|^2$ belong to $L^1(\Omega)$, and since $\chi_{\{L(v)\geq C\frac{a(x,v)}{\varepsilon}\}}$ and $\chi_{\{L(u)\geq \frac{a(x,u)}{\varepsilon}\}}$ almost everywhere converge to zero, being $a(x,s) > 0$ for every $s \in \mathbf{R}$ and almost every $x \in \Omega$, from (2.18) and (2.16) we deduce:

$$\begin{aligned}
\frac{1}{2}\int_\Omega a(x,u)\psi_h(u)|\nabla T_\varepsilon(u-v)|^2\, dx &\leq \varepsilon^2 C \int_{\{0<|u-v|\leq\varepsilon\}} |\nabla v|^2 \frac{L(v)^2}{\alpha(v)}\, dx \\
&+ \int_\Omega a(x,v)|\psi_h(u) - \psi_h(v)|\, |\nabla v|\, |\nabla T_\varepsilon(v - T_{h+1}(u))|\, dx \\
&+ \int_\Omega \left(\int_v^{T_{h+1}(u)} \Phi'(t)\, \psi(t)\, dt\right) \nabla T_\varepsilon(T_{h+1}(u) - v)\, dx + \omega^\varepsilon(h) + \varepsilon^2\omega(\varepsilon).
\end{aligned} \tag{2.19}$$

Let us recall that assumption (2.10) implies that $|\Phi'(t)| \leq 2L_1(t)$ for almost every t in $\mathbf{R}$, and since by (2.12) $L_1(t) \leq \gamma L_1(s)$ if $|t-s| \leq \varepsilon$ we have:

$$\int_\Omega \left(\int_v^{T_{h+1}(u)} \Phi'(t)\, \psi(t)\, dt\right) \nabla T_\varepsilon(T_{h+1}(u) - v)\, dx$$

$$\leq 2\varepsilon\,\gamma \int_{\{|u|\leq h+1\}} L_1(u)\,|\nabla T_\varepsilon(u-v)\,|\,dx$$
$$+2\varepsilon\,\gamma \int_{\{h+1-\varepsilon\leq|v|\leq h+1+\varepsilon\}} L_1(v)\,|\nabla v|\,dx,$$

which yields by Young's inequality and (2.1),

$$\begin{aligned}
&\int_\Omega \left(\int_v^{T_{h+1}(u)} \Phi'(t)\,\psi(t)\,dt\right) \nabla T_\varepsilon(T_{h+1}(u)-v)\,dx \\
&\leq \varepsilon^2\, C \int_{\{0<|u-v|\leq\varepsilon\}} \frac{L_1(u)^2}{\alpha(u)}\,dx \\
&\quad + \varepsilon^2\, C \int_{\{h+1-\varepsilon\leq|v|\leq h+1+\varepsilon\}} a(x,v)|\nabla v|^2\,dx \qquad (2.20)\\
&\quad + \int_{\{|v|>h+1-\varepsilon\}} \frac{L_1(v)^2}{\alpha(v)}\,dx + \frac{1}{4}\int_{\{|u|\leq h+1\}} a(x,u)\,|\nabla T_\varepsilon(u-v)|^2\,dx.
\end{aligned}$$

Using (2.13), last integral in (2.20) can be estimated as follows:

$$\begin{aligned}
&\int_{\{|u|\leq h+1\}} a(x,u)\,|\nabla T_\varepsilon(u-v)|^2\,dx \\
&\leq \int_\Omega a(x,u)\psi_h(u)|\nabla T_\varepsilon(u-v)|^2\,dx \\
&\quad + \int_{\{h\leq|u|\leq h+1\,,|u-v|\leq\varepsilon\}} a(x,u)|\nabla T_\varepsilon(u-v)|^2\,dx \\
&\leq \int_\Omega a(x,u)\psi_h(u)|\nabla T_\varepsilon(u-v)|^2\,dx + \int_{\{h\leq|u|\leq h+1\}} a(x,u)|\nabla u|^2\,dx \\
&\quad + C\int_{\{h\leq|u|\leq h+1\,,|u-v|\leq\varepsilon\}} [a(x,v)+\varepsilon L(v)]\,|\nabla v|^2\,dx.
\end{aligned}$$

Using that, by means of (2.1), we have:

$$\begin{aligned}
&\int_{\{h\leq|u|\leq h+1\,,|u-v|\leq\varepsilon\}} [a(x,v)+\varepsilon L(v)]\,|\nabla v|^2\,dx \\
&\leq C\int_{\{h-\varepsilon\leq|v|\leq h+1+\varepsilon\}} a(x,v)|\nabla v|^2\,dx + C\int_{\{|u|>h\,,\,L(v)>a(x,v)\}} L(v)\,|\nabla v|^2\,dx \\
&\leq C\int_{\{h-\varepsilon\leq|v|\leq h+1+\varepsilon\}} a(x,v)|\nabla v|^2\,dx + C\int_{\{|u|>h\}} \frac{L(v)^2}{\alpha(v)}\,|\nabla v|^2\,dx,
\end{aligned}$$

we can conclude:

$$\begin{aligned}\int_{\{|u|\le h+1\}} a(x,u)\,|\nabla T_\varepsilon(u-v)|^2\,dx &\le \int_\Omega a(x,u)\psi_h(u)|\nabla T_\varepsilon(u-v)|^2\,dx \\ &+ \int_{\{h\le|u|\le h+1\}} a(x,u)|\nabla u|^2\,dx + C\int_{\{h-\varepsilon\le|v|\le h+1+\varepsilon\}} a(x,v)|\nabla v|^2\,dx \\ &+ C\int_{\{|u|>h\}} \frac{L(v)^2}{\alpha(v)}\,|\nabla v|^2\,dx.\end{aligned} \tag{2.21}$$

From (2.20) and (2.21) we obtain, using (2.2) and the fact that $\frac{L(v)^2}{\alpha(v)}\,|\nabla v|^2$ and $\frac{L_1(v)^2}{\alpha(v)}$ belong to $L^1(\Omega)$:

$$\begin{aligned}&\int_\Omega \left(\int_v^{T_{h+1}(u)} \Phi'(t)\,\psi(t)\,dt\right) \nabla T_\varepsilon(T_{h+1}(u)-v)\,dx \\ &\le \varepsilon^2\,C\int_{\{0<|u-v|\le\varepsilon\}} \frac{L_1(u)^2}{\alpha(u)}\,dx \\ &\quad+\frac14\int_\Omega a(x,u)\psi_h(u)|\nabla T_\varepsilon(u-v)|^2\,dx + \omega^\varepsilon(h).\end{aligned} \tag{2.22}$$

By the properties of ψ_h we also have:

$$\begin{aligned}&\int_\Omega a(x,v)|\psi_h(u)-\psi_h(v)|\,|\nabla v|\,|\nabla T_\varepsilon(T_{h+1}(u)-v)|\,dx \\ &\le \int_{\{|u|\le h\,,|v|>h\,,|u-v|\le\varepsilon\}} a(x,v)|\nabla v|\,|\nabla T_\varepsilon(u-v)|\,dx \\ &\quad+\int_{\{h\le|u|\le h+1\,,\,|u-v|\le\varepsilon\}} a(x,v)|\nabla v|\,|\nabla T_\varepsilon(u-v)|\,dx \\ &\quad+\int_{\{|u|>h+1\,,\,|v|\le h+1\,,\,|T_{h+1}(u)-v|\le\varepsilon\}} a(x,v)|\nabla v|^2\,dx.\end{aligned}$$

The same arguments used before (in particular, using (2.9), (2.13), (2.2) and the fact that $\frac{L(v)^2}{\alpha(v)}|\nabla v|^2$ and $\frac{L(u)^2}{\alpha(u)}|\nabla u|^2$ belong to $L^1(\Omega)$) allow to prove that

$$\lim_{h\to\infty}\int_\Omega a(x,v)|\psi_h(u)-\psi_h(v)|\,|\nabla v|\,|\nabla T_\varepsilon(T_{h+1}(u)-v)|\,dx = 0.$$

Therefore, by means of (2.22) and (2.19) we finally get:

$$\begin{aligned}\int_\Omega a(x,u)\psi_h(u)|\nabla T_\varepsilon(v-u)|^2\,dx &\le \varepsilon^2\,C\int_{\{0<|u-v|\le\varepsilon\}} \frac{L_1(u)^2}{\alpha(u)}\,dx \\ &+\varepsilon^2 C\int_{\{0<|u-v|\le\varepsilon\}} |\nabla v|^2\,\frac{L(v)^2}{\alpha(v)}\,dx + \omega^\varepsilon(h) + \varepsilon^2\omega(\varepsilon).\end{aligned} \tag{2.23}$$

Then letting h tend to infinity, by means of Fatou's lemma we find:

$$\int_\Omega a(x,u)|\nabla T_\varepsilon(u-v)|^2\,dx \le \varepsilon^2 C\int_{\{0<|u-v|\le\varepsilon\}}|\nabla v|^2\,\frac{L(v)^2}{\alpha(v)}\,dx$$

$$+\,\varepsilon^2\,C\int_{\{0<|u-v|\le\varepsilon\}}\frac{L_1(u)^2}{\alpha(u)}\,dx+\varepsilon^2\omega(\varepsilon). \tag{2.24}$$

Using the fact that $\operatorname{meas}\{x\,:\,0<|u-v|\le\varepsilon\}$ converges to zero as ε tends to zero we have that

$$\int_{\{0<|u-v|\le\varepsilon\}}|\nabla v|^2\,\frac{L(v)^2}{\alpha(v)}\,dx+\int_{\{0<|u-v|\le\varepsilon\}}\frac{L_1(u)^2}{\alpha(u)}\,dx=\omega(\varepsilon),$$

since $\frac{L(v)^2}{\alpha(v)}|\nabla v|^2$ and $\frac{L_1(u)^2}{\alpha(u)}$ belong to $L^1(\Omega)$. Then we obtain from (2.24):

$$\frac{1}{\varepsilon^2}\int_\Omega a(x,u)|\nabla T_\varepsilon(u-v)|^2\,dx \le\,\omega(\varepsilon),$$

which concludes the proof. □

From Proposition 2.3 we deduce the first uniqueness result for entropy solutions of (1.1) satisfying the condition that $\frac{L(u)^2}{\alpha(u)}|\nabla u|^2$ and $\frac{L_1(u)^2}{\alpha(u)}$ belong to $L^1(\Omega)$. Later we investigate the assumptions which imply this regularity for every solution.

Theorem 2.4 *Assume* (2.1), (2.9)–(2.12), *and that the function* $\alpha(s)$ *appearing in* (2.1) *is such that* $\min_{[-k,k]}\alpha(s)=\alpha(k)$ *for every* $k>0$ *and*

$$\exists c_0>0:\liminf_{|k|\to+\infty}\;|k|\alpha(k)\ge c_0.$$

Then there exists a unique entropy solution u *of* (1.1) *such that* $\frac{L(u)^2}{\alpha(u)}|\nabla u|^2$ *and* $\frac{L_1(u)^2}{\alpha(u)}$ *belong to* $L^1(\Omega)$.

Proof. Let us take a function $\zeta\in C^1(\mathbf{R})$ such that $0\le\zeta\le1$, $\zeta\equiv0$ if $|s|\ge2$ and $\zeta\equiv1$ if $|s|\le1$. Then, since $\zeta(\frac{u}{k})\le1$ we have, from Proposition 2.3:

$$\int_\Omega \alpha(u)|\nabla T_\varepsilon(u-v)|^2\zeta\left(\frac{u}{k}\right)^2\,dx\le\varepsilon^2\omega(\varepsilon),$$

where $\omega(\varepsilon)$ is a term going to zero as ε tends to zero. Since now the integral in the left hand side is taken only in the set where $|u|\le 2k$ we have, using that $\min_{[-k,k]}\alpha(s)=\alpha(k)$:

$$\alpha(2k)\int_\Omega|\nabla T_\varepsilon(u-v)|^2\zeta\left(\frac{u}{k}\right)^2\,dx\le\varepsilon^2\omega(\varepsilon). \tag{2.25}$$

Applying Poincare's inequality to the function $T_\varepsilon(u-v)\zeta(\frac{u}{k})$ and using (2.25) we have

$$
\begin{aligned}
C\int_\Omega |T_\varepsilon(u-v)\zeta\left(\frac{u}{k}\right)|^2\,dx &\le \int_\Omega |\nabla T_\varepsilon(u-v)|^2\zeta\left(\frac{u}{k}\right)^2\,dx \\
&+\frac{1}{k^2}\int_\Omega T_\varepsilon(u-v)^2\zeta'\left(\frac{u}{k}\right)^2|\nabla u|^2\,dx \\
&\le \frac{1}{\alpha(2k)}\varepsilon^2\omega(\varepsilon)+\varepsilon^2\frac{C}{k^2}\int_{\{k\le|u|\le 2k\}}|\nabla u|^2\,dx.
\end{aligned}
$$

Since, by the properties of ζ we have, for every $\delta>0$:

$$
\text{meas}\{|u-v|\ge\varepsilon+\delta, |u|\le k\}\le\frac{1}{\varepsilon^2}\int_\Omega\left|T_\varepsilon(u-v)\zeta\left(\frac{u}{k}\right)\right|^2\,dx,
$$

we deduce:

$$
\text{meas}\{|u-v|\ge\varepsilon+\delta\,, |u|\le k\}\le\frac{C}{\alpha(2k)}\omega(\varepsilon)+\frac{C}{k^2}\int_{\{k\le|u|\le 2k\}}|\nabla u|^2\,dx,
$$

and letting ε go to zero

$$
\text{meas}\{|u-v|>\delta\,, |u|\le k\}\le\frac{C}{k^2}\int_{\{k\le|u|\le 2k\}}|\nabla u|^2\,dx. \tag{2.26}
$$

On the other hand, by definition of entropy solution we have:

$$
\begin{aligned}
&\int_\Omega a(x,u)\nabla u\nabla T_k(u-T_k(u))\,dx-\int_\Omega\Phi(u)\nabla T_k(u-T_k(u))\,dx \\
&=\int_\Omega fT_k(u-T_k(u))\,dx.
\end{aligned}\tag{2.27}
$$

Since

$$
\begin{aligned}
\int_\Omega\Phi(u)\nabla T_k(u-T_k(u))\,dx &= \int_\Omega\Phi(T_{2k}(u))\nabla T_{2k}(u)\chi_{\{k\le|u|\le 2k\}}\,dx \\
&=\int_\Omega\text{div}\left(\int_0^{T_{2k}(u)}\Phi(s)\chi_{\{k\le|s|\le 2k\}}\,ds\right)\,dx=0,
\end{aligned}
$$

we deduce from (2.27):

$$
\int_{\{k\le|u|\le 2k\}}a(x,u)|\nabla u|^2\,dx\le k\int_{\{|u|\ge k\}}|f|\,dx.
$$

Using that $\min_{[-k,k]}\alpha(s)=\alpha(k)$ we get:

$$
\frac{1}{k^2}\int_{\{k\le|u|\le 2k\}}|\nabla u|^2\,dx\le\frac{1}{k\alpha(2k)}\int_{\{|u|\ge k\}}|f|\,dx.
$$

Since $\liminf_{k\to+\infty} k\alpha(k) \geq c_0 > 0$ and using that f is in $L^1(\Omega)$ we deduce that

$$\lim_{k\to+\infty} \frac{1}{k^2} \int_{\{k\leq|u|\leq 2k\}} |\nabla u|^2 \, dx = 0.$$

Thus from (2.26) we obtain, letting k tend to infinity,

$$\text{meas}\{|u-v| > \delta\} = 0 \quad \forall\delta > 0,$$

so that $u = v$. □

Let us see how a first partial uniqueness result follows from Theorem 2.4.

Theorem 2.5 *Let f belong to $L^1(\Omega)$, and assume* (2.1), (2.9)–(2.12), *and that the function $\alpha(s)$ appearing in* (2.1) *is such that $\min\limits_{[-k,k]} \alpha(s) = \alpha(k)$ and*

$$\exists c_0 > 0 : \liminf_{|k|\to+\infty} |k|\alpha(k) \geq c_0.$$

Moreover assume that $\frac{L_1(s)^2}{\alpha(s)} \in L^\infty(\mathbf{R})$ and that $\frac{L(s)^2}{\alpha(s)^2} \in L^1(\mathbf{R})$. Then there exists a unique entropy solution of (1.1).

Proof. Since $\frac{L_1(u)^2}{\alpha(u)} \in L^\infty(\Omega)$, in virtue of Theorem 2.4 we are only left with the proof that if u is an entropy solution then $\frac{L(u)^2}{\alpha(u)}|\nabla u|^2$ belongs to $L^1(\Omega)$. Indeed, this follows directly from (2.3) and the assumption that $\frac{L(s)^2}{\alpha(s)^2} \in L^1(\mathbf{R})$. □

Remark 2.6 The conclusion of the previous theorem underlines how the method used in Proposition 2.3 and introduced in [7] essentially relies on the regularity we have on our solutions u and v. If for instance $\alpha(s)$ is a positive constant $\alpha_0 > 0$, this method gives uniqueness of solutions u such that $L(u)|\nabla u|^2$ belongs to $L^1(\Omega)$. This shows that the conditions on $L(s)$ may be improved in dependence on the regularity of the right hand side, so that, as a limit case, if f lies in $H^{-1}(\Omega)$, which implies that u belongs to $H_0^1(\Omega)$, we can afford to have L bounded, which is the content of the result in [1]. In fact, we are going to prove that the conclusion of the previous theorem is far from optimal, and that the differences between the cases with finite or infinite energy solutions disappear once we transform our problem through a suitable change of unknown.

3 Proof of Theorem 1.4

Let us deal now with the proof of Theorem 1.4, which relies on the fact that the assumptions to get uniqueness found in Theorem 2.5 can be weakened if we transform our original problem (1.1) into a new problem with a degenerate coercivity condition.

Proof of Theorem 1.4

Let us define

$$\tilde{a}(x,s) = \frac{a(x, \frac{1}{\lambda}\log(1+|s|)\mathrm{sign}(s))}{\lambda(1+|s|)}, \qquad \tilde{\Phi}(s) = \Phi\left(\frac{1}{\lambda}\log(1+|s|)\mathrm{sign}(s)\right),$$

where $\lambda > 0$ is to be chosen later, and consider the problem

$$\begin{cases} -\mathrm{div}(\tilde{a}(x,v)\nabla v) + \mathrm{div}(\tilde{\Phi}(v)) = f \text{ in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We know from Corollary 2.2 that u is an entropy solution of (1.1) if and only if $v = (e^{\lambda|u|} - 1)\mathrm{sign}(u)$ is an entropy solution of (3.1). We are going to prove that (3.1) admits only one solution. Indeed, we have that $\tilde{a}(x,s)$ satisfies:

$$\tilde{a}(x,s) \geq \frac{\alpha}{\lambda(1+|s|)}. \tag{3.2}$$

Moreover, for every s, t we have, using (1.5) and the properties of locally Lipschitz continuity of $\log(1+|s|)$ and $\frac{1}{1+|s|}$:

$$|\tilde{a}(x,s) - \tilde{a}(x,t)| \leq C\left[\frac{1}{(1+|s|)^{2-\frac{\mu}{\lambda}}} + \frac{1}{(1+|t|)^{2-\frac{\mu}{\lambda}}}\right]|s-t|,$$

where C only depends on λ and μ. Thus the function $\tilde{a}(x,s)$ has a modulus of locally Lipschitz continuity which is $\tilde{L}(s) = C\frac{1}{(1+|s|)^{2-\frac{\mu}{\lambda}}}$. Moreover, a similar calculation shows that, for $|s-t| < c$

$$|\tilde{\Phi}(s) - \tilde{\Phi}(t)| \leq C\left[\frac{1}{(1+|s|)^{1-\frac{\mu}{\lambda}}} + \frac{1}{(1+|t|)^{1-\frac{\mu}{\lambda}}}\right]|s-t|,$$

so that $\tilde{a}(x,s)$ satisfies assumptions (2.9)–(2.12) with $\tilde{L}(s) = C\frac{1}{(1+|s|)^{2-\frac{\mu}{\lambda}}}$ and $\tilde{L}_1(s) = C\frac{1}{(1+|s|)^{1-\frac{\mu}{\lambda}}}$.

Moreover, since the coercivity modulus of $\tilde{a}(x,s)$ is $\tilde{\alpha}(s) = \frac{\alpha}{\lambda(1+|s|)}$, if $\lambda > 2\mu$ it follows that $\frac{\tilde{L}(s)^2}{\tilde{\alpha}(s)^2}$ belongs to $L^1(\mathbf{R})$ and also $\frac{\tilde{L}_1(s)^2}{\tilde{\alpha}(s)} \in L^\infty(\mathbf{R})$. We are then in the conditions to apply Theorem 2.5 to (3.1) so that we conclude that v is the unique solution of (3.1). □

Of course, our results can also be applied to operators satisfying different coercivity conditions. The proof of Theorem 1.4 suggests the assumptions we can allow to have in these situations.

Corollary 3.1 *Assume that f belongs to $L^1(\Omega)$, $a(x,s)$ is a Carathéodory function and Φ belongs to $C(\mathbf{R},\mathbf{R}^N)$. Let $\alpha \in C(\mathbf{R},\mathbf{R}^+)$ be such that*

$$\exists c_0 > 0 \;:\; \liminf_{|k|\to+\infty} |k|\alpha(k) \geq c_0. \tag{3.3}$$

Setting $A(s)=\int_0^s \alpha(t)\,dt$*, assume that there exists a positive constant* $\mu>0$ *such that* $a(x,s)$ *and* $\Phi(s)$ *satisfy:*

$$\begin{aligned}
&\alpha(s)\le a(x,s)\le \mu\, e^{\mu|A(s)|}, \qquad \forall\, s\in\mathbf{R},\ \text{a.e. } x\in\Omega,\\
&\left|\frac{a(x,s)}{\alpha(s)}-\frac{a(x,t)}{\alpha(t)}\right|\le \mu(e^{\mu|A(s)|}+e^{\mu|A(t)|})|s-t|, \forall\, s,t\in\mathbf{R},\\
&\quad \text{a.e. } x\in\Omega,\\
&|\Phi(s)-\Phi(t)|\le \mu(e^{\mu|A(s)|}+e^{\mu|A(t)|})|s-t|,\ \forall\, s,t\in\mathbf{R},\quad \text{a.e. } x\in\Omega.
\end{aligned}\tag{3.4}$$

Then there exists a unique entropy solution of (1.1)*.*

Proof. Let us set $v=A(u)$. Note that (3.3) implies that $\lim_{s\to\pm\infty}A(s)=\pm\infty$, then from Corollary 2.2 we have that u is an entropy solution of (1.1) if and only if v is an entropy solution of the Dirichlet problem

$$\begin{cases}-\operatorname{div}(\tilde a(x,v)\nabla v)+\operatorname{div}(\tilde\Phi(v))=f \text{ in } \Omega,\\ v=0 \quad \text{on } \partial\Omega,\end{cases}\tag{3.5}$$

with $\tilde a(x,s)=\frac{a(x,A^{-1}(s))}{\alpha(A^{-1}(s))}$ and $\tilde\Phi(s)=\Phi(A^{-1}(s))$. From (3.3) we also deduce that there exists a constant c_1 such that $|A(s)|\ge c_1(1+\log(|s|))$ for $|s|$ large enough, so that, for a possibly different constant c_2 we have that $|A^{-1}(s)|\le c_2e^{c_2|s|}$ for every s in $\mathbf{R}$. Since by (3.3) $\frac{1}{\alpha(A^{-1}(s))}\le\frac{1}{c_0}|A^{-1}(s)|$ if $|s|$ is large, we deduce that $\frac{1}{\alpha(A^{-1}(s))}$ has at most an exponential growth at infinity. Together with (3.4) this implies that there exists $\lambda>0$ such that:

$$\tilde a(x,s)\le\lambda e^{\lambda|s|}\qquad\forall s\in\mathbf{R}.$$

Moreover we also have by (3.4) that

$$\begin{aligned}
|\tilde a(x,s)-\tilde a(x,t)|&\le\mu[e^{\mu|s|}+e^{\mu|t|}]|A^{-1}(s)-A^{-1}(t)|\\
&\le\mu[e^{\mu|s|}+e^{\mu|t|}]\frac{1}{\alpha(A^{-1}(\xi))}\,|s-t|,\quad \xi\in(s,t),
\end{aligned}$$

so that, using again that $\frac{1}{\alpha(A^{-1}(s))}\le c_3e^{c_3|s|}$ for a positive constant c_3, we conclude that there exists $\gamma>0$ such that:

$$\begin{aligned}
&1\le\tilde a(x,s)\le\gamma e^{\gamma|s|},\qquad\forall\, s\in\mathbf{R},\quad\text{a.e. } x\in\Omega,\\
&|\tilde a(x,s)-\tilde a(x,t)|\le\gamma(e^{\gamma|s|}+e^{\gamma|t|})|s-t|,\qquad\forall\, s,\,t\in\mathbf{R},\quad\text{a.e. } x\in\Omega\,.
\end{aligned}$$

Similarly we have that there exists $\delta>0$ such that:

$$|\tilde\Phi(s)-\tilde\Phi(t)|\le\delta(e^{\delta|s|}+e^{\delta|t|})|s-t|,\qquad\forall\, s,\,t\in\mathbf{R},\quad\text{a.e. } x\in\Omega.$$

Thus $\tilde a(x,s)$ and $\tilde\Phi(s)$ satisfy the assumptions of Theorem 1.4, so that v is unique. This implies that u is also unique by means of Corollary 2.2. ∎

Remark 3.2 A particular case treated by Corollary 3.1 is when $\alpha(s) = (1+|s|)^m$, with $m \geq -1$. If $m > 0$, this case has been considered in [4], for instance, with f in $H^{-1}(\Omega)$. Here we extend these results both to the L^1 data and to the class of locally Lipschitz functions $s \mapsto a(x,s)$. In fact, our assumptions allow to consider the case that both $a(x,s)$ and its modulus of locally Lipschitz continuity grow like $\exp(|s|^{m+1})$ at infinity. We also extend the results proved in [15] in case that $-1 < m < 0$. Note that the limit example $m = -1$ is also considered here, in this case we are assuming in (3.4) that $s \mapsto a(x,s)$ and its modulus of locally Lipschitz continuity may grow at infinity like the power $|s|^\beta$, for any $\beta > 0$.

Together with the uniqueness result, the method used for the proof of Theorem 1.4 and Corollary 3.1 also provides a comparison result.

Corollary 3.3 *Assume that Φ belongs to $C(\mathbf{R}, \mathbf{R}^N)$ and $a(x,s)$ is a Carathéodory function. Let $\alpha \in C(\mathbf{R}, \mathbf{R}^+)$ be such that*

$$\exists c_0 > 0 \ : \ \liminf_{|k| \to +\infty} |k| \alpha(k) \geq c_0 .$$

Setting $A(s) = \int_0^s \alpha(t)\, dt$, assume that there exists a positive constant $\mu > 0$ such that $a(x,s)$ and $\Phi(s)$ satisfy (3.4). Let f_1, f_2 belong to $L^1(\Omega)$ and let u_1, u_2 be the entropy solutions of (1.1) with data respectively f_1, f_2. Then if $f_1 \leq f_2$ we have that $u_1 \leq u_2$ (almost everywhere in Ω).

Proof. In fact, this can be proved by simply replacing $T_\varepsilon(T_{h+1}(u_1) - u_2)$ with $T_\varepsilon(T_{h+1}(u_1) - u_2)^+$ in the proof of Proposition 2.3. In that case the conclusion obtained is that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_\Omega a(x,u) |\nabla T_\varepsilon(u_1 - u_2)^+|^2 \, dx = 0,$$

then like in Theorem 2.4 it follows that $u_1 \leq u_2$. As in Corollary 3.1, this argument can be applied after a suitable change of unknown in the problem.

An alternative proof of this comparison result can be done using the result of Corollary 3.1. Indeed, if $f_1 \leq f_2$, we can consider problem (1.1) with data f_{1n}, f_{2n} respectively, where f_{1n}, $f_{2n} \in L^\infty(\Omega)$, $f_{1n} \leq f_{2n}$ and f_{in} strongly converges to f_i in $L^1(\Omega)$, $i = 1, 2$. By classical results we have that the corresponding solutions u_{in} (which are unique) are bounded and $u_{1n} \leq u_{2n}$ for every n. The compactness results proved in [3], [14], [5] imply that u_{in} almost everywhere converges to an entropy solution $\bar{u}_i$ of (1.1) with datum f_i. The uniqueness result implies that $\bar{u}_i = u_i$ so that $u_1 \leq u_2$. ☐

Remark 3.4 The proofs of our results would not change replacing the principal part $-\mathrm{div}(a(x,u)\nabla u)$ with operators in the form $-\mathrm{div}(a(x,u,\nabla u))$ with similar structure of strong monotonicity with respect to ∇u and Lipschitz continuity with respect to u and ∇u. In particular, under the following assumptions:

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) \geq \alpha\, |\xi - \eta|^2, \qquad \alpha > 0,$$
$$|a(x,s,\xi) - a(x,s,\eta)| \leq \beta |\xi - \eta|, \quad \beta > 0$$

$$|a(x,s,\xi)| \le \mu\, e^{\mu|s|}\,|\xi|, \qquad \mu > 0$$
$$|a(x,s,\xi) - a(x,t,\xi)| \le \mu\,(e^{\mu|s|} + e^{\mu|t|})\,|\xi||s-t|$$

for every $s,\ t \in \mathbf{R}$, every $\xi,\ \eta \in \mathbf{R}^N$ and almost every x in Ω the result of Theorem 3.1 still holds true.

Remark 3.5 The results of Theorem 1.4 and Corollary 3.1 still hold true if f is a measure which does not charge the sets of zero harmonic capacity (the standard notion of capacity in $H_0^1(\Omega)$). In fact, in this case we have that f belongs to $L^1(\Omega) + H^{-1}(\Omega)$ by the results in [8], and the proofs of Lemma 2.1, of Proposition 2.3 and of Theorem 2.4 may be applied with very few modifications. In fact, the main difference is that in this case we have to use that if f is a measure which does not charge the sets of zero capacity then the entropy solutions of (1.1) are cap-quasi everywhere finite, precisely the cap-quasi continuous representative of an entropy solution is finite up to a set of zero capacity. This allows to get Lemma 2.1 and Corollary 2.2 and then the other proofs remain unchanged.

Note added in proof. We refer the reader to a forthcoming paper for similar uniqueness results including a refinement of the exponential growth condition (1.5), used here.

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