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Localization for quantum groups

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0. Introduction

In the work [D-cal] we introduced differential operators on noncommutative rings and showed that they enjoy some nice properties (for example compatibility with localizations). We proved that if the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ acts as a Hopf algebra on a ring, then it acts by differential operators.

Let $\mathfrak{g}$ be semisimple and let $U = U_q(\mathfrak{g})$ be the quantized enveloping algebra or the quantum group corresponding to $\mathfrak{g}$. Let U^0 be its "Cartan" part. We would like to consider (noncommutative) schemes with the action of the quantum group U. It was one of our first conclusions that such schemes should be "defined over SpecU^0". That is, one should be working in the category of objects (say rings) which are graded by a fixed abelian group Γ (in our case $\Gamma = Q$ — the root lattice). This leads us to the definition of quantum differential operators, which naturally includes the "grading" action of Γ. These differential operators also have the same nice properties, and one of our results in [D-cal] is the following: Assume that the quantum group acts as a Hopf algebra on a ring R so that R is a diagonalizable U^0-module. Then U acts by quantum differential operators.

In this paper we want to illustrate our theory by constructing the localization for the quantum group U which is parallel to the Beilinson-Bernstein localization construction for the enveloping algebra $U(\mathfrak{g})$. Namely, we construct the quantum flag variety X with a "sheaf" of quantum differential operators and a natural homomorphism from the quantum group to the global differential operators on X.

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We also prove that the functor of global sections is exact on the category of D_f-modules, if f is dominant. In this paper we work out the "generic case", (q generic) and we plan to treat the root of unity case in the near future.

Let us point out some features of our construction. First of all, the "structure sheaf" of X consists of noncommutative rings. Also rings of "functions" on different shifts of the "big cell" in X appear to be nonisomorphic (whereas in the classical case they are all polynomial rings). Moreover, we don't hope to be able to write down these rings (or quantum differential operators on them) in terms of generators and relations. However, we believe that Bernstein's theorem on the dimension of a D-module still holds for quantum D-modules. Some evidence for this was produced in [D-cal] and it is true in the (already nontrivial) case of sl_2. Hence, we expect to have a nice category of holonomic quantum D-modules.

Let us briefly describe the contents of the paper. In Section I we recall some facts about the quantum group. In Section II we construct the deformation of the classical "base affine space" G/u (u is the unipotent radical of a Borel subgroup B in G). In Section III we use the above deformation to construct the desired deformation X of the flag variety G/B. More precisely, we construct the category $(\operatorname{Proj} R)$ of "quasicoherent sheaves" on X. This is a "projective variety" and we define the category of quasicoherent sheaves on it. There is a natural functor of global sections Γ and we define the cohomology as the derived functor of this left exact functor Γ. We define the covering of X by "shifts u^w of the big cell". This gives rise to the Čech cohomology which coincides with the one defined above. By comparing the Čech complex of a coherent sheaf with its classical specialization, we are able to prove a weak ampleness result for the line bundles (Theorem 4), which suffices for the applications to D-modules. In Section IV we define D_f-modules on X and prove that D_f-modules have no higher cohomology if f is dominant (Theorem 5.2 and Proposition 7.1). We then state a conjecture that the global sections of D_f are "the right ones". The Borel-Weil theorem is proved along the way (Theorem 8).

We should mention that the localization for the quantum group sl_2 was constructed "by hand" in [H]. The construction in the general case was attempted by Joseph ([Jo]). His definition of differential operators, however, is not satisfactory. In particular there is no homomorphism from the quantum group to his differential operators. Nevertheless, we were inspired by papers [Jo], [JL1], [JL2], [JL3] and use some of the results from [Jo]. We thank Vinay Deodhar, Alexander Beilinson, Vladimir Drinfeld and Uma Iyer for useful conversations.

I. Preliminaries on quantum groups

1. The quantum group U

Notation: $A = \mathbb{Q}[q, q^{-1}]$, where q is an indeterminate, $\mathcal{A} = A_{(q-1)}$, $k = \mathbb{Q}(q)$, $k \subset K$ — any field.

Let $(a_{ij})_{i,j=1,\dots,n}$ be a Cartan matrix of finite type (*i.e.*, the corresponding Lie algebra $\mathfrak{g}$ is finite dimensional) and choose $d_i \in \{1,2,3\}$ such that $(d_i a_{ij})$ is symmetric. Consider the k-algebra U with generators

$$E_i, F_i, K_i, K_i^{-1}, \quad i = 1, \dots, n$$

and relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = 1, \qquad i,j = 1,\dots,n$$
$$K_iE_jK_i^{-1} = q^{d_ia_{ij}}E_j, \quad K_iF_jK_i^{-1} = q^{-d_ia_{ij}}F_j, \qquad i,j = 1,\dots,n$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \qquad i,j = 1,\dots,n$$
$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^rE_jE_i^s = 0, \qquad i \neq j$$
$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^rF_jF_i^s = 0, \qquad i \neq j.$$

In the last two relations we used brackets to denote Gaussian binomial coefficients. Specifically, we have for $m \in \mathbb{Z}, d, t \in \mathbb{N}$,

$$[m]_d = \frac{q^{md} - q^{-md}}{q^d - q^{-d}}, \quad [t]_d! = [t]_d[t-1]_d \cdots [2]_d[1]_d$$

and

$$\begin{bmatrix} m \\ t \end{bmatrix}_d = \frac{[m]_d[m-1]_d \cdots [m-t+1]_d}{[t]_d!} \in A.$$

In fact U is a Hopf algebra with the comultiplication Δ, the coidentity ε, and the antipode S defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$$
$$\Delta(K_i) = K_i \otimes K_i, \quad i = 1,\dots,n$$

$$\varepsilon(E_i) = 0 = \varepsilon(F_i), \quad \varepsilon(K_i) = 1, \qquad i = 1,\dots,n$$
$$S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i) = K_i^{-1}.$$

The algebra U has a triangular decomposition. Namely, let U^-, U° and U^+ be the subalgebras of U generated by F_i (resp. K_i, K_i^{-1}, resp. E_i), $i = 1,\dots,n$. Then the multiplication map defines the isomorphism

$$U^- \otimes U^\circ \otimes U^+ \xrightarrow{\sim} U.$$

2. The integral form U_A and the Frobenius homomorphism

Put

$$E_i^{(m)} = \frac{E_i^m}{[m]_{d_i}!}, \quad F_i^{(m)} = \frac{F_i^m}{[m]_{d_i}!}, \qquad i = 1, \dots n, \ m \in \mathbb{N},$$

and

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{q^{d_i(c-s+1)} K_i - q^{-d_i(c-s+1)} K_i^{-1}}{q^{d_i s} - q^{-d_i s}}, \qquad i = 1, \dots, n, \ c \in \mathbb{Z}, \ t \in \mathbb{N}.$$

Let U_A be the A-subalgebra of U generated by $E_i^{(m)}$, $F_i^{(m)}$, K_i, K_i^{-1}, $i = 1, \dots, n$, $m \in \mathbb{N}$. It is known ([Lu]) that U_A is a Hopf subalgebra of U. Let U_A^-, U_A^0, U_A^+ be the A-subalgebras of U_A generated by $F_i^{(m)}$ (resp. K_i, K_i^{-1}, $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right]$, resp. $E_i^{(m)}$). Then we have the triangular decomposition

$$U_A = U_A^- \otimes U_A^0 \otimes U_A^+.$$

Let $0 \neq \lambda \in \mathbb{C}$. Let A_λ be the image of A under the homomorphism of $\mathbb{Q}$-algebras $A \to \mathbb{C}$ defined by $q \to \lambda$. Put

$$U_\lambda := U_A \otimes_A A_\lambda.$$

It is known ([Lu]) that U_λ is a Hopf algebra and that the specialization map $U_A \to U_\lambda$ is a Hopf algebra homomorphism. In particular we obtain the Hopf algebra U_1 which turns out to be closely related to the universal enveloping algebra.

2.1. Let $\mathfrak{g} = \mathfrak{g}_\mathbb{Q}$ be the split semisimple Lie algebra corresponding to the Cartan matrix (a_{ij}). Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $Q \subset P \subset \mathfrak{h}^*$ the root lattice and the weight lattice respectively. Let $\pi = \{\alpha_1, \dots, \alpha_n\} \subset Q$ be a set of simple roots, R_+ the set of positive roots, $\{H_\alpha\}_{\alpha \in R_+} \subset \mathfrak{h}$ the corresponding coroots, $Q_+ = \Sigma \mathbb{Z}_+ \alpha_i$ the semigroup of weights that are ≥ 0. Let $\omega_1, \dots, \omega_n \in P$ be the fundamental weights and $P_+ = \Sigma \mathbb{Z}_+ \omega_i$ the semigroup of dominant weights. Let $P(\lambda) \subset P$ be the multiset of weights in the finite dimensional $\mathfrak{g}$-module with the highest weight $\lambda \in P_+$.

Define the nondegenerate symmetric pairing

$$(\ ,\ ) : P \times P \to \mathbb{Q}$$

by

$$(\alpha_i | \alpha_j) := d_i a_{ij}.$$

Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be the Borel subalgebra corresponding to π. Let $f_i, h_i, e_i \quad i = 1, \ldots, n$ be the standard generators of $\mathfrak{g}$ (so that $e_i \in \mathfrak{b}$). Let

$$f_i^{(r)} := f^r/r!,\ e_i^{(r)} := e^r/r!$$

be the corresponding divided powers in the enveloping algebra $U(\mathfrak{g}) = U_{\mathbb{Q}}(\mathfrak{g})$. According to [Lu] we have the following surjective homomorphism of Hopf algebras, called the Frobenius map

$$F : U_1 \to U(\mathfrak{g}), \quad F(E_i^{(r)}) = e_i^{(r)}, \quad F(F_i^{(r)}) = f_i^{(r)}, \quad F(K_i) = F(K_i^{-1}) = 1.$$

The kernel of F is generated by the central (in U_1) elements $K_i - 1$.

3. The center of U and the Harish-Chandra homomorphism

3.0. Let $\tau : U^0 \to k[P]$ be the homomorphism defined by $\tau(K_i) = \alpha_i$. Thus τ is an isomorphism of U^0 onto the group ring $k[Q]$. Let W be the Weyl group. Since it acts on $k[Q]$, it also acts on U^0 via τ. Let ρ be the half sum of positive roots and let $\phi : U^0 \to U^0$ be the automorphism defined by $\phi(K_i) = q^{-(\alpha_i|\rho)}K_i$.

Using the triangular decomposition $U = U^-U^0U^+$, consider the projection $p : U \to U^0$. Let $Z \subset U$ be the center and denote the restriction

$$\xi := \phi \cdot p|_Z : Z \to U^0.$$

As in the case of enveloping algebras, one easily checks that ξ is a ring homomorphism.

Theorem ([Ta], [JL1])**.** *The map ξ is injective and its image is $\tau^{-1}(k[2P] \cap k[Q])^W$.*

3.1. Fix a field $k \subset K$. Put

$$k[P]^* := \operatorname{Hom}_{k-\mathrm{alg}}(k[P], K).$$

We call a character $f \in k[P]^*$ linear if $f(\omega_i) = q^{(\lambda|\omega_i)}$ for some $\lambda \in \mathfrak{h}^*$ (the k-dual). We say that a linear f is integral if $\lambda \in P$.

The Weyl group W acts on $k[P]^*$ via its action on $k[P]$. Consider the P-action on $k[P]^*$ defined by

$$(f + \beta)(\omega_i) = q^{(\beta|\omega_i)}f(\omega_i), \quad f \in k[P]^*, \quad \beta \in P.$$

Given $f \in k[P]^*$ we denote the corresponding central character by

$$\chi_f := f \cdot \tau \cdot \xi : Z \to K.$$

3.2. Definition. A character $f \in k[P]^*$ is called *dominant* if $\chi_f \neq \chi_{f+\beta}$ for all $0 \neq \beta \in Q_+$. We call f *regular dominant* if for all $\gamma \in P_+$ and all $\theta \in P(\gamma)$, $\theta \neq \gamma$ we have $\chi_{f+\gamma} \neq \chi_{f+\theta}$.

Recall that $\lambda \in \mathfrak{h}^*$ is dominant if $\lambda(H_\alpha) \neq -1, -2, \ldots$ for $i = 1, \ldots, n$, $\alpha \in R_+$.

3.3. Lemma. *Let f be a linear character corresponding to $\lambda \in \mathfrak{h}^*$. Then f is dominant (resp. regular dominant) if and only if λ is dominant (resp. regular dominant) (cf.* [BB2]*).*

4. The R-matrix

For U-modules M, N their tensor product, $M \otimes N$ is naturally a U-module via the comultiplication $\Delta : U \to U \otimes U$. Since U is not a cocommutative Hopf algebra, the linear isomorphism

$$' : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto n \otimes m$$

is not a map of U-modules. However, we have the following structure ([CP], Cor. 10.1.20). For any locally finite U-modules V, M there is defined a natural isomorphism of U-modules

$$\mathcal{R}_{M,V} : M \otimes V \xrightarrow{\sim} V \otimes M.$$

In this work we will use the following property of $\mathcal{R}$: Given locally finite U-modules V, M, N, the isomorphism

$$\mathcal{R}_{M\otimes N,V} : M \otimes N \otimes V \xrightarrow{\sim} V \otimes M \otimes N$$

is the product of the isomorphisms

$$\mathrm{id}_M \otimes \mathcal{R}_{N,V} : M \otimes N \otimes V \xrightarrow{\sim} M \otimes V \otimes N,$$

and

$$\mathcal{R}_{M,V} \otimes \mathrm{id}_N : M \otimes V \otimes N \xrightarrow{\sim} V \otimes M \otimes N.$$

5. The highest weight modules

5.0. For a U-module M and $g \in k[P]^*$, denote by M_g the subspace of M on which U^0 acts by the character g (more precisely by $g \cdot \tau$). Nonzero elements of M_g are called *vectors of weight* g or of *weight* λ if g is linear corresponding to $\lambda \in \mathfrak{h}^*$.

Let $f \in k[P]^*$ and denote by K_f the one-dimensional (over K) U^0U^+-module on which U^+ acts trivially and U^0 acts by the character f. The U-module

$$M(f) := U \otimes_{U^0U^+} K_f$$

is called the Verma module with the highest weight f. Note that Z acts on $M(f)$ by the central character $\chi_{f+\rho}$. In case $k \neq K$ we will extend the scalars from k to K, *i.e.*, we will consider $M(f)$ as a module over the quantum group $U \otimes_k K$ instead of U. However to simplify the notation we will denote this bigger algebra again by U. Note that $M(f)$ is a direct sum of K-subspaces

$$M(f) = \bigoplus_{\beta \in Q_+} M(f)_{f-\beta}.$$

It has a unique simple quotient $L(f)$. If f is a linear character corresponding to $\lambda \in \mathfrak{h}^*$, we also write $M(\lambda) = M(f)$, $L(\lambda) = L(f)$.

5.1. In case $\lambda \in P_+$, we take $k = K$ and denote $R(\lambda) = L(\lambda)$. It is a finite dimensional U-module, that satisfies the Weyl character formula. In particular, for each $w \in W$ there exists a unique (up to a nonzero scalar) vector $e_{w\lambda} \in R(\lambda)$ of weight $w\lambda$.

The category of finite dimensional U-modules is semisimple. It follows that there exists a unique (up to a nonzero scalar) surjective map of U-modules

$$R(\lambda) \otimes R(\mu) \to R(\lambda + \mu). \tag{1}$$

We would like to choose these maps in such a way that the P-graded U-module

$$R := \oplus_{\lambda \in P_+} R(\lambda)$$

is a P-graded associative algebra. The following theorem is essentially due to Joseph and Letzter ([JL2]) and it is proved in [Jo], 2.2–2.5.

5.1.1. Theorem. *There exists a choice of maps* (1) *above that makes R a P-graded associative algebra. Moreover, for this algebra structure on R, there exists a choice of highest weight vectors $e_\lambda \in R(\lambda)$ such that $e_\lambda e_\mu = e_{\lambda+\mu}$ in R. In particular, $e_\lambda e_\mu = e_\mu e_\lambda$.*

5.1.2. Remark.

1) By definition U acts on R as a Hopf algebra (that is R is a U-ring), *i.e.*,

$$u(ab) = u_{(1)} a u_{(2)} b,$$

where $a, b \in R$, $u \in U$ and $\Delta(u) = u_{(1)} \otimes u_{(2)} \in U \otimes U$.

2) It is not difficult to show (see for example Lemma 1.7 below) that the algebra R as in above theorem is unique up to an isomorphism.

II. Deformation of G/u

Let G be the simply connected group corresponding to the Lie algebra $\mathfrak{g}$, $u \subset G$ the maximal unipotent subgroup, and G/u the corresponding "base affine space". It is well known that G/u is a quasi-affine variety with the ring of regular functions being the representation ring $\bar{R}$ of $\mathfrak{g}$. Namely, let $\bar{R}(\lambda)$ be the irreducible finite dimensional $\mathfrak{g}$-module with the highest weight $\lambda \in P_+$. Put

$$\bar{R} := \bigoplus_{\lambda \in P_+} \bar{R}(\lambda).$$

Then this $U(\mathfrak{g})$-module is isomorphic to the (commutative) ring of regular functions on G/u. We fix one such isomorphism and thus consider $\bar{R}$ as a P-graded commutative ring with the multiplication given by a projection

$$\bar{R}(\lambda) \otimes \bar{R}(\mu) \to \bar{R}(\lambda + \mu).$$

In the previous section I,5.1 we introduced the analogous representation ring R of the quantum group U (Thm. 5.1.1). This last ring R is a k-algebra that is no longer commutative. Our next goal is to construct a deformation R_A of $\bar{R}$ over A such that the fibre over the generic point Spec$k$ of Spec$A$ is the algebra R. We also show that R_A is naturally a U_A-ring in such a way, that the specialization $R_A \to \bar{R}$ is compatible with the specialization $U_A \to U(\mathfrak{g})$.

1. Construction of R_A

1.0. Consider the representation ring $R = \oplus_{\lambda \in P_+} R(\lambda)$ of U with the choice of highest weight vectors $e_\lambda \in R(\lambda)$ as in Theorem I,5.1.1 above.

1.2. Fix $\lambda \in P_+$. Define an A-submodule $R_A(\lambda)$ of $R(\lambda)$ as follows. Let $R(\lambda)_\mu$ be the weight μ subspace of $R(\lambda)$ for $\mu \in P(\lambda)$ (I,2.1). Then

$$R_A(\lambda) = \bigoplus_{\mu} R_A(\lambda)_\mu, \text{ where } R_A(\lambda)_\mu = R_A(\lambda) \cap R(\lambda)_\mu,$$

and we define $R_A(\lambda)_\mu$ by the descending induction on the natural partial order on μ.

$$R_A(\lambda)_\lambda := A \cdot e_\lambda$$

$$R_A(\lambda)_\mu = \left\{ x \in R(\lambda)_\mu \mid E_i^{(M)} x \in \bigoplus_{\nu > \mu} R_A(\lambda)_\nu \text{ for all } \begin{array}{l} i = 1, \ldots, n \\ M \geq 1 \end{array} \right\}.$$

Proposition.

(a) $R_A(\lambda)$ *is a* U_A*-submodule of* $R(\lambda)$.

(b) $R_A(\lambda)_\mu \subset R_A(\lambda)$ *is a free* A*-module of rank equal to* $\dim_k R(\lambda)_\mu$.

Proof of proposition. (a) Let $x \in R_A(\lambda)_\mu$, $\mu = \sum n_i\omega_i$. Then $E_i^{(N)}x \in R_A(\lambda)$ by definition. Also $K_i x = q^{n_i d_i}x \in Ax \subset R_A(\lambda)$. It remains to show that $F_i^{(M)}x \in R_A(\lambda)$. We will prove this by induction on M and by the descending induction on the weight μ. We need to show that

$$E_j^{(N)} \cdot F_i^{(M)} x \in R_A(\lambda), \quad \text{for } N \geq 1.$$

We use the following formula ([Lu]):

$$E_i^{(N)} F_i^{(M)} = \sum_{\substack{t\geq 0\\ t\leq N, t\leq M}} F_i^{(M-t)} \cdot \begin{bmatrix} K_i; 2t-N-M \\ t \end{bmatrix} \cdot E_i^{(N-t)}$$

If $\mu = \lambda$, then $E_i^{(N)}F_i^{(M)}x = F_i^{(M-N)} \cdot \left[\begin{smallmatrix} K_i; N-M \\ t \end{smallmatrix}\right] x$. It follows from the lemma below that

$$\begin{bmatrix} K_i; N-M \\ t \end{bmatrix} x \in Ax.$$

Hence by induction on M we conclude that

$$F_i^{(M-N)} \begin{bmatrix} K_i; N-M \\ t \end{bmatrix} x \in R_A(\lambda).$$

Thus $F_i^{(M)}x \in R_A(\lambda)$ if $\mu = \lambda$. If $\mu < \lambda$, then we argue by the descending induction on μ using the same formula and the same lemma. This proves (a) in the proposition.

(b) Notice that $U^- \cdot e_\lambda = R(\lambda)$. Hence $U_A^- \cdot e_\lambda$ will generate $R(\lambda)$ over k. Since $U_A^- \cdot e_\lambda \subset R_A(\lambda)$ and $R_A(\lambda) = \oplus_\mu R_A(\lambda)_\mu$ we conclude that $R_A(\lambda)_\mu$ generates $R(\lambda)_\mu$ over k.

Consider the map

$$R_A(\lambda)_\mu \xrightarrow{\oplus E_i} \underset{i}{\oplus} \left(\underset{\nu>\mu}{\oplus} R_A(\lambda)_\nu \right).$$

This map is injective (if $\mu \leq \lambda$). By descending induction on μ we may assume that each $R_A(\lambda)_\nu$, $\nu > \mu$ is a finitely generated A-module. Since A is noetherian, the same is true for $R_A(\lambda)_\mu$. This proves that each $R_A(\lambda)_\mu$ is a finite A-module.

We proved that $R_A(\lambda)_\mu$ is a finite torsion-free A-module. Hence it is free since A is a PID. It also has the right rank. This proves (b) of the proposition.

Lemma. *The expression* $\left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix}\right]$ *when evaluated by* $K_i \mapsto q^{\dim}$*,* $m \in \mathbb{Z}$ *gives an element in* A*.*

1.3. We have constructed a graded U_A-submodule

$$R_A := \oplus R_A(\lambda)$$

of the U-ring R.

Proposition. R_A *is an* A*-subalgebra of* R*, and hence a* U_A*-subring of the* U*-ring* R*.*

Proof of proposition. Let $x \in R_A(\lambda)$, $y \in R_A(\mu)$. We need to show that

$$E_i^{(N)}(xy) \in R_A(\lambda + \mu)$$

for all $N \geq 1$ $i = 1, \dots, n$. We will do this by descending induction on the weights of x, y. If $x = e_\lambda$, $y = e_\mu$, then $e_{\lambda+\mu} = xy \subset R_A(\lambda + \mu)$ by definition. Otherwise recall that R is a U-ring, hence a U_A-ring, that is

$$u(xy) = u_{(1)}(x)u_{(2)}(y)$$

where

$$\Delta u = u_{(1)} \otimes u_{(2)}$$

is the comultiplication.

We have ([Lu])

$$\Delta E_i^{(N)} = \sum_{b=0}^{N} q^{d_i b(N-b)} E_i^{(N-b)} K_i^b \otimes E_i^{(b)}$$

Thus $E_i^{(N)}(xy) = \sum_{b=0}^{N} q^{d_i b(N-b)} E_i^{(N-b)} K_i^b(x) E_i^{(b)}(y)$ and for every $b = 0, \dots, N$, the pair $E_i^{(N-b)} K_i(x)$, $E_i^{(b)}(y)$ will have higher weights than the pair x, y. Hence by induction the product belongs to R_A. This proves the proposition.

1.4. Let $0 \neq \lambda \in \mathbb{C}$. Consider the homomorphism of $\mathbb{Q}$-algebras

$$A \to \mathbb{C}, \; q \mapsto \lambda.$$

Let A_λ be the image of A in $\mathbb{C}$. Consider the algebras

$$R_\lambda := R_A \otimes_A A_\lambda$$
$$U_\lambda := U_A \otimes_A A_\lambda.$$

Then U_λ acts naturally on R_λ. Recall that U_A is a Hopf subalgebra of U and that the specialization map $U_A \to U_\lambda$ is a map of Hopf algebras. Thus R_λ is a U_λ-ring.

1.5. Consider the special case when $\lambda = 1$. We will denote elements $x \otimes 1 \in U_1$ by $\overline{x}$. In U_1 we have

$$0 = (1-1)(\overline{E}_i\overline{F}_i - \overline{F}_i\overline{E}_i) = \overline{K}_i - \overline{K}_i^{-1}.$$

Hence $\overline{K}_i = \overline{K}_i^{-1}$ or $\overline{K}_i^2 = 1$. Elements $\overline{K}_i$ are obviously central in U_1 and it is known ([Lu]) that

$$U_1/\Sigma_i U_1(\overline{K}_i - 1) \simeq U(\mathfrak{g}).$$

Moreover, $\overline{K}_i$ acts trivially on R_1, hence R_1 is a $U(\mathfrak{g})$-ring.

1.6. Proposition. *R_1 as a $U(\mathfrak{g})$-ring is isomorphic to the representation ring $\bar{R}$ of $U(\mathfrak{g})$. In particular the ring R_1 is commutative.*

This follows from the following lemma.

1.7. Lemma. *Let $V = \oplus_{\lambda \in P_+} V(\lambda)$ be a P-graded $U(\mathfrak{g})$-ring such that*

(i) *$U(\mathfrak{g})$ preserves each $V(\lambda)$ and $V(\lambda) \simeq \bar{R}(\lambda)$ as $U(\mathfrak{g})$-modules.*
(ii) *There exists a choice of highest weight vectors $t_\lambda \in V(\lambda)$ such that $t_\lambda \cdot t_\mu = t_{\lambda+\mu}$. Then $V \simeq \bar{R}$ as $U(\mathfrak{g})$-rings.*

Proof of lemma. Let $\{v_\lambda \in \bar{R}(\lambda)\}$ be a collection of highest weight vectors such that $v_\lambda \cdot v_\mu = v_{\lambda+\mu}$. There exists a unique $U(\mathfrak{g})$-module isomorphism

$$\varphi_\lambda : V(\lambda) \xrightarrow{\sim} \bar{R}(\lambda),\ t_\lambda \mapsto v_\lambda.$$

Put

$$\varphi := \oplus \varphi_\lambda : V \xrightarrow{\sim} \bar{R}.$$

We claim that φ is a ring homomorphism. Indeed, consider the multiplication map in V:

$$m_V : V(\lambda) \otimes V(\mu) \to V(\lambda + \mu).$$

$U(\mathfrak{g})$ acts naturally on $V(\lambda) \otimes V(\mu)$ and since V is a $U(\mathfrak{g})$-ring, it follows that m_V is a $U(\mathfrak{g})$-map. But such a $U(\mathfrak{g})$-map is unique up to a scalar, and is specified completely by the image of $t_\lambda \otimes t_\mu$. The same considerations apply to the ring $\bar{R}$. The diagram

$$\begin{array}{ccc} V(\lambda) \otimes V(\mu) & \xrightarrow{m_V} & V(\lambda+\mu) \\ {\scriptstyle \varphi_\lambda \otimes \varphi_\mu}\downarrow & & \downarrow{\scriptstyle \varphi_{\lambda+\mu}} \\ \bar{R}(\lambda) \otimes \bar{R}(\mu) & \xrightarrow{m_{\bar{R}}} & \bar{R}(\lambda+\mu) \end{array}$$

consists of morphisms of $U(\mathfrak{g})$-modules such that

$$\varphi_{\lambda+\mu} \cdot m_V(t_\lambda \otimes t_\mu) = m_{\bar{R}} \cdot \varphi_\lambda \otimes \varphi_\mu(t_\lambda \otimes t_\mu).$$

It follows that the diagram is commutative, *i.e.*, that φ is a homomorphism of rings.

1.8. Corollary. *R_A is an integral domain.*

Proof. Indeed, R_A is a free A-module. Hence the result follows from the fact that $R_1 \simeq \overline{R}$ is an integral domain.

2.0. Let M be an A-module. Consider the $(q-1)$-adic $\mathbb{Z}_-$-filtration F on M:

$$F_{-n}M := (q-1)^n M, \quad n \geq 0.$$

Let $\operatorname{gr} M = \oplus F_n M/F_{n-1}M$ be the associated graded $\mathbb{Q}$-vector space. If $M = A$ then $\operatorname{gr} A \simeq \mathbb{Q}[t]$ as a ring, where t is the symbol of $q-1$ in $F_{-1}A/F_{-2}A$.

For an A-module M, $\operatorname{gr} M$ is therefore a $\mathbb{Q}[t]$-module. If M is a free A-module, then $\operatorname{gr} M \simeq \mathbb{Q}[t] \otimes_{\mathbb{Q}} \operatorname{gr}_0 M$ — a free $\mathbb{Q}[t]$-module. Take in particular the free A-module $M = R_A$. Then

$$\operatorname{gr} R_A \simeq \mathbb{Q}[t] \otimes_{\mathbb{Q}} \bar{R}$$

as a $\mathbb{Q}[t]$-module. Notice that $\operatorname{gr} R_A$ is naturally a ring with $\mathbb{Q}[t]$ and $\bar{R}$ being subrings.

Proposition. *The isomorphism* $\operatorname{gr} R_A \simeq \mathbb{Q}[t] \otimes_{\mathbb{Q}} \bar{R}$ *is in fact a ring isomorphism. In particular* $\operatorname{gr} R_A$ *is a commutative ring (cf. Proposition* 1.6 *above).*

Proof. Since A is in the center of R_A, $\mathbb{Q}[t]$ is in the center of $\operatorname{gr} R_A$. In particular $\mathbb{Q}[t]$ commutes with $\bar{R}$ in $\operatorname{gr} R_A$. This proves the proposition.

2.1. Take $M = U_A$ as a free A-module. The same arguments show that we have an isomorphism of (noncommutative) rings

$$\operatorname{gr} U_A \simeq \mathbb{Q}[t] \otimes_{\mathbb{Q}} \operatorname{gr}_0 U_A = \mathbb{Q}[t] \otimes_{\mathbb{Q}} U_1.$$

The U_A-action on R_A induces the $\mathbb{Q}[t]$-linear $\operatorname{gr} U_A$-action on $\operatorname{gr} R_A \simeq \mathbb{Q}[t] \otimes_{\mathbb{Q}} \bar{R}$, where U_1 acts on $\bar{R}$ as in 1.4 above.

2.2. Put $R_{\mathcal{A}} := R_A \otimes_A \mathcal{A}$. Then $R_{\mathcal{A}}$ has the same $(q-1)$-adic filtration and $\operatorname{gr} R_{\mathcal{A}} = \operatorname{gr} R_A$.

2.3. Proposition. *The rings $R_{\mathcal{A}}$, R are graded noetherian, i.e., every graded ideal is finitely generated.*

Proof. It suffices to prove the lemma for the ring $R_{\mathcal{A}}$. Indeed, given a graded ideal $I \subset R$, we have

$$R \otimes_{R_{\mathcal{A}}} (I \cap R_{\mathcal{A}}) = I.$$

Hence, if $I \cap R_{\mathcal{A}}$ is a finite $R_{\mathcal{A}}$-module, then I is a finite R-module.

Consider $\mathcal{A}$-submodules of $R_{\mathcal{A}}$ with the induced (submodule) filtration, so that the functor gr is exact on the category of such submodules. Let $I \subset J \subset R_{\mathcal{A}}$,

$I \neq J$ be two graded ideals. In view of Proposition 2.0 above, it suffices to prove that $\operatorname{gr} I \neq \operatorname{gr} J$.

Let $\lambda \in P$ be such that $J_\lambda / I_\lambda \neq 0$, and assume that $\operatorname{gr} I_\lambda = \operatorname{gr} J_\lambda$. Notice that I_λ and J_λ are finite $\mathcal{A}$-modules. Hence, in order to obtain a contradiction it suffices to prove that $I_\lambda/(q-1)I_\lambda = J_\lambda/(q-1)J_\lambda$ and apply the Nakayama lemma. It suffices to prove that $J_\lambda = I_\lambda + (q-1)J_\lambda$.

By the Artin-Rees lemma ([AM], Thm. 10.11) the filtration on J_λ (as a submodule of $R_{\mathcal{A}}(\lambda)$) and its own $(q-1)$-adic filtration have finite difference. In particular, there exists $d \in \mathbb{Z}_-$ such that for all $j \in J_\lambda$ with the degree $d(j) < d$ we have $j \in (q-1)J_\lambda$. Let $j \in J_\lambda$ be an element with the lowest degree such that $j \notin I_\lambda + (q-1)J_\lambda$. Since $\operatorname{gr} I_\lambda = \operatorname{gr} J_\lambda$ there exists $i \in I_\lambda$ such that $d(i-j) < d(j)$. By induction $i - j \in I_\lambda + (q-1)J_\lambda$. Hence also $j \in I_\lambda + (q-1)J_\lambda$. This contradiction proves the proposition.

2.3.1. Remark. Joseph proved that R is noetherian by a different method (see [Jo]).

3. Localization of filtered rings

3.1. Later on we will need some results on the compatibility of Ore localizations with the passage to graded rings. Namely, let B be a unital ring with a $\mathbb{Z}$-filtration

$$\cdots \subset F_n B \subset F_{n+1} B \subset \ldots \quad n \in \mathbb{Z}$$

such that $\bigcup F_n B = B$. Assume for simplicity that $\bigcap F_n B = 0$. Let $\operatorname{gr} B = \oplus F_n B / F_{n+1} B$ be the associated graded ring. We assume that $\operatorname{gr} B$ is a domain (hence also B is a domain). For $0 \neq b \in B$ define its degree $d(b) = \min\{n \in \mathbb{Z} | b \in F_n,\ b \notin F_{n-1}\}$, $\deg 0 = -\infty$. Let $\bar{b}$ denote the symbol of b in $F_{d(b)}/F_{d(b)-1}$. Since $\operatorname{gr} B$ is a domain, the symbol map $B \to \operatorname{gr} B$ is multiplicative, and $d(ab) = d(a) + d(b)$.

Let S be a left Ore set in B ($0 \notin S$, $1 \in S$). Let $[S^{-1}]B$ be the corresponding localization. This is also naturally a $\mathbb{Z}$-filtered fing. Namely, put

$$d(s,b) := d(b) - d(s)$$

(well defined since $\operatorname{gr} B$ is a domain) and

$$F_n[S^{-1}]B = \{(s,b) | d(s,b) \leq n\}.$$

Notice that $\cap F_n[S^{-1}]B = 0$. Let $\operatorname{gr}[S^{-1}]B$ be the associated graded ring.

Lemma. *Let $\bar{S} \subset \operatorname{gr} B$ be the symbol image in $\operatorname{gr} B$ of the Ore set S. Then $\bar{S}$ is a left Ore set in $\operatorname{gr} B$.*

Using the lemma we can define the (graded) localization $[\bar{S}^{-1}] \operatorname{gr} B$. Define a map

$$\theta : \operatorname{gr}[S^{-1}]B \to [\bar{S}^{-1}] \operatorname{gr} B, \quad \overline{(s,b)} \mapsto (\bar{s}, \bar{b}).$$

3.2. Proposition. *The above map θ is well defined and is an isomorphism of graded rings.*

3.3. More generally assume that M is a filtered B-module, *i.e.*, M is a B-module with a $\mathbb{Z}$-filtration

$$\cdots \subset F_nM \subset F_{n+1}M \subset \dots \qquad n \in \mathbb{Z},$$

such that

$$F_mB \cdot F_nM \subset F_{m+n}M.$$

Then $\operatorname{gr} M$ is a graded $\operatorname{gr} B$-module. Assume that $\bigcup F_nM = M$ and $\bigcap F_nM = 0$. Also assume that if $0 \neq \bar{b} \in \operatorname{gr} B$, $0 \neq \bar{m} \in \operatorname{gr} M$, then $\bar{b}\bar{m} \neq 0$. Consider also the localized $[S^{-1}]B$-module $[S^{-1}]M = [S^{-1}]B \otimes_B M$. Given $m \in M$, $(s,m) \in [S^{-1}]M$, we define $d(m)$, $d(s,m)$ exactly as in the ring case above. This again is well defined and we obtain a filtered $[S^{-1}]B$-module $[S^{-1}]M$, such that $\bigcup F_n[S^{-1}]M = [S^{-1}]M$ and $\bigcap F_n[S^{-1}]M = 0$. Again we have the natural map

$$\theta : \operatorname{gr}[S^{-1}]M \to [S^{-1}] \operatorname{gr} M, \qquad \overline{(s,m)} \mapsto (\bar{s}, \bar{m}).$$

Proposition. *The above map θ is well defined and is an isomorphism of graded $\operatorname{gr}[S^{-1}]B \simeq [\bar{S}^{-1}] \operatorname{gr} B$-modules.*

3.4. In Section III we will consider localizations of $R_{\mathcal{A}}$-modules M, which satisfy the assumptions in 3.3 above with respect to the $(q-1)$-adic filtration. We will localize by Ore sets $S_i \subset R_{\mathcal{A}}$ consisting of elements of degree 0 with respect to $(q-1)$-adic filtration on $R_{\mathcal{A}}$. In fact, we will consider successive localizations

$$M_{i_1 \dots i_k} := [S_{i_k}^{-1}]R_{\mathcal{A}} \otimes_{R_{\mathcal{A}}} [S_{i_{k-1}}^{-1}]R_{\mathcal{A}} \otimes \cdots \otimes_{R_{\mathcal{A}}} [S_{i_1}^{-1}]R_{\mathcal{A}} \otimes_{R_{\mathcal{A}}} M.$$

These are left $[S_{i_k}^{-1}]R_{\mathcal{A}}$- modules. These modules have a natural $(q-1)$-adic filtration and we can consider the corresponding associated graded $\operatorname{gr} M_{i_1 \dots i_k}$.

On the other hand, we have the following graded analogue

$$\bar{M}_{i_1 \dots i_k} = [\bar{S}_{i_k}^{-1}] \operatorname{gr} R_{\mathcal{A}} \otimes_{\operatorname{gr} R_{\mathcal{A}}} \cdots \otimes_{\operatorname{gr} R_{\mathcal{A}}} [\bar{S}_{i_1}^{-1}] \operatorname{gr} R_{\mathcal{A}} \otimes_{\operatorname{gr} R_{\mathcal{A}}} \operatorname{gr} M.$$

The following proposition is proved by induction on k using Proposition 3.3.

Proposition. *There exists a natural isomorphism of graded $\operatorname{gr}[S_{i_k}^{-1}] R_{\mathcal{A}} \simeq [\bar{S}_{i_k}^{-1}] \operatorname{gr} R_{\mathcal{A}}$-modules*

$$\operatorname{gr} M_{i_1 \dots i_k} \simeq \bar{M}_{i_1 \dots i_k}.$$

III. Deformation of G/B

0. In Section II we constructed the deformation R_A of the P-graded ring $\bar{R}$. Since the commutative ring $\bar{R}$ is the ring of regular functions on the quasiaffine variety G/u, we consider R_A as the (noncommutative) ring of regular functions on the corresponding quantum deformation of G/u. In this work we will be interested in the "generic case" (q an indeterminate). Hence from now on, we will work over $\mathcal{A}$ or k. That is we will consider the algebras $R_{\mathcal{A}}$ and R.

In the classical case the P-graded structure of the ring $\bar{R}$ allows one to reconstruct the abelian category of quasicoherent sheaves on $\bar{X} = G/B$ – the flag variety of G, namely,

$$\mathrm{Qcoh}_{\bar{X}} \simeq \mathrm{Proj}\,\bar{R}.$$

Thus it is natural to define the category $\mathrm{Qcoh}_{X_{\mathcal{A}}}$ (resp. Qcoh_X) of quasicoherent sheaves on the deformation $X_{\mathcal{A}}$ (resp. X) of the flag variety $\bar{X}$ as $\mathrm{Proj}\,R_{\mathcal{A}}$ (resp. $\mathrm{Proj}\,R$). Moreover, we will think of $X_{\mathcal{A}}$, X as *being* the categories $\mathrm{Proj}\,R_{\mathcal{A}}$, $\mathrm{Proj}\,R$ respectively.

Unfortunately, there seems to be no good notion of the corresponding deformation of the structure sheaf of the flag variety $\bar{X}$. However there is an "affine covering" of $X_{\mathcal{A}}$ and therefore we have a local description of a quasicoherent sheaf on $X_{\mathcal{A}}$ in terms of its Čech complex (of nonalternating cochains) (3.4).

Since the category $\mathrm{Proj}\,R_{\mathcal{A}}$ is defined as the localization of the category $\mathrm{Mod} - R_{\mathcal{A}}$ of graded $R_{\mathcal{A}}$-modules (see 1 below), it has a natural cohomology theory $\{H^i\}$. The above mentioned Čech complex defines the Čech cohomology $\{\check{H}^i\}$. We show that $H^i \simeq \check{H}^i$ (3.8). We use the Čech cohomology to compare the cohomology of coherent sheaves on $X_{\mathcal{A}}$ with their classical specializations. This way we prove a weak version of the usual ampleness result (Thm. 4.).

The same constructions and results hold for R instead of $R_{\mathcal{A}}$.

Our ultimate goal is to develop the theory of D-modules on X and to construct the localization for quantum groups. This is done in the next Section IV.

1. The category $\mathrm{Proj}\,R_{\mathcal{A}}$

Let $\mathrm{Mod} - R_{\mathcal{A}}$ be the abelian category of graded left $R_{\mathcal{A}}$-modules. That is, an object in $\mathrm{Mod} - R_{\mathcal{A}}$ is a left P-graded R-module $M = \oplus_{\lambda \in P} M_\lambda$, and morphisms are homomorphisms of degree zero. The group P has a natural partial order. Namely, if $\lambda = \sum a_i\omega_i,\ \mu = \sum b_i\omega_i$, then $\lambda \geq \mu$ iff $a_i \geq b_i$ for all i.

Let $M \in \mathrm{Ob}\,\mathrm{Mod} - R_{\mathcal{A}},\ m \in M$. We call m a torsion element if $R_{\mathcal{A}}(\lambda)m = 0$ for $\lambda \gg 0$. Let $M^{\mathrm{tor}} \subset M$ be the submodule consisting of torsion elements. We call M a torsion module if $M^{\mathrm{tor}} = M$. Let $\mathrm{Tor} \subset \mathrm{Mod} - R_{\mathcal{A}}$ be the full subcategory consisting of torsion modules. This is a Serre subcategory and we denote the quotient category by

$$\operatorname{Proj} R_{\mathcal{A}} := \operatorname{Mod} - R_{\mathcal{A}}/\operatorname{Tor}.$$

Let $\operatorname{Mod}^f - R_{\mathcal{A}}$ be the full subcategory of $\operatorname{Mod} - R_{\mathcal{A}}$ consisting of finitely generated modules. Since $R_{\mathcal{A}}$ is noetherian (II,2.3), this is an abelian category. Put

$$\operatorname{Proj}^f R_{\mathcal{A}} := \operatorname{Mod}^f - R_{\mathcal{A}}/\operatorname{Tor}.$$

This is a full subcategory of $\operatorname{Proj} R_{\mathcal{A}}$.

Note that the group P acts by autoequivalences of the categories $\operatorname{Mod} - R_{\mathcal{A}}$, $\operatorname{Proj}^f R_{\mathcal{A}}$, $\operatorname{Proj} R_{\mathcal{A}}$. Namely for $\lambda \in P$, $M \in \operatorname{Ob}\operatorname{Mod} - R_{\mathcal{A}}$, define $M[\lambda] \in \operatorname{Ob}\operatorname{Mod} - R_{\mathcal{A}}$ by

$$M[\lambda]_\mu := M_{\mu+\lambda}.$$

We define categories $\operatorname{Mod} - R$, $\operatorname{Proj} R$, $\operatorname{Proj}^f R$ similarly.

2. Ore localizations of the ring $R_{\mathcal{A}}$

Let $\mathcal{A}^* \subset \mathcal{A}$ be the multiplicative subset of invertible elements. Fix $w \in W$. For $\lambda \in P_+$ choose a nonzero w-extremal vector $e_{w\lambda} \in R_{\mathcal{A}}(\lambda)$. It spans the free rank one $\mathcal{A}$-submodule consisting of vectors of weight $w\lambda$. Put

$$e_w := \{\mathcal{A}^* e_{w\lambda}\}_\lambda \subset R_{\mathcal{A}}.$$

This follows from the Weyl character formula that $e_{w\lambda} \cdot e_{w\mu} \in \mathcal{A}^* e_{w(\lambda+\mu)}$. Hence e_w is a multiplicative subset in $R_{\mathcal{A}}$. It was proved by Joseph ([Jo]) that e_w is a (left and right) Ore subset in $R_{\mathcal{A}}$. (Actually Joseph proves the statement for the ring R, but the same proof works for $R_{\mathcal{A}}$ and even for R_A.)

Let $\bar{e}_w \subset \bar{R}$ be the image of e_w under the canonical specialization map $R_{\mathcal{A}} \to \bar{R}$. Consider the localization $\bar{e}_w^{-1}\bar{R}$. This is a P-graded ring. It is easy to see that its zero component $(\bar{e}_w^{-1}\bar{R})_0$ is the ring of regular functions on the w-translate u^w of the big cell in $\bar{X} = G/B$. (The whole ring $\bar{e}_w^{-1}\bar{R}$ is the ring of functions on the preimage of u^w under the projection $G/u \to G/B$.) Thus the graded localized ring $e_w^{-1}R_{\mathcal{A}}$ and its zero graded part $(e_w^{-1}R_{\mathcal{A}})_0$ should be considered as deformations of the corresponding rings of functions. However, if we try to deform the ring of functions on the intersection $u^{w_1} \cap u^{w_2}$ in terms of the rings $(e_{w_1}^{-1}R_{\mathcal{A}})_0$, $(e_{w_2}^{-1}R_{\mathcal{A}})_0$, we run into difficulties since the product of Ore sets $e_{w_1}e_{w_2}$ is not an Ore set in general. Our next definition of the standard complex (3.1) of $M \in \operatorname{Ob}\operatorname{Proj} R_{\mathcal{A}}$ is therefore the closest analogue of a quasicoherent sheaf on the deformed flag variety $X_{\mathcal{A}}$.

3. Cohomology of R-modules

3.0. Let us introduce some notation. Let

$$\varphi^* : \mathrm{Mod} - R_{\mathcal{A}} \to \mathrm{Proj}\, R_{\mathcal{A}}$$

denote the localization functor. Let $\{w_1, \dots, w_N\} = W$ be the elements of the Weyl group W. Let $\mathrm{Mod} - e_{w_i}^{-1} R_{\mathcal{A}}$ denote the category of left graded modules over the P-graded ring $e_{w_i}^{-1} R_{\mathcal{A}}$. Let

$$g_i^* : \mathrm{Mod} - R_{\mathcal{A}} \to \mathrm{Mod} - e_{w_i}^{-1} R_{\mathcal{A}}$$

denote the obvious localization functor. Then g_i^* factors though $\mathrm{Proj}\, R_{\mathcal{A}}$, and thus we obtain the localization functor

$$f_i^* : \mathrm{Proj}\, R_{\mathcal{A}} \to \mathrm{Mod} - e_{w_i}^{-1} R_{\mathcal{A}},$$

such that $g_i^* = f_i^* \varphi^*$.

Denote by φ_*, g_{i*}, f_{i*} the corresponding right adjoint functors. Functors φ^*, g_i^*, f_i^* are exact; φ_*, g_{i*}, f_{i*} are left exact and fully faithful ([GZ]). We have

$$\varphi^* \varphi_* = \mathrm{Id}_{\mathrm{Proj}\, R_{\mathcal{A}}},$$

$$f_i^* f_{i*} = \mathrm{Id}_{\mathrm{Mod} - e_{w_i}^{-1} R_{\mathcal{A}}} = g_i^* g_{i*}$$

and

$$g_{i*} = \varphi_* f_{i*}.$$

Note that g_{i*} is the functor of restrictions of scalars from $e_{w_i}^{-1} R_{\mathcal{A}}$ to $R_{\mathcal{A}}$, hence it is exact. Therefore, the functor

$$f_{i*} = \varphi^* \varphi_* f_{i*} = \varphi^* g_{i*}$$

is also exact.

3.1 The standard complex. Put $\ell_i := f_{i*} f_i^*$ and consider the adjunction morphism

$$\sigma_i : \mathrm{Id}_{\mathrm{Proj}\, R_{\mathcal{A}}} \to \ell_i.$$

Thus we get the diagram of functors

$$\mathrm{Id}_{\mathrm{Proj}\, R_{\mathcal{A}}} \xrightarrow{\tau_0^1} \bigoplus_i \ell_i \;\begin{array}{c}\xrightarrow{\tau_1^1} \\ \xrightarrow[\tau_1^2]{} \end{array}\; \bigoplus_{i,j} \ell_i \cdot \ell_j \;\begin{array}{c}\xrightarrow{\tau_2^1} \\ \xrightarrow{\tau_2^2} \\ \xrightarrow{\tau_2^3} \end{array}\; \bigoplus_{i,j,k} \ell_i \cdot \ell_j \cdot \ell_k \cdots$$

where $\tau_\alpha^\beta : \ell_{i_1} \cdot \ell_{i_2} \cdot \dots \cdot \ell_{i_\alpha} \to \oplus_j \ell_{i_1} \cdot \dots \cdot \ell_{i_{\beta-1}} \cdot \ell_j \cdot \ell_{i_\beta} \cdot \dots \cdot \ell_{i_\alpha}$ is the morphism of functors $\tau_\alpha^\beta := \oplus_j \ell_{i_1} \cdots \ell_{i_{\beta-1}} \cdot \sigma_j \cdot \ell_{i_\beta} \cdots \ell_{i_\alpha}$. Put $d_\alpha := \oplus_\beta (-1)^{\beta+1} \tau_\alpha^\beta$.

Denote

$$S^{i-1} := \underset{j_1,\dots,j_i}{\oplus} \ell_{j_1} \cdot \ldots \cdot \ell_{j_i}.$$

We obtain the functoral diagram

$$\mathrm{Id}_{\mathrm{Mod}-R_{\mathcal{A}}} \xrightarrow{d_0} S^0 \xrightarrow{d_1} S^1 \to \cdots$$

called the *standard complex* corresponding to the collection $\{f_i^*\}$.

3.1.1. Claim. *We have $d_{i+1} \cdot d_i = 0$, i.e. the standard complex is a complex.*

Proof. This follows from the fact that all σ_i are morphisms of functors.

3.2. Theorem. *The standard complex is exact.*

Proof. By Proposition 1.4 in [R] we only have to check that the collection $\{f_{i*}\}$ is a *covering* of $\mathrm{Proj}\, R_{\mathcal{A}}$, *i.e.*, if for a morphism s in $\mathrm{Proj}\, R_{\mathcal{A}}$ each $f_i^*(s)$ is invertible then s is invertible. This follows from the following lemma.

3.3. Lemma. *Fix $\mu \in P_+$. Then for $\lambda \gg 0$ we have*

$$\sum_{w \in W} R_{\mathcal{A}}(\lambda) e_{w\mu} = R_{\mathcal{A}}(\lambda + \mu).$$

Similarly for R.

Proof of the lemma. It suffices to prove the lemma for $R_{\mathcal{A}}$. Since $\mathcal{A}$ is a local ring and both sides of the desired equality are finitely generated $\mathcal{A}$-modules, it suffices to prove the equality of the residues

$$\sum_{w \in W} \bar{R}(\lambda) \bar{e}_{w\mu} = \bar{R}(\lambda + \mu),$$

which follows from the classical fact that the shifts u^w of the big cell cover the flag variety $\bar{X}$.

3.4. For $M \in \mathrm{Ob}\, \mathrm{Proj}\, R_{\mathcal{A}}$ we put $C^i(M) := \varphi_* S^i(M)$ and call

$$\check{C}(M) := C^0(M) \xrightarrow{d_1} C^1(M) \to \cdots$$

the Čech complex of M. This is a complex in $\mathrm{Mod} - R_{\mathcal{A}}$.

3.5. Definition. Let $M \in \operatorname{Ob}\operatorname{Proj} R_{\mathcal{A}}$. Denote

$$\check{h}^i(M) := H^i(\check{C}(M))$$

and

$$\check{H}^i(M) := \check{h}^i(M)_0,$$

the zero component of the graded $R_{\mathcal{A}}$-module $\check{h}^i(M)$. We call the $\mathcal{A}$-module $\check{H}^i(M)$ the i-th *Čech cohomology* group of M. Note that the collection $\{\check{h}^i\}$ (resp. $\{\check{H}^i\}$) is a δ-functor from $\operatorname{Proj} R_{\mathcal{A}}$ to $\operatorname{Mod} - R_{\mathcal{A}}$ (resp. to $\mathcal{A}$-modules). Also

$$\check{H}^i(M[\lambda]) = \check{h}^i(M)_\lambda.$$

3.6. Lemma. *The category* $\operatorname{Proj} R_{\mathcal{A}}$ *has enough injectives.*

Proof. The category $\operatorname{Mod} - R_{\mathcal{A}}$ is a Grothendieck category ([BD]). By Proposition 5.39 in [BD] a localization of a Grothendieck category is also a Grothendieck category. By Corollary 6.32 in [BD] a Grothendieck category has enough injectives.

3.7. Definition. Consider the left exact functor $\varphi_* : \operatorname{Proj} R_{\mathcal{A}} \to \operatorname{Mod} - R_{\mathcal{A}}$. The last lemma allows us to define its right derived functors $R^i\varphi_*$, $i \geq 0$. For $M \in \operatorname{Ob}\operatorname{Proj} R_{\mathcal{A}}$ put

$$h^i(M) := R^i\varphi_*(M)$$

and

$$H^i(M) := h^i(M)_0,$$

the zero component of $h^i(M)$.

3.8. Theorem. *The functors* $\check{h}^i$ *and* h^i *are isomorphic. In particular, the functors* $\check{H}^i$ *and* H^i *are isomorphic.*

Proof. Since functors f_i^*, f_{i*} are exact, the collection $\{f_i^*\}$ is a finite *biflat* covering of $\operatorname{Proj} R_{\mathcal{A}}$ (in the terminology of [R]). Moreover, the functors $\varphi_* f_{i*} = g_{i*}$ are exact. Therefore our theorem follows from Theorem 2.2 in [R]. More precisely, in the proof of Theorem 2.2 in [R] it was established that for $M \in \operatorname{Ob}\operatorname{Proj} R$ we have $h^i(S^j(M)) = 0$ for $i > 0$, $j \geq 0$.

3.9. Remarks.

1. Since φ_* is left exact we have $\varphi_* = \check{h}^0$.
2. Take $M \in \operatorname{Ob}\operatorname{Proj} R_{\mathcal{A}}$. For $i > 0$, the graded $R_{\mathcal{A}}$-module $\check{h}^i(M)$ is torsion. Indeed, apply the exact functor φ^* to $\check{C}(M)$. Since $\varphi^*\varphi_* = \operatorname{Id}_{\operatorname{Proj} R_{\mathcal{A}}}$ we get $S^\cdot(M)$, which is exact (except at $S^0(M)$) by Theorem 3.2.
3. For $N \in \operatorname{Ob}\operatorname{Mod} - R_{\mathcal{A}}$ we will denote

$$\check{h}^i(N) := \check{h}^i(\varphi^* N), \qquad h^i(N) := h^i(\varphi^* N);$$

$$\check{H}^i(N) := \check{H}^i(\varphi^* N), \qquad H^i(N) := H^i(\varphi^* N).$$

4. Let $N \in \operatorname{Ob}\operatorname{Mod} - R_{\mathcal{A}}$. Since $\check{h}^0(N) = \varphi_* \varphi^*(N)$ we have the adjunction morphism
$$N \to \check{h}^0(N).$$
Its kernel is N^{tor}.
5. All definitions and assertions of this section are valid for R instead of $R_{\mathcal{A}}$.
6. We also denote $\Gamma(M) := H^0(M)$.

4. Ampleness of $R_{\mathcal{A}}[\lambda]$, for $\lambda \gg 0$

Theorem. *Let $0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{N} \to 0$ be an exact sequence in $\operatorname{Proj}^f R$. Then for $\lambda \gg 0$ the sequence*

$$0 \to \Gamma(\mathcal{K}[\lambda]) \to \Gamma(\mathcal{M}[\lambda]) \to \Gamma(\mathcal{N}[\lambda]) \to 0$$

is also exact.

Proof. Consider a finite free R-module $T' = \oplus R[\lambda_i]$. It is enough to prove that a surjective morphism $p' : T' \to N'$ in $\operatorname{Mod} - R$ induces a surjection $\Gamma(T'[\lambda]) \to \Gamma(N'[\lambda])$ for $\lambda \gg 0$. Choose a free $R_{\mathcal{A}}$-submodule $T \subset T'$ such that $T' = k \otimes_{\mathcal{A}} T$. Put $p := p'|_T$ and $N := \operatorname{im}(p) \subset N'$. Thus we obtain a surjection $p : T \to N$ in $\operatorname{Mod} - R_{\mathcal{A}}$. It suffices to prove that for $\lambda \gg 0$ this map induces a surjection $\Gamma(T[\lambda]) \to \Gamma(N[\lambda])$. Let $M = \operatorname{Ker}(p) \in \operatorname{Mod}^f - R_{\mathcal{A}}$. Thus we have an exact sequence in $\operatorname{Mod}^f - R_{\mathcal{A}}$
$$0 \to M \to T \to N \to 0.$$

We first study the cohomology of M.

4.1. Proposition. *Let $\lambda \gg 0$. Then*

a) *$\alpha : M_\lambda \to \check{H}^0(M[\lambda])$ is an isomorphism,*

b) *for $i > 0$ the $\mathcal{A}$-module $\check{H}^i(M[\lambda])$, is a k-module.*

Similarly for T instead of M.

It suffices to prove the proposition for M. The proof of the proposition will be based on several lemmas.

4.1.1. Lemma. *Consider the $(q-1)$-adic filtration on M, so that $\operatorname{gr} M$ is a $\operatorname{gr} R_{\mathcal{A}}$-module. Let $0 \neq \bar{b} \in \operatorname{gr} R_{\mathcal{A}}$, $0 \neq \bar{m} \in \operatorname{gr} M$, then $\bar{b}\bar{m} \neq 0$.*

Proof. Since N is a torsion-free $\mathcal{A}$-module, it follows that the $(q-1)$-adic filtration on M coincides with the one induced from T. Hence $\operatorname{gr} M \subset \operatorname{gr} T$ and the lemma follows, because $\operatorname{gr} R_{\mathcal{A}}$ is a domain (II,1.8).

4.1.2. Corollary. *The $(q-1)$-adic filtration on M satisfies the assumptions in* II,3.3. *In particular, Proposition* II,3.3 *(with $B = R_{\mathcal{A}}$) and Proposition* II,3.4 *hold for M.*

4.1.3. Consider the Čech complex $\check{C}(M)$ with the $(q-1)$-adic filtration. It follows from the last Corollary 4.1.2 that $\operatorname{gr}\check{C}(M) \simeq \check{C}(\operatorname{gr} M)$, where $\check{C}(\operatorname{gr} M)$ is the Čech complex of $\operatorname{gr} M$ considered as a $\operatorname{gr} R_{\mathcal{A}}$-module (where we consider localizations of the ring $\operatorname{gr} R_{\mathcal{A}} \simeq \mathbb{Q}[t] \otimes_{\mathbb{Q}} \bar{R}$ (II,2.0) with respect to the symbols $\bar{e}_{w_i}$ of Ore sets $e_{w_i} \subset R_{\mathcal{A}}$). Note that $\bar{e}_{w_i} \subset \operatorname{gr}_0 R_{\mathcal{A}} \simeq \bar{R}$. Denote $\operatorname{gr}_0 M = \bar{M}$. Since M is a free $\mathcal{A}$-module, $\operatorname{gr} M \simeq \mathbb{Q}[t] \otimes_{\mathbb{Q}} \bar{M}$. We get that $\check{C}(\operatorname{gr} M) = \mathbb{Q}[t] \otimes_{\mathbb{Q}} \check{C}(\bar{M})$, where $\check{C}(\bar{M})$ is the Čech complex of $\bar{M}$ considered as an $\bar{R}$ module. In particular, the residue complex $\operatorname{gr}\check{C}(M)/(q-1)$ is isomorphic to $\check{C}(\bar{M})$. This last complex is P graded and for $\lambda \in P$ the subcomplex $\check{C}(\bar{M})_\lambda$ computes the cohomology of the coherent sheaf $\bar{\mathcal{F}}$ on the flag variety $\bar{X}$, which corresponds to the graded $\bar{R}$-module $\bar{M}[\lambda]$. In particular, for $\lambda \gg 0$ we have

$$\bar{M}_\lambda \simeq \check{H}^0(\bar{M}[\lambda])$$

and

$$\check{H}^i(\bar{M}[\lambda]) = 0, \quad i > 0.$$

4.1.4. Remark. Clearly, the $\mathcal{A}$-module $\check{C}(M)$ has no torsion. In particular, the $(q-1)$-adic filtration on $h^0(\check{C}(M))$ coincides with the one induced from $\check{C}^0(M)$. Also $\bigcap (q-1)^n \check{C}(M) = 0$.

4.1.5. Lemma. *The $\mathcal{A}$-module $\check{C}^0(M)$ is free and the natural map $M \to \check{C}^0(M)$ is a split injection of free $\mathcal{A}$-modules.*

4.1.6. Corollary. *The map $\alpha : M \to h^0(\check{C}(M))$ is a split injection of free $\mathcal{A}$-modules.*

Proof of Lemma 4.1.5. Fix $w \in W$. It suffices to prove that the natural map $M \to e_w^{-1}M$ is a split injection of free $\mathcal{A}$-modules. Since

$$e_w^{-1}M = \oplus_\mu (e_w^{-1}M)_\mu,$$

it suffices to show that the map $M_\mu \to (e_w^{-1}M)_\mu$ is such. Fix $\mu \in P$. For $\nu \in P$ denote by $e_\nu^{-1}M(\mu+\nu)$ the span of fractions $(e_{w\nu}, x)$, $\quad x \in M_{\mu+\nu}$. The multiplication by $e_{w\nu'}$ defines a natural embedding

$$e_{w\nu'} : e_{w\nu}^{-1}M_{\mu+\nu} \hookrightarrow e_{w(\nu+\nu')}^{-1}M_{\mu+\nu+\nu'},$$

which is a split embedding of free $\mathcal{A}$-modules. Notice that these embeddings identify

$$(e_w^{-1}M)_\mu = \lim_{\nu} e_{w\nu}^{-1}M_{\mu+\nu}.$$

Hence $(e_w^{-1}M)_\mu$ is a free $\mathcal{A}$-module and the embedding $M[\mu] \to (e_w^{-1}M)_\mu$ is split. This proves the lemma.

4.1.7. Lemma. *Let $\lambda \gg 0$. Then $\alpha : M_\lambda \to \check{H}^0(M[\lambda])$ is an isomorphism.*

Proof. Consider the diagram of maps

$$\begin{array}{ccc} M_\lambda & \xrightarrow{\alpha} & \check{H}^0(M[\lambda]) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ \bar{M}_\lambda & \xrightarrow{\beta} & \check{H}^0(\bar{M}[\lambda]) \end{array}$$

where α and β are the natural maps and β is an isomorphism (4.1.3). The maps s, t are the residue maps modulo $q-1$. It follows from 4.1.3,4 that this diagram is commutative. By definition s is surjective. Hence α is a map of $\mathcal{A}$-modules, which induces a surjection of the residues. By Lemma 4.1.5, α is a split injection of free $\mathcal{A}$-modules. Hence α is an isomorphism. This proves the lemma.

4.1.8. Lemma. *Let $i > 0$, $\lambda \gg 0$. Then the $\mathcal{A}$-module $\check{H}^i(M[\lambda])$ is in fact a k-module.*

Proof. It suffices to prove that $\check{H}^i(M[\lambda])$ has no $(q-1)$-torsion and is $(q-1)$-divisible. Recall (4.1.3) that the residue complex $\check{C}(M)_\lambda/(q-1) = \check{C}(\bar{M})_\lambda$ has no cohomology in positive degrees if $\lambda \gg 0$. For $a \in \check{C}(M)$ we will denote by $\bar{a}$ its image in $\check{C}(\bar{M})$.

We first prove that $\check{H}^i(M[\lambda])$ is divisible. Let $a \in \check{C}^i(M)_\lambda$, $d_{i+1}(a) = 0$. By Remark 4.1.4 we may assume that $0 \neq \bar{a} \in \check{C}^i(\bar{M})$. Hence $a = d_i(b) + (q-1)c$ for some b and c. This proves that the $\mathcal{A}$-module $\check{H}^i(M)[\lambda]$ consists of elements infinitely divisible by $(q-1)$.

Let $a \in \check{C}^i(M)_\lambda$, $d_{i+1}(a) = 0$, and $(q-1)a = d_i(b)$. If $b = (q-1)e$ then $a = d_i e$. So we may assume that the residue $\bar{b}$ of b in $\check{C}^{i-1}(\bar{R})$ is not zero. Here we consider two cases

1) $i > 1$. In this case $b = d_{i-1}c + (q-1)f$. Hence $d_i b = d_i(q-1)f$. Therefore $a = d_i f$.

2) $i = 0$. Since $\bar{M}_\lambda \simeq \check{H}^0(\bar{M}[\lambda])$ there exists $g \in \check{C}^0(M)_\lambda$ such that $\bar{g} = \bar{b}$ and $d_1 g = 0$. It follows that $g - b = (q-1)s$ and hence again $d_1 s = a$.

This shows that $\check{H}^i(M)[\lambda]$ has no $(q-1)$-torsion and completes the proof of the lemma.

The last two lemmas prove Proposition 4.1.

4.1.9. Remark. The argument in the proof of Lemma 4.1.7 also proves the following assertion:

$$R(\lambda) \xrightarrow{\sim} \check{H}^0(R[\lambda]) \quad \text{for} \quad \lambda \in P_+$$

and

$$\check{H}^0(R[\lambda]) = 0 \quad for \ \ \lambda \notin P_+.$$

4.2. Recall that we have an exact sequence

$$0 \to M \to T \to N \to 0$$

in $\mathrm{Mod}^f - R_{\mathcal{A}}$. We may and will assume that $N^{\mathrm{tor}} = 0$.

4.2.1. Corollary. *For $\lambda \gg 0$ the image of the map $\delta : \check{H}^0(N[\lambda]) \to \check{H}^1(M[\lambda])$ is a k-module.*

Proof. We have an exact sequence of $\mathcal{A}$-modules

$$\check{H}^0(N[\lambda]) \xrightarrow{\delta} \check{H}^1(M[\lambda]) \to \check{H}^1(T[\lambda]),$$

where the last two terms are k-modules by Proposition 4.1. This implies that $\mathrm{im}\,(\delta)$ is also a k-module.

4.3. Consider the map of graded $R_{\mathcal{A}}$-modules

$$\tilde{p} : h^0(\check{C}(T)) \to h^0(\check{C}(N))$$

induced by the surjection $p : T \to N$. For $\lambda \gg 0$ we have $T_\lambda = \check{H}^0(T[\lambda])$ (4.1) and hence $(\mathrm{Im}\tilde{p})_\lambda$ is a finite $\mathcal{A}$-submodule of $h^0(\check{C}(N))_\lambda$. Consider the graded $R_{\mathcal{A}}$-module $C := \mathrm{Coker}\tilde{p}$. By Corollary 4.2.1, C_λ is a k-module for $\lambda \gg 0$. Therefore the next lemma proves Theorem 4.

4.3.1. Lemma. *Let $\mu \in P$ be such that C_μ is a k-module. Then $C_\mu = 0$.*

Proof. Assume $C_\mu \neq 0$. Fix $0 \neq m_1 \in C_\mu$ and let $C_{m_1} := km_1 \subset C_\mu$. Let $m \in h^0(\check{C}(N))_\mu$ be the preimage of m_1.

Since $h^0(\check{C}(N))$ is an $R_{\mathcal{A}}$-submodule of $\oplus_{w \in W} e_w^{-1} N$, there exists $w \in W$ such that $e_{w\nu} m \neq 0$ for all $\nu \in P_+$. Fix one such w. Recall that $C = C^{\mathrm{tor}}$ is a torsion $R_{\mathcal{A}}$-module (3.9 Remark 2). Thus $e_{w\nu} m \in (\mathrm{im}\,\tilde{p})_{\mu+\nu}$ for $\nu \gg 0$. But $\mathrm{im}\,\tilde{p}_\lambda$ is a finite $\mathcal{A}$-module for $\lambda \gg 0$. It follows that the image of the multiplication map

$$e_{w\nu} : C_{m_1} \to C_{\mu+\nu}$$

is a nonzero torsion $\mathcal{A}$-submodule for $\nu \gg 0$. In particular the $\mathcal{A}$-module $C_{\mu+\nu}$ has nonzero torsion for all $\nu \gg 0$. This contradicts the fact that C_λ is a k-module for $\lambda \gg 0$ and thus proves the lemma.

IV. D-modules

1. The q-differential operators $D = D_q(R)$

1.0. Let us recall some facts about quantum (or q-)differential operators from [D-cal]. Let Γ be an abelian group with a bicharacter $\beta : \Gamma \times \Gamma \to K^*$. Let $B =$

$\oplus_{a\in\Gamma} B_a$ be a Γ-graded K-algebra. In this situation we defined the ring of quantum differential operators $D_q(B) \subset \mathrm{End}_K(B)$ with the canonical homomorphism of algebras $B \to D_q(B)$ (B acts by left multiplication on B). This is a Γ-graded K-algebra with the canonical filtration

$$D_q^0(B) \subset D_q^1(B) \subset \cdots = D_q(B)$$

by the "order" of differential operator. The subalgebra $D_q^0(B)$ is generated by left and right multiplication by elements of B and the "grading" action σ of Γ defined by

$$\sigma(a)|_{B_b} = \beta(a,b) \cdot id_{B_b}, \quad a, b \in \Gamma.$$

1.0.1. Assume that B is a domain. Let $S \subset B$ be a left and right Ore set consisting of Γ-homogeneous elements, and let $[S^{-1}]B$ be the corresponding Γ-graded localization. Then the following holds.

1) The $D_q(B)$-action extends canonically to $[S^{-1}]B$ via a natural homomorphism of K-algebras

$$D_q(B) \to D_q([S^{-1}]B).$$

2) Given a graded left $D_q(B)$-module M, consider it as a B-module. Let $[S^{-1}]M$ be the localization of M with respect to the Ore set S. Then $[S^{-1}]M$ is also a $D_q(B)$-module and the natural map $M \to [S^{-1}]M$ is a map of $D_q(B)$-modules.

1.1. Consider the root lattice Q with the bicharacter

$$\beta' : Q \times Q \to K^*, \quad (a,b) \mapsto q^{(a|b)}.$$

(I,2.1). Recall the isomorphism of K-algebras $\tau : U^0 \to K[Q]$. Notice that the quantum group U is Q-graded with

$$\deg(K_i) = 0, \quad \deg(E_i) = \alpha_i, \quad \deg(F_i) = -\alpha_i,$$

and the corresponding grading action $\sigma : Q \to \mathrm{Aut}(U)$ is the adjoint action of U^0 on U:

$$\sigma(a)u = \tau^{-1}(a)u(\tau^{-1}(a))^{-1}, \quad a \in Q, u \in U.$$

Assume that Q is a subgroup of an abelian group Γ and β' is the restriction to $Q \times Q$ of a bicharacter $\beta : \Gamma \times \Gamma :\to K^*$. Since U is Q-graded, it is also Γ-graded. Let B be a Γ-graded K-algebra as in 1.0. Assume that U acts on B as a Hopf algebra (*i.e.*, B is a U-ring):

$$u(ab) = u_{(1)}au_{(2)}b,$$

where $a, b \in B$, $\Delta u = u_{(1)} \otimes u_{(2)} \in U$. Denote this action by $\rho : U \to \mathrm{End}_K(B)$. Assume that this action is compatible with the grading, *i.e.*,

$$\sigma|_Q = \rho \cdot \tau^{-1} : Q \to \mathrm{Aut}(B).$$

Under these assumptions we established the following facts in [D-cal].

1. U acts on B by q-differential operators, *i.e.*, we have the algebra homomorphism

$$U \to D_q(B).$$

Let $u \in U$, $a \in B$. Considering a, u as elements in $D_q(B)$ we have the relation

$$ua = u_{(1)}(a)u_{(2)}.$$

2. Assume that B is a domain and let $B \to [S^{-1}]B$ be an Ore localization as in 1.0.1. Then the composition of the canonical homomorphisms

$$U \to D_q(B) \to D_q([S^{-1}]B)$$

makes $[S^{-1}]B$ a U-ring. In particular, for $u \in U$, $a \in [S^{-1}]B$ considered as elements of $D_q([S^{-1}]B)$, we have

$$ua = u_{(1)}(a)u_{(2)}.$$

1.2. We apply this theory to our main example $B = R$. Namely, let $\Gamma = P \times P$ (P — the weight lattice) with the bicharacter

$$\beta((a,b),(c,d)) = q^{(a|c)+(b|d)}.$$

The algebra R is naturally Γ-graded. Namely, let v_μ be a weight vector of weight μ in $R(\lambda)$. Then v_μ has degree $(\mu, \lambda) \in \Gamma$. The corresponding grading action of Γ on R is defined by

$$\sigma((a,b))v_\mu = q^{(a|\mu)+(b|\lambda)}v_\mu.$$

Denote the corresponding q-differential operators $D_q(R)$ simply by D.

Consider Q as a subgroup of Γ by $Q = (Q, 0) \subset \Gamma$. We know that U acts on R as a Hopf algebra and notice that this action is compatible with the grading (1.1). It follows that we have an algebra homomorphism

$$U \to D.$$

1.2.1. Consider the action of the group algebra $K[P]$ on R given by the grading action of $(0, P) \subset \Gamma$. This action is compatible with the action of the center $Z \subset U$ via the projection $p : Z \to U^0$ corresponding to the triangular decomposition $U = U^- U^0 U^+$ (I,3). Namely, consider $K[P]$ as a Z-module via the projection

$$Z \xrightarrow{p} U^0 \simeq K[Q] \subset K[P].$$

Then the ring homomorphism

$$U \otimes_K K[P] \to D$$

factors through

$$U \otimes_Z K[P] \to D.$$

From now on when we refer to R (or D) as a graded ring, we will always mean its usual graded structure (with respect to the group $(0, P)$):

$$R = \oplus_{\lambda \in P_+} R(\lambda).$$

The meaning of a graded R-module is therefore the same as in Section III above. In particular, D is a graded left R-module and we have the canonical homomorphism

$$U \otimes_Z K[P] \to D_0.$$

2. Category $\operatorname{Proj} D$

2.1. Let Mod $- D$ be the abelian category of graded left D-modules. We call a D-module M a torsion module if M is such when considered as a left R-module. Let Tor be the full subcategory of Mod $- D$ consisting of torsion D-modules. We put

$$\operatorname{Proj} D := \text{Mod } - D/\text{Tor}.$$

2.2. Let $M \in \operatorname{Ob} \operatorname{Proj} D$, $\lambda \in P$. Define $M[\lambda] \in \operatorname{Ob} \operatorname{Proj} D$ by the formula

$$M[\lambda]_\mu := M_{\lambda+\mu}$$

with the same D-module structure. This defines an action of P by auto-equivalences of the category $\operatorname{Proj} D$.

3. Cohomology of D-modules

3.0. Let $\psi^* : \mathrm{Mod} - D \to \mathrm{Proj}\, D$ be the exact localization functor and let $\psi_* : \mathrm{Proj}\, D \to \mathrm{Mod} - D$ be its fully faithful right adjoint ([GZ]). The functor ψ_* is left exact. Let Σ_D (resp. Σ_R) be the class of morphisms s in $\mathrm{Mod} - D$ (resp. in $\mathrm{Mod} - R$), such that $\psi^*(s)$ (resp. $\phi^*(s)$) is invertible.

Denote by $F_* : \mathrm{Mod} - D \to \mathrm{Mod} - R$ the functor of restriction of scalars. By definition, a morphism t in $\mathrm{Mod} - D$ is in Σ_D if and only if $F_*(t)$ is in Σ_R. Hence there exists a (unique) functor $\tilde{F}_* : \mathrm{Proj}\, D \to \mathrm{Proj}\, R$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{Mod} - D & \xrightarrow{F_*} & \mathrm{Mod} - R \\ {\scriptstyle \psi^*}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ \mathrm{Proj}\, D & \xrightarrow{\tilde{F}_*} & \mathrm{Proj}\, R\,. \end{array}$$

Consider the functor $F^* := D \otimes_R \cdot : \mathrm{Mod} - R \to \mathrm{Mod} - D$, which is the left adjoint of F_*.

3.1. Proposition.

a) *There exists a (unique) functor* $\tilde{F}^* : \mathrm{Proj}\, R \to \mathrm{Proj}\, D$ *which makes the following diagram commutative*

$$\begin{array}{ccc} \mathrm{Mod} - D & \xleftarrow[F^*]{} & \mathrm{Mod} - R \\ {\scriptstyle \psi^*}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ \mathrm{Proj}\, D & \xleftarrow{\tilde{F}^*} & \mathrm{Proj}\, R\,. \end{array}$$

b) *The functor* $\tilde{F}^*$ *is the left adjoint to* $\tilde{F}_*$.

Proof. a) It suffices to show that $F^*(\Sigma_R) \subset \Sigma_D$ or, equivalently, that $F_* F^*(\Sigma_R) \subset \Sigma_R$. Recall the localization functors

$$g_i^* : \mathrm{Mod} - R \to \mathrm{Mod} - e_{w_i}^{-1} R$$

from Section III,3.0. Let Σ_i be the class of morphisms s in $\mathrm{Mod} - R$ such that $g_i^*(s)$ is invertible. It follows from Lemma 3.3 in Section III, that $\Sigma_R = \cap \Sigma_i$. Hence it suffices to prove that $F_* F^*(\Sigma_i) \subset \Sigma_i$. But this follows from Proposition 4.2.1 in [D-cal].

b) Note that since

$$\phi_* \tilde{F}_* \simeq F_* \psi_*, \quad \tilde{F}^* \phi^* \simeq \psi^* F^*$$

the functors $\tilde{F}^*\phi^*$ and $\phi_*\tilde{F}_*$ are adjoint. Hence there is an isomorphism of bifunctors

$$\mathrm{Hom}_{\,\mathrm{Mod}-R}(-,\phi_*\tilde{F}_*(-)) \simeq \mathrm{Hom}_{\,\mathrm{Proj}\,D}(\tilde{F}^*\phi^*(-),-) \qquad (1)$$

from $\mathrm{Mod}-R \times \mathrm{Proj}\,D$ to $K-\mathrm{mod}$. But $\phi^*\phi_* \simeq \mathrm{Id}$, hence the RHS in (1) is isomorphic to $\mathrm{Hom}_{\,\mathrm{Proj}\,D}(\tilde{F}^*(-),-)$. Since ϕ_* is fully faithful, the LHS in (1) is isomorphic to $\mathrm{Hom}_{\,\mathrm{Proj}\,R}(-,\tilde{F}_*(-))$.

3.2. Corollary. *We have* $F_*\psi_* \simeq \phi_*\tilde{F}_*$.

Proof. Indeed, we have the adjoint pairs $(\psi^*F^*, F_*\psi_*)$, $(\tilde{F}^*\phi^*, \phi_*\tilde{F}_*)$ (Prop. 3.1,b), and $\psi^*F^* \simeq \tilde{F}^*\phi^*$.

3.1. Lemma. *The category* $\mathrm{Proj}\,D$ *has enough injectives.*

Proof. Same as that of Lemma 3.6 in Section III.

3.2. Definition. Consider the zero degree part D_0 of the graded ring D and denote by $\mathrm{mod}-D_0$ the category of left D_0-modules. Consider the left exact functor $\psi_* : \mathrm{Proj}\,D \to \mathrm{Mod}-D$. The last lemma allows us to define its right derived functors $R^i\psi_*$, $i \geq 0$. For $M \in \mathrm{Ob}\,\mathrm{Proj}\,D$ we put

$$h^i(M) := R^i\psi_*(M) \in \mathrm{Ob}\,\mathrm{Mod}-D$$
$$H^i(M) := h^i(M)_0 \in \mathrm{mod}-D_0.$$

In particular, $\psi_*(M) = h^0(M)$.

3.3. Remark. Fix $M \in \mathrm{Ob}\,\mathrm{Proj}\,D$, and consider $\tilde{F}_*M \in \mathrm{Ob}\,\mathrm{Proj}\,R$ (Prop. 3.1). Then Corollary 3.2 implies that $h^i(M)$ considered as a graded R-module is isomorphic to $h^i(\tilde{F}_*M)$ (Section III,3.7). In particular, $H^i(M)$ is isomorphic to $H^i(\tilde{F}_*M)$ as a K-vector space.

3.4. Definition. Let $M \in \mathrm{Ob}\,\mathrm{Proj}\,D$. Consider M as an R-module. For $w \in W$ consider the localization $e_w^{-1}M = e_w^{-1}R \otimes_R M$. This is a graded $e_w^{-1}R$-module and also a D-module; the natural morphism $M \to e_w^{-1}M$ is a morphism of D-modules (1.0.1,2). This shows that the standard complex $M \to S^\cdot(M)$ (3.2,3.4) is a complex in the category $\mathrm{Proj}\,D$. Hence the complex $\psi_*S^\cdot(M)$ lies in $\mathrm{Mod}-D$. We call it the *Čech complex* of M and denote it by $\check{C}(M)$. We put

$$\check{h}^i(M) := H^i(\check{C}(M)) \in \mathrm{Ob}\,\mathrm{Mod}-D$$

and

$$\check{H}^i(M) := \check{h}^i(M)_0 \in \mathrm{Ob}\,\mathrm{mod}-D_0.$$

We call $\check{H}^i(M)$ the i-th Čech cohomology of M. Note that the collection $\{\check{h}^i\}$ (resp. $\{\check{H}^i\}$) is a δ-functor from $\mathrm{Proj}\,D$ to $\mathrm{Mod}-D$ (resp. to $\mathrm{mod}-D_0$). Also

$$\check{H}^i(M[\lambda]) = \check{h}^i(M)_\lambda.$$

3.5. Remark. Fix $M \in \operatorname{Ob}\operatorname{Proj} D$, and consider $\tilde{F}_* \in \operatorname{Ob}\operatorname{Proj} R$ (Prop. 3.1). Then we have an isomorphism of graded R-modules $F_*\check{h}^i(M) \simeq \check{h}^i(\tilde{F}_*(M))$. In particular, $\check{H}^i(M)$ is isomorphic to $\check{H}^i(\tilde{F}_*M)$ as a K-vector space.

3.6. Proposition. *The functors h^i and $\check{h}^i$ from* $\operatorname{Proj} D$ *to* $\operatorname{Mod} - D$ *are isomorphic. In particular, the functors H^i and $\check{H}^i$ from* $\operatorname{Proj} D$ *to* $\operatorname{mod} - D_0$ *are isomorphic.*

Proof. Fix $M \in \operatorname{Ob}\operatorname{Proj} D$. The standard complex $S^\cdot(M)$ is a resolution of the identity functor $\operatorname{Id}_{\operatorname{Proj} D}$ (Section III, Theorem 3.2). Thus it suffices to show that $h^i(S^j(M)) = 0$ for $i > 0$, $j \geq 0$. But $F_*h^i(S^j(M)) = h^i(\tilde{F}_*(S^j(M)))$ (Remark 3.3) and so the statement follows from the fact that $h^i(\tilde{F}_*(S^j(M))) = h^i(S^j(\tilde{F}_*(M))) = 0$ for $i > 0$, $j \geq 0$ (see the proof of Theorem 3.8 in Section III).

3.7. Remarks.

1. Since ψ_* is left exact, we have $\psi_* = h^0 = \check{h}^0$.
2. For $N \in \operatorname{Ob}\operatorname{Mod} - D$ we will denote

$$h^i(N) := h^i(\psi^* N), \qquad \check{h}^i(N) := \check{h}^i(\psi^* N)$$
$$H^i(N) := H^i(\psi^* N), \qquad \check{H}^i(N) := \check{H}^i(\psi^* N).$$

3. For $M \in \operatorname{Ob}\operatorname{Proj} D$, we also denote $\Gamma(M) := H^0(M)$.

4. The category $\operatorname{Proj}_f D$

Fix $f \in K[P]^*$ (we identify $P = (0, P) \subset \Gamma(D_0)$). Consider the full subcategory $\operatorname{Mod}_f - D \subset \operatorname{Mod} - D$ consisting of objects M such that $K[P]$ acts on M_μ by the character $f + \mu - \rho$ (I,3.1), *i.e.*,

$$\omega_i\big|_{M_\mu} = q^{(\omega_i|\mu-\rho)} f(\omega_i) \cdot \operatorname{id}_{M_\mu}$$

Note, that the center Z of U will act on M_μ by the central character $\chi_{f+\mu}$ (I,3.1).

Put

$$\operatorname{Proj}_f D := \operatorname{Mod}_f - D/\operatorname{Tor}.$$

This is a full subcategory of $\operatorname{Proj} D$. For $\lambda \in P$ the functor $[\lambda] : \operatorname{Proj} D \to \operatorname{Proj} D$ induces an equivalence

$$[\lambda] : \operatorname{Proj}_f D \xrightarrow{\sim} \operatorname{Proj}_{f+\lambda} D.$$

The Čech complex $\check{C}(M)$ of $M \in \operatorname{Ob}\operatorname{Proj}_f D$ also belongs to $\operatorname{Mod}_f - D$, and the functor

$$h^0(\cdot) : \operatorname{Proj}_f D \to \operatorname{Mod}_f - D, \qquad M \mapsto h^0(M)$$

is the right adjoint to the localization functor

$$\psi^* : \mathrm{Mod}_f - D \to \mathrm{Proj}_f D.$$

Note that $K[P]$ lies in the center of the ring D_0 (1.2.1). Let $J_{f-\rho} \subset K[P]$ be the kernel of the homomorphism $f - \rho : K[P] \to K$. Put $D_f := D_0/D_0 J_{f-\rho}$. For $M \in \mathrm{Ob}\,\mathrm{Proj}_f D$, $\Gamma(M)$ is a D_f-module.

5. The localization functor $\mathcal{L}$

5.0. Given a left D_0-module N, consider the left graded D-module $D \otimes_{D_0} N$. This defines a right exact localization functor

$$\mathcal{L} : \mathrm{Mod} - D_0 \to \mathrm{Proj}\, D, \qquad \mathcal{L}(N) := \psi^*(D \otimes_{D_0} N).$$

Vice versa, given $M \in \mathrm{Ob}\,\mathrm{Proj}\, D$, its global sections $\Gamma(M)$ is a D_0-module. This defines a left exact functor

$$\Gamma : \mathrm{Proj}\, D \to \mathrm{Mod} - D_0.$$

Remark. $(\mathcal{L}, \Gamma)$ is an adjoint pair.

5.1. The functors $\mathcal{L}$ and Γ above induce a pair of adjoint functors

$$\mathcal{L} : \mathrm{Mod} - D_f \to \mathrm{Proj}_f D$$

and

$$\Gamma : \mathrm{Proj}_f D \to \mathrm{Mod} - D_f.$$

5.2. Theorem. *Assume that the character f is dominant* (I,3.2). *Then the functor*

$$\Gamma : \mathrm{Proj}_f D \to \mathrm{Mod} - D_f$$

is exact.

5.3. Conjectures.

1. The natural map $D_0 \to \Gamma(D)$ is an isomorphism.
2. Let $\check{U}$ be the “simply connected” version of the quantum group U, that is, we replace $U^0 \simeq k[Q]$ by the bigger algebra $k[P]$. Then $\check{U} \otimes_Z K[P] \simeq D_0$.

   Let $f \in K[P]^*$. Then we expect the following.
3. The natural map $D_f \to \Gamma(D/DJ_{f-\rho})$ is an isomorphism.
4. Let $I_f \subset \check{U}$ be the annihilator of the Verma module $M(f - \rho)$. Then $\check{U}/I_f \simeq D_f$.
5. Assume that f is regular dominant. Then $\Gamma(M) \neq 0$ if $0 \neq M \in \mathrm{Ob}\,\mathrm{Proj}_f D$.

We immediately obtain the following standard corollary.

5.4. Corollary. *Assume that the conjectures* 3 *and* 5 *above holds. Let* $f \in K[P]^*$ *be regular dominant. Then the functors*

$$\mathcal{L} : \text{Mod} - D_f \to \text{Proj}_f D$$

and

$$\Gamma : \text{Proj}_f D \to \text{Mod} - D_f$$

are inverse equivalences of categories.

6. Proof of Theorem 5.2

6.1. The R-bimodule F^μ.

Fix $\mu \in P_+$ and consider the finite dimensional U-module $R(\mu)$. Consider the free graded right R-module

$$F^\mu := R(\mu) \underset{K}{\otimes} R.$$

Let us define the structure of a left R-module on F^μ as follows. The $\mathcal{R}$-matrix (I,4) defines an isomorphism of U-modules

$$\mathcal{R} = \mathcal{R}_{R,R(\mu)} : R \otimes R(\mu) \xrightarrow{\sim} R(\mu) \otimes R.$$

Let $m : R \otimes R \to R$ denote the multiplication in the ring R. Define the left multiplication

$$R \otimes F^\mu \to F^\mu$$

as the composition

$$R \otimes R(\mu) \otimes R \xrightarrow{\mathcal{R} \otimes \text{id}} R(\mu) \otimes R \otimes R \xrightarrow{\text{id} \otimes m} R(\mu) \otimes R.$$

The property of the $\mathcal{R}$-matrix discussed in I,4 ensures that this is indeed a left R-module structure on F^μ. The left and right R-multiplications obviously commute; hence F^μ is an R-bimodule.

6.1.1. Remark. Notice that F^μ as a left R-module is also free. Indeed, the R-matrix defines an isomorphism

$$\mathcal{R}^{-1}_{R,R(\mu)} : R(\mu) \otimes R \xrightarrow{\sim} R \otimes R(\mu)$$

of the left R-module $F^\mu = R(\mu) \otimes R$ with the free left R-module $R \otimes R(\mu)$.

6.1.2. Remark. Notice that F^μ has a natural structure of a U-module:

$$u(a \otimes b) := u_{(1)}a \otimes u_{(2)}b \qquad u \in U.$$

The left and right R-multiplications on F^μ are compatible with this U-module structure on F^μ. Namely,

$$u(rf) = u_{(1)}(r)u_{(2)}(f), \qquad u(fr) = u_{(1)}(f)u_{(2)}(r),$$

where $u \in U$, $r \in R$, $f \in F^\mu$.

6.2 The filtration of F^μ

6.2.0. Let $\nu_1, \dots, \nu_k \in P_+$ be all *different* weights of $R(\mu)$ in the order, which is compatible with the natural partial order on P determined by Q_+. In particular, $\nu_k = \mu$ and ν_1 is the lowest weight of $R(\mu)$. Let m_i be the multiplicity of ν_i in $R(\mu)$. It follows from the Weyl character formula that for, $\lambda \gg 0$, there is an isomorphism of U-modules

$$R(\mu) \otimes_K R(\lambda) \simeq \oplus_i(\oplus_{m_i} R(\lambda + \nu_i)).$$

Thus for $\lambda \gg 0$

$$F^\mu_\lambda \simeq \oplus_i(\oplus_{m_i} R(\lambda + \nu_i))$$

as U-modules. For every $i = 1, \dots, k$, define a graded U-submodule $F^i \subset F^\mu$ by

$$F^i_\lambda = \oplus_{j \le i}(\oplus_{m_i} R(\lambda + \nu_j)) \subset F^\mu_\lambda, \quad \text{if} \quad \lambda \gg 0 \quad \text{and} \quad F^i_\lambda = 0 \quad \text{otherwise.}$$

It follows from Remark 6.1.2 above that F^i is an R-subbimodule of F^μ.

6.2.1. Proposition. *Fix $1 \le i \le k$. Consider the R-bimodule $R[\nu_i] \otimes_K K^{m_i} = \oplus_{m_i} R[\nu_i]$.*

a) *There exists an isomorphism (in* Proj *)*

$$\alpha : R[\nu_i] \otimes_K K^{m_i} \xrightarrow{\sim} F_i/F_{i-1}$$

of graded right R-modules. This is an isomorphism of U-modules.

b) *Under this isomorphism α the left R-module structures on F_i/F_{i-1} and $R[\nu_i] \otimes_K K^{m_i}$ compare as follows. There exists a representation $\sigma : P \to Gl(K^{m_i})$ of the abelian group P such that*

$$r\alpha(f \otimes v) = \alpha(rf \otimes \sigma(\nu)v),$$

where $r \in R(\nu)$, $f \otimes v \in R[\nu_i] \otimes K^{m_i}$.

Proof. a). For $\lambda \gg 0$ U-modules $(F_i/F_{i-1})_\lambda$ and $R(\lambda + \nu_i) \otimes K^{m_i}$ are isomorphic. Denote by $H_\lambda \subset (F_i/F_{i-1})_\lambda$, $L_\lambda \subset R[\nu_i]_\lambda \otimes K^{m_i}$, the K-subspaces of highest weight vectors, *i.e.*, vectors of weight $\lambda + \nu_i$. Recall that for each $\nu \in P_+$ we have a choice of a highest weight vector $e_\nu \in R(\nu)$, such that $e_{\nu_1} e_{\nu_2} = e_{\nu_1 + \nu_2}$ in R. Clearly, the right multiplication map

$$L_\lambda \otimes e_\nu \to L_{\lambda + \nu}$$

is an isomorphism. Since the right R-module F^μ is free, therefore F^i is torsion free, and so the right multiplication map

$$H_\lambda \otimes e_\nu \to H_{\lambda + \nu}$$

is also an isomorphism. Fix $\lambda_0 \gg 0$ and choose a linear isomorphism

$$\beta_{\lambda_0} : L_{\lambda_0} \xrightarrow{\sim} H_{\lambda_0}.$$

There exists a unique collection of linear isomorphisms

$$\beta_\lambda : L_\lambda \xrightarrow{\sim} H_\lambda, \qquad \lambda \geq \lambda_0,$$

such that β_λ's commute with the right multiplication by e_ν for each $\nu \in P_+$.

The isomorphism β_λ extends to a unique isomorphism of U-modules

$$\alpha_\lambda : R[\nu_i]_\lambda \otimes K^{m_i} \xrightarrow{\sim} (F_i/F_{i-1})_\lambda, \qquad \lambda \geq \lambda_0.$$

The morphism of right multiplication

$$(F_i/F_{i-1})_\lambda \otimes R(\lambda) \to (F_i/F_{i-1})_{\lambda+\nu}$$

is completely determined by the induced isomorphism

$$H_\lambda \otimes e_\nu \to H_{\lambda+\nu}.$$

Similarly for $R[\nu_i]_\lambda \otimes K^{m_i}$. It follows that the isomorphism (in Proj) of U-modules

$$\alpha := \oplus_{\lambda \geq \lambda_0} \alpha_\lambda : R[\nu_i] \otimes K^{m_i} \to F_i/F_{i-1}$$

is an isomorphism of right R-modules. This proves a).

b). This is proved using similar arguments plus the fact that right and left R-multiplications commute.

6.3. Fix $M \in \mathrm{Ob}\,\mathrm{Proj}_f D$

6.3.0. Consider the left R-module $F^\mu \otimes_R M$. It has a natural U-action which is compatible with the R-module structure. It follows that this U-action extends to any localization of this R-module with respect to an Ore set in R (via the comultiplication in U). In particular the Čech complex $\check{C}(F^\mu \otimes_R M)$ has a natural U-action.

6.3.1. Note that the R-module $F^\mu \otimes_R M$ is filtered by the bimodule filtration $\{F^i\}$ of F^μ (6.2.0). We put

$$M^i := F^i/F^{i-1} \otimes_R M.$$

The isomorphism α from Proposition 6.2.1,a) induces an isomorphism

$$\tau : M^i \xrightarrow{\sim} M[\nu_i] \otimes K^{m_i}$$

of U-modules. Recall the representation $\sigma : P \to Gl(K^{m_i})$ (Proposition 6.2.1,b)) and consider the automorphism

$$\delta : M[\nu_i] \otimes K^{m_i} \xrightarrow{\sim} M[\nu_i] \otimes K^{m_i}$$

defined by

$$\delta(m \otimes v) = m \otimes \sigma(\lambda)^{-1} v$$

for $m \in M[\nu_i]_\lambda$. Then by Proposition 6.2.1,b) the isomorphism of U-modules

$$\delta \cdot \tau : M^i \xrightarrow{\sim} M[\nu_i] \otimes K^{m_i}$$

is an isomorphism of left R-modules. It follows that the U- and the R-module structures on M^i extend to a D-module structure. Moreover, $M^i \in \mathrm{Ob}\,\mathrm{Proj}_{f+\nu_i} D$. Hence the Čech complex $\check{C}(M^i)$ also belongs to $\mathrm{Proj}_{f+\nu_i} D$ (4). This implies the following corollary.

6.3.2. Corollary. *Consider the Čech complex $\check{C}(F^\mu \otimes_R M)$ with the filtration by the subcomplexes $\check{C}(F^i \otimes M)$. Consider the natural U-action on this filtered complex* (6.3.0,1) *and in particular on its zero graded component $\check{C}(F^\mu \otimes M)_0$. Then the center Z of U acts on $\check{C}(F^i \otimes M/F^{i-1} \otimes M)_0 \simeq \check{C}(M^i)_0$ by the central character $\chi_{f+\nu_i}$.*

6.4. We keep the notations of 6.2 and 6.3 above. Notice that $M^1 \simeq M[\nu_1]$ as R- and U-modules (6.3.1), where ν_1 is the lowest weight of $R(\mu)$. Hence we have an embedding of left R-modules $i_\mu : M \hookrightarrow F^\mu \otimes_R M[-\nu_1]$, which induces the corresponding embedding of Čech complexes

$$i_\mu : \check{C}(M) \hookrightarrow \check{C}(F^\mu \otimes_R M[-\nu_1]).$$

Also $M^k \simeq M[\mu]$ (6.3.1). Hence we have a canonical surjection of left R-modules $p_\mu : F^\mu \otimes_R M \to M[\mu]$, which induces the corresponding surjection of Čech complexes

$$p_\mu : \check{C}(F^\mu \otimes_R M) \to \check{C}(M[\mu]).$$

Note that i_μ and p_μ are U-morphisms.

The following two lemmas are immediate consequences of Definition 3.2 in Section I and of the previous Corollary 6.3.2.

6.4.1. Lemma. *Let $f \in K[P]^*$ be dominant and $M \in \mathrm{Od}\,\mathrm{Proj}_f D$. Then the embedding*

$$i_\mu : \check{C}(M)_0 \hookrightarrow \check{C}(F^\mu \otimes_R M[-\nu_1])_0.$$

of complexes has a canonical splitting. Namely, the image of i_μ is the generalized χ_f-eigenspace of $\check{C}(F^\mu \otimes_R M[-\nu_1])_0$ for the Z-action.

6.4.2. Lemma. *Let $f \in K[P]^*$ be dominant and regular, $M \in \mathrm{Ob}\,\mathrm{Proj}_f D$. Then the surjection*

$$p_\mu : \check{C}(F^\mu \otimes_R M)_0 \to \check{C}(M[\mu])_0$$

has a canonical splitting (again given by the eigenspace decomposition of the Z-action).

6.5. *Proof of Theorem* 5.2. Let $f \in K[P]^*$ be dominant. Let

$$M \to N \to 0 \tag{1}$$

be an exact sequence in $\mathrm{Mod}_f - D$. Let $a \in \Gamma(N)$. We will prove that a is in the image of $\Gamma(M)$. We can find an exact subsequence of (1)

$$\overline{M} \to \overline{N} \to 0 \tag{2}$$

of finitely generated R-submodules and such that $a \in \Gamma(\overline{N})$. Consider the finitely generated left R-modules $F^\mu \otimes (\overline{M})$ and $F^\mu \otimes (\overline{N})$ (6.1). By Theorem 4 in Section III there exists $\nu \gg 0$ in P_+ such that the map

$$\Gamma(F^\mu \otimes \overline{M}[\nu]) \to \Gamma(F^\mu \otimes \overline{N}[\nu])$$

is surjective. Consider the commutative diagram

$$\begin{array}{ccc} \Gamma(\overline{M}) & \longrightarrow & \Gamma(\overline{N}) \\ \big\downarrow & & \big\downarrow \\ \Gamma(M) & \xrightarrow{\ \gamma\ } & \Gamma(N)\,. \end{array} \tag{3}$$

Let $\mu \in P_+$ be such that $-\nu$ is the lowest weight in $R(\mu)$. Then by 6.4 the diagram (3) has a canonical embedding i_μ in the commutative square

$$\begin{array}{ccc} \Gamma(F^\mu \otimes \overline{M}[\nu]) & \xrightarrow{\ \alpha\ } & \Gamma(F^\mu \otimes \overline{N}[\nu]) \\ \big\downarrow & & \big\downarrow \\ \Gamma(F^\mu \otimes M[\nu]) & \xrightarrow{\ \beta\ } & \Gamma(F^\mu \otimes N[\nu])\,. \end{array}$$

By our assumption $i_\mu(a) \in \Gamma(F^\mu \otimes \overline{N}[\nu])$ is in the image of α. Hence $i_\mu(a) \in \Gamma(F^\mu \otimes N[\nu])$ is in the image of β. Since the splitting of i_μ in Lemma 6.4.1 is functorial, it follows that that $a \in \mathrm{im}\,(\gamma)$. This proves Theorem 5.2.

6.6. Remark. Let $f \in K[P]^*$ be regular dominant. Let $M \in \mathrm{Ob}\,\mathrm{Proj}_f D$ be such that $\Gamma(M) = 0$. Assume that $M \neq 0$. There exists $\mu \gg 0$ in P_+ such that $\Gamma(M(\mu)) \neq 0$. By Lemma 6.4.2 the surjection $p_\mu : R(\mu) \otimes_K M \to M(\mu)$ splits. Hence $\Gamma(R(\mu) \otimes_K M) \neq 0$. If we could derive from this that $\Gamma(M) \neq 0$, then we would prove conjecture 5 in 5.3.

7. Cohomology of objects in $\mathrm{Proj}_f D$

7.1. Proposition. *Let $f \in K[P]^*$ be dominant, $M \in \mathrm{Proj}_f D$. Then*

$$H^i(M) = 0, \quad \textit{for} \quad i > 0.$$

Proof. The cohomology $H^i(M)$ can be computed using the Čech complex $\check{C}(M)$ (prop. 3.6). But $\check{C}(M)$ lies in the category $\mathrm{Mod}_f - D$ (4). Hence the proposition follows from Theorem 5.2.

7.2. Corollary. *Let $\lambda \in P_+ - \rho$. Then*

$$H^i(R[\lambda]) = 0 \quad \textit{for} \quad i > 0.$$

Proof. Fix $\lambda \in P_+ - \rho$. Let $f \in K[P]^*$ be the corresponding character (Section I, 3.1). By Lemma 3.3 in Section I $f + \rho$ is dominant. The D-module $R[\lambda]$ belongs to $\mathrm{Mod}_{f+\rho} - D$. Hence the corollary follows from Proposition 7.1.

8. Borel-Weil Theorem

Theorem. *Fix $\lambda \in P_+$. Then the U-module $\Gamma(R[\lambda])$ is isomorphic to $R(\lambda)$.*

Proof. See Remark 4.1.9 in Section III.

References

[AM] M.F. Atiyah and I.G. Macdonald. *Introduction to Commutative Algebra.* Addison-Wesley, Reading, Massachusetts, 1969.

[BB] A.A. Beilinson and J. Bernstein. Localisation de $\mathfrak{g}$-modules. *C.R. Acad. Sci. Paris Ser. I Math.* **292** (1981), 15–18.

[BB2] A.A. Beilinson and J. Bernstein. A proof of Jantzen conjectures. *Advances in Soviet Mathematics* **16** part 1, 1-50.

[BD] I. Bucur and A. Deleanu. Introduction to the Theory of Categories and Functors. *Pure and Applied Math.* **XIX**.

[D-cal] V.A. Lunts and A.L. Rosenberg. Differential calculus on noncommutative rings. *Selecta math., New ser.* **3** (1997), 335–359.

[GZ] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory.* Springer-Verlag, 1967.

[H] T.J. Hodges. Ring-theoretical aspects of the Bernstein-Beilinson theorem. *LNM* **1448** (1990), 155–163.

[JL1] A. Joseph and G. Letzter. Local finiteness of the adjoint action for quantized enveloping algebras. *J. Algebra* **153** (1992), 289–318.

[JL2] A. Joseph and G. Letzter. Separation of variables for quantized enveloping algebras. *Amer. J. Math.* **116** (1994), 125–177.

[JL3] A. Joseph and G. Letzter. Verma module annihilators and quantized enveloping algebras. *Ann. Scient. Ec. Norm. Sup.* 4^e *serie, t.28* (1995), 493–526.

[Jo] A. Joseph. Faithfully flat embeddings for minimal primitive quotients of quantized enveloping algebras. Israel Math. Conf. Proc. Vol 7, 1993..

[Lu] G. Lusztig. Quantum groups at roots of 1. *Geometriae Dedicata* **35** (1990), 89–114.

[Ro] A.L. Rosenberg. Noncommutative schemes. *Compositio Mathematica* **112** (1998), 93–125.

[Ta] T. Tanisaki. Killing forms, Harish-Chandra isomorphisms, and universal R-matrices for quantum algebras. *Intern. Journal of Modern Physics A Vol.7, Suppl. 1B* (1992), 941–961.

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