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Frobenius manifolds and Virasoro constraints

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Abstract. For an arbitrary Frobenius manifold, a system of Virasoro constraints is constructed. In the semisimple case these constraints are proved to hold true in the genus one approximation. In particular, the genus ≤ 1 Virasoro conjecture of T. Eguchi, K. Hori, M. Jinzenji, and C.-S. Xiong and of S. Katz is proved for smooth projective varieties having semisimple quantum cohomology.

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1. Introduction

In a 2D topological field theory (TFT) one is interested in the computation of correlators of the chiral primary fields $\phi_1 = 1, \phi_2, \dots, \phi_n$ and of their gravitational descendents $\tau_0(\phi_\alpha) = \phi_\alpha$, $\tau_1(\phi_\alpha)$, $\tau_2(\phi_\alpha)$, $\dots$, $\alpha = 1, \dots, n$. Due to topological invariance these correlators are just numbers $< \tau_p(\phi_\alpha)\, \tau_q(\phi_\beta) \cdots >_g$ depending only on the labels (α, p), (β, q), $\dots$ of the fields and on the genus g of a 2D surface where the fields live. The free energy of the theory is the generating function of these numbers

$$\mathcal{F}(T) = \sum_{g=0}^{\infty} \mathcal{F}_g(T), \tag{1.1}$$

where T is the infinite vector of coupling constants $T^{\alpha,p}$, $\alpha = 1, \dots, n$, $p = 0, 1, 2, \dots$ and the genus g part is defined by

$$\mathcal{F}_g(T) = \left\langle e^{\sum_{\alpha=1}^{n} \sum_{p=0}^{\infty} \tau_p(\phi_\alpha) T^{\alpha,p}} \right\rangle_g. \tag{1.2}$$

The correlators of the fields for different genera g satisfy complicated identities by now completely settled only for the simplest case of pure topological gravity (only one primary field $\phi_1 = 1$). Of course, the shape of these identities and their derivations depend on the concrete model of 2D TFT. There are, however, some universal identities that can probably be used as the defining relations of some class of 2D TFTs.

Physicists construct 2D TFTs by twisting N=2 supersymmetric theories [32], [39]. The main consequence of this general approach [7] is a remarkable system of differential equations for the function

$$F(t^1, \ldots, t^n) := \mathcal{F}_0(T)\big|_{T^{\alpha,p}=0 \text{ for } p>0,\ T^{\alpha,0}=t^\alpha,\ \alpha=1,\ldots,n}. \tag{1.3}$$

These are WDVV associativity equations. They were also derived by mathematicians [31], [35], [37] who were mainly concerned with the topological sigma-models known in the mathematical literature as *quantum cohomology*. The mathematical derivations were based on a careful study of intersection theory on the moduli spaces of Riemann surfaces with markings and on the moduli spaces of stable maps of these surfaces to smooth projective varieties. (In this case it is more convenient to consider the correlators as elements of the so-called Novikov ring, see Example 2.2 below.) This method proved to be successful also in the analysis of the structure of genus one free energy [5], [21], [24].

The coordinate-free form of WDVV is the notion of a Frobenius manifold (see Sect. 2 below) proposed in [10]. The main motivation for the study of Frobenius manifolds in [10] was the idea to construct all the building of a given 2D TFT starting from the corresponding Frobenius manifolds, i.e., starting from a solution (1.3) of WDVV. Other motivations arrived when Frobenius manifolds appeared in the theory of Gromov-Witten invariants, in K. Saito's theory of primitive forms for isolated singularities, in the geometry of invariants of reflection groups and of their extensions etc. (see details in [11], [12], [13], [16], [26], [36]). Physically, all these differently looking mathematical objects are just different models of 2D TFT. Our idea is that, on the mathematical side, the theory of Frobenius manifolds is the common denominator unifying these rather distant mathematical theories.

Remarkably, the original programme for reconstructing a 2D TFT (i.e., in our setting, of the full free energy $\mathcal{F}(T)$) starting from a solution (1.3) to WDVV seems to become realistic under the assumption of semisimplicity of the Frobenius manifold (physically semisimple are the models of 2D TFT with all massive perturbations [2]). Namely, there is a universal procedure to reconstruct the components $\mathcal{F}_0(T)$ and $\mathcal{F}_1(T)$ starting from an arbitrary semisimple Frobenius manifold. In particular, this procedure reproduces all topological recursion relations of [5], [21], [27] for correlators in topological sigma-models, i.e., the relations between the intersection numbers of Gromov-Witten and Mumford-Morita-Miller cycles on the moduli spaces of stable maps of punctured Riemann surfaces of the genus $g \le 1$ to smooth projective varieties. For genus 0 the reconstruction procedure of $\mathcal{F}_0(T)$ was found in [8], [11]. Reconstruction of the genus 1 component $\mathcal{F}_1(T)$ for an arbitrary semisimple Frobenius manifold was obtained in our paper [17].

In the present paper we suggest new evidence supporting our vision of semisimple Frobenius manifolds as the mathematical basement of the building of 2D TFT. For an arbitrary Frobenius manifold we construct a system of linear differential operators of the form

$$L_m = \sum_{(\alpha,p),\,(\beta,q)} a_m^{\alpha,p;\,\beta,q} \frac{\partial^2}{\partial T^{\alpha,p}\partial T^{\beta,q}} + \sum_{(\alpha,p),\,(\beta,q)} b_{m\,\alpha,p}^{\;\;\beta,q}\, \tilde T^{\alpha,p} \frac{\partial}{\partial T^{\beta,q}}$$
$$+ \sum_{(\alpha,p),\,(\beta,q)} c^m_{\alpha,p;\,\beta,q}\, \tilde T^{\alpha,p}\, \tilde T^{\beta,q} + \text{const}\ \delta_{m,0}, \quad m \geq -1 \tag{1.4}$$

with

$$\tilde T^{\alpha,p} = T^{\alpha,p} \quad \text{for} \quad (\alpha,p) \neq (1,1), \quad \tilde T^{1,1} = T^{1,1} - 1 \tag{1.5}$$

such that, in the semisimple case, the partition function

$$Z(T) = e^{\sum_{g=0}^{\infty} \mathcal{F}_g(T)} \tag{1.6}$$

satisfies, within $g \leq 1$ approximation, an infinite system of linear equations

$$L_m Z(T) = 0, \quad m \geq -1. \tag{1.7}$$

Here $a_m^{\alpha,p;\,\beta,q}$, $b_{m\,\alpha,p}^{\;\;\beta,q}$, $c^m_{\alpha,p;\,\beta,q}$ are certain constants computed via the monodromy data of the Frobenius manifold. Under a certain nondegeneracy assumption about the monodromy data, the construction of the operators L_m can be extended to an arbitrary integer m. The operators L_m satisfy the commutation relations of the Virasoro algebra

$$[L_i, L_j] = (i-j)L_{i+j} + n\frac{i^3 - i}{12}\delta_{i+j,0}. \tag{1.8}$$

Because of this the equations (1.7) are called *Virasoro constraints*.

Our construction contains, as a particular case, the Virasoro constraints in 2D topological gravity (in this case $n = 1$, the Frobenius manifold is trivial) established by M. Kontsevich in his proof [30] of Witten's conjecture [40] describing topological gravity by KdV hierarchy (see also [41]). It generalizes also the Virasoro constraints conjectured in [18] in the framework of quantum cohomology. Recently the quantum cohomology Virasoro constraints of [18] were proved [33] in the genus zero approximation. However, the genus zero result does not suffice to fix uniquely all the coefficients of the Virasoro operators and, therefore, it does not suffice to prove the commutation relations (1.8).

One may ask where do the Virasoro constraints come from? In our opinion, validity of the constraints proved for $g \leq 1$ in the general setting in the present paper is a manifestation of a general involvement of integrable bihamiltonian hierarchies of KdV-type into an arbitrary 2D TFT. Due to an idea of [40], the unknown functions of the hierarchy are the following correlators

$$t_\alpha(T) := \sum_{g=0}^{\infty} \left\langle \tau_0(\phi_\alpha)\tau_0(\phi_1)e^{\sum_{\beta,\,q}\tau_q(\phi_\beta)T^{\beta,q}} \right\rangle_g, \quad \alpha = 1, \dots, n \tag{1.9}$$

as the functions of the "times" $T^{\beta,q}$. The differential equations for the correlators can be considered as a hierarchy of commuting evolutionary PDEs

$$\partial_{T^{\alpha,p}} t = \partial_X K_{\alpha,p}(t, t_X, t_{XX}, \dots) \tag{1.10}$$

on the loop space

$$L(M) = \left\{ (t^1(X), \dots, t^n(X)) \,|\, X \in S^1 \right\}.$$

Here $X = T^{1,0}$ is one of the couplings. The partition function (1.6) is the tau-function of a particular solution to the hierarchy specified by the *string equation*

$$L_{-1} Z \equiv \sum T^{\alpha,p+1} \frac{\partial Z}{\partial T^{\alpha,p}} + \frac{1}{2}\eta_{\alpha\beta} T^{\alpha,0} T^{\beta,0} Z - \frac{\partial Z}{\partial X} = 0. \tag{1.11}$$

The Virasoro operators correspond to the symmetries of the hierarchy. The operator L_{-1} corresponds to a Galilean-type symmetry; L_0 corresponds to the scaling symmetry; other symmetries can be derived using the recursion procedure (see, e.g., [25] and references therein) due to the bihamiltonian structure of the hierarchy.

This scheme works perfectly well in the particular case of topological gravity where the hierarchy (1.10) coincides with KdV [30]. For other models of 2D TFT, the hierarchy and the bihamiltonian structure are to be constructed. (This was done for $g = 0$ in [8], [11] and for $g = 1$ in our paper [17], see below.) So, the programme of reconstruction of a 2D TFT from a given Frobenius manifold contains a subprogramme of realization of the following identification

$$\text{semisimple Frobenius manifolds} \;\equiv\; \begin{array}{l}\text{moduli of integrable bihamiltonian}\\ \text{hierarchies of KdV type.}\end{array}$$

The last important idea to be explained in this introduction is a technical trick of introducing of a small parameter ϵ in the genus expansions (1.6) and (1.9). In the physical literature the small parameter ϵ is called *the string coupling constant*. Introduction of the parameter ϵ is based on the observation, essentially due to [5], that the genus g part of the free energy is a homogeneous function of the degree $2 - 2g$ of the shifted couplings (1.5). So, after the rescaling

$$\tilde{T}^{\alpha,p} \mapsto \epsilon \tilde{T}^{\alpha,p}, \tag{1.12}$$

the genus expansion (1.6) reads

$$Z(T;\epsilon) = \exp \sum_{g=0}^{\infty} \epsilon^{2g-2} \mathcal{F}_g(T). \tag{1.13}$$

This trick is very convenient when separating the contributions of different genera. Observe that the partition function (1.13) is annihilated by the operator

$$D := \sum \tilde{T}^{\alpha,p} \partial_{T^{\alpha,p}} + \epsilon \partial_\epsilon + \frac{n}{24}. \tag{1.14}$$

From the point of view of integrable hierarchies the genus expansion corresponds to the small dispersion expansion of (1.10) (we also rescale $X \mapsto \epsilon X$)

$$\partial_{T^{\alpha,p}} t = \partial_X K^{(0)}_{\alpha,p}(t) + \epsilon^2 \partial_X K^{(1)}_{\alpha,p}(t, t_X, t_{XX}) + \dots. \tag{1.15}$$

It turns out [17] that for an arbitrary semisimple Frobenius manifold, the first two terms of the bihamiltonian hierarchy can be constructed (see [8], [11] for $K^{(0)}_{\alpha,p}$ and [17] for $K^{(1)}_{\alpha,p}$). Moreover, they are determined uniquely by the properties of topological correlators.

Applying the Virasoro operators (1.4), where one is to do the same rescaling (1.12), to the partition function (1.13) one obtains a series

$$L_m Z(T;\epsilon) = \left(\sum_{g=0}^{\infty} \mathcal{A}_{m,g} \epsilon^{2g-2} \right) Z(T;\epsilon)$$

where the coefficients $\mathcal{A}_{m,g}$ are expressed via $\mathcal{F}_0, \dots, \mathcal{F}_g$ and their derivatives. In particular,

$$\begin{aligned} \mathcal{A}_{m,0} &= \sum a_m^{\alpha,p;\,\beta,q}\, \partial_{T^{\alpha,p}} \mathcal{F}_0\, \partial_{T^{\beta,q}} \mathcal{F}_0 + \sum b_{m\,\alpha,p}^{\ \ \beta,q}\, \tilde{T}^{\alpha,p}\, \partial_{T^{\beta,q}} \mathcal{F}_0 \\ &\quad + \sum c^m_{\alpha,p;\,\beta,q}\, \tilde{T}^{\alpha,p}\, \tilde{T}^{\beta,q}, \end{aligned} \tag{1.16}$$

$$\begin{aligned} \mathcal{A}_{m,1} &= \sum a_m^{\alpha,p;\,\beta,q} \left(\partial_{T^{\alpha,p}} \partial_{T^{\beta,q}} \mathcal{F}_0 + \partial_{T^{\alpha,p}} \mathcal{F}_1\, \partial_{T^{\beta,q}} \mathcal{F}_0 + \partial_{T^{\alpha,p}} \mathcal{F}_0\, \partial_{T^{\beta,q}} \mathcal{F}_1 \right) \\ &\quad + \sum b_{m\,\alpha,p}^{\ \ \beta,q}\, \tilde{T}^{\alpha,p}\, \partial_{T^{\beta,q}} \mathcal{F}_1 + \operatorname{const} \delta_{m,0}. \end{aligned} \tag{1.17}$$

The main result of this paper, besides the construction of the Virasoro operators (1.4), is

Main Theorem.

1. *For an arbitrary Frobenius manifold the genus zero Virasoro constraints*

$$\mathcal{A}_{m,0} = 0, \quad m \geq -1 \tag{1.18}$$

hold true.

2. *For an arbitrary semisimple Frobenius manifold the genus* 1 *Virasoro constraints*

$$\mathcal{A}_{m,1} = 0, \quad m \geq -1 \tag{1.19}$$

hold true.

In the setting of quantum cohomology of a variety X having $H^{\mathrm{odd}}(X) = 0$, the first part of our theorem gives a new proof of the main result of [33]. The second part, particularly, proves the Virasoro conjecture of Eguchi, Hori, Jinzenji and Xiong [18], [19] and of S. Katz in the genus one approximation for smooth projective varieties X having semisimple quantum cohomology. According to the conjecture of [13] the semisimplicity takes place for those Fano varieties X on which a full system of exceptional objects, in the sense of [1], exists in the derived category of coherent sheaves $\mathrm{Der}^b(\mathrm{Coh}(X))$. In the cohomology of such a variety X only $H^{k,k}(X) \neq 0$. Because of this we do not consider in the present paper the Virasoro constraints for Frobenius supermanifolds neither the extension of the Virasoro algebra due to the presence of nonanalytic cycles, i.e., those belonging to $H^{k,l}(X)$ with $k \neq l$. Here $H(X) = \oplus H^{k,l}(X)$ is the Hodge decomposition.

Our theorem also fixes uniquely all the coefficients of the Virasoro operators conjectured in [18], [19]. As a byproduct of the proof of the second part of the theorem, we discover a simple way of expressing elliptic Gromov-Witten invariants via rational ones.

The structure of the paper is as follows. In Sect. 2 we present a Sugawara-type (see [29]) construction for the needed Virasoro operators. In Sect. 3 we recall the necessary information about semisimple Frobenius manifolds and the construction of $\mathcal{F}_0(T)$ and $\mathcal{F}_1(T)$. In Sect. 4 we sketch the proof of the main theorem.

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2. A family of representations of the Virasoro algebra

We describe first the parameters of the family. Let V be n-dimensional complex vector space equipped with a symmetric nondegenerate bilinear form $< \, , \, >$. The parameters of our Virasoro algebras consist of:

1. An antisymmetric linear operator

$$\mu : V \to V, \quad < \mu(a), b > + < a, \mu(b) >= 0. \tag{2.1}$$

For the sake of simplicity, in this paper we will consider only the case of diagonalizable operators μ. For any eigenvalue $\lambda \in \text{spec } \mu$, denote V_λ the subspace of V consisting of all eigenvectors with the eigenvalue λ. Let $\pi_\lambda : V \to V_\lambda$ be the projector,

$$\pi_\lambda|_{V_\lambda} = \text{id}, \quad \pi_\lambda(V_{\lambda'}) = 0 \quad \text{for} \quad \lambda' \neq \lambda. \tag{2.2}$$

For any linear operator $A : V \to V$ and for any integer k, denote

$$A_k := \sum_{\lambda \in \text{spec } \mu} \pi_{\lambda+k} A \pi_\lambda. \tag{2.3}$$

By construction

$$A_k(V_\lambda) \subset V_{\lambda+k}. \tag{2.4}$$

2. A μ-*nilpotent* linear operator $R : V \to V$. By definition this means that

$$R = R_1 \oplus R_2 \oplus \dots \tag{2.5}$$

and the components R_k (defined as in (2.3)) for any k satisfy

$$[\mu, R_k] = k R_k \tag{2.6}$$

and also the following symmetry conditions

$$< R_k(a), b > +(-1)^k < a, R_k(b) >= 0, \quad k = 1, 2, \dots, . \tag{2.7}$$

These can be recast into the form

$$\{R(a), b\} + \{a, R(b)\} = 0 \quad \text{for any} \quad a, b \in V \tag{2.8}$$

where the bilinear form $\{ \, , \, \}$ on V is defined by

$$\{a, b\} := \langle e^{\pi i \mu} a, b \rangle. \tag{2.9}$$

Remark 2.1. The operator R must be zero if the operator μ is *nonresonant*. By definition this means that any two eigenvalues of μ do not differ by a nonzero integer. In the opposite case the operator μ is called *resonant*.

We define an equivalence relation between two sets $\big(V^{(i)}, < \, , \, >^{(i)}, \mu^{(i)}, R^{(i)}\big)$, $i = 1, 2$ of the parameters. An equivalence is established by an isomorphism

$$G : V^{(1)} \to V^{(2)} \tag{2.10}$$

of the linear spaces satisfying

$$G = G_0 \oplus G_1 \oplus \dots,$$

G_0 is an isometry

$$\big\langle G_0(a), G_0(b)\big\rangle^{(2)} = \big\langle a, b\big\rangle^{(1)}$$

satisfying

$$G_0 \mu^{(1)} = \mu^{(2)} G_0$$

other components G_k with $k > 0$ satisfy

$$\begin{gathered} \mu^{(2)} G_k - G_k \mu^{(1)} = k G_k \\ G\, R^{(1)} = R^{(2)}\, G \\ \big\{G(a), G(b)\big\}^{(2)} = \{a, b\}^{(1)} \quad \text{for any} \quad a,\, b \in V^{(1)}. \end{gathered}$$

The bilinear forms $\{\ ,\ \}^{(1)}$ and $\{\ ,\ \}^{(2)}$ on the spaces $V^{(1)}$ and $V^{(2)}$ resp. are defined by (2.9).

Remark 2.2. As was explained in [12], the equivalence class of a quadruple $(V, < \, , \, >, \mu, R)$ is the set of monodromy data at $z = 0$ of the linear differential equation

$$z \frac{dy}{dz} = A(z) y \tag{2.11}$$

where the $n \times n$ matrix $A(z)$ satisfies

$$A(z) = \sum_{k \geq 0} A_k z^k, \quad A_0 = \mu, \tag{2.12}$$

$$A^T(-z)\eta + \eta A(z) = 0 \tag{2.13}$$

for constant symmetric nondegenerate matrix η. Hear η is the Gram matrix of the bilinear form $< \, , \, >$ w.r.t. a basis in V. We denote the matrices of the operators μ, R w.r.t. this basis by the same letters μ, R. The monodromy data, according to the definition in [3], are defined as the equivalence class of the system (2.11) w.r.t. gauge transformations of the form

$$y \mapsto \Theta(z)y, \quad \Theta(z) = \sum_{k\geq 0} \Theta_k z^k, \quad \Theta^T(-z)\eta\Theta(z) = \eta. \tag{2.14}$$

In particular, this means that the system (2.11) possesses a fundamental matrix of the form

$$Y(z) = \Theta(z) z^{\mu} z^{R} \tag{2.15}$$

where the matrices $\Theta(z)$, η, μ, R satisfy the above conditions. For two equivalent quadruples of the monodromy data, the correspondent matrices $\Theta^{(1)}(z)$ and $\Theta^{(2)}(z)$ are related by right multiplication by the matrix $G_0 + zG_1 + z^2G_2 + \dots$.

We proceed now to the construction of the Virasoro algebra for any data $(V, < \ , \ >, \mu, R)$. Choose a basis $e_1, \dots, e_n$ in V. Denote

$$\eta_{\alpha\beta} =< e_\alpha, e_\beta > . \tag{2.16}$$

Introduce the inverse matrix

$$\left(\eta^{\alpha\beta}\right) = \left(\eta_{\alpha\beta}\right)^{-1}. \tag{2.17}$$

We start with introducing of a Heisenberg algebra of $(V, < \ , \ >)$. It has generators $\mathbf{1}$ and a_k^α, $\alpha = 1, \dots, n$, $k \in \mathbf{Z}$ obeying the commutation relations

$$[\mathbf{1}, a_k^\alpha] = 0, \quad \left[a_k^\alpha, a_l^\beta\right] = (-1)^k \eta^{\alpha\beta} \delta_{k+l+1,0} \cdot \mathbf{1}. \tag{2.18}$$

The normal ordering is defined by

$$: a_k^\alpha a_l^\beta := \begin{cases} a_l^\beta a_k^\alpha, & \text{if } \ l < 0, \ k \geq 0 \\ a_k^\alpha a_l^\beta, & \text{otherwise.} \end{cases} \tag{2.19}$$

Introduce vector-valued operators

$$a_k = a_k^\alpha e_\alpha, \quad k \in \mathbf{Z}. \tag{2.20}$$

For any integer m we define matrices in $\mathrm{End}(V)$ as

$$P_m(\mu, R) := \begin{cases} \left[e^{R\,\partial_x} \prod_{j=0}^{m} \left(x + \mu + j - \frac{1}{2}\right)\right]_{x=0}, & m \geq 0 \\ 1, & m = -1 \\ \left[e^{R\,\partial_x} \prod_{j=1}^{-m-1} \left(x + \mu - j - \frac{1}{2}\right)^{-1}\right]_{x=0}, & m < -1. \end{cases} \tag{2.21}$$

Remark 2.3. The matrices $P_m(\mu, R)$ for $m < -1$ are defined only if the spectrum of μ does not contain half-integers.

Now we define the operators of the Virasoro algebra by a Sugawara-type construction

$$L_m = \frac{1}{2}\sum_{k,\,l}(-1)^{k+1} : \left\langle a_l, [P_m(\mu - k, R)]_{m-1-l-k}\, a_k \right\rangle : \\ + \frac{1}{4}\delta_{m,0}\ \mathrm{tr}\left(\frac{1}{4} - \mu^2\right)\mathbf{1}, \quad m \in \mathbf{Z}. \tag{2.22}$$

Here the components $[\ ]_q$ for any integer q are defined in (2.3). Due to Remark 2.3 the operators L_m with $m \geq -1$ are always defined, another half, $L_{m<-1}$, is defined only under the assumption spec $\mu \cap \frac{1}{2} + \mathbf{Z} = \emptyset$.

Theorem 1. *The operators* (2.22)*, when well-defined, satisfy the Virasoro commutation relations* (1.8) *with the central charge n.*

Proof. For an arbitrary constant z, the following identity is easily checked to hold true:

$$\eta\,[P_m(\mu + z + 1, R)]_q = (-1)^{m+q+1}\,[P_m(\mu - m - z + q, R)]_q^T\,\eta. \tag{2.23}$$

Using this identity we can express the L_m operators in the form

$$L_m = (1 - \delta_{m,0})\sum_{0\leq i\leq m-1}\ \sum_{0\leq k\leq m-i-1}\frac{1}{2}(-1)^{k+1}\left\langle [P_m(\mu - k, R)]_i\, a_k, a_{m-i-k-1}\right\rangle \\ + \sum_{k\geq 0}\ \sum_{0\leq i\leq m+k}(-1)^k\left\langle [P_m(\mu + k + 1, R)]_i\, a_{-k-1}, a_{m+k-i}\right\rangle \\ + \sum_{k,l\geq 0}\frac{1}{2}(-1)^k\left\langle [P_m(\mu + k + 1, R)]_{l+k+m+1}\, a_{-k-1}, a_{-l-1}\right\rangle \\ + \frac{1}{4}\delta_{m,0}\ \mathrm{tr}\left(\frac{1}{4} - \mu^2\right)\mathbf{1}, \quad m \geq 0, \tag{2.24}$$

$$L_m = \sum_{i\geq 0}\ \sum_{k\geq i-m}(-1)^k\left\langle [P_m(\mu + k + 1, R)]_i\, a_{-k-1}, a_{m+k-i}\right\rangle \\ + \sum_{i\geq 0}\ \sum_{0\leq k\leq i-m-1}\frac{1}{2}(-1)^k\left\langle [P_m(\mu + k + 1, R)]_i\, a_{-k-1}, a_{m+k-i}\right\rangle, \quad m \leq -1. \tag{2.25}$$

From the above expression of the L_m operators and the identity (2.23), it follows, for any $m, m' \geq 0, m + m' \neq 0$, that the commutator $[L_m, L_{m'}]$ can be written as

$$\sum_{0\leq q\leq m+m'-1} \sum_{0\leq k\leq m+m'-q-1} \sum_{i+i'=q} \frac{1}{2}(-1)^{m+i+k+1}\times$$

$$\times \left\langle [P_m(\mu - m - m' + q + k + 1, R)]_i \, a_{m+m'-q-k-1}, [P_{m'}(\mu - k, R)]_{i'} \, a_k \right\rangle$$

$$- \sum_{0\leq q\leq m+m'-1} \sum_{0\leq k\leq m+m'-q-1} \sum_{i+i'=q} \frac{1}{2}(-1)^{m'+i'+k+1}\times$$

$$\times \left\langle [P_{m'}(\mu - m - m' + q + k + 1, R)]_{i'} \, a_{m+m'-q-k-1}, [P_m(\mu - k, R)]_i \, a_k \right\rangle$$

$$+ \sum_{k\geq 0} \sum_{0\leq q\leq m+m'+k} \sum_{i+i'=q} (-1)^{m+i+k}\times$$

$$\times \left\langle [P_{m'}(\mu + k + 1, R)]_{i'} \, a_{-k-1}, [P_m(\mu - m - m' + q - k, R)]_i \, a_{m+m'-q+k} \right\rangle$$

$$- \sum_{k\geq 0} \sum_{0\leq q\leq m+m'+k} \sum_{i+i'=q} (-1)^{m'+i'+k}\times$$

$$\times \left\langle [P_m(\mu + k + 1, R)]_i \, a_{-k-1}, [P_{m'}(\mu - m - m' + q - k, R)]_{i'} \, a_{m+m'-q+k} \right\rangle$$

$$+ \sum_{k,k'\geq 0} \sum_{0\leq i\leq m+m'+k+k'+1} \frac{1}{2}(-1)^{m+k'+i}\times$$

$$\times \left\langle [P_m(\mu + k + 1, R)]_i \, a_{-k-1}, [P_{m'}(\mu + k' + 1, R)]_{m+m'+k+k'-i+1} \, a_{-k'-1} \right\rangle$$

$$- \sum_{k,k'\geq 0} \sum_{0\leq i'\leq m+m'+k+k'+1} \frac{1}{2}(-1)^{m'+k+i'}\times$$

$$\times \left\langle [P_{m'}(\mu + k' + 1, R)]_{i'} \, a_{-k'-1}, [P_m(\mu + k + 1, R)]_{m+m'+k+k'-i'+1} \, a_{-k-1} \right\rangle$$

$$- \frac{1}{2} \sum_{0\leq i\leq m-1} \sum_{0\leq k\leq m-i-1} \mathrm{tr}\left([P_m(\mu - k, R)]_i \, [P_{m'}(\mu + m - i - k, R)]_{m+m'-i}\right)$$

$$+ \frac{1}{2} \sum_{0\leq i'\leq m'-1} \sum_{0\leq k'\leq m'-i'-1} \mathrm{tr}\left([P_{m'}(\mu - k', R)]_{i'} \, [P_m(\mu + m' - i' - k', R)]_{m+m'-i'}\right).$$

The traces which appeared in the last two sums vanish (for $m + m' > 0$) due to the fact that

$$\left[\mu, [R^k]_l\right] = l\,[R^k]_l. \tag{2.26}$$

The remaining terms are then summed up to give the desired $(m-m')\,L_{m+m'}$ due to the following identity:

$$\begin{aligned}
&\sum_{i+i'=q} (-1)^{m+i} \left[P_m(\mu - m - m' + q + z + 1, R)\right]_i^T \eta \left[P_{m'}(\mu - z, R)\right]_{i'} \\
&- \sum_{i+i'=q} (-1)^{m'+i'} \left[P_{m'}(\mu - m - m' + q + z + 1, R)\right]_{i'}^T \eta \left[P_m(\mu - z, R)\right]_i \\
&= (m-m')\,\eta\,[P_{m+m'}(\mu - z, R)]_q, \quad \text{for} \quad m, m' \geq 0.
\end{aligned} \tag{2.27}$$

We can compute the commutator $[L_m, L_{m'}]$ for other cases of $m, m' \in \mathbf{Z}$ in a similar way. The theorem is proved. □

The algebra generated by the operators L_m will be denoted by $\mathrm{Vir}(V, <\ ,\ >, \mu, R)$. It can be readily seen that two equivalent sets of data produce isomorphic Virasoro algebras.

A particular realization of the Heisenberg algebra will be in the space of functions of an infinite number of variables $T^{\alpha,p}$, $\alpha = 1, \dots, n$, $p = 0, 1, 2, \dots$. We put

$$a_k^\alpha = \begin{cases} \eta^{\alpha\beta} \frac{\partial}{\partial T^{\beta,k}}, & k \geq 0 \\ (-1)^{k+1} T^{\alpha,-k-1}, & k < 0. \end{cases} \tag{2.28}$$

We also introduce vectors

$$T^k = T^{\alpha,k} e_\alpha \in V \otimes \mathbf{C}[[T]], \quad k \geq 0$$

and covectors

$$\partial_{T^k} = e^\alpha \partial_{T^{\alpha,k}} \in V^* \otimes \mathrm{Der}\left(\mathbf{C}[[T]]\right), \quad k \geq 0,$$

where $e^1, \dots, e^n \in V^*$ is the basis dual to $e_1 \dots, e_n \in V$. Denote by the same symbol $<\ ,\ >$ the natural pairing $V \otimes V^* \to \mathbf{C}$ and also the bilinear form on V^* dual to that on V. Then the first few Virasoro operators look as follows:

$$L_{-1} = \sum_{p\geq 1} \left\langle T^p, \partial_{T^{p-1}} \right\rangle + \frac{1}{2} \left\langle T^0, T^0 \right\rangle \tag{2.29}$$

$$\begin{aligned}
L_0 = &\sum_{p \geq 0} \left\langle \left(p + \frac{1}{2} + \mu\right) T^p, \partial_{T^p} \right\rangle + \sum_{p\geq 1} \sum_{1 \leq r \leq p} \left\langle R_r T^p, \partial_{T^{p-r}} \right\rangle \\
&+ \frac{1}{2} \sum_{p,\, q} (-1)^q \left\langle R_{p+q+1} T^p, T^q \right\rangle + \frac{1}{4} \mathrm{tr}\left(\frac{1}{4} - \mu^2\right).
\end{aligned} \tag{2.30}$$

To write down the formulae for L_1 and $L_{\pm 2}$, we introduce the matrices $R_{k,l} \in \mathrm{End}(V)$ putting

$$\begin{aligned} R_{0,0} &= 1 \\ R_{k,0} &= 0, && k > 0 \\ R_{k,l} &= \left[R^l\right]_k = \sum_{i_1+\cdots+i_l=k} R_{i_1} \ldots R_{i_l}, && l > 0. \end{aligned}$$

Then

$$\begin{aligned} L_1 = &\sum_{p\geq 0} \left\langle \left(p+\frac{1}{2}+\mu\right)\left(p+\frac{3}{2}+\mu\right) T^p, \partial_{T^{p+1}} \right\rangle \\ &+ \sum_{p\geq 0}\sum_{1\leq r\leq p+1} \left\langle R_r \left(2p+2+2\mu\right) T^p, \partial_{T^{p-r+1}} \right\rangle \\ &+ \sum_{p\geq 1}\sum_{2\leq r\leq p+1} \left\langle R_{r,2} T^p, \partial_{T^{p-r+1}} \right\rangle + \frac{1}{2}\left\langle \partial_{T^0}\left(\frac{1}{2}+\mu\right), \partial_{T^0}\left(\frac{1}{2}+\mu\right)\right\rangle \\ &+ \sum_{p,\, q\geq 0} (-1)^q \left\langle R_{p+q+2}\left(p+\mu+1\right) T^p, T^q \right\rangle \\ &+ \frac{1}{2}\sum_{p,\, q\geq 0} (-1)^q \left\langle R_{p+q+2,2} T^p, T^q \right\rangle, \end{aligned} \tag{2.31}$$

$$\begin{aligned} L_2 = &\sum_{p\geq 0} \left\langle \left(p+\frac{1}{2}+\mu\right)\left(p+\frac{3}{2}+\mu\right)\left(p+\frac{5}{2}+\mu\right) T^p, \partial_{T^{p+2}} \right\rangle \\ &+ \sum_{p\geq 0}\sum_{1\leq r\leq p+2} \left\langle R_r \left[3\left(p+\frac{1}{2}+\mu\right)^2 + 6\left(p+\frac{1}{2}+\mu\right)+2\right] T^p, \partial_{T^{p-r+2}} \right\rangle \\ &+ \sum_{p\geq 0}\sum_{2\leq r\leq p+2} \left\langle R_{r,2}\left(3p+\frac{9}{2}+3\mu\right) T^p, \partial_{T^{p-r+2}} \right\rangle \\ &+ \sum_{p\geq 1}\sum_{3\leq r\leq p+2} \left\langle R_{r,3} T^p, \partial_{T^{p-r+2}} \right\rangle \\ &+ \left\langle \partial_{T^1}\left(\frac{1}{2}-\mu\right), \partial_{T^0}\left(\frac{1}{2}-\mu\right)\left(\frac{3}{2}-\mu\right)\right\rangle \\ &+ \frac{1}{2}\left\langle \partial_{T^0} R_1, \partial_{T^0}\left(\frac{1}{4}+3\mu-3\mu^2\right)\right\rangle \\ &+ \frac{1}{2}\sum_{p,\, q\geq 0} (-1)^q \left\langle \left(R_{p+q+3,2} + 3R_{p+q+3,2}\left(p+\mu+\frac{3}{2}\right) \right.\right. \\ &+ \left.\left. R_{p+q+3}\left[\frac{3}{4}(2p+2\mu+3)^2 - 1\right]\right) T^p, T^q \right\rangle, \end{aligned} \tag{2.32}$$

$$
\begin{aligned}
L_{-2} = & \sum_{p\geq 2}\left\langle \left(p-\frac{1}{2}+\mu\right)^{-1} T^p, \partial_{T^{p-2}}\right\rangle \\
& + \sum_{k\geq 1}\sum_{l\geq k}\sum_{p\geq l+2}(-1)^k\left\langle R_{l,k}\left(p-\frac{1}{2}+\mu\right)^{-k-1} T^p, \partial_{T^{p-l-2}}\right\rangle \\
& + \left\langle \left(\frac{1}{2}-\mu\right)^{-1} T^0, T^1\right\rangle \\
& + \frac{1}{2}\sum_{k\geq 1}\sum_{l\geq k}\sum_{0\leq p\leq l+1}(-1)^{l+p+k+1}\times \\
& \times\left\langle R_{l,k}\left(p-\frac{1}{2}+\mu\right)^{-k-1} T^p, T^{l-p+1}\right\rangle. \qquad (2.33)
\end{aligned}
$$

Example 2.1. For $n=1$ it must be $\mu=0$, $R=0$. So

$$L_m = \frac{1}{2}\sum_k (-1)^{k+1}P_m(-k) : a_k a_{m-1-k} : + \frac{1}{16}\delta_{m,0}$$

where

$$P_m(x) = \begin{cases} \prod_{j=0}^m \left(x+\frac{2j-1}{2}\right), & m\geq 0 \\ 1, & m=-1 \\ \prod_{j=1}^{-m-1}\left(x-\frac{2j+1}{2}\right)^{-1}, & m<-1. \end{cases}$$

We obtain the well known realization of the Virasoro algebra in the theory of KdV hierarchy and in 2D topological gravity [6], [20].

Example 2.2. Let $V = H^*(X;\mathbf{C})$ where X is a smooth projective variety of complex dimension d having $H^{\mathrm{odd}}(X)=0$. The bilinear form $<\ ,\ >$ is specified by the Poincaré pairing

$$<\omega_1,\omega_2> = \int_X \omega_1\wedge\omega_2.$$

The matrix μ is diagonal in the homogeneous basis

$$e_\alpha \in H^{2q_\alpha}(X), \quad \mu(e_\alpha) = \left(q_\alpha - \frac{d}{2}\right)e_\alpha.$$

The matrix $R = R_1$ (i.e., $R_2 = R_3 = \cdots = 0$) is the matrix of multiplication by the first Chern class $c_1(X)$. In this case the operators L_m coincide with the Virasoro operators of [19] (introduced in this paper for the particular case of quantum cohomology of $X = \mathbf{C}P^d$). Recall that the chiral primary fields of the quantum cohomology of X (of the topological sigma-model coupled to gravity with the target space X, using physical language) are in one-to-one correspondence with a homogeneous basis in $H^*(X)$,

$$\phi_1 = 1 \in H^0(X), \quad \phi_\alpha \in H^{2q_\alpha}(X).$$

The topological correlators are defined by

$$\begin{gathered}\left\langle \tau_{n_1}(\phi_{\alpha_1}) \dots \tau_{n_k}(\phi_{\alpha_k}) \right\rangle_g \\ := \sum_{\beta \in H_2(X,\mathbf{Z})} q^\beta \int_{\left[\bar{\mathcal{M}}_{g,k}(X,\beta)\right]^{\mathrm{virt}}} c_1(L_1)^{n_1} \cup \mathrm{ev}_1^*(\phi_{\alpha_1}) \cup \dots \cup c_1(L_k)^{n_k} \cup \mathrm{ev}_k^*(\phi_{\alpha_k}).\end{gathered}$$

In this formula q^β belongs to the Novikov ring $\mathbf{Z}$-spanned by monomials $q^\beta := q_1^{b_1} \dots q_r^{b_r}$ for a basis $q_1, \dots, q_r$ of $H_2(X,\mathbf{Z})$ where $\beta = \sum b_i q_i$. The integration is taken over the virtual moduli space $\left[\bar{\mathcal{M}}_{g,k}(X,\beta)\right]^{\mathrm{virt}}$ of the degree $\beta \in H_2(X,\mathbf{Z})$ stable maps from k-marked curves of genus g to X; $c_1(L_i)$ is the first Chern class of the tautological line bundle L_i over $\left[\bar{\mathcal{M}}_{g,k}(X,\beta)\right]^{\mathrm{virt}}$ whose fiber over each stable map is defined by the cotangent space of underlying curve at the i-th marked point. Finally, ev_i is the evalution map from $\left[\bar{\mathcal{M}}_{g,k}(X,\beta)\right]^{\mathrm{virt}}$ to X defined by evaluating each stable map at the i-th marked point. According to Remark 2.3, the operators L_m with $m \geq -1$ are always well-defined. The full Virasoro algebra exists for even complex dimension d. See also [22] where certain higher genus Virasoro constraints were studied.

We define now another embedding of Virasoro algebra into the universal enveloping of the Heisenberg algebra (2.18). Let us choose an eigenvector e_1 of the operator μ,

$$\mu(e_1) = \mu_1 e_1.$$

Lemma 1. *The shift $a_k \mapsto \tilde{a}_k$ where*

$$\tilde{a}_k = \begin{cases} a_k, & k \neq -2 \\ a_k - e_1 \mathbf{1}, & k = -2 \end{cases} \tag{2.34}$$

in the formula (2.22) *preserves the Virasoro commutation relations* (1.8)*.*

The proof is straightforward.

The resulting Virasoro algebra will be denoted $\mathrm{Vir}\,(V, <\ ,\ >, \mu, e_1, R)$. It depends only on the equivalence class of the data $(V, <\ ,\ >, \mu, e_1, R)$ under the transformations (2.10) such that the operator G_0 respects the marked eigenvectors $G_0(e_1^{(1)}) = e_1^{(2)}$.

At the end of this section we construct a free field representation of the Virasoro algebra. This construction is analogous to [19].

Let us introduce the operator-valued vector function of z

$$\phi(z;\mu) = -\sum_{k\in\mathbf{Z}} z^{-k-\frac{1}{2}+\mu} \frac{\Gamma\left(\frac{1}{2}-\mu+k\right)}{\Gamma\left(\frac{1}{2}-\mu\right)} a_k \tag{2.35}$$

and the current

$$j(z;\mu) = \partial_z \phi(z;\mu). \tag{2.36}$$

Then the Virasoro operators are the coefficients of the expansion of the stress tensor

$$\begin{aligned} T(z) = \sum_{k\in\mathbf{Z}} L_k z^{-k-2} &= \frac{1}{4z^2}\,\mathrm{tr}\left(\frac{1}{4}-\mu^2\right) \\ &+ \frac{1}{2} : \left\langle \Gamma\left(\frac{1}{2}-\mu+x\right) j(z;\mu-x), e^{R\overleftrightarrow{\partial}_x}\Gamma^{-1}\left(\frac{1}{2}+\mu+x\right) j(z;\mu+x)\right\rangle_{x=0} : . \end{aligned} \tag{2.37}$$

Here the operator $e^{R\overleftrightarrow{\partial}_x}$ is defined by the equation

$$\left\langle f(x), e^{R\overleftrightarrow{\partial}_x} g(x)\right\rangle = \left\langle e^{R^*\partial_x} f(x), e^{R\,\partial_x} g(x)\right\rangle, \tag{2.38}$$

where R^* is the conjugate operator $< R^*(a), b >=< a, R(b) >$. Observe that

$$R^* = (R_1 + R_2 + R_3 + \dots)^* = R_1 - R_2 + R_3 - \dots . \tag{2.39}$$

3. From Frobenius manifolds to partition functions

WDVV equations of associativity involve the problem of finding a quasihomogeneous, up to at most quadratic polynomial, function $F(t)$ of the variables $t = (t^1, \dots, t^n)$ and of a constant nondegenerate symmetric matrix $\left(\eta^{\alpha\beta}\right)$ such that the following combinations of the third derivatives

$$c^\gamma_{\alpha\beta}(t) := \eta^{\gamma\epsilon}\partial_\epsilon\partial_\alpha\partial_\beta F(t) \tag{3.1}$$

for any t are structure constants of an asociative algebra

$$A_t = \mathrm{span}\,(e_1, \dots, e_n), \quad e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(t) e_\gamma, \quad \alpha, \beta = 1, \dots, n$$

with the unity $e = e_1$ (summation w.r.t. repeated indices will be assumed). Frobenius manifolds were introduced in [10] as the coordinate free form of WDVV. We recall now the definition.

A *Frobenius algebra* (over the field $k = \mathbf{C}$) is a pair $(A, <\ ,\ >)$, where A is a commutative associative k-algebra with a unity e, $<\ ,\ >$ is a symmetric nondegenerate *invariant* bilinear form $A \times A \to k$, i.e., $< a \cdot b, c >=< a, b \cdot c >$ for any a, b, $c \in A$. A *gradation of charge* d on A is a k-derivation $Q : A \to A$ such that

$$< Q(a), b > + < a, Q(b) >= d < a, b >, \quad d \in k. \tag{3.2}$$

More generally, a graded Frobenius algebra $(A, <\ ,\ >)$ with a charge $d \in k$ over a graded commutative associative k-algebra R by definition is endowed with two k-derivations $Q_R : R \to R$ and $Q_A : A \to A$ satisfying the properties

$$Q_A(\alpha a) = Q_R(\alpha)a + \alpha Q_A(a), \quad \alpha \in R,\ a \in A \tag{3.3}$$

$$<Q_A(a), b > + < a, Q_A(b) > -Q_R < a, b >= d < a, b >, \quad a,\ b \in A. \tag{3.4}$$

A *Frobenius structure* of charge d on the manifold M is the structure of a Frobenius algebra on the tangent spaces $T_tM = (A_t, <\ ,\ >_t)$ depending (smoothly, analytically etc.) on the point $t \in M$. It must satisfy the following axioms:

FM1. The metric $<\ ,\ >_t$ on M is flat (but not necessarily positive definite). Denote ∇ the Levi-Civita connection for the metric. The unity vector field e must be covariantly constant, $\nabla e = 0$.

FM2. Let c be the 3-tensor $c(u, v, w) :=< u \cdot v, w >$, u, v, $w \in T_tM$. The 4-tensor $(\nabla_z c)(u, v, w)$ must be symmetric in u, v, w, $z \in T_tM$.

FM3. A linear vector field $E \in \mathrm{Vect}(M)$ must be fixed on M, i.e., $\nabla\nabla E = 0$, such that the derivations $Q_{\mathrm{Func}(M)} := E$, $\quad Q_{\mathrm{Vect}(M)} := \mathrm{id} + \mathrm{ad}_E$ introduce in $\mathrm{Vect}(M)$ the structure of graded Frobenius algebra of the given charge d over the graded ring $\mathrm{Func}(M)$ of (smooth, analytic etc.) functions on M. We call E the *Euler vector field*.

A manifold M equipped with a Frobenius structure is called a Frobenius manifold.

Locally, in the flat coordinates $t^1, \ldots, t^n$ for the metric $<\ ,\ >_t$, a FM is described by a solution $F(t)$ of the WDVV associativity equations, where $\partial_\alpha\partial_\beta\partial_\gamma F(t) = < \partial_\alpha \cdot \partial_\beta, \partial_\gamma >$, and vice versa. We will call $F(t)$ *the potential* of the FM (physicists call it *primary free energy*; in the setting of quantum cohomology it is called the Gromov-Witten potential [31]). Observe that the unity e is an eigenvector of ∇E with the eigenvalue 1. The eigenfunctions of the operator ∂_E, $\partial_E f(t) = p\, f(t)$ are called *quasihomogeneous* functions of degree p on the Frobenius manifold. In main applications the $(1, 1)$-tensor ∇E is diagonalizable. The potential $F(t)$ is a quasihomogeneous function of the degree $3-d$ well-defined up to a quadratic polynomial.

A generalization of the above definition to Frobenius supermanifolds was proposed in [31]. We will not consider it in the present article.

An important geometrical structure on a Frobenius manifold is the *deformed flat connection* $\tilde\nabla$ introduced in [8]. It is defined by the formula

$$\tilde\nabla_u v := \nabla_u v + z\, u\cdot v. \tag{3.5}$$

Here u, v are two vector fields on M, z is the parameter of the deformation. (In [23] another normalization is used $\tilde\nabla \mapsto \hbar\tilde\nabla$, $\hbar = z^{-1}$.) We extend this to a meromorphic connection on the direct product $M\times \mathbf{C}$, $z\in\mathbf{C}$, by the formula

$$\tilde\nabla_{d/dz} v = \partial_z v + E\cdot v - \frac{1}{z}\mu\, v \tag{3.6}$$

with

$$\mu := \frac{1}{2}(2-d)\cdot\mathbf{1} - \nabla E, \tag{3.7}$$

other covariant derivatives are trivial. Here u, v are tangent vector fields on $M\times\mathbf{C}$ having zero components along $\mathbf{C}\ni z$. The curvature of $\tilde\nabla$ is equal to zero [11]. So, there locally exist n independent functions $\tilde t_1(t;z),\dots,\tilde t_n(t;z)$, $z\neq 0$, such that

$$\tilde\nabla\, d\tilde t_\alpha(t;z) = 0, \quad \alpha = 1,\dots,n. \tag{3.8}$$

We call these functions *deformed flat coordinates.*

Remark 3.1. In the setting of quantum cohomology, the part of the system (3.8) not containing $\tilde\nabla_{d/dz}$ is called the *quantum differential equations* [23].

The component $\tilde\nabla_{d/dz}$ of the system (3.8) for the vectors $\xi = \nabla\tilde t$ reads

$$z\partial_z\xi = (\mu + z\mathcal{U})\xi \tag{3.9}$$

where

$$\mathcal{U} = \left(\mathcal{U}^\alpha_\beta(t)\right), \quad \mathcal{U}^\alpha_\beta(t) := E^\epsilon(t) c^\alpha_{\epsilon\beta}(t) \tag{3.10}$$

is the matrix of the operator of multiplication by the Euler vector field. The matrix-valued function

$$A(z) = \mu + z\mathcal{U} \tag{3.11}$$

satisfies the symmetry (2.13)

$$\langle A(-z)a, b\rangle + \langle a, A(z)b\rangle = 0 \tag{3.12}$$

for any two vectors a, b. The equivalence class of the system (3.9) w.r.t. the gauge transformations $\xi \mapsto \Theta(z)\xi$ with $\Theta(z)$ of the form (2.14) is called *the monodromy data at* $z = 0$ of the Frobenius manifold. As was shown in [12] (for the case of diagonalizable matrix μ, the general case is similar), the monodromy data are uniquely determined by a quadruple $(V, <\ ,\ >, \mu, R)$ and by a marked eigenvector $e_1 := e$ of μ, $\mu(e_1) = -\frac{d}{2}e_1$, of the form described in Section 1 above. Here $V = T_tM$ with the bilinear form $<\ ,\ >$ and with the antisymmetric operator μ given by (3.7). The μ-nilpotent matrix R gives an additional set of parameters in the case of resonant μ. The monodromy data do not depend on $t \in M$ [12].

In particular, in the setting of quantum cohomology the monodromy data has the form of Example 2.2 above (see [12]). From the structure (2.15) of the fundamental system of solutions we obtain the existence of a system of deformed flat coordinates of the form

$$\left(\tilde{t}_1(t;z), \dots, \tilde{t}_n(t;z)\right) = \left(\theta_1(t;z), \dots, \theta_n(t;z)\right) z^\mu z^R \tag{3.13}$$

where the functions $\theta_1(t;z)$, … , $\theta_n(t;z)$ are analytic at $z = 0$ and they satisfy the conditions

$$\theta_\alpha(t;0) = \eta_{\alpha\beta}t^\beta, \quad \alpha = 1, \dots, n \tag{3.14}$$

$$\left\langle \nabla\theta_\alpha(t;-z), \nabla\theta_\beta(t;z)\right\rangle = \eta_{\alpha\beta}. \tag{3.15}$$

In the notations of (3.13) the columns of the matrix $\Theta(t;z) = \left(\Theta^\alpha_\beta(t;z)\right)$ are the gradients $\nabla\theta_1(t;z)$, … , $\nabla\theta_n(t;z)$,

$$\Theta^\alpha_\beta(t;z) = \nabla^\alpha\theta_\beta(t;z) = \delta^\alpha_\beta + O(z). \tag{3.16}$$

The deformed flat coordinates (3.13) are determined uniquely up to transformations of the form (2.10).

Let us denote $\theta_{\alpha,p}(t)$ the coefficients of Taylor expansions

$$\theta_\alpha(t;z) = \sum_{p=0}^{\infty} \theta_{\alpha,p}(t) z^p. \tag{3.17}$$

We introduce also, due to (3.15), the matrix-valued function

$$\Omega(t;z,w) = \left(\Omega_{\alpha\beta}(t;z,w)\right) = \frac{1}{z+w}\left[\Theta^T(t;z)\,\eta\,\Theta(t;w) - \eta\right] \tag{3.18}$$

and its Taylor coefficients

$$\Omega_{\alpha\beta}(t;z,w) = \sum_{p,\,q=0}^{\infty} \Omega_{\alpha,p;\,\beta,q}(t) z^p w^q. \tag{3.19}$$

The coefficients $\Omega_{\alpha,p;\,\beta,q}(t)$ can be expressed via scalar products of the gradients of the functions $\theta_{\alpha,m}(t)$, $\theta_{\beta,m}(t)$ with various m

$$\Omega_{\alpha,p;\,\beta,q}(t) = \sum_{m=0}^{q} (-1)^m \big\langle \nabla\theta_{\alpha,p+m+1}, \nabla\theta_{\beta,q-m}\big\rangle. \tag{3.20}$$

They also satisfy the following identities

$$\Omega_{\alpha,p+1;\beta,q} + \Omega_{\alpha,p;\beta,q+1} = \Omega_{\alpha,p;\sigma,0}\,\eta^{\sigma\rho}\,\Omega_{\rho,0;\beta,q}. \tag{3.21}$$

For the gradients of the entries of the matrix $\Omega(t;z,w)$, one has

$$\nabla\Omega_{\alpha\beta}(t;z,w) = \nabla\theta_\alpha(t;z)\cdot\nabla\theta_\beta(t;w). \tag{3.22}$$

From this and (3.16) it readily follows that the functions

$$t^\alpha := \eta^{\alpha\beta}\Omega_{\beta,0;\,1,0} \tag{3.23}$$

can serve as the flat coordinates on M and

$$\partial_\beta t^\alpha = \delta^\alpha_\beta. \tag{3.24}$$

In particular, the unity vector field is

$$e = e_1 = \frac{\partial}{\partial t^1}.$$

Moreover, if we define the function F on M by

$$F(t) = \frac{1}{2}\big[\Omega_{1,1;\,1,1}(t) - 2t^\alpha\Omega_{\alpha,0;\,1,1}(t) + t^\alpha t^\beta\Omega_{\alpha,0;\,\beta,0}(t)\big] \tag{3.25}$$

with the flat coordinates of the form (3.23), then

$$\big\langle \partial_\alpha\cdot\partial_\beta, \partial_\gamma\big\rangle = \partial_\alpha\partial_\beta\partial_\gamma F(t).$$

So $F(t)$ is the potential of the Frobenius manifold. It will be also useful to know that the derivatives of the potential defined by (3.25) are

$$\partial_\alpha F = t^\beta\Omega_{\alpha,0;\beta,0}(t) - \Omega_{\alpha,0;1,1}(t) \tag{3.26}$$

$$\partial_\alpha\partial_\beta F = \Omega_{\alpha,0;\beta,0}(t). \tag{3.27}$$

Now we are able to construct the genus zero approximation to the integrable hierarchy (1.10) and to compute the genus zero free energy $\mathcal{F}_0(T)$. The genus zero hierarchy reads

$$\partial_{T^{\alpha,p}} t = \partial_X K^{(0)}_{\alpha,p}(t), \quad K^{(0)}_{\alpha,p}(t) = \nabla\theta_{\alpha,p+1}(t). \tag{3.28}$$

This is a Hamiltonian hierarchy on the loop space $L(M)$ equipped with the Poisson bracket $\{\ ,\ \}_1^{(0)}$

$$\left\{t^\alpha(X), t^\beta(Y)\right\}_1^{(0)} = \eta^{\alpha\beta}\delta'(X-Y). \tag{3.29}$$

Here $\delta'(X)$ is the derivative of the delta-function on the circle. The Hamiltonian reads

$$H_{\alpha,p}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} \theta_{\alpha,p+1}\left(t(X)\right)\, dX. \tag{3.30}$$

In particular,

$$\partial_{T^{1,0}} t = \partial_X t.$$

So we will identify X and $T^{1,0}$.

The second Hamiltonian structure of the hierarchy (3.28), found in [9], has the form

$$\left\{t^\alpha(X), t^\beta(Y)\right\}_2^{(0)} = g^{\alpha\beta}\big(t(X)\big)\delta'(X-Y) + \Gamma_\gamma^{\alpha\beta}(t)t_X^\gamma\delta(X-Y). \tag{3.31}$$

Here

$$g^{\alpha\beta}(t) = \eta^{\alpha\gamma}\mathcal{U}_\gamma^\beta(t) \tag{3.32}$$

is *the intersection form* of the Frobenius manifold M (see [11]). It defines on an open subset of M, where $\det\big(g^{\alpha\beta}(t)\big) \neq 0$, a new flat metric. The coefficients

$$\Gamma_\gamma^{\alpha\beta}(t) = c_\gamma^{\alpha\epsilon}(t)\left(\frac{1}{2}-\mu\right)_\epsilon^\beta \tag{3.33}$$

are related to the Christoffel coefficients $\Gamma_{\epsilon\gamma}^\beta(t)$ of the Levi-Civita connection of the metric (3.32)

$$\Gamma_\gamma^{\alpha\beta}(t) = -g^{\alpha\epsilon}(t)\Gamma_{\epsilon\gamma}^\beta(t). \tag{3.34}$$

The Poisson brackets (3.29) and (3.31) are *compatible*. This means [34] that any linear combination

$$\{\ ,\ \}_2^{(0)} - \lambda\{\ ,\ \}_1^{(0)}$$

with an arbitrary value of λ is again a Poisson bracket. All these statements follow from the general theory of [15] of Poisson brackets of hydrodynamic type (i.e., those of the form (3.31) and from the theorem [11] that the metrics $g^{\alpha\beta}(t)$ and $\eta^{\alpha\beta}$ form *a flat pencil* (see also [14]). In particular, denoting $\hat\nabla$ the Levi-Civita covariant gradient for the metric (3.32), we obtain the following important formula

$$\hat\nabla d\tilde t \equiv \left(\hat\nabla^\alpha\partial_\beta\tilde t\right) = d\left(\partial_z - \frac{1}{2\,z}\right)\nabla\tilde t \tag{3.35}$$

for an arbitrary deformed flat coordinate $\tilde t = \tilde t(t;z)$ of the deformed connection $\tilde\nabla$.

The bihamiltonian structure (3.29) and (3.31) gives the possibility [34] to construct the genus zero hierarchy applying the recursion operator

$$\mathcal{R} = \left(\mathcal{R}^\alpha_\beta\right), \quad \mathcal{R}^\alpha_\beta = \mathcal{U}^\alpha_\beta(t) + \Gamma^{\alpha\epsilon}_\gamma(t) t^\gamma_X \eta_{\epsilon\beta} \partial_X^{-1}. \tag{3.36}$$

We now specify a particular solution of the hierarchy. Due to the obvious invariance of the equations of the hierarchy w.r.t. the transformations

$$T^{\alpha,p} \mapsto c\,T^{\alpha,p}, \quad t \mapsto t$$

and w.r.t. shifts along the times $T^{\beta,q}$, we specify the symmetric solution $t^{(0)}(T)$ as follows

$$\sum_{\alpha,p} \tilde{T}^{\alpha,p} \partial_{T^{\alpha,p}} t^{(0)} = 0, \tag{3.37}$$

where the shifted times are defined in (1.5). This solution can be found from the fixed point equation

$$t = \nabla \Phi_T(t) \tag{3.38}$$

where

$$\Phi_T(t) = \sum T^{\alpha,p} \theta_{\alpha,p}(t) \tag{3.39}$$

in the form

$$t^{(0)}(T) = T^0 + \sum_{q>0} T^{\beta,q} \nabla\theta_{\beta,q}(T^0) + \sum_{p,\,q>0} T^{\beta,q} T^{\gamma,p} \nabla\theta_{\beta,q-1}(T^0) \cdot \nabla\theta_{\gamma,p}(T^0) + \dots .$$

This is a power series in $T^{\alpha,p>0}$ with coefficients that depend on $T^{\alpha,0}$ through certain functions defined on the Frobenius manifold, where we are to substitute $t^\alpha \mapsto T^{\alpha,0}$ for any α. Finally, the genus zero free energy has the form

$$\mathcal{F}_0(T) = \frac{1}{2} \sum \Omega_{\alpha,p;\,\beta,q}\left(t^{(0)}(T)\right) \tilde{T}^{\alpha,p} \tilde{T}^{\beta,q}. \tag{3.40}$$

From this formula it follows that

$$\Omega_{\alpha,p;\,\gamma,r}(t)|_{t=t^{(0)}(T)} = \left\langle \tau_p(\phi_\alpha)\tau_r(\phi_\gamma) e^{\sum \tau_q(\phi_\beta) T^{\beta,q}} \right\rangle_0 = \partial_{T^{\alpha,p}} \partial_{T^{\gamma,r}} \mathcal{F}_0(T). \tag{3.41}$$

The genus 1 free energy has the following structure, according to [5]

$$\mathcal{F}_1(T) = \left[G(t) + \frac{1}{24} \log\det M^\alpha_\beta(t, t_X) \right]_{t=t^{(0)}(T),\ t_X = \partial_{T^{1,0}} t^{(0)}(T)}. \tag{3.42}$$

Here the matrix $M^\alpha_\beta(t, t_X)$ is given by the formula

$$M^\alpha_\beta(t, t_X) = c^\alpha_{\beta\gamma}(t) t^\gamma_X$$

and $G(t)$ is some function on the Frobenius manifold M. In the case of quantum cohomology this function $G(t)$ was identified in [21] as the generating function of elliptic Gromov-Witten invariants. As it was shown in [17], this function can be computed in terms of the Frobenius structure by a universal formula provided that the semisimplicity condition for M holds true.

To explain here this formula we recall first necessary constructions of the theory [11] of semisimple FMs.

A point $t \in M$ is called *semisimple* if the algebra on T_tM is semisimple. A connected FM M is called semisimple if it has at least one semisimple point. Classification of semisimple FMs can be reduced, by a nonlinear change of coordinates, to a system of ordinary differential equations. First we will describe these new coordinates.

Denote $u_1(t), \ldots, u_n(t)$ the roots of the characteristic polynomial of the operator $\mathcal{U}(t)$ of multiplication by the Euler vector field $E(t)$ ($n = \dim M$). Denote $M^0 \subset M$ the open subset where all the roots are pairwise distinct. It turns out [8] that M^0 is nonempty and the functions $u_1(t), \ldots, u_n(t)$ are independent local coordinates on M^0. In these coordinates

$$\partial_i \cdot \partial_j = \delta_{ij}\partial_i, \quad \text{where} \quad \partial_i := \partial/\partial u_i, \tag{3.43}$$

and

$$E = \sum_i u_i \partial_i. \tag{3.44}$$

The local coordinates $u_1, \ldots, u_n$ on M^0 are called *canonical*.

We will now rewrite WDVV in the canonical coordinates. Staying in a small ball on M^0, let us order the canonical coordinates and choose the signs of the square roots

$$\psi_{i1} := \sqrt{< \partial_i, \partial_i >}, \quad i = 1, \ldots, n. \tag{3.45}$$

The orthonormal frame of the normalized idempotents ∂_i establishes a local trivialization of the tangent bundle TM^0. The deformed flat connection $\tilde\nabla$ in TM^0 is recast into the following flat connection in the trivial bundle $M^0 \times \mathbf{C} \times \mathbf{C}^n$

$$\tilde\nabla_i \mapsto \mathcal{D}_i = \partial_i - z\, E_i - V_i, \qquad \tilde\nabla_{d/dz} \mapsto \mathcal{D}_z = \partial_z - U - \frac{1}{z}V, \tag{3.46}$$

other components are obvious. Here the $n \times n$ matrices E_i, U, $V = (V_{ij})$ read

$$(E_i)_{kl} = \delta_{ik}\delta_{il}, \quad U = \mathrm{diag}\,(u_1, \ldots, u_n), \quad V = \Psi\, \mu\, \Psi^{-1} = -V^T \tag{3.47}$$

where the matrix $\Psi = (\psi_{i\alpha})$ satisfying $\Psi^T\Psi = \eta$ is defined by

$$\psi_{i\alpha} := \psi_{i1}^{-1}\frac{\partial t_\alpha}{\partial u_i}, \quad i,\, \alpha = 1, \dots, n. \tag{3.48}$$

The skew-symmetric matrices V_i are determined by the equations

$$[U, V_i] = [E_i, V]. \tag{3.49}$$

Flatness of the connection (3.46)

$$[\mathcal{D}_i, \mathcal{D}_j] = 0, \quad [\mathcal{D}_i, \mathcal{D}_z] = 0$$

reads as the system of commuting time-dependent Hamiltonian flows on the Lie algebra $so(n) \ni V$ equipped with the standard linear Poisson bracket

$$\frac{\partial V}{\partial u_i} = \big\{V, H_i(V;u)\big\}, \quad i = 1, \dots, n \tag{3.50}$$

with the quadratic Hamiltonians

$$H_i(V;u) = \frac{1}{2}\sum_{j\neq i}\frac{V_{ij}^2}{u_i - u_j}, \quad i = 1, \dots, n. \tag{3.51}$$

The monodromy of the operator $\tilde{\nabla}_{d/dz}$ (i.e., the monodromy at the origin, the Stokes matrix, and the central connection matrix, see definitions in [11], [12]) does not change with small variations of a point $u = (u_1, \dots, u_n) \in M$. Using this isomonodromicity property, one can parametrize semisimple Frobenius manifolds by the monodromy data of the deformed flat connection (see [12]). Recall that in our construction of the Virasoro constraints, we use just the monodromy at the origin defined for an arbitrary Frobenius manifold, not only for a semisimple one.

We are ready now to give the formula for the function $G(t)$ called in [17] the G-function of the Frobenius manifold. We define, following [28], the tau-function of the isomonodromy deformation (3.50) by a quadrature

$$d\log\tau = \sum_{i=1}^{n} H_i(V(u);u)du_i. \tag{3.52}$$

The G-function of the FM is defined by

$$G = \log(\tau/J^{1/24}) \tag{3.53}$$

where

$$J = \det\left(\partial t^\alpha/\partial u_i\right) = \pm\prod_{i=1}^{n}\psi_{i1}(u). \tag{3.54}$$

This function is defined on some covering $\hat{M}^0$ of $M^0 \subset M$. As was proved in [17], if the Frobenius manifold M is the quantum cohomology of a smooth projective variety X, and M is semisimple, then the G-function of M coincides with the generating function of elliptic Gromov-Witten invariants of X.

The genus 0 hierarchy (3.28) can be extended to the genus 1 (see (1.15)). The explicit formula for the correction $K^{(1)}(t, t_X, t_{XX})$ together with an extension of the bihamiltonian structure (3.29), (3.31) was obtained in [17]. We will not give here these formulae.

We are now ready to return to Virasoro constraints. Let us consider the algebra $\mathrm{Vir}\,(V, <\ ,\ >, \mu, e_1, R)$ where $(V, <\ ,\ >, \mu, e_1, R)$ is the monodromy at the origin of the Frobenius manifold M. The realization (2.22), (2.28) of the algebra, where one is to do the shift (2.34) corresponding to (1.5), acts on the algebras of "functions"

$$\mathbf{A}_0 := \mathrm{Func}(M) \otimes \mathbf{C}\big[[T^{\alpha,p>0}]\big]$$

or on

$$\mathbf{A}_1 := \mathrm{Func}(\hat{M}^0) \otimes \mathbf{C}\big[[T^{\alpha,p>0}]\big].$$

Here we embed the algebras of smooth functions $\mathrm{Func}(M)$ or $\mathrm{Func}(\hat{M}^0)$ into $\mathbf{A}_0$ or $\mathbf{A}_1$ resp. mapping any function $f(t)$ to $f(T^0)$. Observe that $\mathcal{F}_0(T) \in \mathbf{A}_0$, $\mathcal{F}_1(T) \in \mathbf{A}_1$.

So, our Main Theorem says that, for an arbitrary Frobenius manifold, the function $\mathcal{F}_0(T)$ given by (3.40) satisfies the differential equations (1.18), and for an arbitrary semisimple Frobenius manifold the function $\mathcal{F}_1(T)$ given by (3.42) satisfies the differential equations (1.19).

The proof of this theorem will be given in the next section.

4. From symmetries of the hierarchy to Virasoro constraints

Definition 4.1. A *symmetry* of the hierarchy (3.28) is an evolutionary system

$$\partial_s t = S\,(t, t_X, \dots; X, T)$$

commuting with all equations of the hierarchy

$$\partial_s \partial_{T^{\alpha,p}} = \partial_{T^{\alpha,p}} \partial_s.$$

All the symmetries of the hierarchy form a Lie algebra w.r.t. the commutator.

Example 4.1. Any of the flows of the hierarchy is a symmetry.

Example 4.2. The rescaling combined with the shift along $T^{1,1}$

$$\partial_s t = \sum \tilde{T}^{\alpha,p} \partial_{T^{\alpha,p}} t$$

is a symmetry of the hierarchy.

Recall that the solution $t^{(0)}(T)$ is invariant w.r.t. this symmetry. From this it readily follows that the genus zero free energy is a homogeneous function of degree 2 of the variables $\tilde{T}^{\alpha,p}$. Indeed, the coefficients of the quadratic form (3.40) being functions on $t^{(0)}(T)$ are constant along the vector field $\sum \tilde{T}^{\alpha,p}\, \partial_{T^{\alpha,p}}$.

Example 4.3. For *Galilean symmetry*

$$S_{-1} = \sum_{p\geq 1} \tilde{T}^{\alpha,p} \partial_{T^{\alpha,p-1}} t + e_1. \tag{4.1}$$

Here, as above, e_1 is the unity vector field on the Frobenius manifold.

To produce other symmetries we will apply the recursion relation (3.36) to the symmetry (4.1). We obtain a new symmetry

$$\begin{aligned} S_0 = \mathcal{R}\, S_{-1} = &\left[\sum_{p\geq 0} \left\langle \left(p + \mu + \frac{1}{2} \right) \tilde{T}^p, \partial_{T^p} \right\rangle + \sum_{p\geq 1} \sum_{1\leq r\leq p} \left\langle R_r \tilde{T}^p, \partial_{T^{p-r}} \right\rangle \right] t \\ &+ E(t). \end{aligned} \tag{4.2}$$

As we will see, this symmetry corresponds to the quasihomogeneity transformations generated by the Euler vector field $E(t)$. Iterating this process we obtain symmetries with non-local terms in the r.h.s.

$$\begin{aligned} S_1 = &\left[\sum_{p\geq 0} \left\langle \left(p + \mu + \frac{1}{2} \right) \left(p + \mu + \frac{3}{2} \right) \tilde{T}^p, \partial_{T^{p+1}} \right\rangle \right. \\ &+ 2 \sum_{p\geq 0} \sum_{1\leq r\leq p+1} \left\langle R_r (p + \mu + 1) \tilde{T}^p, \partial_{T^{p-r+1}} \right\rangle \\ &\left. + \sum_{p\geq 1} \sum_{2\leq r\leq p+1} \left\langle R_{r,2} \tilde{T}^p, \partial_{T^{p-r+1}} \right\rangle \right] t \\ &+ \sum \partial_{T^{\alpha,0}} t \left(\frac{1}{4} - \mu^2 \right)^{\alpha}_{\beta} \partial_X^{-1} t^\beta + E^2(t), \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
S_2 = \Bigg[& \sum_{p\geq 0} \left\langle \left(p+\mu+\frac{1}{2}\right)\left(p+\mu+\frac{3}{2}\right)\left(p+\mu+\frac{5}{2}\right)\tilde{T}^p, \partial_{T^{p+2}} \right\rangle \\
& + \sum_{p\geq 0}\sum_{1\leq r\leq p+2} \left\langle R_r \left[3\left(p+\mu+\frac{1}{2}\right)^2 + 6\left(p+\mu+\frac{1}{2}\right)+2\right]\tilde{T}^p, \partial_{T^{p-r+2}} \right\rangle \\
& + 3\sum_{p\geq 0}\sum_{2\leq r\leq p+2} \left\langle R_{r,2}\left(p+\mu+\frac{3}{2}\right)\tilde{T}^p, \partial_{T^{p-r+2}} \right\rangle \\
& + \sum_{p\geq 1}\sum_{3\leq r\leq p+2} \left\langle R_{r,3}\tilde{T}^p, \partial_{T^{p-r+2}} \right\rangle \Bigg] t \\
& + \partial_{T^{\alpha,1}} t \left(\frac{3}{8}+\frac{1}{4}\mu-\frac{3}{2}\mu^2-\mu^3\right)^\alpha_\beta \partial_X^{-1} t^\beta \\
& + \partial_{T^{\alpha,0}} t \left[R_1\left(\frac{1}{4}-3\mu-3\mu^2\right)\right]^\alpha_\beta \partial_X^{-1} t^\beta \\
& + \partial_{T^{\alpha,0}} t \left(\frac{3}{8}-\frac{1}{4}\mu-\frac{3}{2}\mu^2+\mu^3\right)^\alpha_\beta \eta^{\beta\gamma}\partial_X^{-1}\partial_\gamma F(t) + E^3(t).
\end{aligned}
\tag{4.4}
$$

Proposition 1. *The vector fields S_m with $-1 \leq m \leq 2$ are symmetries of the hierarchy* (3.28).

Actually, there is an unpleasant problem to choose the integration constants in the derivation of these formulae. So we are to prove directly that S_m for $-1 \leq m \leq 2$ are symmetries of the hierarchy. To do this we will first formulate the following simple observation [31].

Lemma 2. *On an arbitrary Frobenius manifold the vector fields*

$$v_m = E^{m+1}(t), \quad m \geq 0, \quad v_{-1} = e \tag{4.5}$$

satisfy the Virasoro commutation relations

$$[v_i, v_j] = (j-i)v_{i+j}, \quad i, j \geq -1 \tag{4.6}$$

with respect to the Lie bracket.

Remark 4.1. On a semisimple Frobenius manifold, the claim of Lemma 2 is obvious since, as it follows from (3.43), (3.44)

$$v_m = \sum_{i=1}^n u_i^{m+1}\partial_i.$$

In this case the vector fields v_m are defined also for any $m \in \mathbf{Z}$ outside the discriminant of the Frobenius manifold (see [12]) and they satisfy the commutation relations of the Virasoro algebra with zero central charge.

It turns that (half of) the Virasoro algebra described in Lemma 2 acts on the polynomial ring generated by the functions $\Omega_{\alpha,p;\,\beta,q}(t)$. This follows from

Proposition 2. *The derivatives of the coefficients $\Omega_{\alpha,p;\,\beta,q}(t)$ along the vector fields (4.5) are polynomials of the same functions with constant coefficients depending only on the monodromy at the origin η, μ, R. These derivatives are uniquely determined by the following formulae for the derivatives of the generating matrix-valued function $\Omega = \Omega(t; z, w)$*

$$\partial_e \Omega = (z+w)\,\tilde\Omega(z,w), \tag{4.7}$$

$$\partial_E \Omega = \left(D_z + \frac12\right)^T \tilde\Omega(z,w) + \tilde\Omega(z,w)\left(D_w + \frac12\right), \tag{4.8}$$

$$\begin{aligned}\partial_{E^2}\Omega = {} & \left(D_z + \frac12\right)^T \left(D_z + \frac32\right)^T z^{-1}\,\tilde\Omega(z,w)\\ & + \tilde\Omega(z,w)\,w^{-1}\left(D_w + \frac32\right)\left(D_w + \frac12\right)\\ & + \tilde\Omega(z,0)\left(\frac14 - \mu^2\right)\eta^{-1}\,\tilde\Omega(0,w),\end{aligned} \tag{4.9}$$

$$\begin{aligned}\partial_{E^3}\Omega = {} & \left(D_z + \frac12\right)^T \left(D_z + \frac32\right)^T \left(D_z + \frac52\right)^T z^{-2}\,\tilde\Omega(z,w)\\ & + \tilde\Omega(z,w)\,w^{-2}\left(D_w + \frac52\right)\left(D_w + \frac32\right)\left(D_w + \frac12\right)\\ & + (\partial_w\tilde\Omega)(z,0)\,\eta^{-1}\left(\frac12 + \mu\right)^T\left(\frac12 - \mu\right)^T\left(\frac32 - \mu\right)^T\tilde\Omega(0,w)\\ & + \tilde\Omega(z,0)\left(\frac32 - \mu\right)\left(\frac12 - \mu\right)\left(\frac12 + \mu\right)\eta^{-1}\,(\partial_z\tilde\Omega)(0,w)\\ & + \tilde\Omega(z,0)\left(\frac14 + 3\mu - 3\,\mu^2\right)R_1\,\eta^{-1}\,\tilde\Omega(0,w).\end{aligned} \tag{4.10}$$

where

$$\tilde\Omega(z,w) := \Omega(t;z,w) + \frac{\eta}{z+w}, \tag{4.11}$$

the right operator D_w acts on the matrices as follows

$$\Omega D_w = w\frac{\partial\Omega}{\partial w} + \Omega B(w), \tag{4.12}$$

the operator D_z^T *acts by a similar formula on the left,*

$$D_z^T \Omega = z \frac{\partial \Omega}{\partial z} + B^T(z)\Omega \tag{4.13}$$

and the matrix-valued polynomial B *is defined by*

$$B(z) := \mu + \sum_{k \geq 1} R_k z^k. \tag{4.14}$$

Proof. From the definition (3.16) of the matrix $\Theta(t;z)$ and the definition (3.8) of the deformed flat coordinates, it follows that

$$\partial_v \Theta = z\, \mathcal{C}(v)\Theta$$

where $\mathcal{C}(v)$ is the operator of multiplication by the vector field v tangent to M. From this we obtain

$$\partial_E \Theta = z\mathcal{U}\Theta, \tag{4.15}$$

By using (3.9) we get

$$\partial_E \Theta = \Theta D_z - \mu\, \Theta. \tag{4.16}$$

Besides, from the definition it follows that

$$\Theta^T(t;z)\eta\Theta(t;w) - \eta = (z+w)\Omega(t;z,w). \tag{4.17}$$

We will omit the explicit dependence on $t \in M$ in subsequent calculations.

The formula (4.7) follows from (4.17) and from $\partial_e \Theta = z\Theta$. Differentiating (4.17) along E and applying (4.16) we arrive at the formula (4.8). We can compute the same derivative in an alternative way

$$\begin{aligned} \partial_E \left(\Theta^T(z)\eta\Theta(w)\right) &= z\, \Theta^T(z)\, \mathcal{U}^T \eta\, \Theta(w) + w\, \Theta^T(z)\, \eta\, \mathcal{U}\, \Theta(w) \\ &= (z+w)\Theta^T(z)\, \eta\, \mathcal{U}\, \Theta(w) \\ &= \frac{z+w}{w} \Theta^T(z)\, \eta\, \partial_E \Theta(w). \end{aligned} \tag{4.18}$$

Applying (4.16) and comparing with

$$\partial_E \left(\Theta^T(z)\eta\Theta(w)\right) = D_z^T \Theta(z)\, \eta\, \Theta(w) + \Theta^T(z)\, \eta\, \Theta(w) D_w,$$

we arrive at

$$\begin{aligned} \Theta^T(z)\eta\mu\Theta(w) &= \frac{z\Theta^T(z)\eta\Theta(w)D_w - wD_z^T\Theta^T(z)\eta\Theta(w)}{z+w} \\ &= z\, \tilde{\Omega}(z,w)\, D_w - w\, D_z^T \tilde{\Omega}(z,w). \end{aligned} \tag{4.19}$$

Now, to derive (4.9) we differentiate (4.17) along E^2. This can be written as follows:

$$\begin{aligned} \partial_{E^2}\Theta^T(z)\eta\Theta(w) &= z\Theta^T(z)\left[\mathcal{U}^2\right]^T \eta\Theta(w) + w\Theta^T(z)\eta\mathcal{U}^2\Theta(w) \\ &= (z+w)\Theta^T(z)\mathcal{U}^T\eta\mathcal{U}\Theta(w) \\ &= \frac{z+w}{zw}\partial_E\Theta^T(z)\eta\partial_E\Theta(w). \end{aligned}$$

Substituting (4.17) and using (4.19) we obtain (4.9).

To prove the formula (4.10) we need the following lemma:

Lemma 3. *The following formulae hold true:*

$$\begin{aligned}\partial_{E^2}\Theta(z) &= \Theta(z)\left(D_z-\frac{1}{2}\right)\left(D_z-\frac{3}{2}\right) z^{-1}+\mathcal{C}\,\left(\frac{1}{4}-\mu^2\right)\Theta(z)\\ &\quad + z\left(\frac{1}{2}-\mu\right)\left(\frac{3}{2}-\mu\right)\eta^{-1}\,\left(\partial_w\Omega(z,0)\right)^T-2(\mu-1)R_1\Theta(z)\\ &\quad - z^{-1}\left(\frac{1}{2}-\mu\right)\left(\frac{3}{2}-\mu\right),\end{aligned}\tag{4.20}$$

$$\begin{aligned}\Theta^T(z)\,\eta\,\mu\,\mathcal{C} &= -z^{-1}(D_z^T-1)\,\Theta^T(z)\,\eta+z\,\partial_w\Omega(z,0)\,(1+\mu)\\ &\quad+\Theta^T(z)\,\eta\,R_1-z^{-1}\eta\,(1+\mu),\end{aligned}\tag{4.21}$$

where the matrix $\mathcal{C}$ *is defined as*

$$\mathcal{C}=\Omega(t;0,0)\eta^{-1}.$$

Proof. From the definition (3.10) of $\mathcal{U}$ and (4.8) we obtain

$$\mathcal{U}=(1-\mu)\,\mathcal{C}+\mathcal{C}\,\mu+R_1,\tag{4.22}$$

$$\partial_E\mathcal{U}=(1-\mu)\,\mathcal{U}+\mathcal{U}\,\mu.\tag{4.23}$$

By using (3.21) we get

$$\mathcal{C}\,\Theta=z^{-1}\,(\Theta-1)+z\,\eta^{-1}\left(\partial_w\Omega(z,0)\right)^T.\tag{4.24}$$

From (4.15) and (4.23) it follows that

$$\begin{aligned}\partial_E\partial_E\Theta &= z\,\mathcal{U}\,\partial_E\Theta+z\,(\partial_E\mathcal{U})\,\Theta=z\,\partial_{E^2}\Theta+z\,((1-\mu)\,\mathcal{U}+\mathcal{U}\,\mu)\,\Theta\\ &= z\,\partial_{E^2}\Theta+2\,(1-\mu)\,\partial_E\Theta-z\,(1-\mu)\,\mathcal{U}\,\Theta+z\,\mathcal{U}\,\mu\,\Theta.\end{aligned}\tag{4.25}$$

Insert (4.22) into the above expression, we arrive at

$$\begin{aligned}\partial_E\partial_E\Theta &= z\,\partial_{E^2}\Theta+2\,(1-\mu)\,\partial_E\Theta-z(1-\mu)[(1-\mu)\,\mathcal{C}+\mathcal{C}\,\mu+R_1]\,\Theta\\ &\quad+z\,[(1-\mu)\,\mathcal{C}+\mathcal{C}\,\mu+R_1]\,\mu\,\Theta\\ &= z\,\partial_{E^2}\Theta+2\,(1-\mu)\,\partial_E\Theta-z\,(1-\mu)^2\,\mathcal{C}\,\Theta+z\,\mathcal{C}\,\mu^2\,\Theta+2\,z\,(\mu-1)R_1\,\Theta.\end{aligned}$$

From the above expression and by using the formula (4.16) we obtain

$$\begin{aligned}z\partial_{E^2}\Theta &= (\Theta\,D_z-\mu\,\Theta)\,D_z-\mu\,(\Theta\,D_z-\mu\,\Theta)-2(1-\mu)(\Theta\,D_z-\mu\,\Theta)\\ &\quad+z(1-\mu)^2\,\mathcal{C}\,\Theta-z\,\mathcal{C}\,\mu^2\,\Theta-2\,z\,(\mu-1)\,R_1\,\Theta\\ &= \Theta\,D_z^2-2\,\Theta\,D_z+(2\mu-\mu^2)\,\Theta+z\,\mathcal{C}\,(\frac{1}{4}-\mu^2)\,\Theta\\ &\quad+z\,(\frac{1}{2}-\mu)(\frac{3}{2}-\mu)\,\mathcal{C}\,\Theta-2\,z\,(\mu-1)\,R_1\,\Theta.\end{aligned}$$

By using formula (4.24) we finally get

$$\begin{aligned}
z\partial_{E^2}\Theta &= \Theta\, D_z^2 - 2\,\Theta\, D_z + (2\mu-\mu^2)\,\Theta + z\,\mathcal{C}\,\left(\frac{1}{4}-\mu^2\right)\,\Theta \\
&+ \left(\frac{1}{2}-\mu\right)\left(\frac{3}{2}-\mu\right)(\Theta-1) \\
&+ z^2\,\left(\frac{1}{2}-\mu\right)\left(\frac{3}{2}-\mu\right)\,\eta^{-1}\,(\partial_w\Omega(z,0))^T - 2\,z\,(\mu-1)R_1\Theta \\
&= \Theta\,\left(D_z-\frac{1}{2}\right)\left(D_z-\frac{3}{2}\right) + z\,\mathcal{C}\,(\frac{1}{4}-\mu^2)\,\Theta \\
&+ z^2\,\left(\frac{1}{2}-\mu\right)\left(\frac{3}{2}-\mu\right)\,\eta^{-1}\,(\partial_w\Omega(z,0))^T - 2\,z\,(\mu-1)R_1\Theta \\
&- \left(\frac{1}{2}-\mu\right)\left(\frac{3}{2}-\mu\right),
\end{aligned}$$

which leads to (4.20).

To prove (4.21), let us note that

$$\frac{\partial\Theta(z)}{\partial z}|_{z=0} = \mathcal{C}, \quad \Theta(z)|_{z=0} = I. \tag{4.26}$$

Now differentiating (4.19) w.r.t. w and putting $w=0$, we arrive at (4.21). Lemma is proved. □

Now let us prove formula (4.10). Using (4.15) and (4.17) we get

$$\begin{aligned}
\partial_{E^3}\Omega &= (z+w)^{-1}\partial_{E^3}\left(\Theta^T(z)\,\eta\,\Theta(w)\right) \\
&= (z+w)^{-1}\left(z\,\Theta^T(z)\,(\mathcal{U}^3)^T\eta\,\Theta(w) + w\,\Theta^T(z)\,\mathcal{U}^3\,\eta\,\Theta(w)\right) \\
&= (z+w)^{-1}\left[\partial_{E^2}\left(\Theta^T(z)\,\eta\,\mathcal{U}\,\Theta(w)\right) - \Theta^T(z)\,\eta\,\left(\partial_{E^2}\mathcal{U}\right)\,\Theta(w)\right] \\
&= (z+w)^{-1}\left[(z+w)^{-1}\,\partial_{E^2}\partial_E\left(\Theta^T(z)\,\eta\,\Theta(w)\right) - \Theta^T(z)\,\eta\,\left(\partial_{E^2}\mathcal{U}\right)\,\Theta(w)\right].
\end{aligned}$$

From (4.17), (4.22) and the fact that

$$\partial_{E^2}\mathcal{C} = \mathcal{U}^2,$$

we obtain

$$\begin{aligned}
\partial_{E^3}\Omega &= (z+w)^{-1}\partial_{E^2}\partial_E\Omega - (z+w)^{-1}\Theta^T(z)\,\eta\,\left[(1-\mu)\,\mathcal{U}^2+\mathcal{U}^2\,\mu\right]\,\Theta(w) \\
&= (z+w)^{-1}\left(\partial_{E^2}\partial_E\Omega - 2\,\partial_{E^2}\Omega + w^{-1}\Theta^T(z)\,\eta\,\left(\frac{1}{2}+\mu\right)\,\partial_{E^2}\Theta(w)\right. \\
&\left. + \,z^{-1}\left(\partial_{E^2}\Theta^T(z)\right)\,\eta\,\left(\frac{1}{2}-\mu\right)\,\Theta(w)\right).
\end{aligned}$$

By using formulae (4.8), (4.9), (4.17), (4.19), (4.20) and (4.21) we obtain (4.10) from the above formula by a straightforward calculation. The proposition is proved. □

Corollary 1 (see [12], Exercise 2.11.). *In the flat coordinates* (3.23) *the Euler vector field reads*

$$E(t) = \sum (1 + \mu_1 - \mu)^\alpha_\beta t^\beta \partial_\alpha + (R_1)^\alpha_1 \partial_\alpha \tag{4.27}$$

(recall that $d = -2\mu_1$). The potential $F(t)$ defined by (3.25) *satisfies the following quasihomogeneity condition*

$$\partial_E F(t) = (3 + 2\mu_1)F(t) + \frac{1}{2}\langle t, R_1 t\rangle + \langle e_1, R_2 t\rangle - \frac{1}{2}\langle e_1, R_3 e_1\rangle . \tag{4.28}$$

By using Proposition 2, it is straightforward to prove Proposition 1, we omit the derivation here. Instead, in order to prove the genus zero Virasoro constraints, we will simultaneously show that the solution $t^{(0)}(T)$ is a stationary point of the symmetries S_m for $-1 \le m \le 2$. To be more specific, we are to say how to define the nonlocal terms in S_1 and S_2. Using (3.41), (3.21), (3.23) and (3.26) we define the nonlocal terms as

$$\partial_X^{-1} t^\beta := \eta^{\beta\epsilon}\partial_{T^{\epsilon,0}}\mathcal{F}_0 \tag{4.29}$$

$$\partial_X^{-1}\partial_\gamma F := \partial_{T^{\gamma,1}}\mathcal{F}_0. \tag{4.30}$$

Proposition 3. *For $-1 \le m \le 2$ we have*

$$S_m\big|_{t=t^{(0)}(T)} = 0 \tag{4.31}$$

$$\mathcal{A}_{m,0} = 0, \tag{4.32}$$

where in S_1 and S_2 one is to substitute (4.29) *and* (4.30).

Proof. Let us begin with S_{-1}. Differentiating (3.38) w.r.t. $X = T^{1,0}$ and using the explicit form (3.28) of $\partial_{T^{\alpha,p-1}}$, we obtain

$$\partial_X t = \nabla\theta_{1,0} + \sum T^{\alpha,p}\partial_X(\nabla\theta_{\alpha,p}) = e + \sum T^{\alpha,p}\partial_{T^{\alpha,p-1}} t. \tag{4.33}$$

Here we used $\nabla\theta_{1,0} = e$. We obtain (4.31) for $m = -1$.

Let us now derive the first Virasoro constraint (i.e., the string equation)

$$\mathcal{A}_{-1,0} \equiv \sum \tilde{T}^{\alpha,p+1}\partial_{T^{\alpha,p}}\mathcal{F}_0(T) + \frac{1}{2}\left\langle T^0, T^0\right\rangle = 0. \tag{4.34}$$

To this end we apply the operator

$$\sum \tilde{T}^{\alpha,p+1}\partial_{T^{\alpha,p}}$$

to the function $\mathcal{F}_0(T)$ (see (3.40)). Since $t^{(0)}(T)$ is a stationary point of the symmetry S_{-1}, the coefficients $\Omega_{\alpha,p;\,\beta,q}(t^{(0)}(T))$ are constants along S_{-1}. Using this and the formula (4.7), we easily arrive at (4.34).

Let us proceed with the derivation of the S_0 symmetry of $t^{(0)}$ and of the correspondent L_0 Virasoro constraint. This is the first case where we are to fix certain integration constants (see the formula (3.36) for the recursion operator).

First, applying the equation (4.8) to $\partial_E \Omega_{\beta,p;\,\gamma,0}(t)$, multiplying the result by $\eta^{\alpha\gamma}$ and differentiating along ∂_ν, we obtain

$$\begin{aligned} & g^{\alpha\gamma}\partial_\gamma\partial_\nu\theta_{\beta,p} + \Gamma^{\alpha\gamma}_\nu \partial_\gamma\theta_{\beta,p} \\ &= \left(p+\mu+\frac{1}{2}\right)^\gamma_\beta \partial^\alpha\partial_\nu\theta_{\gamma,p+1} + \sum_{k=0}^{p-1}(R_{p-k})^\gamma_\beta\,\partial^\alpha\partial_\nu\theta_{\gamma,k+1}. \end{aligned} \tag{4.35}$$

Multiplying this by t^ν_X we obtain, for $p>0$

$$\begin{aligned} & g^{\alpha\gamma}\partial_{T^{\beta,p-1}}t_\gamma + t^\nu_X\Gamma^{\alpha\gamma}_\nu\partial_\gamma\theta_{\beta,p} \\ &= \left(p+\mu+\frac{1}{2}\right)^\gamma_\beta \partial_{T^{\gamma,p}}t^\alpha + \sum_{k=0}^{p-1}(R_{p-k})^\gamma_\beta\,\partial_{T^{\gamma,k}}t^\alpha. \end{aligned} \tag{4.36}$$

Multiplying by $\tilde{T}^{\beta,p}$ and using S_{-1} we arrive, after summation w.r.t. β and $p\geq 1$, at

$$\begin{aligned} & -g^{\alpha\gamma}\eta_{\gamma\,1} + t^\nu_X\Gamma^{\alpha\gamma}_\nu \sum_{p\geq 1}\tilde{T}^{\beta,p}\,\partial_\gamma\theta_{\beta,p} \\ &= \sum_{p\geq 1}\tilde{T}^{\beta,p}\left[\left(p+\mu+\frac{1}{2}\right)^\gamma_\beta \partial_{T^{\gamma,k}} + \sum_{k=0}^{p-1}(R_{p-k})^\gamma_\beta\,\partial_{T^{\gamma,k}}\right] t^\alpha. \end{aligned} \tag{4.37}$$

From the definition of $g^{\alpha\beta}$, it follows that the first term in the l.h.s. equals $E^\alpha(t)$. The sum on the l.h.s. is equal to $-\tilde{T}^{\beta,0}\partial_\gamma\theta_{\beta,0} = -\tilde{T}^{\beta,0}\eta_{\beta\gamma}$ due to the specification $\nabla\Phi_{\tilde{T}}(t)=0$ of the solution $t^{(0)}$. So on the l.h.s. we obtain

$$\begin{aligned} -E^\alpha - \tilde{T}^{\beta,0}t^\epsilon_X\Gamma^{\alpha\gamma}_\epsilon\eta_{\gamma\beta} &= -E^\alpha - \tilde{T}^{\beta,0}t^\nu_X c^\alpha_{\nu\gamma}\left(\frac{1}{2}+\mu\right)^\gamma_\beta \\ &= -E^\alpha - \tilde{T}^{\beta,0}\left(\frac{1}{2}+\mu\right)^\gamma_\beta \partial_{T^{\gamma,0}}t^\alpha. \end{aligned} \tag{4.38}$$

This proves that $S_0\big|_{t^{(0)}}=0$. Thus, applying the operator

$$\sum_{p\geq 0}\left\langle \left(p+\mu+\frac{1}{2}\right)\tilde{T}^p, \partial_{T^p}\right\rangle + \sum_{p\geq 1}\sum_{1\leq r\leq p}\left\langle R_r\tilde{T}^p, \partial_{T^{p-r}}\right\rangle$$

to the function (3.40), we may assume that the coefficients $\Omega_{\alpha,p;\,\beta,q}$ are constants along the symmetry S_0. After simple calculations with the use of (4.8), we obtain the genus zero L_0 Virasoro constraint

$$\left[\sum_{p\geq 0}\left\langle\left(p+\mu+\frac{1}{2}\right)\tilde{T}^p,\partial_{T^p}\right\rangle+\sum_{p\geq 1}\sum_{1\leq r\leq p}\left\langle R_r\tilde{T}^p,\partial_{T^{p-r}}\right\rangle\right]\mathcal{F}_0 \\ +\frac{1}{2}\sum_{p,\,q}(-1)^q\left\langle R_{p+q+1}\tilde{T}^p,\tilde{T}^q\right\rangle=0. \tag{4.39}$$

Note that the term $1/4\,\mathrm{tr}\,(1/4-\mu^2)$ in L_0 (see (2.30)) does not enter in this equation. It will appear in the genus 1 Virasoro constraint.

Similar calculations complete the proof of the proposition. The proposition is proved. □

Corollary 2. *The genus zero Virasoro constraint*

$$\mathcal{A}_{m,0}=0$$

hold true for any $m\geq -1$.

So, the first part of the Main Theorem is proved.

Remark 4.2. In [18] it was actually proved only that the $\partial_{T^{\alpha,0}}$-derivatives of the genus zero Virasoro constraints hold true (only the case $R=R_1$, valid in quantum cohomology, was under consideration). From this the authors of [18] infer the validity of the Virasoro constraints choosing zero to be the integration constant. A posteriori, in this case, $R=R_1$, the integration constant is zero indeed. However, for a general Frobenius manifold, the approach of [18] does not work since the integration constant does not vanish.

We proceed now to the genus one Virasoro constraints. Starting from this point we assume the Frobenius manifold to be semisimple.

Proposition 4. *The derivatives of the G-function along the powers of the Euler vector field are given by the following formulae*

$$\partial_e G=0, \tag{4.40}$$

$$\partial_E G=\frac{n\,d}{48}-\frac{1}{4}\,\mathrm{tr}\;\mu^2, \tag{4.41}$$

$$\partial_{E^k}G=-\frac{1}{4}\,\mathrm{tr}\;\left(\mu\left(\mu\mathcal{U}^{k-1}+\mathcal{U}\mu\mathcal{U}^{k-2}+\cdots+\mathcal{U}^{k-1}\mu\right)\right) \\ -\frac{1}{24}\left\langle\left(\mu\mathcal{U}^{k-2}+\mathcal{U}\mu\mathcal{U}^{k-3}+\cdots+\mathcal{U}^{k-2}\mu\right)E-\frac{d}{2}\mathcal{U}^{k-2}\,E,H\right\rangle, \\ k\geq 2 \tag{4.42}$$

where

$$H = c_\nu^{\nu\alpha}\partial_\alpha. \tag{4.43}$$

Proof. The formula (4.40) makes part of the definition of the G-function. The formula (4.41) was proved in [17]. To prove (4.42) we first compute the logarithmic derivatives of the isomonodromic tau-function along E^k. Using (3.47) and (3.52) we have

$$\begin{aligned}\partial_{E^k}\log\tau &= \frac{1}{2}\sum_{i\neq j}\frac{u_i^k}{u_i-u_j}V_{ij}^2 = \frac{1}{4}\sum_{i\neq j}\frac{u_i^k-u_j^k}{u_i-u_j}V_{ij}^2\\ &= \frac{1}{4}\sum_{i,\,j=1}^{n}\sum_{m=0}^{k-1}u_i^m u_j^{k-m-1}\psi_{i\alpha}\mu_\beta^\alpha\psi_j^\beta\psi_{i\lambda}\mu_\nu^\lambda\psi_j^\nu.\end{aligned} \tag{4.44}$$

Using

$$\Psi^T\Psi = \eta, \quad \Psi\mathcal{U}\Psi^{-1} = \mathrm{diag}\,(u_1,\dots,u_n) \tag{4.45}$$

and antisymmetry (2.1) of the operator μ, we obtain

$$\partial_{E^k}\log\tau = -\frac{1}{4}\,\mathrm{tr}\,\left(\mu\left(\mu\mathcal{U}^{k-1}+\mathcal{U}\mu\mathcal{U}^{k-2}+\dots+\mathcal{U}^{k-1}\mu\right)\right). \tag{4.46}$$

The next step is to derive the formula for the derivatives of the matrix $\Psi = (\psi_{i\alpha})$. Here we use the following equations (see [11])

$$\partial_i\Psi = V_i\Psi \tag{4.47}$$

where the matrix V_i was defined in (3.49). From this it follows that

$$\partial_{E^k}\Psi = V^{(k)}\Psi \tag{4.48}$$

with the matrix $V^{(k)}$ of the form

$$V_{ij}^{(k)} = \frac{u_i^k-u_j^k}{u_i-u_j}V_{ij}. \tag{4.49}$$

Doing the calculations similar to the above and using the fact that the (co)vector (4.43)

$$H_\alpha = c_{\nu\alpha}^\nu = \sum_{i=1}^n\frac{\psi_{i\alpha}}{\psi_{i1}} \tag{4.50}$$

we easily obtain the second line in the formula (4.42). The proposition is proved. $\square$

Remark 4.3. For a semisimple Frobenius manifold, one can uniquely reconstruct the first derivatives G-function from the Frobenius structure solving the system (4.41), (4.42) for $k = 2, \dots, n-1$ together with $\partial_e G = 0$. Indeed, in the canonical coordinates the coefficients of the linear system is the Vandermonde matrix of $u_1, \dots, u_n$. In particular, for any smooth projective variety X with semisimple quantum cohomology, this give a practical way to express elliptic Gromov-Witten invariants via rational ones (see examples in the Concluding Remarks below).

We now compute derivatives of the second part

$$F^{(1)} = \frac{1}{24} \log\det M^\alpha_\beta(t, t_X) \tag{4.51}$$

in the genus 1 free energy (3.42). We will use the following formula for this function (see [17])

$$F^{(1)} = \frac{1}{24}\left(\log\prod_{i=1}^n \sigma_i - \log\prod_{i=1}^n \psi_{i1}\right) \tag{4.52}$$

where

$$\sigma_i = \psi_{i\alpha} t_X^\alpha. \tag{4.53}$$

It follows that

$$\frac{\partial F^{(1)}}{\partial t_X^\alpha} = \frac{1}{24}\sum_{i=1}^n \frac{\psi_{i\alpha}}{\sigma_i}. \tag{4.54}$$

Let us define the operator of a Lie derivative of a function $F^{(1)}(t, t_X)$ along a vector field $v = v^\alpha(t)\partial_\alpha$ on M

$$\mathrm{Lie}_v\, F^{(1)}(t, t_X) = v^\alpha \frac{\partial F^{(1)}}{\partial t^\alpha} + (v^\alpha)_X \frac{\partial F^{(1)}}{\partial t_X^\alpha}. \tag{4.55}$$

Lemma 4. *The following formulae hold true for the derivatives of the function* (4.52)

$$\mathrm{Lie}_e\, F^{(1)}(t, t_X) = 0 \tag{4.56}$$

$$\mathrm{Lie}_E\, F^{(1)}(t, t_X) = \frac{n}{24} \tag{4.57}$$

$$\mathrm{Lie}_{E^k}\, F^{(1)}(t, t_X) = \frac{k}{24}\left\langle E^{k-1}, H\right\rangle \tag{4.58}$$

$$t_X^\alpha \frac{\partial F^{(1)}(t, t_X)}{\partial t_X^\alpha} = \frac{n}{24}. \tag{4.59}$$

Here H is defined in (4.43).

The proof easily follows from (4.45), (4.48), (4.50), (4.52).

We proceed now to the derivation of the genus one Virasoro constraints. First, the L_{-1} Virasoro constraint reads

$$\begin{aligned}\mathcal{A}_{-1,1} &\equiv \sum_{p\geq 1} \tilde{T}^{\alpha,p} \frac{\partial \mathcal{F}_1}{\partial T^{\alpha,p-1}} \\ &= \sum_{p\geq 1} \tilde{T}^{\alpha,p} \frac{\partial t^\gamma}{\partial T^{\alpha,p-1}} \frac{\partial \mathcal{F}_1}{\partial t^\gamma} + \sum_{p\geq 1} \tilde{T}^{\alpha,p} \partial_X \left(\frac{\partial t^\gamma}{\partial T^{\alpha,p-1}} \right) \frac{\partial \mathcal{F}_1}{\partial t_X^\gamma} \\ &= -\delta_1^\gamma \frac{\partial \mathcal{F}_1}{\partial t^\gamma} = -\operatorname{Lie}_e \mathcal{F}_1 = 0. \end{aligned} \tag{4.60}$$

In this computation we used vanishing of S_{-1} on the solution $t^{(0)}$.

Let us now prove validity of the genus 1 L_0 Virasoro constraint

$$\begin{aligned}\mathcal{A}_{0,1} &\equiv \left[\sum_{p\geq 0} \left\langle \left(p+\mu+\frac{1}{2}\right) \tilde{T}^p, \partial_{T^p} \right\rangle \right. \\ &\left. + \sum_{p\geq 1} \sum_{1\leq r\leq p} \left\langle R_r \tilde{T}^p, \partial_{T^{p-r}} \right\rangle \right] \mathcal{F}_1 + \frac{1}{4} \operatorname{tr} \left(\frac{1}{4} - \mu^2 \right) \\ &= \frac{\partial \mathcal{F}_1}{\partial t^\gamma} \left[\sum_{p\geq 0} \left\langle \left(p+\mu+\frac{1}{2}\right) \tilde{T}^p, \partial_{T^p} \right\rangle \right. \\ &\left. + \sum_{p\geq 1} \sum_{1\leq r\leq p} \left\langle R_r \tilde{T}^p, \partial_{T^{p-r}} \right\rangle \right] t^\gamma \\ &+ \frac{\partial \mathcal{F}_1}{\partial t_X^\gamma} \left[\sum_{p\geq 0} \left\langle \left(p+\mu+\frac{1}{2}\right) \tilde{T}^p, \partial_{T^p} \right\rangle \right. \\ &\left. + \sum_{p\geq 1} \sum_{1\leq r\leq p} \left\langle R_r \tilde{T}^p, \partial_{T^{p-r}} \right\rangle \right] t_X^\gamma + \frac{1}{4} \operatorname{tr} \left(\frac{1}{4} - \mu^2 \right) \\ &= -\operatorname{Lie}_E \mathcal{F}_1 - \left(\frac{1}{2} + \mu \right)_1^\nu \frac{\partial t^\gamma}{\partial T^{\nu,0}} \frac{\partial \mathcal{F}_1}{\partial t_X^\gamma} + \frac{1}{4} \operatorname{tr} \left(\frac{1}{4} - \mu^2 \right) \\ &= -\frac{n}{24} - \frac{n\, d}{48} + \frac{1}{4} \operatorname{tr} \mu^2 - \frac{n}{24} \left(\frac{1-d}{2} \right) + \frac{n}{16} - \frac{1}{4} \operatorname{tr} \mu^2 = 0. \end{aligned} \tag{4.61}$$

In this computation we used the vanishing of the symmetry S_0 on the solution $t^{(0)}$, and also the formulae (4.41), (4.57), (4.59) and the fact that $\mu_1^\nu = -\frac{d}{2} \delta_1^\nu$.

In a similar, although more involved way, using the vanishing of the symmetries S_1 and S_2 on the solution $t^{(0)}$, one can prove validity of the genus one L_1 and L_2 Virasoro constraints.

Now, since we proved the Virasoro constraints L_m, up to the genus 1, for $-1 \leq m \leq 2$, from the commutation relation

$$[L_m, L_1] = (m-1)L_{m+1}$$

we derive that these constraints hold true also for any $m \geq -1$. Main Theorem is proved.

5. Concluding remarks

1. We consider the results of this paper as a strong support of the conjectural relation between semisimple Frobenius manifolds and integrable hierarchies of the KdV type (see Introduction above). It gives also a practical algorithm to reconstruct the integrable hierarchy starting from a given semisimple Frobenius manifold. The algorithm is based on some more strong requirement that for the tau-function $\tau(T)$ of an *arbitrary* solution $t_\alpha = t_\alpha(T)$ of the hierarchy, the function

$$\tau(T) + \delta\tau(T) := \tau(T) + \sigma L_m \tau(T) + O(\sigma^2)$$

for any m is again the tau-function of another solution of the hierarchy in the linear approximation in the small parameter σ. Recall that, by definition of the tau-function,

$$t_\alpha(T) = \frac{\partial^2 \log \tau(T)}{\partial T^{\alpha,0} \partial X}, \quad \alpha = 1, \ldots, n.$$

In other words, we postulate that our Virasoro operators correspond to the symmetries

$$t_\alpha \mapsto t_\alpha + \sigma S_m[t]_\alpha + O(\sigma^2), \quad S_m[t]_\alpha := \frac{\partial^2}{\partial X \partial T^{\alpha,0}} \left(\frac{L_m \tau}{\tau} \right)$$

of the hierarchy

$$\partial_{T^{\alpha,p}} t = \partial_X K^{(0)}_{\alpha,p}(t) + \epsilon^2 \partial_X K^{(1)}_{\alpha,p}(t, t_X, t_{XX}) + \epsilon^4 \partial_X K^{(2)}_{\alpha,p}(t, t_X, t_{XX}, t_{XXX}, t^{IV}) + \ldots$$

in all orders in ϵ. This algorithm will give us a recursion procedure for computing the coefficients $K^{(r)}(t, t_X, \ldots, t^{(2r)})$ for any r. The terms $K^{(0)}$ and $K^{(1)}$ of the hierarchy have already been constructed in [8], [11] and [17] resp. We are going to study the structure of higher terms in a subsequent publication.

2. As we have already mentioned (see Remark 4.3 above) the formulae (4.40), (4.41), (4.42) give a simple way to compute the elliptic Gromov-Witten invariants of those smooth projective varieties X for which the quantum cohomology is semisimple. We give here two examples of application of this method. The equations (4.40), (4.41), (4.42) for the derivatives of the G-function can be recast into an elegant form using a generating function

$$\begin{aligned}\partial_{[e-zE]^{-1}}G &= \sum_{k=0}^{\infty} z^k \partial_{E^k} G \\ &= \frac{z}{24}\left(\left\langle \mu\left(\frac{H}{e-zE}\right), \frac{e}{e-zE}\right\rangle - 6 \text{ tr } \left(\mu\,(1-z\mathcal{U})^{-1}\right)^2\right) \end{aligned}\tag{5.1}$$

Here z is an indeterminate.

Example 5.1 *G-function for quantum cohomology on $CP^1 \times CP^1$.*

The primary free energy for $CP^1 \times CP^1$ is given by [4]

$$F = \frac{1}{2}\,(t^1)^2\,t^4 + t^1\,t^2\,t^3 + (t^4)^{-1}\,f(z_1, z_2);\tag{5.2}$$

here

$$z_1 = t^2 + 2\,\log(t^4), \quad z_2 = t^3 + 2\,\log(t^4),\tag{5.3}$$

and

$$f(z_1, z_2) = \sum_{k,l\geq 0, k+l\geq 1} \frac{N_{k,l}^{(0)}}{(2(k+l)-1)!}\, e^{k\,z_1 + l\,z_2},\tag{5.4}$$

where $N_{k,l}^{(0)}$ are the numbers of rational curves on $CP^1 \times CP^1$ with bidegree (k, l) which pass through $2(k+l)-1$ points. They are defined recursively by the following formula with the initial condition $N_{1,0}^{(0)} = 1$:

$$\begin{aligned} N_{k,l}^{(0)} = \sum (k_1\,l_2 + k_2\,l_1)\,l_2\,\Bigg(k_1 &\begin{pmatrix} 2(k+l)-4 \\ 2(k_1+l_1)-2 \end{pmatrix} \\ -k_2 &\begin{pmatrix} 2(k+l)-4 \\ 2(k_1+l_1)-3 \end{pmatrix}\Bigg)\, N_{k_1,l_1}^{(0)} N_{k_2,l_2}^{(0)}, \end{aligned}$$

here the summation is taken over k_i, l_i with $k_1 + k_2 = k$, $l_1 + l_2 = l$, $k_i \geq 0$, $l_i \geq 0$.

The derivatives $G_i := \partial G/\partial t^i$ of the G-function for $CP^1 \times CP^1$ are determined from the system (recall that G_1 is always equal to 0)

$$\begin{aligned} 2G_2 + 2G_3 - t^4 G_4 &= -\frac{1}{3}, \\ \big[f_2 - f_{22} - f_{12}\big]G_2 + \big[f_1 - f_{12} - f_{11}\big]G_3 + 4\,t^4 G_4 &= \frac{1}{12}\big(f_{122} + f_{112}\big), \end{aligned}$$

$$\begin{aligned}&\left[4\,f_2-12\,f_{22}-4\,f_1-f_{22}\,f_1-8\,f_{12}-f_2\,f_{12}+2\,f_{22}\,f_{12}+{f_{12}}^2+4\,f_{11}+f_{22}\,f_{11}\right]G_2\\&+\left[-4\,f_2+4\,f_{22}+4\,f_1-8\,f_{12}-f_1\,f_{12}+{f_{12}}^2-12\,f_{11}-f_2\,f_{11}+f_{22}\,f_{11}+2\,f_{12}\,f_{11}\right]G_3\\&\quad=\frac{1}{3}\,(2\,f_{22}+4\,f_{12}+2\,f_{11})-\frac{1}{12}\,(f_{12}\,f_{122}+f_{122}\,f_{11}+f_{22}\,f_{112}+f_{12}\,f_{112})\,,\end{aligned}$$

where $f_i=\frac{\partial f}{\partial z_i},\ f_{ij}=\frac{\partial^2 f}{\partial z_i\partial z_j},\ f_{ijk}=\frac{\partial^3 f}{\partial z_i\partial z_j\partial z_k}$. It turns out that the G-function has the form

$$G=-\frac{1}{12}\,t^2-\frac{1}{12}\,t^3+\sum_{k,l\geq 1}\frac{N^{(1)}_{k,l}}{(2k+2l)!}\,e^{k\,z_1+l\,z_2} \tag{5.5}$$

where $N^{(1)}_{k,l}=N^{(1)}_{l,k}$ are constants. We list in Table 1 the numbers $N^{(0)}_{k,l}$ and $N^{(1)}_{k,l}$ with $k+l\leq 14$. For $1\leq k\leq 14$ we have

$$\begin{aligned}&N^{(0)}_{k,0}=N^{(0)}_{0,k}=\delta_{k,1},\\&N^{(0)}_{k,1}=N^{(0)}_{1,k}=1,\\&N^{(1)}_{k,1}=N^{(1)}_{1,k}=0.\end{aligned}$$

This agrees with the definition of GW invariants (for any k). The remaining numbers are put in the table below. A comparison with the results of [38] suggests that the numbers $N^{(1)}_{k,l}$ coincide with the numbers of irreducible elliptic curves in the class $kS+lF$ on the rational ruled surface $F_0\simeq \mathbf{C}P^1\times\mathbf{C}P^1$ (in the notation of [38], see Table 1 there).

Example 5.2 *G-function for quantum cohomology on CP^3.*

The primary free energy for CP^3 is given by [4]

$$F=\frac{1}{2}\,(t^1)^2\,t^4+t^1\,t^2\,t^3+\frac{1}{6}\,(t^2)^3+f(z_1,z_2), \tag{5.6}$$

where

$$z_1=\frac{t^4}{(t^3)^2},\quad z_2=t^2+4\,\log(t^3) \tag{5.7}$$

and

$$f(z_1,z_2)=\sum_{k\geq 1}\ \sum_{0\leq l\leq 2\,k}\frac{N^{(0)}_{4k-2l,l}}{(4\,k-2\,l)!\,l!}\,z_1^l\,e^{k\,z_2}. \tag{5.8}$$

The numbers $N^{(0)}_{4k-2l,l}$ are the numbers of rational curves of degree k which pass through l points and $4k-2l$ lines in general position on CP^3.

(k,l)	$N^{(0)}_{k,l}$	$N^{(1)}_{k,l}$
$(2,2)$	12	1
$(3,2)$	96	20
$(4,2)$	640	240
$(3,3)$	3510	1920
$(5,2)$	3840	2240
$(4,3)$	87544	87612
$(6,2)$	21504	17920
$(5,3)$	1763415	2763840
$(4,4)$	6508640	12017160
$(7,2)$	114688	129024
$(6,3)$	30940512	69488120
$(5,4)$	348005120	1009712640
$(8,2)$	589824	860160
$(7,3)$	492675292	1495782720
$(6,4)$	15090252800	62820007680
$(5,5)$	43628131782	199215950976
$(9,2)$	2949120	5406720
$(8,3)$	7299248880	28742077000
$(7,4)$	565476495360	3183404098560
$(6,5)$	4114504926336	26965003723840
$(10,2)$	14417920	32440320
$(9,3)$	102276100605	506333947840
$(8,4)$	19021741768704	138871679557632
$(7,5)$	318794127432450	2824624505793600
$(6,6)$	780252921765888	7337244206710400
$(11,2)$	69206016	187432960
$(10,3)$	1370760207040	8327258171820
$(9,4)$	588743395737600	5402199925555200
$(8,5)$	21377025195016320	245508475513868160
$(7,6)$	115340307031443456	1465539494120378880
$(12,2)$	327155712	1049624576
$(11,3)$	17716885497906	129517853380160
$(10,4)$	17053897886924800	191937248700825600
$(9,5)$	1282815980041107375	18505625758298112000
$(8,6)$	14211230949697683456	233887641913890478080
$(7,7)$	30814236194426422332	528646007400035492736

Table 1: List of some numbers $N^{(0)}_{k,l}$ and $N^{(1)}_{k,l}$ for $CP^1 \times CP^1$

The derivatives of the G-function for CP^3 is determined by the system

$$4G_2 - t_3G_3 - 2t_4G_4 = -1,$$

$$\left[4\, f_{22} - 2\, f_2 - {z_1}^2\, f_{11}\right] G_2 + t_3\, \left[-4 + f_{22} + \frac{1}{2} z_1\, f_{12}\right] G_3 + 2\, {t_3}^2 G_4$$
$$= \frac{f_{22} - 2\, f_{222}}{6} + \frac{{z_1}^2\, f_{112}}{12}$$

$$\left[16\, f_2 - 64\, f_{22} - 8\, f_2\, f_{22} + 24\, {f_{22}}^2 - 9\, f_1 - 24\, z_1\, f_1 + 6\, z_1\, f_{22}\, f_1 + 14\, f_{12} + 64\, z_1\, f_{12}\right.$$
$$\left. - 12\, z_1\, f_{22}\, f_{12} + 3\, {z_1}^2\, f_1\, f_{12} - 8\, {z_1}^2\, {f_{12}}^2 + z_1\, \left(-5 - 16\, z_1 + 2\, z_1\, f_{22} + 3\, {z_1}^2\, f_{12}\right) f_{11}\right] G_2$$
$$+ \frac{t_3}{4} \left[-16\, f_2 + 8\, {f_{22}}^2 + 3\, f_1 + 6\, f_{12} + 16\, z_1\, f_{12} - 2\, {z_1}^2\, {f_{12}}^2 - z_1\, \left(1 + 8\, z_1\right)\, f_{11}\right] G_3$$
$$+ {t_3}^2\, \left[-8 + f_2 + \frac{3}{2} z_1\, f_1\right] G_4$$
$$= \frac{1}{24} \left[-12\, f_2 + 72\, f_{22} + 16\, {f_{22}}^2 - 96\, f_{222} - 12\, f_2\, f_{222} + 16\, f_{22}\, f_{222} + 24\, f_1 + 18\, z_1\, f_1\right.$$
$$+ 12\, z_1\, f_{222}\, f_1 + 16\, f_{12} - 84\, z_1\, f_{12} - 12\, z_1\, f_{22}\, f_{12} - 28\, z_1\, f_{222}\, f_{12}$$
$$+ 96\, z_1\, f_{122} + 6\, z_1\, f_2\, f_{122} - 12\, z_1\, f_{22}\, f_{122} - 6\, {z_1}^2\, f_1\, f_{122}$$
$$+ 20\, {z_1}^2\, f_{12}\, f_{122} + 3\, z_1\, f_{11} + 12\, {z_1}^2\, f_{11} + 6\, {z_1}^2\, f_{222}\, f_{11} - 3\, {z_1}^3\, f_{122}\, f_{11}$$
$$\left. - {z_1}^2\, \left(24 - 2\, f_{22} + 3\, z_1\, f_{12}\right)\, f_{112}\right].$$

From this system it follows that the G-function has the form

$$G = -\frac{1}{4}\, t^2 + \sum_{k \geq 1} \sum_{0 \leq l \leq 2\, k} \frac{N^{(1)}_{4k-2l,l}}{(4\, k - 2\, l)!\, l!}\, z_1^l\, e^{k\, z_2}. \tag{5.9}$$

We can determine the numbers $N^{(1)}_{k,l}$ recursively from the expression of the gradient $\frac{\partial G}{\partial t^2}$ in terms of the primary free energy. The numbers $N^{(1)}_{4k-2l,l} + \frac{2k-1}{12}\, N^{(0)}_{4k-2l,l}$ represent the numbers of elliptic curves of degree k passing through l points and $4k-2l$ lines in the general position, as was shown by Getzler and Pandharipande [21].

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