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Trace formula in noncommutative geometry and the zeros of the Riemann zeta function

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Abstract. We give a spectral interpretation of the critical zeros of the Riemann zeta function as an absorption spectrum, while eventual noncritical zeros appear as resonances. We give a geometric interpretation of the explicit formulas of number theory as a trace formula on the noncommutative space of Adele classes. This reduces the Riemann hypothesis to the validity of the trace formula and eliminates the parameter δ of our previous approach.

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Contents

Introduction

We shall give in this paper a spectral interpretation of the zeros of the Riemann zeta function and a geometric framework in which one can transpose the ideas of algebraic geometry involving the action of the Frobenius and the Lefchetz formula. The spectral interpretation of the zeros of zeta will be as an absorption spectrum, i.e., as missing spectral lines. All zeros will play a role in the spectral side of the trace formula, but while the critical zeros will appear perse, the noncritical ones will appear as resonances and enter in the trace formula through their harmonic potential with respect to the critical line. Thus the spectral side is entirely canonical, and by proving positivity of the Weil distribution, we shall show that its equality with the geometric side, i.e., the global trace formula, is equivalent to the Riemann Hypothesis for all L-functions with Grössencharakter.

We shall model our discussion on the Selberg trace formula, but it differs from the latter in several important respects. We shall first explain in particular why a crucial negative sign in the analysis of the statistical fluctuations of the zeros of zeta indicates that the spectral interpretation should be as an absorption spectrum, or equivalently should be of a cohomological nature. As it turns out, the geometric framework involves an innocent looking space, the space X of Adele classes, where two adeles which belong to the same orbit of the action of $GL_1(k)$ (k a global field), are considered equivalent. The group $C_k = GL_1(A)/GL_1(k)$ of Idele classes (which is the class field theory counterpart of the Galois group) acts by multiplication on X.

Our first preliminary result (Theorem 1 of Section III) gives a spectral interpretation of the critical zeros of zeta and L functions on a global field k from the action of the Idele class group on a space of square integrable functions on the space $X = A/k^*$ of Adele classes. Corollary 2 gives the corresponding computation of the spectral trace. This result is only preliminary because it requires the use of an unnatural parameter δ which plays the role of a Sobolev exponent and allows us to see the absorption spectrum as a point spectrum.

Our second preliminary result is a formal computation (Section VI) of the character of the representation of the Idele class group on the above L^2 space. This formal computation gives the Weil distribution which is the essential ingredient of the Riemann-Weil explicit formula. At this point (which was the situation in [Co]), the main problems are to give a rigorous meaning to the formal trace computation and to eliminate the unwanted parameter δ.

These two problems will be solved in the present paper. We first prove a trace formula (Theorem 3 of Section V) for the action of the multiplicative group K^* of a local field K on the Hilbert space $L^2(K)$, and (Theorem 4 of Section VII) a trace formula for the action of the multiplicative group C_S of Idele classes associated to a finite set S of places of a global field k, on the Hilbert space of square integrable functions $L^2(X_S)$, where X_S is the quotient of $\prod_{v\in S} k_v$ by the action of the group O_S^* of S-units of k. In both cases we obtain exactly the terms of the Weil explicit formulas which belong to the finite set of places. This result is quite important since the space X_S is highly nontrivial as soon as the cardinality of S is larger or equal to 3. Indeed this quotient space is nontype I in the sense of Noncommutative Geometry, and it is reassuring that the trace formula continues to hold there.

We check in detail (Theorem 6 of Appendix II) that the rewriting of the Weil explicit formulas which is predicted by the global trace formula is correct.

Finally, we eliminate in Section VIII using ideas that are common both to the Selberg trace formula and to the standard explanation of the absorption lines in physics, the unpleasant parameter δ which appeared as a label of the function spaces of Section III. We write the global trace formula as an analogue of the Selberg trace formula. The validity of the trace formula for any finite set of places follows from Theorem 4 of Section VII, but in the global case is left open and shown (Theorem 5 of Section VIII) to be equivalent to the validity of the Riemann Hypothesis for all L functions with Grössencharakter. This equivalence, together with the plausibility of a direct proof of the trace formula along the lines of Theorem 4 (Section VII) constitute the main result of this paper. The elimination of the parameter δ is the main improvement of the present paper with respect to [Co].

It is an old idea, due to Pólya and Hilbert, that in order to understand the location of the zeros of the Riemann zeta function, one should find a Hilbert space $\mathcal{H}$ and an operator D in $\mathcal{H}$ whose spectrum is given by the nontrivial zeros of the zeta function. The hope then is that suitable selfadjointness properties of D (of $i\left(D-\frac{1}{2}\right)$ more precisely) or positivity properties of $\Delta = D(1-D)$ will be easier to handle than the original conjecture. The main reasons why this idea should be taken seriously are first the work of A. Selberg ([Se]) in which a suitable Laplacian Δ is related in the above way to an analogue of the zeta function, and secondly the theoretical ([M] [B] [KS]) and experimental evidence ([O] [BG]) on the fluctuations of the spacing between consecutive zeros of zeta. The number of zeros of zeta whose imaginary part is less than $E > 0$,

$$N(E) = \# \text{ of zeros } \rho, \quad 0 < \text{Im } \rho < E \tag{1}$$

has an asymptotic expression ([R]) given by

$$N(E) = \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + \frac{7}{8} + o(1) + N_{\text{osc}}(E) \tag{2}$$

where the oscillatory part of this step function is

$$N_{\text{osc}}(E) = \frac{1}{\pi} \text{ Im } \log \zeta \left(\frac{1}{2} + iE\right) \tag{3}$$

assuming that E is not the imaginary part of a zero and taking for the logarithm the branch which is 0 at $+\infty$.

One shows (cf. [Pat]) that $N_{\rm osc}(E)$ is $O(\log E)$. In the decomposition (2) the two terms $\langle N(E)\rangle = N(E) - N_{\rm osc}(E)$ and $N_{\rm osc}(E)$ play an independent role. The first one $\langle N(E)\rangle$ which gives the average density of zeros just comes from Stirling's formula and is perfectly controlled. The second $N_{\rm osc}(E)$ is a manifestation of the randomness of the actual location of the zeros, and to eliminate the role of the density, one returns to the situation of uniform density by the transformation

$$x_j = \langle N(E_j)\rangle \quad (E_j \text{ the } j^{\rm th} \text{ imaginary part of zero of zeta}). \tag{4}$$

Thus the spacing between two consecutive x_j is now 1 on average and the only information that remains is in the statistical fluctuation. As it turns out ([M] [O]) these fluctuations are the same as the fluctuations of the eigenvalues of a random hermitian matrix of very large size.

H. Montgomery [M] proved (assuming RH) a weakening of the following conjecture (with $\alpha, \beta > 0$),

$$\begin{aligned} &\mathrm{Card}\,\Big\{(i,j)\,;\, i,j \in 1,\ldots,M\,;\, x_i - x_j \in [\alpha,\beta]\Big\} \\ &\qquad \sim M \int_\alpha^\beta \left(1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2\right) du. \end{aligned} \tag{5}$$

This law (5) is precisely the same as the correlation between eigenvalues of hermitian matrices of the gaussian unitary ensemble ([M]). Moreover, numerical tests due to A. Odlyzko ([O] [BG]) have confirmed with great precision the behaviour (5) as well as the analogous behaviour for more than two zeros. In [KS], N. Katz and P. Sarnak proved an analogue of the Montgomery-Odlyzko law for zeta and L-functions of function fields over curves.

It is thus an excellent motivation to try and find a natural pair $(\mathcal{H}, D)$ where naturality should mean for instance that one should not even have to define the zeta function, let alone its analytic continuation, in order to obtain the pair (in order for instance to avoid the joke of defining $\mathcal{H}$ as the ℓ^2 space built on the zeros of zeta).

I. Quantum chaos and the hypothetical Riemann flow

Let us first describe following [B] the direct attempt to construct the Pólya-Hilbert space from quantization of a classical dynamical system. The original motivation for the theory of random matrices comes from quantum mechanics. In this theory the quantization of the classical dynamical system given by the phase space X and hamiltonian h gives rise to a Hilbert space $\mathcal{H}$ and a selfadjoint operator H whose spectrum is the essential physical observable of the system. For complicated systems the only useful information about this spectrum is that, while the average part of the counting function,

$$N(E) = \#\text{ eigenvalues of } H \text{ in } [0, E] \tag{1}$$

is computed by a semiclassical approximation mainly as a volume in phase space, the oscillatory part,

$$N_{\text{osc}}(E) = N(E) - \langle N(E) \rangle \tag{2}$$

is the same as for a random matrix, governed by the statistic dictated by the symmetries of the system.

In the absence of a magnetic field, i.e., for a classical hamiltonian of the form,

$$h = \frac{1}{2m}\, p^2 + V(q) \tag{3}$$

where V is a real-valued potential on configuration space, there is a natural symmetry of classical phase space, called time reversal symmetry,

$$T(p, q) = (-p, q) \tag{4}$$

which preserves h, and entails that the correct ensemble on the random matrices is not the above GUE but rather the gaussian orthogonal ensemble: GOE. Thus the oscillatory part $N_{\text{osc}}(E)$ behaves in the same way as for a random *real symmetric* matrix.

Of course H is just a specific operator in $\mathcal{H}$ and, in order that it behave *generically*, it is necessary (cf. [B]) that the classical hamiltonian system (X, h) be *chaotic* with isolated *periodic orbits* whose instability exponents (i.e., the logarithm of the eigenvalues of the Poincaré return map acting on the transverse space to the orbits) are different from 0.

One can then ([B]) write down an asymptotic semiclassical approximation to the oscillatory function $N_{\text{osc}}(E)$

$$N_{\text{osc}}(E) = \frac{1}{\pi}\ \text{Im} \int_0^\infty \text{Trace}\,(H - (E + i\eta))^{-1}\, i\, d\eta \tag{5}$$

using the stationary phase approximation of the corresponding functional integral. For a system whose configuration space is 2-dimensional, this gives ([B] (15)),

$$N_{\text{osc}}(E) \simeq \frac{1}{\pi} \sum_{\gamma_p} \sum_{m=1}^{\infty} \frac{1}{m} \, \frac{1}{2\,\text{sh}\left(\frac{m\lambda_p}{2}\right)}\ \sin(S_{\text{pm}}(E)) \tag{6}$$

where the γ_p are the primitive periodic orbits, the label m corresponds to the number of traversals of this orbit, while the corresponding instability exponents are $\pm\lambda_p$. The phase $S_{\rm pm}(E)$ is up to a constant equal to $m\,E\,T_\gamma^\#$ where $T_\gamma^\#$ is the period of the primitive orbit γ_p.

The formula (6) gives very precious information ([B]) on the hypothetical "Riemann flow" whose quantization should produce the Pólya-Hilbert space. The point is that the Euler product formula for the zeta function yields (cf. [B]) a similar asymptotic formula for $N_{\rm osc}(E)$ (3),

$$N_{\rm osc}(E) \simeq \frac{-1}{\pi} \sum_p \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{p^{m/2}} \sin(m\,E\,\log p)\,. \tag{7}$$

Comparing (6) and (7) gives the following information,

(A) The periodic primitive orbits should be labelled by the prime numbers $p = 2,3,5,7,\ldots$, their periods should be the $\log p$ and their instability exponents $\lambda_p = \pm\log p$.

Moreover, since each orbit is only counted once, the Riemann flow should not possess the symmetry T of (4) whose effect would be to duplicate the count of orbits. This last point excludes in particular the geodesic flows since they have the time reversal symmetry T. Thus we get

(B) The Riemann flow cannot satisfy time reversal symmetry.

However there are two important mismatches (cf. [B]) between the two formulas (6) and (7). The first one is the overall *minus sign* in front of formula (7), the second one is that though $2\,{\rm sh}\left(\frac{m\lambda_p}{2}\right) \sim p^{m/2}$ when $m\to\infty$, we do not have an equality for finite values of m.

These are two fundamental difficulties, and in order to overcome them we shall use the well known strategy of extending the problem to the case of arbitrary *global fields*. By specializing to the function field case, we shall then obtain additional precious information.

II. Algebraic Geometry and global fields of nonzero characteristic

The basic properties of the Riemann zeta function extend to zeta functions associated to an arbitrary global field, and it is unliquely that one can settle the problem of the spectral interpretation of the zeros, let alone find the Riemann flow, for the particular case of the global field $\mathbb{Q}$ of rational numbers without at the same time settling these problems for all global fields. The conceptual definition of such fields k, is the following:

A field k is a global field iff it is discrete *and* cocompact *in a (nondiscrete) locally compact semisimple abelian ring A.*

As it turns out, A then depends functorially on k and is called the Adele ring of k, often denoted by k_A. Thus though the field k itself has no interesting topology, there is a canonical and highly nontrivial topological ring which is canonically associated to k. When the characteristic p of a global field k is > 0, the field k is the function field of a nonsingular algebraic curve Σ defined over a finite field $\mathbb{F}_q$ included in k as its maximal finite subfield, called the field of constants. One can then apply the ideas of algebraic geometry, first developed over $\mathbb{C}$, to the geometry of the curve Σ and obtain a geometric interpretation of the basic properties of the zeta function of k; the dictionary contains in particular the following lines

Spectral interpretation of the zeros	Eigenvalues of action of Frobenius on ℓ-adic cohomology	
Functional equation	Riemann Roch theorem (Poincaré duality)	
		(1)
Explicit formulas of number theory	Lefchetz formula for the Frobenius	
Riemann hypothesis	Castelnuovo positivity	

Since $\mathbb{F}_q$ is not algebraically closed, the points of Σ defined over $\mathbb{F}_q$ do not suffice and one needs to consider $\bar{\Sigma}$, the points of Σ on the algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$, which is obtained by adjoining to $\mathbb{F}_q$ the roots of unity of order prime to q. This set of points is a countable union of periodic orbits under the action of the Frobenius automorphism; these orbits are parametrized by the set of places of k and their periods are indeed given by the analogues of the $\log p$ of (A). Being a countable set it does not qualify as an analogue of the Riemann flow and it only aquires an interesting structure from algebraic geometry. The minus sign which was problematic in the above discussion admits here a beautiful resolution since the analogue of the Pólya-Hilbert space is given, if one replaces $\mathbb{C}$ by $\mathbb{Q}_\ell$ the field of ℓ-adic numbers $\ell \neq p$, by the cohomology group

$$H^1_{\mathrm{et}}\,(\bar{\Sigma}, \mathbb{Q}_\ell) \tag{2}$$

which appears with an overall minus sign in the Lefchetz formula

$$\sum (-1)^j \text{ Trace } \varphi^* / H^j = \sum_{\varphi(x)=x} 1. \tag{3}$$

For the general case this suggests

(C) The Pólya-Hilbert space $\mathcal{H}$ should appear from its negative $\ominus\mathcal{H}$.

In other words, the spectral interpretation of the zeros of the Riemann zeta function should be as an absorption spectrum rather than as an emission spectrum, to borrow the language of spectroscopy.

The next thing that one learns from this excursion in characteristic $p > 0$ is that in that case one is not dealing with a flow but rather with a single transformation. In fact taking advantage of abelian covers of Σ and of the fundamental isomorphism of class field theory, one finds that the natural group that should replace $\mathbb{R}$ for the general Riemann flow is the Idele class group:

$$C_k = \mathrm{GL}_1(A)/k^* . \tag{D}$$

We can thus collect the information (A) (B) (C) (D) that we have obtained so far and look for the Riemann flow as an action of C_k on an hypothetical space X.

III. Spectral interpretation of critical zeros

There is a third approach to the problem of the zeros of the Riemann zeta function, due to G. Pólya [P] and M. Kac [K] and pursued further in [J] [BC]. It is based on statistical mechanics and the construction of a quantum statistical system whose *partition function* is the Riemann zeta function. Such a system was naturally constructed in [BC] and it does indicate using the first line of the dictionary of Noncommutative Geometry (namely the correspondence between quotient spaces and noncommutative algebras) what the space X should be in general:

$$X = A/k^* \tag{1}$$

namely the quotient of the space A of adeles, $A = k_A$ by the action of the multiplicative group k^*,

$$a \in A, \quad q \in k^* \to aq \in A. \tag{2}$$

This space X already appears in a very implicit manner in the work of Tate and Iwasawa on the functional equation. It is a noncommutative space in that, even at the level of measure theory, it is a tricky quotient space. For instance at the measure theory level, the corresponding von Neumann algebra,

$$R_{01} = L^\infty(A) \rtimes k^* \tag{3}$$

where A is endowed with its Haar measure as an additive group, is the hyperfinite factor of type II_∞.

The idele class group C_k acts on X by

$$(j, a) \to ja \qquad \forall\, j \in C_k,\ a \in X \tag{4}$$

and it was exactly necessary to divide A by k^* so that (4) makes good sense.

We shall come back later to the analogy between the action of C_k on R_{01} and the action of the Galois group of the maximal abelian extension of k.

What we shall do now is to construct the Hilbert space L^2_δ of functions on X with growth indexed by $\delta > 1$. Since X is a quotient space we shall first learn in the usual manifold case how to obtain the Hilbert space $L^2(M)$ of square integrable functions on a manifold M by working only on the universal cover $\widetilde{M}$ with the action of $\Gamma = \pi_1(M)$. Every function $f \in C_c^\infty(\widetilde{M})$ gives rise to a function $\widetilde{f}$ on M by

$$\widetilde{f}(x) = \sum_{\pi(\widetilde{x})=x} f(\widetilde{x}) \tag{5}$$

and all $g \in C_c^\infty(M)$ appear in this way. Moreover, one can write the Hilbert space inner product $\int_M \widetilde{f}_1(x)\, \widetilde{f}_2(x)\, dx$, in terms of f_1 and f_2 alone. Thus $\|\widetilde{f}\|^2 = \int \left|\sum_{\gamma\in\Gamma} f(\gamma x)\right|^2 dx$ where the integral is performed on a fundamental domain for Γ acting on $\widetilde{M}$. This formula defines a pre-Hilbert space norm on $C_c^\infty(\widetilde{M})$ and $L^2(M)$ is just the completion of $C_c^\infty(\widetilde{M})$ for that norm. Note that any function of the form $f - f_\gamma$ has vanishing norm and hence disappears in the process of completion. In our case of $X = A/k^*$, we thus need to define the analogous norm on the Bruhat-Schwartz space $\mathcal{S}(A)$ of functions on A (cf. Appendix I for the general definition of the Bruhat-Schwartz space). Since 0 is fixed by the action of k^*, the expression $\sum_{\gamma\in k^*} f(\gamma x)$ does not make sense for $x = 0$ unless we require that $f(0) = 0$. Moreover, when $|x| \to 0$, the above sums approximate, as Riemann sums, the product of $|x|^{-1}$ by $\int f\, dx$ for the additive Haar measure; thus we also require $\int f\, dx = 0$. We can now define the Hilbert space $L^2_\delta(X)_0$ as the completion of the codimension 2 subspace

$$\mathcal{S}(A)_0 = \left\{ f \in \mathcal{S}(A) \ ; \ f(0) = 0, \ \int f\, dx = 0 \right\} \tag{6}$$

for the norm $\| \ \|_\delta$ given by

$$\|f\|^2_\delta = \int \Big| \sum_{q\in k^*} f(qx) \Big|^2 (1 + \log^2 |x|)^{\delta/2}\, |x|\, d^*x \tag{7}$$

where the integral is performed on A^*/k^* and d^*x is the multiplicative Haar measure on A^*/k^*. The ugly term $(1 + \log^2 |x|)^{\delta/2}$ is there to control the growth of the functions on the noncompact quotient. We shall see how to remove it later in Section VII. Note that $|qx| = |x|$ for any $q \in k^*$.

The key point is that we use the measure $|x|\, d^*x$ instead of the additive Haar measure dx. Of course for a local field K, one has $dx = |x|\, d^*x$, but this fails in the above global situation. Instead one has,

$$dx = \lim_{\varepsilon\to 0} \varepsilon\, |x|^{1+\varepsilon}\, d^*x. \tag{8}$$

One has a natural representation of C_k on $L^2_\delta(X)_0$ given by

$$(U(j)\,f)\,(x) = f(j^{-1}\,x) \qquad \forall\, x \in A, \quad j \in C_k \tag{9}$$

and the result is independent of the choice of a lift of j in $J_k = \mathrm{GL}_1(A)$ because the functions $f - f_q$ are in the kernel of the norm. The conditions (6) which define $\mathcal{S}(A)_0$ are invariant under the action of C_k and give the following action of C_k on the 2-dimensional supplement of $\mathcal{S}(A)_0 \subset \mathcal{S}(A)$; this supplement is $\mathbb{C} \oplus \mathbb{C}(1)$ where $\mathbb{C}$ is the trivial C_k module (corresponding to $f(0)$) while the Tate twist $\mathbb{C}(1)$ is the module

$$(j, \lambda) \to |j|\,\lambda \tag{10}$$

coming from the equality

$$\int f(j^{-1}\,x)\,d\,x = |j| \int f(x)\,dx. \tag{11}$$

In order to analyze the representation (9) of C_k on $L^2_\delta(X)_0$, we shall relate it to the left regular representation of the group C_k on the Hilbert space $L^2_\delta(C_k)$ obtained from the following Hilbert space square norm on functions,

$$\|\xi\|^2_\delta = \int_{C_k} |\xi(g)|^2\,(1 + \log^2|g|)^{\delta/2}\,d^*g. \tag{12}$$

Here we have normalized the Haar measure of the multiplicative group C_k, with module

$$|\ | : C_k \to \mathbb{R}^*_+ \tag{13}$$

in such a way that (cf. [W3])

$$\int_{|g| \in [1,\Lambda]} d^*g \sim \log \Lambda \quad \text{when} \quad \Lambda \to +\infty. \tag{14}$$

The left regular representation V of C_k on $L^2_\delta(C_k)$ is

$$(V(a)\,\xi)\,(g) = \xi(a^{-1}\,g) \qquad \forall\, g, a \in C_k. \tag{15}$$

Note that because of the weight $(1 + \log^2|x|)^{\delta/2}$, this representation is *not* unitary but it satisfies the growth estimate

$$\|V(g)\| = 0\,(\log|g|)^{\delta/2} \quad \text{when} \quad |g| \to \infty \tag{16}$$

which follows from the inequality (valid for $u, v \in \mathbb{R}$)

$$\rho(u+v) \leq 2^{\delta/2}\,\rho(u)\,\rho(v), \quad \rho(u) = (1+u^2)^{\delta/2}\,. \tag{17}$$

We let E be the linear isometry from $L^2_\delta(X)_0$ into $L^2_\delta(C_k)$ given by the equality,

$$E(f)\,(g) = |g|^{1/2} \sum_{q\in k^*} f(qg) \qquad \forall\, g \in C_k. \tag{18}$$

By comparing (7) with (12) we see that E is an isometry and the factor $|g|^{1/2}$ is dictated by comparing the measures $|g|\, d^*g$ of (7) with d^*g of (12).

One has $E(U(a)\,f)\,(g) = |g|^{1/2} \sum_{k^*} (U(a)\,f)\,(qg) = |g|^{1/2} \sum_{k^*} f(a^{-1}\; qg) = |a|^{1/2}\,|a^{-1}\,g|^{1/2} \sum_{k^*} f(q\,a^{-1}\,g) = |a|^{1/2}\,(V(a)\,E(f))\;(g)$. Thus,

$$E\,U(a) = |a|^{1/2}\,V(a)\,E. \tag{19}$$

This equivariance shows that the range of E in $L^2_\delta(C_k)$ is a closed invariant subspace for the representation V.

The following theorem and its corollary show that the cokernel $\mathcal{H} = L^2_\delta(C_k)/\mathrm{Im}\,(E)$ of the isometry E plays the role of the Pólya-Hilbert space. Since $\mathrm{Im}\;E$ is invariant under the representation V, we let W be the corresponding representation of C_k on $\mathcal{H}$.

The abelian locally compact group C_k is (noncanonically) isomorphic to $K \times N$ where

$$K = \{g \in C_k \;;\; |g| = 1\}, \quad N = \text{ range } |\;| \subset \mathbb{R}^*_+. \tag{20}$$

For number fields one has $N = \mathbb{R}^*_+$, while for fields of nonzero characteristic $N \simeq \mathbb{Z}$ is the subgroup $q^{\mathbb{Z}} \subset \mathbb{R}^*_+$ (where $q = p^\ell$ is the cardinality of the field of constants).

We choose (noncanonically) an isomorphism

$$C_k \simeq K \times N. \tag{21}$$

By construction the representation W satisfies (using (16)),

$$\|W(g)\| = 0(\log|g|)^{\delta/2} \tag{22}$$

and its restriction to K is unitary. Thus $\mathcal{H}$ splits as a canonical direct sum of pairwise orthogonal subspaces,

$$\mathcal{H} = \underset{\chi\in\widehat{K}}{\oplus} \mathcal{H}_\chi, \quad \mathcal{H}_\chi = \left\{\xi \;;\; W(g)\,\xi = \chi\,(g)\,\xi, \quad \forall\, g \in K\right\} \tag{23}$$

where χ runs through the Pontrjagin dual group of K, which is the discrete abelian group $\widehat{K}$ of characters of K. Using the noncanonical isomorphism (21), i.e., the corresponding inclusion $N \subset C_k$, one can now restrict the representation W to any of the sectors $\mathcal{H}_\chi$. When $\mathrm{char}\,(k) > 0$, then $N \simeq \mathbb{Z}$ and the condition (22) shows that the action of N on $\mathcal{H}_\chi$ is given by a single operator with *unitary* spectrum. (One uses the spectral radius formula $|\,\mathrm{Spec}\;w| = \overline{\mathrm{Lim}}\,\|\,w^n\|^{1/n}$.) When

Char $(k) = 0$, we are dealing with an action of $\mathbb{R}_+^* \simeq \mathbb{R}$ on $\mathcal{H}_\chi$ and the condition (22) shows that this representation is generated by a closed unbounded operator D_χ with purely imaginary spectrum. The resolvent $R_\lambda = (D_\chi - \lambda)^{-1}$ is given, for $\mathrm{Re}\,\lambda > 0$, by the equality

$$R_\lambda = \int_0^\infty W_\chi(e^s)\, e^{-\lambda s}\, ds \tag{24}$$

and for $\mathrm{Re}\,\lambda < 0$ by,

$$R_\lambda = \int_0^\infty W_\chi(e^{-s})\, e^{\lambda s}\, ds \tag{25}$$

while the operator D_χ is defined by

$$D_\chi\, \xi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\, (W_\chi(e^\varepsilon) - 1)\, \xi. \tag{26}$$

Theorem 1. *Let $\chi \in \widehat{K}$, $\delta > 1$, $\mathcal{H}_\chi$ and D_χ be as above. Then D_χ has discrete spectrum,* $\mathrm{Sp}\, D_\chi \subset i\,\mathbb{R}$ *is the set of imaginary parts of zeros of the L function with Grössencharakter $\widetilde{\chi}$ which have real part equal to $\frac{1}{2}$; $\rho \in \mathrm{Sp}\, D \Leftrightarrow L\left(\widetilde{\chi}, \frac{1}{2} + \rho\right) = 0$ and $\rho \in i\,\mathbb{R}$, where $\widetilde{\chi}$ is the unique extension of χ to C_k which is equal to 1 on N. Moreover the multiplicity of ρ in $\mathrm{Sp}\, D$ is equal to the largest integer $n < \frac{1+\delta}{2}$, $n \leq$ multiplicity of $\frac{1}{2} + \rho$ as a zero of L.*

Theorem 1 has a similar formulation when the characteristic of k is nonzero. The following corollary is valid for global fields k of arbitrary characteristic.

Corollary 2. *For any Schwartz function $h \in \mathcal{S}(C_k)$ the operator $W(h) = \int W(g)\, h(g)\, d^*g$ in $\mathcal{H}$ is of trace class, and its trace is given by*

$$\mathrm{Trace}\; W(h) = \sum_{\substack{L\left(\widetilde{\chi}, \frac{1}{2}+\rho\right)=0 \\ \rho \in i\,\mathbb{R}/N^\perp}} \widehat{h}(\widetilde{\chi}, \rho)$$

where the multiplicity is counted as in Theorem 1 and where the Fourier transform $\widehat{h}$ of h is defined by,

$$\widehat{h}(\widetilde{\chi}, \rho) = \int_{C_k} h(u)\, \widetilde{\chi}(u)\, |u|^\rho\, d^*u.$$

Note that we did not have to define the L functions, let alone their analytic continuation, before stating the theorem, which shows that the pair

$$(\mathcal{H}_\chi, D_\chi) \tag{27}$$

certainly qualifies as a Pólya-Hilbert space.

The case of the Riemann zeta function corresponds to the trivial character $\chi = 1$ for the global field $k = \mathbb{Q}$ of rational numbers.

In general the zeros of the L functions can have multiplicity but one expects that for a fixed Grössencharakter χ this multiplicity is bounded, so that for a large enough value of δ the spectral multiplicity of D will be the right one. When the characteristic of k is > 0 this is certainly true.

If we modify the choice of noncanonical isomorphism (21) this modifies the operator D by

$$D' = D - i\,s, \tag{28}$$

where $s \in \mathbb{R}$ is determined by the equality

$$\widetilde{\chi}'(g) = \widetilde{\chi}(g)\,|g|^{i\,s} \qquad \forall\, g \in C_k. \tag{29}$$

The coherence of the statement of the theorem is insured by the equality

$$L(\widetilde{\chi}', z) = L(\widetilde{\chi}, z + i\,s) \qquad \forall\, z \in \mathbb{C}. \tag{30}$$

When the zeros of L have multiplicity and δ is large enough, the operator D is *not* semisimple and has a nontrivial Jordan form (cf. Appendix I). This is compatible with the almost unitary condition (22) but not with skew symmetry for D.

The proof of Theorem 1, explained in Appendix I, is based on the distribution theoretic interpretation by A. Weil [W2] of the idea of Tate and Iwasawa on the functional equation. Our construction should be compared with [Bg] and [Z].

As we expected from (C), the Pólya-Hilbert space $\mathcal{H}$ appears as a cokernel. Since we obtain the Hilbert space $L^2_\delta(X)_0$ by imposing two linear conditions on $\mathcal{S}(A)$,

$$0 \to \mathcal{S}(A)_0 \to \mathcal{S}(A) \xrightarrow{L} \mathbb{C} \oplus \mathbb{C}(1) \to 0 \tag{31}$$

we shall define $L^2_\delta(X)$ so that it fits in an exact sequence of C_k-modules

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0. \tag{32}$$

We can then use the exact sequence of C_k-modules

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(C_k) \to \mathcal{H} \to 0 \tag{33}$$

together with Corollary 2 to compute in a formal manner what the character of the module $L^2_\delta(X)$ should be. Using (32) and (33) we obtain,

$$\text{“Trace”}\ (U(h)) = \widehat{h}(0) + \widehat{h}(1) - \sum_{\substack{L(\chi,\rho)=0 \\ \mathrm{Re}\,\rho = \frac{1}{2}}} \widehat{h}(\chi,\rho) + \infty\, h(1) \tag{34}$$

where $\widehat{h}(\chi, \rho)$ is defined by Corollary 2 and

$$U(h) = \int_{C_k} U(g)\, h(g)\, d^* g \tag{35}$$

while the test function h is in a suitable function space. Note that the trace on the left hand side of (34) only makes sense after a suitable regularization since the left regular representation of C_k is not of trace class. This situation is similar to the one encountered by Atiyah and Bott ([AB]) in their proof of the Lefchetz formula. We shall first learn how to compute in a formal manner the above trace from the fixed points of the action of C_k on X. In Section VII, we shall show how to regularize the trace and completely eliminate the parameter δ.

IV. The distribution trace formula for flows on manifolds

In order to understand how the left hand side of III(34) should be computed, we shall first give an account of the proof of the usual Lefchetz formula by Atiyah-Bott ([AB]) and describe the computation of the distribution theoretic trace for flows on manifolds, which is a variation on the theme of [AB] and is due to Guillemin-Sternberg [GS]. We refer to Appendix III for a more detailed coordinate independent treatment following [GS].

Let us start with a diffeomorphism φ of a smooth compact manifold M and assume that the graph of φ is transverse to the diagonal in $M \times M$. One can then easily define and compute the distribution theoretic trace of the operator $U : C^\infty(M) \to C^\infty(M)$,

$$(U\xi)(x) = \xi(\varphi(x)). \tag{1}$$

Indeed let $k(x, y)$ be the Schwartz distribution on $M \times M$ such that

$$(U\xi)(x) = \int k(x, y)\, \xi(y)\, dy. \tag{2}$$

The distributional trace of U is simply

$$\text{“Trace”}\,(U) = \int k(x, x)\, dx, \tag{3}$$

Near the diagonal and in local coordinates one gets

$$k(x, y) = \delta(y - \varphi(x)) \tag{4}$$

where δ is the Dirac distribution.

Since, by hypothesis, the fixed points of φ are isolated, one can compute the trace (3) as a finite sum $\sum_{x,\varphi(x)=x}$ and get the contribution of each fixed point $x \in M$, $\varphi(x)=x$, as

$$\frac{1}{|1-\varphi'(x)|} \tag{5}$$

where $\varphi'(x)$ is the Jacobian of φ and $|A| = |\det A|$. One just uses the invertibility of $\mathrm{id} - \varphi'(x)$ to change variables in the integral,

$$\int \delta(y-\varphi(y))\,dy. \tag{6}$$

One thus gets (cf. [AB]),

$$\text{“Trace” } (U) = \sum_{x,\varphi(x)=x} \frac{1}{|1-\varphi'(x)|}. \tag{7}$$

This computation immediately extends to the action of φ on sections of an equivariant vector bundle E such as the bundle $\wedge^k T^*$ whose sections, $C^\infty(M,E)$ are the smooth forms of degree k. The alternate sum of the corresponding distribution theoretic traces is the *ordinary* trace of the action of φ on the de Rham cohomology, thus yielding the usual Lefchetz formula,

$$\sum (-1)^j \text{ Trace } \varphi^*/H^j = \sum_{\varphi(x)=x} \text{sign } \det(1-\varphi'(x))\,. \tag{8}$$

Let us refer to the appendix for more pedantic notations which show that the distribution theoretic trace is coordinate independent.

We shall now write down the analogue of formula (7) in the case of a flow $F_t = \exp(tv)$ of diffeomorphisms of M, where $v \in C^\infty(M,T)$ is a vector field on M. We get a one parameter group of operators acting on $C^\infty(M)$,

$$(U_t\,\xi)(x) = \xi(F_t(x)) \qquad \forall\,\xi \in C^\infty(M)\,,\ x\in M\,,\ t\in\mathbb{R}\,, \tag{9}$$

and we need the formula for,

$$\rho(h) = \text{“Trace”} \left(\int h(t)\,U_t\,dt\right), \quad h \in C_c^\infty(\mathbb{R})\,,\ h(0)=0. \tag{10}$$

The condition $h(0)=0$ is required because we cannot expect that the identity map F_0 be transverse to the diagonal. In order to define ρ as a distribution evaluated on the test function h, we let f be the following map:

$$f : X = M\times\mathbb{R} \to Y = M, \quad f(x,t) = F_t(x). \tag{11}$$

The graph of f is the submanifold Z of $X \times Y$,

$$Z = \{(x,t,y) \; ; \; y = F_t(x)\}. \tag{12}$$

One lets φ be the diagonal map,

$$\varphi(x,t) = (x,t,x), \quad \varphi : M \times \mathbb{R} \to X \times Y \tag{13}$$

and one assumes the transversality $\varphi \pitchfork Z$ outside $M \times (0)$.

Let τ be the distribution,

$$\tau = \varphi^*(\delta(y - F_t(x))\, dy), \tag{14}$$

and let q be the second projection,

$$q(x,t) = t \in \mathbb{R}. \tag{15}$$

Then by definition ρ is the pushforward $q_*(\tau)$ of the distribution τ. One checks (cf. Appendix III) that $q_*(\tau)$ is a generalized function.

Exactly as in the case of a single transformation, the contributions to (10) will come from the fixed points of F_t. The latter will come either from a *zero* of the vector field v, (i.e., $x \in M$ such that $v_x = 0$) or from a *periodic orbit* γ of the flow and we call $T_\gamma^\#$ the length of such a periodic orbit. Under the above transversality hypothesis, the formula for (10) is (cf. [GS], [G] and the Appendix III),

$$\begin{aligned} &\text{“Trace”} \left(\int h(t)\, U_t\, dt \right) \\ &= \sum_{x, v_x = 0} \int \frac{h(t)}{|1 - (F_t)_*|}\, dt + \sum_\gamma \sum_T T_\gamma^\# \frac{1}{|1 - (F_{T/})_*|}\, h(T) \end{aligned} \tag{16}$$

where in the second sum γ is a periodic orbit with length $T_\gamma^\#$, and T varies in $\mathbb{Z}\, T_\gamma^\#$ while $(F_{T/})_*$ is the Poincaré return map, i.e., the restriction of the tangent map to the transversal of the orbit.

One can rewrite (16) in a better way as,

$$\text{“Trace”} \left(\int h(t)\, U_t\, dt \right) = \sum_\gamma \int_{I_\gamma} \frac{h(u)}{|1 - (F_u)_*|}\, d^*u, \tag{17}$$

where the zeros $x \in M$, $v_x = 0$, are considered also as periodic orbits γ, while $I_\gamma \subset \mathbb{R}$ is the isotopy subgroup of any $x \in \gamma$, and d^*u is the unique Haar measure in I_γ such that the covolume of I_γ is equal to 1, i.e., such that for the unique Haar measure $d\mu$ of total mass 1 on $\mathbb{R}/I$ and any $f \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} f(t)\, dt = \int_{\mathbb{R}/I} \left(\int_I f(u+s)\, d^*u \right) d\mu(s). \tag{18}$$

Also we still write $(F_u)_*$ for the restriction of the tangent map to F_u to the transverse space of the orbits.

To understand what $(F_t)_*$ looks like at a zero of v we can replace $v(x)$ for x near x_0 by its tangent map. For simplicity we take the one dimensional case, with $v(x) = x\,\frac{\partial}{\partial x}$, acting on $\mathbb{R} = M$.

One has $F_t(x) = e^t\,x$. Since F_t is linear the tangent map $(F_t)_*$ is

$$(F_t)_* = e^t \tag{19}$$

and (12) becomes

$$\text{“Trace”}\left(\int h(t)\,U_t\,dt\right) = \int \frac{h(t)}{|1-e^t|}\,dt. \tag{20}$$

Thus, for this flow, the distribution trace formula is

$$\text{“Trace”}\,(U(h)) = \int \frac{h(u)}{|1-u|}\;d^*u \tag{21}$$

where we used the multiplicative notation so that $\mathbb{R}_+^*$ acts on $\mathbb{R}$ by multiplication, while $U(h) = \int U(v)h(v)\;d^*v$ and d^*v is the Haar measure of the group $\mathbb{R}_+^*$.

One can treat in a similar way the action, by multiplication, of the group of nonzero complex numbers on the manifold $\mathbb{C}$.

We shall now investigate the more general case of an arbitrary local field.

V. The action $(\lambda, x) \to \lambda\,x$ of K^* on a local field K

We let K be a local field and consider the map,

$$f : K \times K^* \to K, \quad f(x,\lambda) = \lambda\,x \tag{1}$$

together with the diagonal map,

$$\varphi : K \times K^* \to K \times K^* \times K, \quad \varphi(x,\lambda) = (x,\lambda,x) \tag{2}$$

as in IV(11) and (12) above.

When K is Archimedean we are in the framework of manifolds and we can associate to f a δ-section with support $Z =$ Graph (f),

$$\delta_Z = \delta(y - \lambda\,x)\,dy. \tag{3}$$

Using the projection $q(x,\lambda) = \lambda$ from $K \times K^*$ to K^*, we then consider as above the generalized function on K^* given by

$$q_*(\varphi^*\,\delta_Z). \tag{4}$$

The formal computation of this generalized function of λ is

$$\int \delta(x-\lambda\,x)\,dx = \int \delta((1-\lambda)x)\,dx = \int \delta(y)\,d((1-\lambda)^{-1}\,y)$$
$$= |1-\lambda|^{-1}\int \delta(y)\,dy = |1-\lambda|^{-1}.$$

We want to justify it by computing the convolution of the Fourier transforms of $\delta(x-y)$ and $\delta(y-\lambda\,x)$ since this is the correct way of defining the product of two distributions in this local context. Let us first compute the Fourier transform of $\delta(ax+by)$ where $(a,b)\in K^2(\neq 0)$. The pairing between K^2 and its dual K^2 is given by

$$\langle (x,y),(\xi,\eta)\rangle = \alpha(x\,\xi + y\,\eta)\in U(1), \tag{5}$$

where α is a fixed nontrivial character of the additive group K.

Let $(c,d)\in K^2$ be such that $ad-bc=1$ and consider the linear invertible transformation of K^2,

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \tag{6}$$

The Fourier transform of $\varphi\circ L$ is given by

$$(\varphi\circ L)^{\wedge} = |\det L|^{-1}\,\widehat{\varphi}\circ (L^{-1})^t. \tag{7}$$

Here one has $\det L = 1$ and $(L^{-1})^t$ is

$$(L^{-1})^t = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}. \tag{8}$$

One first computes the Fourier transform of $\delta(x)$, the additive Haar measure dx is normalized so as to be selfdual, and in one variable, $\delta(x)$ and 1 are Fourier transforms of each other. Thus

$$(\delta\otimes 1)^{\wedge} = 1\otimes\delta. \tag{9}$$

Using (7) one gets that the Fourier transform of $\delta(ax+by)$ is $\delta(-b\,\xi + a\,\eta)$. Thus we have to compute the convolution of the two generalized functions $\delta(\xi+\eta)$ and $\delta(\xi+\lambda\,\eta)$. Now

$$\int f(\xi,\eta)\,\delta(\xi+\eta)\,d\xi\,d\eta = \int f(\xi,-\xi)\,d\xi$$

and

$$\int f(\xi,\eta)\,\delta(\xi+\lambda\,\eta)\,d\xi\,d\eta = \int f(-\lambda\,\eta,\eta)\,d\eta.$$

Thus we are dealing with two measures carried respectively by two distinct lines. Their convolution evaluated on $f\in C_c^\infty(K^2)$ is $\int f(\alpha+\beta)\,d\mu(\alpha)\,d\nu(\beta) =$

$\int\int f((\xi,-\xi)+(-\lambda\,\eta,\eta))\,d\xi\,d\eta = \int\int f(\xi-\lambda\,\eta,-\xi+\eta)\,d\xi\,d\eta = \left(\int\int f(\xi',\eta')\right.$ $\left.d\xi'\,d\eta'\right)\times|J|^{-1}$ where J is the determinant of the matrix $\begin{bmatrix} 1 & -\lambda \\ -1 & 1 \end{bmatrix} = L$, so that $\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = J\begin{bmatrix} \xi \\ \eta \end{bmatrix}$. One has $J = 1-\lambda$ and thus the convolution of the generalized functions $\delta(\xi+\eta)$ and $\delta(\xi+\lambda\,\eta)$ gives as expected the constant function

$$|1-\lambda|^{-1}\,1. \tag{10}$$

Correspondingly, the product of the distribution $\delta(x-y)$ and $\delta(y-\lambda\,x)$ gives $|1-\lambda|^{-1}\,\delta_0$ so that

$$\int \delta(x-y)\,\delta(y-\lambda\,x)\,dx\,dy = |1-\lambda|^{-1}. \tag{11}$$

In this local case the Fourier transform alone was sufficient to make sense of the relevant product of distributions. In fact this would continue to make sense if we replace $\delta(y-\lambda\,x)$ by $\int h(\lambda^{-1})\,\delta(y-\lambda\,x)\,d^*\,\lambda$ where $h(1)=0$.

We shall now treat in detail the more delicate general case where $h(1)$ is arbitrary.

We shall prove a precise general result (Theorem 3) which handles the lack of transversality when $h(1)\neq 0$. We deal directly with the following operator in $L^2(K)$,

$$U(h) = \int h(\lambda)\,U(\lambda)\,d^*\lambda, \tag{12}$$

where the scaling operator $U(\lambda)$ is defined by

$$(U(\lambda)\,\xi)(x) = \xi(\lambda^{-1}\,x) \qquad \forall\,x\in K \tag{13}$$

and where the multiplicative Haar measure $d^*\lambda$ is normalized by

$$\int_{|\lambda|\in[1,\Lambda]} d^*\lambda \sim \log\Lambda \qquad \text{when } \Lambda\to\infty. \tag{14}$$

To understand the "trace" of $U(h)$ we shall proceed as in the Selberg trace formula ([Se]) and use a cutoff. For this we use the orthogonal projection P_Λ onto the subspace

$$P_\Lambda = \{\xi\in L^2(K)\,;\ \xi(x)=0 \qquad \forall x,\quad |x|>\Lambda\}. \tag{15}$$

Thus, P_Λ is the multiplication operator by the function ρ_Λ, where $\rho_\Lambda(x)=1$ if $|x|\le\Lambda$, and $\rho(x)=0$ for $|x|>\Lambda$. This gives an infrared cutoff, and to get an ultraviolet cutoff we use $\widehat{P}_\Lambda = FP_\Lambda F^{-1}$ where F is the Fourier transform (which depends upon the basic character α). We let

$$R_\Lambda = \widehat{P}_\Lambda\,P_\Lambda. \tag{16}$$

The main result of this section is then

Theorem 3. *Let K be a local field with basic character α. Let $h \in \mathcal{S}(K^*)$ have compact support. Then $R_\Lambda U(h)$ is a trace class operator and when $\Lambda \to \infty$, one has*

$$\text{Trace}\,(R_\Lambda U(h)) = 2h(1)\log'\Lambda + \int' \frac{h(u^{-1})}{|1-u|}\,d^*u + o(1)$$

where $2\log'\Lambda = \int_{\lambda\in K^*,\,|\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda$, and the principal value $\int'$ is uniquely determined by the pairing with the unique distribution on K which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform vanishes at 1.

Proof. We normalize as above the additive Haar measure to be the selfdual one on K. Let the constant $\rho > 0$ be determined by the equality

$$\int_{1\le|\lambda|\le\Lambda} \frac{d\lambda}{|\lambda|} \sim \rho\log\Lambda \quad \text{when} \quad \Lambda\to\infty \tag{17}$$

so that $d^*\lambda = \rho^{-1}\frac{d\lambda}{|\lambda|}$. Let L be the unique distribution, extension of $\rho^{-1}\frac{du}{|1-u|}$ whose Fourier transform vanishes at 1, $\widehat{L}(1) = 0$. One then has by definition

$$\int' \frac{h(u^{-1})}{|1-u|}\,d^*u = \left\langle L,\, \frac{h(u^{-1})}{|u|}\right\rangle, \tag{18}$$

where $\frac{h(u^{-1})}{|u|} = 0$ for u^{-1} outside the support of h.

Let $T = U(h)$. We can write the Schwartz kernel of T as

$$k(x,y) = \int h(\lambda^{-1})\,\delta(y-\lambda x)\,d^*\lambda. \tag{19}$$

Given any such kernel k we introduce its symbol,

$$\sigma(x,\xi) = \int k(x,x+u)\,\alpha(u\xi)\,du \tag{20}$$

as its partial Fourier transform. The Schwartz kernel $r^t_\Lambda(x,y)$ of the transpose R^t_Λ is given by

$$r^t_\Lambda(x,y) = \rho_\Lambda(x)\,(\widehat{\rho_\Lambda})\,(x-y). \tag{21}$$

Thus, the symbol σ_Λ of R^t_Λ is simply

$$\sigma_\Lambda(x,\xi) = \rho_\Lambda(x)\,\rho_\Lambda(\xi). \tag{22}$$

The operator R_Λ is of trace class and one has

$$\text{Trace}\,(R_\Lambda T) = \int k(x,y)\,r^t_\Lambda(x,y)\,dx\,dy. \tag{23}$$

Using the Parseval formula we thus get

$$\text{Trace}\,(R_\Lambda\,T) = \int_{|x|\le\Lambda,|\xi|\le\Lambda} \sigma(x,\xi)\,dx\,d\xi. \tag{24}$$

Now the symbol σ of T is given by

$$\sigma(x,\xi) = \int h(\lambda^{-1}) \left(\int \delta(x+u-\lambda x)\,\alpha(u\xi)\,du\right) d^*\lambda. \tag{25}$$

One has

$$\int \delta(x+u-\lambda x)\,\alpha(u\xi)\,du = \alpha((\lambda-1)\,x\xi)\,; \tag{26}$$

thus (25) gives

$$\sigma(x,\xi) = \rho^{-1}\int_K g(\lambda)\,\alpha(\lambda x\xi)\,d\lambda \tag{27}$$

where

$$g(\lambda) = h((\lambda+1)^{-1})\,|\lambda+1|^{-1}. \tag{28}$$

Since h is smooth with compact support on K^*, the function g belongs to $C_c^\infty(K)$.

Thus $\sigma(x,\xi) = \rho^{-1}\,\widehat{g}(x\xi)$ and

$$\text{Trace}\,(R_\Lambda\,T) = \rho^{-1}\int_{|x|\le\Lambda,|\xi|\le\Lambda} \widehat{g}(x\xi)\,dx\,d\xi. \tag{29}$$

With $u = x\xi$ one has $dx\,d\xi = du\,\frac{dx}{|x|}$ and, for $|u|\le\Lambda^2$,

$$\rho^{-1}\int_{\frac{|u|}{\Lambda}\le|x|\le\Lambda} \frac{dx}{|x|} = 2\log'\Lambda - \log|u| \tag{30}$$

(using the precise definition of $\log'\Lambda$ to handle the boundary terms). Thus we can rewrite (29) as

$$\text{Trace}\,(R_\Lambda\,T) = \int_{|u|\le\Lambda^2} \widehat{g}(u)\,(2\log'\Lambda - \log|u|)\,du. \tag{31}$$

Since $g \in C_c^\infty(K)$, one has

$$\int_{|u|\ge\Lambda^2} |\widehat{g}(u)|\,du = O(\Lambda^{-N}) \qquad \forall\, N \tag{32}$$

and similarly for $|\widehat{g}(u)\,\log|u||$. Thus

$$\text{Trace}\,(R_\Lambda\,T) = 2\,g(0)\,\log'\Lambda - \int \widehat{g}(u)\,\log|u|\,du\; + o(1). \tag{33}$$

Now for any local field K and basic character α, if we take for the Haar measure da the selfdual one, the Fourier transform of the distribution $\varphi(u) = -\log|u|$ is given outside 0 by

$$\widehat{\varphi}(a) = \rho^{-1}\,\frac{1}{|a|}, \tag{34}$$

with ρ determined by (17). To see this one lets P be the distribution on K given by

$$P(f) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in \mathrm{Mod}\,(K)}} \left(\int_{|x|\geq\varepsilon} f(x)\,d^*x + f(0)\log\varepsilon\right). \tag{35}$$

One has $P(f_a) = P(f) - \log|a|\,f(0)$ which is enough to show that the function $\widehat{P}(x)$ is equal to $-\log|x|+$ cst, and $\widehat{\varphi}$ differs from P by a multiple of δ_0.

Thus the Parseval formula gives, with the convention of Theorem 3,

$$-\int \widehat{g}(u)\,\log|u|\,du = \frac{1}{\rho}\int' g(a)\,\frac{da}{|a|}\,. \tag{36}$$

Replacing a by $\lambda - 1$ and applying (28) gives the desired result. $\square$

We shall show in Appendix II that the privileged principal value, which depends upon the basic character α, is the same as in Weil's explicit formulas.

VI. The global case and the formal trace computation

We shall now consider the action of C_k on X and write down the analogue of IV(17) for the distribution trace formula.

Both X and C_k are defined as quotients and we let

$$\pi : A \to X\,, \quad c : \mathrm{GL}_1(A) \to C_k \tag{1}$$

be the corresponding quotient maps.

As above we consider the graph Z of the action

$$f : X \times C_k \to X\,, \quad f(x,\lambda) = \lambda x \tag{2}$$

and the diagonal map

$$\varphi : X \times C_k \to X \times C_k \times X \qquad \varphi(x,\lambda) = (x,\lambda,x). \tag{3}$$

We first investigate the fixed points, $\varphi^{-1}(Z)$, i.e., the pairs $(x,\lambda) \in X \times C_k$ such that $\lambda x = x$. Let $x = \pi(\tilde{x})$ and $\lambda = c(j)$. Then the equality $\lambda x = x$ means that $\pi(j\tilde{x}) = \pi(\tilde{x})$. Thus there exists $q \in k^*$ such that with $\tilde{j} = qj$, one has

$$\tilde{j}\tilde{x} = \tilde{x}. \tag{4}$$

Recall now that A is the restricted direct product $A = \prod_{\mathrm{res}} k_v$ of the local fields k_v obtained by completion of k with respect to the place v. The equality (4) means that $\tilde{j}_v \tilde{x}_v = \tilde{x}_v$; thus, if $\tilde{x}_v \neq 0$, for all v it follows that $\tilde{j}_v = 1\ \forall v$ and $\tilde{j} = 1$. This shows that the projection of $\varphi^{-1}(Z) \cap C_k \backslash \{1\}$ on X is the union of the hyperplanes

$$\cup H_v\,;\ H_v = \pi(\tilde{H}_v)\,, \quad \tilde{H}_v = \{x\,;\ x_v = 0\}. \tag{5}$$

Each $\tilde{H}_v$ is closed in A and is invariant under multiplication by elements of k^*. Thus each H_v is a closed subset of X and one checks that it is the closure of the orbit under C_k of any of its generic points

$$x\,,\ x_u = 0 \quad \iff \quad u = v. \tag{6}$$

For any such point x, the isotropy group I_x is the image in C_k of the multiplicative group k_v^*,

$$I_x = k_v^* \tag{7}$$

by the map $\lambda \in k_v^* \to (1, \dots, 1, \lambda, 1, \dots)$. This map already occurs in class field theory (cf. [W1]) to relate local Galois theory to the global one.

Both groups k_v^* and C_k are commensurable to $\mathbb{R}_+^*$ by the module homomorphism, which is proper with cocompact range,

$$G \xrightarrow{|\ |} \mathbb{R}_+^*. \tag{8}$$

Since the restriction to k_v^* of the module of C_k is the module of k_v^*, it follows that

$$I_x \text{ is a cocompact subgroup of } C_k. \tag{9}$$

This allows us to normalize the respective Haar measures in such a way that the covolume of I_x is 1. This is in fact insured by the canonical normalization of the Haar measures of modulated groups ([W3]),

$$\int_{|g| \in [1,\Lambda]} d^*g \sim \log \Lambda \quad \text{when} \quad \Lambda \to +\infty. \tag{10}$$

It is important to note that though I_x is cocompact in C_k, the orbit of x is not closed and one needs to close it, the result being H_v. We shall learn how to justify this point later in Section VII, in the similar situation of the action of C_S on X_S. We can now in view of the results of the two preceding sections, write down the contribution of each H_v to the distributional trace.

Since $\tilde{H}_v$ is a hyperplane, we can identify the transverse space N_x to H_v at x with the quotient

$$N_x = A/\tilde{H}_v = k_v, \tag{11}$$

namely the additive group of the local field k_v. Given $j \in I_x$ one has $j_u = 1\ \forall u \neq v$, and $j_v = \lambda \in k_v^*$. The action of j on A is linear and fixes x; thus the action on the transverse space N_x is given by

$$(\lambda, a) \to \lambda a \quad \forall a \in k_v. \tag{12}$$

We can thus proceed with some faith and write down the contribution of H_v to the distributional trace in the form

$$\int_{k_v^*} \frac{h(\lambda)}{|1-\lambda|}\, d^*\lambda \tag{13}$$

where h is a test function on C_k which vanishes at 1. We now have to take care of a discrepancy in notation with the third section (formula 9), where we used the symbol $U(j)$ for the operation

$$\big(U(j)f\big)(x) = f(j^{-1}x), \tag{14}$$

whereas we use j in the above discussion. This amounts to replacing the test function $h(u)$ by $h(u^{-1})$, and we thus obtain as a formal analogue of IV(17) the following expression for the distributional trace

$$\text{“Trace”}\ (U(h)) = \sum_v \int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u. \tag{15}$$

Now the right-hand side of (15) is, when restricted to the hyperplane $h(1) = 0$, the distribution obtained by André Weil [W3] as the synthesis of the explicit formulas of number theory for all L-functions with Grössencharakter. In particular we can rewrite it as

$$\hat{h}(0) + \hat{h}(1) - \sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho) + \infty\ h(1) \tag{16}$$

where this time the restriction $\mathrm{Re}\,(\rho) = \frac{1}{2}$ has been eliminated.

Thus, equating (34) of Section III and (16) for $h(1) = 0$ would yield the desired information on the zeros. Of course, this do requires first eliminating the role of δ, and (as in [AB]) to prove that the distributional trace coincides with the ordinary operator theoretic trace on the cokernel of E. This is achieved for the usual setup of the Lefchetz fixed point theorem by the use of families.

A very important property of the right hand side of (15) (and of IV(17) in general) is that if the test function $h, h(1) = 0$ is positive,

$$h(u) \geq 0 \quad \forall\ u \in C_k \tag{17}$$

then the right-hand side is *positive*. This indicated from the very start that in order to obtain the Pólya-Hilbert space from the Riemann flow, it is *not* quantization that should be involved but simply the passage to the L^2 space, $X \to L^2(X)$. Indeed the positivity of IV(17) is typical of *permutation matrices* rather than of quantization. This distinction plays a crucial role in the above discussion of the trace formula, in particular the expected trace formula is not a semi-classical formula but a Lefchetz formula in the spirit of [AB].

The above discussion is *not* a rigorous justification of this formula. The first obvious obstacle is that the distributional trace is only formal and to give it a rigorous meaning tied up to Hilbert space operators, one needs as in Section V, to perform a cutoff. The second difficulty comes from the presence of the parameter δ as a label for the Hilbert space, while δ does not appear in the trace formula. As we shall see in the next two sections, the cutoff will completely eliminate the role of δ, and we shall nevertheless show (by proving positivity of the Weil distribution) that the validity of the (δ independent) trace formula is equivalent to the Riemann Hypothesis for all Grössencharakters of k.

VII. Proof of the trace formula in the S-local case

In the formal trace computation of Section VI, we skipped over the difficulties inherent in the tricky structure of the space X. In order to understand how to handle trace formulas on such spaces, we shall consider the slightly simpler situation which arises when one only considers a finite set S of places of k. As soon as the cardinality of S is larger than 3, the corresponding space X_S do shares most of the tricky features of the space X. In particular it is no longer of type I in the sense of Noncommutative Geometry.

We shall nevertheless prove a precise general result (Theorem 4) which shows that the above handling of periodic orbits and of their contribution to the trace is the correct one. It will in particular show why the orbit of the fixed point 0, or of elements $x \in A$, such that x_v vanishes for at least two places do not contribute to the trace formula.

At the same time, we shall handle as in Section V, the lack of transversality when $h(1) \neq 0$.

Let us first describe the reduced framework for the trace formula. We let k be a global field and S a finite set of places of k containing all infinite places. The group O_S^* of S-units is defined as the subgroup of k^*,

$$O_S^* = \{q \in k^*, \quad |q_v| = 1, \quad \forall v \notin S\}. \tag{1}$$

It is cocompact in J_S^1 where

$$J_S = \prod_{v \in S} k_v^* \tag{2}$$

and,

$$J_S^1 = \{j \in J_S, \quad |j| = 1\}. \tag{3}$$

Thus the quotient group $C_S = J_S/O_S^*$ plays the same role as C_k, and acts on the quotient X_S of $A_S = \prod_{v\in S} k_v$ by O_S^*.

To keep in mind a simple example, one can take $k = \mathbb{Q}$, while S consists of the three places 2, 3, and ∞. One checks in this example that the topology of X_S is not of type I since for instance the group $O_S^* = \{\pm 2^n 3^m;\, n,m \in \mathbb{Z}\}$ acts ergodically on $\{0\} \times \mathbb{R} \subset A_S$.

We normalize the multiplicative Haar measure $d^*\lambda$ of C_S by

$$\int_{|\lambda|\in[1,\Lambda]} d^*\lambda \sim \log \Lambda \quad \text{when} \quad \Lambda \to \infty, \tag{4}$$

and normalize the multiplicative Haar measure $d^*\lambda$ of J_S so that it agrees with the above on a fundamental domain D for the action of O_S^* on J_S.

There is no difficulty in defining the Hilbert space $L^2(X_S)$ of square integrable functions on X_S. We proceed as in Section III (without the δ), and complete (and separate) the Schwartz space $\mathcal{S}(A_S)$ for the pre-Hilbert structure given by

$$\|f\|^2 = \int \Big| \sum_{q\in O_S^*} f(qx) \Big|^2 |x|\, d^*x \tag{5}$$

where the integral is performed on C_S or equivalently on a fundamental domain D for the action of O_S^* on J_S. To show that (5) makes sense, one proves that for $f \in \mathcal{S}(A_S)$, the function $E_0(f)(x) = \sum_{q\in O_S^*} f(qx)$ is bounded above by a power of Log $|x|$ when $|x|$ tends to zero. To see this when f is the characteristic function of $\{x \in A_S, |x_v| \le 1\,, \forall v \in S\}$, one uses the cocompactness of O_S^* in J_S^1, to replace the sum by an integral. The latter is then comparable to

$$\int_{u_i\ge 0, \sum u_i = -\,\text{Log}\,|x|} \prod du_i, \tag{6}$$

where the index i varies in S. The general case follows.

The scaling operator $U(\lambda)$ is defined by

$$(U(\lambda)\,\xi)(x) = \xi(\lambda^{-1}\,x) \qquad \forall\, x \in A_S \tag{7}$$

and the same formula, with $x \in X_S$, defines its action on $L^2(X_S)$. Given a smooth compactly supported function h on C_S, $U(h) = \int h(g)U(g)dg$ makes sense as an operator acting on $L^2(X_S)$.

We shall now show that the Fourier transform F on $\mathcal{S}(A_S)$ does extend to a unitary operator on the Hilbert space $L^2(X_S)$.

Lemma 1.

a) *For any $f_i \in \mathcal{S}(A_S)$ the series $\sum_{O_S^*} \langle f_1, U(q)\, f_2\rangle_A$ of inner products in $L^2(A_S)$ converges geometrically on the abelian finitely generated group O_S^*.*

Moreover its sum is equal to the inner product of f_1 and f_2 in the Hilbert space $L^2(X_S)$.

b) *Let $\alpha = \prod \alpha_v$ be a basic character of the additive group A_S and F the corresponding Fourier transformation. The map $f \to F(f)$, $f \in \mathcal{S}(A_S)$ extends uniquely to a unitary operator in the Hilbert space $L^2(X_S)$.*

Proof. The map $L : O_S^* \to \mathbb{R}^S$, given by $L(u)_v = \log |u_v|$, has a finite kernel and its range is a lattice in the hyperplane $H = \{(y_v), \sum_S y_v = 0\}$. On H one has $\mathrm{Sup}_S\, y_v \geq 1/2n \sum |y_v|$, where $n = \mathrm{card}\,(S)$. Thus one has the inequality

$$\mathrm{Sup}_S\, |q_v| \geq \exp(d(q,1) \qquad \forall q \in O_S^* \tag{8}$$

for a suitable word metric d on O_S^*.

Let $K_n = \{x \in A_S;\, |x_v| \leq n\ \forall v \in S\}$ and k_n be the characteristic function of K_n. Let (λ_n) be a sequence of rapid decay such that

$$|f_i(x)| \leq \sum \lambda_n\, k_n(x) \qquad \forall x \in A_S. \tag{9}$$

One has for a suitable constant c,

$$|\left\langle k_n, U(q^{-1})\, k_n \right\rangle| \leq c\, n^m (\,\mathrm{Sup}_S\, |q_v|)^{-1} \tag{10}$$

where $m = \mathrm{Card}\,(S)$.

Using (9) we thus see that $\langle f_1, U(q)\, f_2 \rangle_A$ decays exponentially on O_S^*. Applying Fubini's theorem yields the equality,

$$\int \Big| \sum_{q \in O_S^*} f(qx) \Big|^2 |x|\, d^*x = \sum_{O_S^*} \langle f, U(q)\, f \rangle_A\,. \tag{11}$$

This proves a).

To prove b), one just uses (11) and the equalities $\langle Ff,\, Ff \rangle_A = \langle f,\, f \rangle_A$ and $F(U(q)\,f) = U(q^{-1})F(f)$. □

Now exactly as above for the case of local fields (Theorem V.3), we need to use a cutoff. For this we use the orthogonal projection P_Λ onto the subspace,

$$P_\Lambda = \{\xi \in L^2(X_S)\,;\; \xi(x) = 0 \qquad \forall x, \quad |x| > \Lambda\}. \tag{12}$$

Thus, P_Λ is the multiplication operator by the function ρ_Λ, where $\rho_\Lambda(x) = 1$ if $|x| \leq \Lambda$, and $\rho(x) = 0$ for $|x| > \Lambda$. This gives an infrared cutoff and to get an ultraviolet cutoff we use $\widehat{P}_\Lambda = F P_\Lambda F^{-1}$ where F is the Fourier transform (Lemma 1) which depends upon the choice of the basic character $\alpha = \prod \alpha_v$. We let

$$R_\Lambda = \widehat{P}_\Lambda\, P_\Lambda. \tag{13}$$

The main result of this section is then

Theorem 4. *Let A_S be as above, with basic character $\alpha = \prod \alpha_v$. Let $h \in \mathcal{S}(C_S)$ have compact support. Then when $\Lambda \to \infty$, one has*

$$\text{Trace}\,(R_\Lambda\, U(h)) = 2h(1)\log' \Lambda + \sum_{v\in S} \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

where $2\log'\Lambda = \int_{\lambda\in C_S,\,|\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda$, each k_v^* is embedded in C_S by the map $u \to (1,1,\dots,u,\dots,1)$ and the principal value $\int'$ is uniquely determined by the pairing with the unique distribution on k_v which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform relative to α_v vanishes at 1.

Proof. We normalize as above the additive Haar measure dx to be the selfdual one on the abelian group A_S. Let the constant $\rho > 0$ be determined by the equality, (where the fundamental domain D is as above),

$$\int_{\lambda\in D,\,1\le|\lambda|\le\Lambda} \frac{d\lambda}{|\lambda|} \sim \rho \log \Lambda \quad \text{when} \quad \Lambda \to \infty.$$

so that $d^*\lambda = \rho^{-1}\frac{d\lambda}{|\lambda|}$.

We let f be a smooth compactly supported function on J_S such that

$$\sum_{q\in O_S^*} f(qg) = h(g) \qquad \forall\, g \in C_S. \tag{14}$$

The existence of such an f follows from the discreteness of O_S^* in J_S. We then have the equality $U(f) = U(h)$, where

$$U(f) = \int f(\lambda)\, U(\lambda)\, d^*\lambda. \tag{15}$$

To compute the trace of $U(h)$ acting on functions on the quotient space X_S, we shall proceed as in the Selberg trace formula ([Se]). Thus for an operator T, acting on functions on A_S, which commutes with the action of O_S^* and is represented by an integral kernel,

$$T(\xi) = \int k(x,y)\xi(y)\, dy, \tag{16}$$

the trace of its action on $L^2(X_S)$ is given by

$$Tr(T) = \sum_{q\in O_S^*} \int_D k(x,qx)\, dx, \tag{17}$$

where D is as above a fundamental domain for the action of O_S^* on the subset J_S of A_S, whose complement is negligible. Let $T = U(f)$. We can write the Schwartz kernel of T as

$$k(x,y) = \int f(\lambda^{-1})\,\delta(y - \lambda x)\,d^*\lambda. \tag{18}$$

By construction one has

$$k(qx,qy) = k(x,y) \qquad q \in O_S^*. \tag{19}$$

For any $q \in O_S^*$, we shall evaluate the integral,

$$I_q = \int_{x\in D} k(qx,y) r_\Lambda^t(x,y)\,dy\,dx \tag{20}$$

where the Schwartz kernel $r_\Lambda^t(x,y)$ for the transpose R_Λ^t is given by

$$r_\Lambda^t(x,y) = \rho_\Lambda(x)\,(\widehat{\rho_\Lambda})\,(x-y). \tag{21}$$

To evaluate the above integral, we let $y = x + a$ and perform a Fourier transform in a. For the Fourier transform in a of $r_\Lambda^t(x, x+a)$, one gets,

$$\sigma_\Lambda(x,\xi) = \rho_\Lambda(x)\,\rho_\Lambda(\xi). \tag{22}$$

For the Fourier transform in a of $k(qx, x+a)$, one gets

$$\sigma(x,\xi) = \int f(\lambda^{-1}) \left(\int \delta(x + a - \lambda qx)\,\alpha(a\xi)\,da\right) d^*\lambda. \tag{23}$$

One has

$$\int \delta(x + a - \lambda qx)\,\alpha(a\xi)\,da = \alpha((\lambda q - 1)\,x\xi); \tag{24}$$

thus (23) gives

$$\sigma(x,\xi) = \rho^{-1} \int_{A_S} g_q(u)\,\alpha(ux\xi)\,du \tag{25}$$

where

$$g_q(u) = f(q(u+1)^{-1})\,|u+1|^{-1}. \tag{26}$$

Since f is smooth with compact support on A_S^*, the function g_q belongs to $C_c^\infty(A_S)$.

Thus $\sigma(x,\xi) = \rho^{-1}\,\widehat{g}_q(x\xi)$ and, using the Parseval formula we get

$$I_q = \int_{x\in D,\,|x|\le\Lambda,|\xi|\le\Lambda} \sigma(x,\xi)\,dx\,d\xi. \tag{27}$$

This gives

$$I_q = \rho^{-1} \int_{x \in D,\, |x| \leq \Lambda, |\xi| \leq \Lambda} \widehat{g}_q(x\xi)\, dx\, d\xi. \tag{28}$$

With $u = x\xi$ one has $dx\, d\xi = du\, \frac{dx}{|x|}$ and, for $|u| \leq \Lambda^2$,

$$\rho^{-1} \int_{x \in D,\, \frac{|u|}{\Lambda} \leq |x| \leq \Lambda} \frac{dx}{|x|} = 2 \log' \Lambda - \log |u| \tag{29}$$

(using the precise definition of $\log' \Lambda$ to handle the boundary terms). Thus we can rewrite (28) as

$$\text{Trace } (R_\Lambda\, T) = \sum_{q \in O_S^*} \int_{|u| \leq \Lambda^2} \widehat{g}_q(u)\, (2 \log' \Lambda - \log |u|)\, du. \tag{30}$$

Now $\log |u| = \sum_{v \in S} \log |u_v|$, and we shall first prove that

$$\sum_{q \in O_S^*} \int \widehat{g}_q(u)\, du = h(1), \tag{31}$$

while for any $v \in S$,

$$\sum_{q \in O_S^*} \int \widehat{g}_q(u)\, (-\log |u_v|)\, du = \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u. \tag{32}$$

In fact all the sums in q will have only finitely many nonzero terms. It will then remain to control the error term, namely to show that,

$$\sum_{q \in O_S^*} \int \widehat{g}_q(u)\, (\log |u| - 2 \log' \Lambda)^+\, du = 0(\Lambda^{-N}) \tag{33}$$

for any N, where we used the notation $x^+ = 0$ if $x \leq 0$ and $x^+ = x$ if $x > 0$. Now recall that

$$g_q(u) = f(q(u+1)^{-1})\, |u+1|^{-1},$$

so that $\int \widehat{g}_q(u)\, du = g_q(0) = f(q)$. Since f has compact support in A_S^*, the intersection of O_S^* with the support of f is finite and by (14) we get the equality (31).

To prove (32), we consider the natural projection pr_v from $\prod_{l \in S} k_l^*$ to $\prod_{l \neq v} k_l^*$. The image $pr_v(O_S^*)$ is still a discrete subgroup of $\prod_{l \neq v} k_l^*$, (since k_v^* is cocompact in C_S); thus there are only finitely many $q \in O_S^*$ such that k_v^* meets the support of f_q, where $f_q(a) = f(qa)$ for all a.

For each $q \in O_S^*$ one has, as in Section V,

$$\int \widehat{g}_q(u)\,(-\log|u_v|)\,du = \int_{k_v^*}' \frac{f_q(u^{-1})}{|1-u|}\,d^*u, \tag{34}$$

and from what we have just seen, this vanishes except for finitely many $q's$, so that by (14) we get the equality (32). Let us prove (33). Let $\varepsilon_\Lambda(u) = (\log|u| - 2\log'\Lambda)^+$, and let

$$\delta_q(\Lambda) = \int \widehat{g}_q(u)\,\varepsilon_\Lambda(u)\,du \tag{35}$$

be the error term. We shall prove

Lemma 2. *For any Λ the series $\sum_{O_S^*}|\delta_q(\Lambda)|$ converges geometrically on the abelian finitely generated group O_S^*. Moreover its sum $\sigma(\Lambda)$ is $O(\Lambda^{-N})$ for any N.*

Proof. Let (cf. (8)), d be a suitable word metric on O_S^* such that

$$\mathrm{Sup}_S\,|q_v| \geq \exp(d(q,1)) \qquad \forall q \in O_S^*. \tag{36}$$

Let $\xi \in \mathcal{S}(A_S)$ be defined by $\xi(x) = f(x^{-1})|x^{-1}|$ for all $x \in A_S^*$ and extended by 0 elsewhere. One has $g_q(x) = \xi(q^{-1}(1+x))$ for all $x \in A_S$, so that $\widehat{g}_q(u) = \int g_q(x)\alpha(u\,x)\,dx = \alpha(-u)\,\widehat{\xi}(q\,u)$. Now, $\delta_q(\Lambda) = \int \widehat{g}_q(u)\,\varepsilon_\Lambda(u)\,du = \int \widehat{\xi}(q\,u)\,\alpha(-u)\,\varepsilon_\Lambda(u)\,du = \int \widehat{\xi}(y)\,\alpha(-q^{-1}\,y)\,\varepsilon_\Lambda(y)\,dy$, since $\varepsilon_\Lambda(q\,u) = \varepsilon_\Lambda(u)$ for all u.

Thus we get, using the symbol $\overline{F}\eta$ for the inverse Fourier transform of η, the equality,

$$\delta_q(\Lambda) = \overline{F}(\varepsilon_\Lambda\widehat{\xi})(q^{-1}). \tag{37}$$

Let $\alpha \in]0, 1/2[$ and consider the norm

$$\|\eta\| = \mathrm{Sup}_{x\in A_S}\,|F(\eta)(x)\ \mathrm{Sup}_S\,|x_v|^\alpha|. \tag{38}$$

In order to estimate (38), we fix a smooth function ψ on $\mathbb{R}$, equal to 1 in a neighborhood of 0 and with support in $[-1,1]$, and introduce the convolution operators

$$(C_{\alpha,v} * \eta)(x) = \int_{k_v} \psi(|\varepsilon|)(\eta(x+\varepsilon) - \eta(x))\frac{d\varepsilon}{|\varepsilon|^{1+\alpha}}, \tag{39}$$

and the norms,

$$\|\eta\|_{(1,\alpha,v)} = \|C_{\alpha,v} * \eta\|_1, \tag{40}$$

where $\|\ \|_1$ is the L^1 norm.

The Fourier transform on k_v of the distribution $C_{\alpha,v}$ behaves like $|x_v|^\alpha$ for $|x_v| \to \infty$. Thus, using the equality $F(C_{\alpha,v} * \eta) = F(C_{\alpha,v})\, F(\eta)$, and the control of the sup norm of $F(g)$ by the L^1 norm of g, we get an inequality of the form

$$\mathrm{Sup}_{x\in A_S} |F(\eta)(x)\ \mathrm{Sup}_S |x_v|^\alpha| \le c_\alpha \sum_S \|\eta\|_{(1,\alpha,v)}. \tag{41}$$

Let us now show that for any $\eta \in \mathcal{S}(A_S)$, and $\alpha < 1/2$, one has

$$\|\varepsilon_\Lambda\, \eta\|_{(1,\alpha,v)} = O(\Lambda^{-N}), \tag{42}$$

for any N.

One has $|(\varepsilon_\Lambda(x+\varepsilon)\eta(x+\varepsilon) - \varepsilon_\Lambda(x)\eta(x)) - \varepsilon_\Lambda(x)(\eta(x+\varepsilon) - \eta(x))| \le |(\varepsilon_\Lambda(x+\varepsilon) - \varepsilon_\Lambda(x))||\eta(x+\varepsilon)|$. Moreover using the inequality

$$|a^+ - b^+| \le |a-b|, \tag{43}$$

we see that $|(\varepsilon_\Lambda(x+\varepsilon) - \varepsilon_\Lambda(x))| \le |\log|x_v+\varepsilon| - \log|x_v||$, for $\varepsilon \in k_v$. Let then

$$c'_\alpha = \int_{k_v} \log|1+y| \frac{dy}{|y|^{1+\alpha}}. \tag{44}$$

It is finite for all places $v \in S$ provided $\alpha < 1/2$, and one has

$$\int_{k_v} \psi(|\varepsilon|)(|\log|x+\varepsilon| - \log|x||)\frac{d\varepsilon}{|\varepsilon|^{1+\alpha}} \le c'_\alpha |x|^{-\alpha}. \tag{45}$$

Thus one obtains the inequality

$$|C_{\alpha,v} * \varepsilon_\Lambda\, \eta - \varepsilon_\Lambda\, (C_{\alpha,v} * \eta)|(x) \le c'_\alpha |x_v|^{-\alpha}\ \mathrm{Sup}_{\varepsilon\in k_v, |\varepsilon|\le 1}\ |\eta(x+\varepsilon)|. \tag{46}$$

Since the function $|x_v|^{-\alpha}$ is locally integrable, for $\alpha < 1$, one has for $\eta \in \mathcal{S}(A_S)$, and any N,

$$\int_{X_\Lambda} |x_v|^{-\alpha}\ \mathrm{Sup}_{\varepsilon\in k_v, |\varepsilon|\le 1}\ |\eta(x+\varepsilon)| dx = O(\Lambda^{-N}), \tag{47}$$

where $X_\Lambda = \{\, y+\varepsilon;\ |y| \ge \Lambda,\ \varepsilon \in k_v,\ |\varepsilon| \le 1 \,\}$.

Moreover one has for any N,

$$\|\varepsilon_\Lambda\, (C_{\alpha,v} * \eta)\|_1 = O(\Lambda^{-N}). \tag{48}$$

Thus, using (46), we obtain the inequality (42).

Taking $\eta = \widehat{\xi}$ and using (41), we thus get numbers δ_Λ, such that $\delta_\Lambda = O(\Lambda^{-N})$ for all N and

$$|\overline{F}(\varepsilon_\Lambda \widehat{\xi})\ \mathrm{Sup}_S |x_v|^\alpha|| \le \delta_\Lambda \qquad \forall x \in A_S \quad \forall \Lambda. \tag{49}$$

Taking $x = q \in O_S^*$, and using (36) and (37), we thus get

$$|\delta_q(\Lambda)| \le \delta_\Lambda \exp(-d(q,1)) \qquad \forall q \in O_S^*, \tag{50}$$

which is the desired inequality. □

VIII. The trace formula in the global case, and elimination of δ

The main difficulty created by the parameter δ in Theorem 1 is that the formal trace computation of Section VI is independent of δ, and thus cannot give in general the expected value of the trace of Theorem 1, since in the latter each critical zero ρ is counted with a multiplicity equal to the largest integer $n < \frac{1+\delta}{2}$, $n \leq$ multiplicity of ρ as a zero of L. In particular for L functions with multiple zeros, the δ-dependence of the spectral side is nontrivial. It is also clear that the function space $L^2_\delta(X)$ artificially eliminates the noncritical zeros by the introduction of the δ.

As we shall see, all these problems are eliminated by the cutoff. The latter will be performed directly on the Hilbert space $L^2(X)$ so that the only value of δ that we shall use is $\delta = 0$. All zeros will play a role in the spectral side of the trace formula, but while the critical zeros will appear perse, the noncritical ones will appear as resonances and enter in the trace formula through their harmonic potential with respect to the critical line. Thus the spectral side is entirely canonical and independent of δ, and by proving positivity of the Weil distribution, we shall show that its equality with the geometric side, i.e., the global analogue of Theorem 4, is equivalent to the Riemann Hypothesis for all L-functions with Grössencharakter.

The abelian group A of adeles of k is its own Pontrjagin dual by means of the pairing

$$\langle a, b\rangle = \alpha(ab) \tag{1}$$

where $\alpha : A \to U(1)$ is a nontrivial character which vanishes on $k \subset A$. Note that such a character is *not canonical*, but that any two such characters α and $\alpha^{'}$ are related by k^*

$$\alpha^{'}(a) = \alpha(qa) \quad \forall a \in A. \tag{2}$$

It follows that the corresponding Fourier transformations on A are related by

$$\hat{f}^{'} = \hat{f}_q. \tag{3}$$

This is yet another reason why it is natural to mod out by functions of the form $f - f_q$, i.e., to consider the quotient space X.

We fix the additive character α as above, $\alpha = \prod \alpha_v$ and let d be a differental idele

$$\alpha(x) = \alpha_0(d\,x) \quad \forall x \in A, \tag{4}$$

where $\alpha_0 = \prod \alpha_{0,v}$ is the product of the local normalized additive characters (cf. [W1]). We let S_0 be the finite set of places where α_v is ramified.

We shall first concentrate on the case of positive characteristic, i.e., of function fields, both because it is technically simpler and also because it allows us to keep track of the geometric significance of the construction (cf. Section II).

In order to understand how to perform in the global case, the cutoff $R_\Lambda = \widehat{P}_\Lambda\, P_\Lambda$ of Section VII, we shall first analyze the relative position of the pair of projections

$\widehat{P}_\Lambda$, P_Λ when $\Lambda \to \infty$. Thus, we let $S \supset S_0$ be a finite set of places of k, large enough so that $\mathrm{mod}\,(C_S) = \mathrm{mod}\,(C_k) = q^{\mathbb{Z}}$ and that for any fundamental domain D for the action of O_S^* on J_S, the product $D \times \prod R_v^*$ is a fundamental domain for the action of k^* on J_k.

Both $\widehat{P}_\Lambda$ and P_Λ commute with the decomposition of $L^2(X_S)$ as the direct sum of the subspaces, indexed by characters χ_0 of $C_{S,1}$,

$$L^2_{\chi_0} = \{\xi \in L^2(X_S)\,;\, \xi(a^{-1}x) = \chi_0(a)\,\xi(x), \qquad \forall\, x \in X_S\,, a \in C_{S,1}\} \tag{5}$$

which corresponds to the projections $P_{\chi_0} = \int \overline{\chi_0}(a)\,U(a)\,d_1\,a$, where $d_1\,a$ is the Haar measure of total mass 1 on $C_{S,1}$.

Lemma 1. *Let χ_0 be a character of $C_{S,1}$. Then for Λ large enough $\widehat{P}_\Lambda$ and P_Λ commute on the Hilbert space $L^2_{\chi_0}$.*

Proof. Let $\mathcal{U}_S$ be the image in C_S of the open subgroup $\prod R_v^*$. It is a subgroup of finite index l in $C_{S,1}$. Let us fix a character χ of $\mathcal{U}_S$ and consider the finite direct sum of the Hilbert spaces $L^2_{\chi_0}$ where χ_0 varies among the characters of $C_{S,1}$ whose restriction to $\mathcal{U}_S$ is equal to χ,

$$L^2(X_S)_\chi = \{\xi \in L^2(X_S)\,;\, \xi(a^{-1}x) = \chi(a)\,\xi(x), \quad \forall\, x \in X_S, \quad a \in \mathcal{U}_S\}. \tag{6}$$

The corresponding orthogonal projection is $U(h_\chi)$, where $h_\chi \in \mathcal{S}(C_S)$ is such that

$$\mathrm{Supp}\,(h_\chi) = \mathcal{U}_S \qquad h_\chi(x) = \lambda\,\overline{\chi}(x) \qquad \forall\, x \in \mathcal{U}_S, \tag{7}$$

and the constant $\lambda = l/\log(q)$ corresponds to our standard normalization of the Haar measure on C_S. Let as in Section VII, $f \in \mathcal{S}(J_S)$ with support $\prod R_v^*$ be such that $U(f) = U(h)$ and let $\xi \in \mathcal{S}(A_S)$ be defined by $\xi(x) = f(x^{-1})|x^{-1}|$ for all $x \in A_S^*$ and extended by 0 elsewhere.

Since ξ is locally constant, its Fourier transform has compact support and the equality (37) of Section VII shows that for Λ large enough, one has the equality

$$\mathrm{Trace}\,(\widehat{P}_\Lambda\,P_\Lambda\,U(h_\chi)) = 2h_\chi(1)\log'\Lambda + \sum_{v\in S} \int'_{k_v^*} \frac{h_\chi(u^{-1})}{|1-u|}\,d^*u. \tag{8}$$

With $\Lambda = q^N$, one has $2\log'\Lambda = (2N+1)\log(q)$ so that

$$2h_\chi(1)\log'\Lambda = (2N+1)l. \tag{9}$$

The character χ of $\prod R_v^*$ is a product, $\chi = \prod \chi_v$ and if one uses the standard additive character α_0 to take the principal value one has, (cf. [W1] Appendix IV),

$$\int'_{R*_v} \frac{\chi_v(u)}{|1-u|}\,d^*u = -f_v\log(q_v) \tag{10}$$

where f_v is the order of ramification of χ_v. We thus get

$$\int'_{k_v^*} \frac{h_\chi(u^{-1})}{|1-u|}\, d^*u = \; -f_v \, \deg(v)\, l + l\, \frac{\log(|d_v|)}{\log(q)} \tag{11}$$

where $q_v = q^{\deg(v)}$, and since we use the additive character α_v, we had to take into account the shift $\log(|d_v|)\, h_\chi(1)$ in the principal value.

Now one has $|d| = \prod |d_v| = q^{2-2g}$, where g is the genus of the curve. Thus we get

$$\text{Trace}\,(\widehat{P}_\Lambda\, P_\Lambda\, U(h_\chi)) = \; (2N+1)l \; - f\, l + (2-2g)\, l \tag{12}$$

where $f = \sum_S f_v \, \deg(v)$ is the order of ramification of χ, i.e., the degree of its conductor.

Let $B_\Lambda = \; \text{Im}\,(P_\Lambda) \cap \text{Im}\,(\widehat{P}_\Lambda)$ be the intersection of the ranges of the projections P_Λ and $\widehat{P}_\Lambda$, and let B_Λ^χ be its intersection with $L^2(X_S)_\chi$. We shall exhibit for each character χ of $\mathcal{U}_S$ a vector $\eta_\chi \in L^2(X_S)_\chi$ such that

$$U(g)(\eta_\chi) \in \; B_\Lambda \qquad \forall g \in C_S, |g| \leq \Lambda, |g^{-1}| \leq q^{2-2g-f}\, \Lambda, \tag{13}$$

while the vectors $U(g)(\eta_\chi)$ are linearly independent for $g \in D_S$, where D_S is the quotient of C_S by the open subgroup $\mathcal{U}_S$.

With $\Lambda = q^N$ as above, the number of elements g of D_S such that $|g| \leq \Lambda, |g^{-1}| \leq q^{2-2g-f}\, \Lambda$ is precisely equal to $(2N+1)l \; - f\, l + (2-2g)\, l$, which allows us to conclude that the projections $\widehat{P}_\Lambda$ and P_Λ commute in $L^2(X_S)_\chi$ and that the subspace B_Λ^χ is the linear span of the $U(g)(\eta_\chi)$.

Let us now construct the vectors $\eta_\chi \in L^2(X_S)_\chi$. With the notations of [W1] Proposition VII.13, we let

$$\eta_\chi = \prod_S \phi_v \tag{14}$$

be the standard function associated to $\chi = \prod \chi_v$ so that for unramified v, ϕ_v is the characteristic function of R_v, while for ramified v it vanishes outside R_v^* and agrees with $\overline{\chi}_v$ on R_v^*. By construction the support of η_χ is contained in $R = \prod R_v$. Thus one has $U(g)(\eta_\chi) \in \; \text{Im}\,(P_\Lambda)$ if $|g| \leq \Lambda$. Similarly by [W1] Proposition VII.13, we get that $U(g)(\eta_\chi) \in \; \text{Im}\,(\widehat{P}_\Lambda)$ as soon as $|g^{-1}| \leq q^{2-2g-f}\, \Lambda$. This shows that η_χ satisfies (13) and it remains to show that the vectors $U(g)(\eta_\chi)$ are linearly independent for $g \in D_S$.

Let us start with a nontrivial relation of the form

$$\Big\| \sum \lambda_g U(g)(\eta_\chi) \Big\| = \; 0 \tag{15}$$

where the norm is taken in $L^2(X_S)$, (cf. VII. 5). Let then $\xi_\chi = \prod_S \phi_v \otimes 1_R$ where $R = \prod_{v \notin S} R_v$. Let us assume first that $\chi \neq 1$. Then ξ_χ gives an element of $L^2_\delta(X)_0$

which is cyclic for the representation U of C_k in the direct sum of the subspaces $L^2_{\delta,\chi_0}(X)_0$ where χ_0 varies among the characters of $C_{k,1}$ whose restriction to $\mathcal{U}$ is equal to χ.

Now (15) implies that in $L^2_\delta(X)_0$ one has $\sum \lambda_g U(g)(\xi_\chi) = 0$. By the cyclicity of ξ_χ, one then gets $\sum \lambda_g U(g) = 0$ on any $L^2_{\delta,\chi_0}(X)_0$ which gives a contradiction (cf. Appendix 1, Lemma 3).

The proof for $\chi = 1$ is similar but requires more care since $1_R \notin \mathcal{S}_0(A)$. □

We can thus rewrite Theorem 4 in the case of positive characteristic as

Corollary 2. *Let Q_Λ be the orthogonal projection on the subspace of $L^2(X_S)$ spanned by the $f \in \mathcal{S}(A_S)$ which vanish as well as their Fourier transform for $|x| > \Lambda$. Let $h \in \mathcal{S}(C_S)$ have compact support. Then when $\Lambda \to \infty$, one has*

$$\text{Trace}\,(Q_\Lambda\, U(h)) = 2h(1)\log' \Lambda + \sum_{v\in S} \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

where $2\log'\Lambda = \int_{\lambda\in C_S,\, |\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda$, *and the other notations are as in Theorem* VII.4.

In fact the proof of Lemma 1 shows that the subspaces B_Λ stabilize very quickly, so that the natural map $\xi \to \xi \otimes 1_R$ from $L^2(X_S)$ to $L^2(X'_S)$ for $S \subset S'$ maps B^S_Λ onto $B^{S'}_\Lambda$.

We thus get from Corollary 2 an S-independent global formulation of the cutoff and of the trace formula. We let $L^2(X)$ be the Hilbert space $L^2_\delta(X)$ of Section III for the trivial value $\delta = 0$ which of course eliminates the unpleasant term from the inner product, and we let Q_Λ be the orthogonal projection on the subspace B_Λ of $L^2(X)$ spanned by the $f \in \mathcal{S}(A)$ which vanish as well as their Fourier transform for $|x| > \Lambda$. As we mentionned earlier, the proof of Lemma 1 shows that for S and Λ large enough (and fixed character χ), the natural map $\xi \to \xi \otimes 1_R$ from $L^2(X_S)_\chi$ to $L^2(X)_\chi$ maps B^S_Λ onto B_Λ.

It is thus natural to expect that the following global analogue of the trace formula of Corollary 2 actually holds, i.e., that when $\Lambda \to \infty$, one has

$$\text{Trace}\,(Q_\Lambda\, U(h)) = 2h(1)\log' \Lambda + \sum_{v} \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1) \tag{16}$$

where $2\log'\Lambda = \int_{\lambda\in C_k,\, |\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda$, and the other notations are as in Theorem VII.4.

We can prove directly that (16) holds when h is supported by $C_{k,1}$ but are not able to prove (16) directly for arbitrary h (even though the right hand side of the formula only contains finitely many nonzero terms since $h \in \mathcal{S}(C_k)$ has compact support). What we shall show however is that the trace formula (16) implies the positivity of the Weil distribution, and hence the validity of RH for k. Remember that we are still in positive characteristic where RH is actually a theorem of A. Weil. It will thus be important to check the actual equivalence between the validity of RH and the formula (16). This is achieved by

Theorem 5. *Let k be a global field of positive characteritic and let Q_Λ be the orthogonal projection on the subspace of $L^2(X)$ spanned by the $f \in \mathcal{S}(A)$ such that $f(x)$ and $\widehat{f}(x)$ vanish for $|x| > \Lambda$. Let $h \in \mathcal{S}(C_k)$ have compact support. Then the following conditions are equivalent:*

a) *When $\Lambda \to \infty$, one has*

$$\text{Trace}\,(Q_\Lambda\, U(h)) = 2h(1) \log' \Lambda + \sum_v \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)\,;$$

b) *All L functions with Grössencharakter on k satisfy the Riemann Hypothesis.*

Proof. To prove that a) implies b), we shall prove (assuming a)) the positivity of the Weil distribution (cf. Appendix 2),

$$\Delta = \log |d^{-1}|\, \delta_1 + D - \sum_v D_v\,. \tag{17}$$

First, by Theorem III.1 applied for $\delta = 0$, the map E,

$$E(f)\,(g) = |g|^{1/2} \sum_{q\in k^*} f(qg) \qquad \forall\, g \in C_k\,, \tag{18}$$

defines a surjective isometry from $L^2(X)_0$ to $L^2(C_k)$ such that

$$E\,U(a) = |a|^{1/2}\, V(a)\, E\,, \tag{19}$$

where the left regular representation V of C_k on $L^2(C_k)$ is given by

$$(V(a)\,\xi)\,(g) = \xi(a^{-1}\,g) \qquad \forall\, g, a \in C_k\,. \tag{20}$$

Let S_Λ be the subspace of $L^2(C_k)$ given by

$$S_\Lambda = \{\xi \in L^2(C_k)\,;\; \xi(g) = 0, \qquad \forall g, \quad |g| \notin [\Lambda^{-1}, \Lambda]\}\,. \tag{21}$$

We shall denote by the same letter the corresponding orthogonal projection.

Let $B_{\Lambda,0}$ be the subspace of $L^2(X)_0$ spanned by the $f \in \mathcal{S}(A)_0$ such that $f(x)$ and $\widehat{f}(x)$ vanish for $|x| > \Lambda$ and let $Q_{\Lambda,0}$ be the corresponding orthogonal projection. Let $f \in \mathcal{S}(A)_0$ be such that $f(x)$ and $\widehat{f}(x)$ vanish for $|x| > \Lambda$. Then $E(f)\,(g)$ vanishes for $|g| > \Lambda$, and the equality (Appendix 1),

$$E(f)(g) = E(\widehat{f})\left(\frac{1}{g}\right) \qquad f \in \mathcal{S}(A)_0, \tag{22}$$

shows that $E(f)\,(g)$ vanishes for $|g| < \Lambda^{-1}$.

This shows that $E(B_{\Lambda,0}) \subset S_\Lambda$, so that if we let $Q'_{\Lambda,0} = E\,Q_{\Lambda,0}\,E^{-1}$, we get the inequality

$$Q'_{\Lambda,0} \leq S_\Lambda \tag{23}$$

and for any Λ, the following distribution on C_k is of positive type,

$$\Delta_\Lambda(f) = \text{ Trace } ((S_\Lambda - Q'_{\Lambda,0})\,V(f)), \tag{24}$$

i.e., one has,

$$\Delta_\Lambda(f * f^*) \geq 0, \tag{25}$$

where $f^*(g) = \overline{f}(g^{-1})$ for all $g \in C_k$.

Let then $f(g) = |g|^{-1/2}\,h(g^{-1})$, so that by (19) one has $E\,U(h) = V(\tilde{f})E$ where $\tilde{f}(g) = f(g^{-1})$ for all $g \in C_k$. By Lemma 3 of Appendix 2 one has

$$\sum_v D_v(f) - \log|d^{-1}| = \sum_v \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u. \tag{26}$$

One has Trace $(S_\Lambda\, V(f)) = 2f(1)\log'\Lambda$; thus using a) we see that the limit of Δ_Λ when $\Lambda \to \infty$ is the Weil distribution Δ (cf. (17)). The term D in the latter comes from the nuance between the subspaces B_Λ and $B_{\Lambda,0}$. This shows using (24) that the distribution Δ is of positive type so that b) holds (cf. [W3]).

Let us now show that b) implies a). We shall compute from the zeros of L-functions and independently of any hypothesis the limit of the distributions Δ_Λ when $\Lambda \to \infty$.

We choose (non canonically) an isomorphism

$$C_k \simeq C_{k,1} \times N\,. \tag{27}$$

where $N =$ range $|\ | \subset \mathbb{R}_+^*$, $N \simeq \mathbb{Z}$ is the subgroup $q^{\mathbb{Z}} \subset \mathbb{R}_+^*$.

For $\rho \in \mathbb{C}$ we let $d\mu_\rho(z)$ be the harmonic measure of ρ with respect to the line $i\,\mathbb{R} \subset \mathbb{C}$. It is a probability measure on the line $i\,\mathbb{R}$ and coincides with the Dirac mass at ρ when ρ is on the line.

The implication b)$\Rightarrow$a) follows immediately from the explicit formulas (Appendix 2) and the following lemma.

Lemma 3. *The limit of the distributions* Δ_Λ *when* $\Lambda \to \infty$ *is given by*

$$\Delta_\infty(f) = \sum_{\substack{L\left(\widetilde{\chi}, \frac{1}{2}+\rho\right)=0 \\ \rho \in B/N^\perp}} N\Big(\widetilde{\chi}, \frac{1}{2} + \rho\Big) \int_{z \in i\,\mathbb{R}} \widehat{f}(\widetilde{\chi}, z) d\mu_\rho(z)\,,$$

where B *is the open strip* $B = \{\rho \in \mathbb{C};\, \mathrm{Re}\,(\rho) \in]\frac{-1}{2}, \frac{1}{2}[\}$, $N(\widetilde{\chi}, \frac{1}{2} + \rho)$ *is the multiplicity of the zero,* $d\mu_\rho(z)$ *is the harmonic measure of* ρ *with respect to the line* $i\,\mathbb{R} \subset \mathbb{C}$, *and the Fourier transform* $\widehat{f}$ *of* f *is defined by*

$$\widehat{f}(\widetilde{\chi}, \rho) = \int_{C_k} f(u)\, \widetilde{\chi}(u)\, |u|^\rho\, d^*\, u\,.$$

Proof. Let $\Lambda = q^N$. The proof of Lemma 1 gives the lower bound $(2N+1) - f + (2-2g)$ for the dimension of $B_{\Lambda,\chi}$ in terms of the order of ramification f of the character χ of $C_{k,1}$, where we assume first that $\chi \neq 1$. We have seen moreover that $E(B_{\Lambda,\chi}) \subset S_{\Lambda,\chi}$ while the dimension of $S_{\Lambda,\chi}$ is $2N+1$.

Now by Lemma 3 of Appendix 1, every element $\xi \in E(B_{\Lambda,\chi})$ satisfies the conditions

$$\int \xi(x)\, \chi(x)\, |x|^\rho\, d^*x = 0 \qquad \forall \rho \in B/N^\perp, \quad L\left(\chi, \frac{1}{2} + \rho\right) = 0\,. \tag{28}$$

This gives $2g - 2 + f$ linearly independent conditions (for N large enough), using [W1] Theorem VII.6, and shows that they actually characterize the subspace $E(B_{\Lambda,\chi})$ of $S_{\Lambda,\chi}$.

This reduces the proof of the lemma to the following simple computation: One lets F be a finite subset (possibly with multiplicity) of $\mathbb{C}^*$ and E_N the subspace of $S_N = \{\xi \in l^2(\mathbb{Z});\, \xi(n) = 0\ \forall n, |n| > N\}$ defined by the conditions $\sum \xi(n)\, z^n = 0$ $\forall z \in F$. One then has to compute the limit when $N \to \infty$ of Trace $((S_N - E_N)\, V(f))$ where V is the regular representation of $\mathbb{Z}$ (so that $V(f) = \sum f_k\, V^k$ where V is the shift, $V(\xi)_n = \xi_{n-1}$). One then checks that the unit vectors $\eta_z \in S_N$, $z \in F$, $\eta_z(n) = \overline{z}^n\, (|z^{2N+1}| - |z^{-(2N+1)}|)^{-\frac{1}{2}} (|z| - |z^{-1}|)^{\frac{1}{2}}\ \forall n \in [-N, N]$, are asymptotically orthogonal and span $(S_N - E_N)$ (when F has multiplicity one has to be more careful). The conclusion follows from

$$\mathrm{Lim}_{\ N \to \infty} \langle V(f)\eta_z,\, \eta_z \rangle = \int_{|u|=1} P_z(u)\, \widehat{f}(u)\, du, \tag{29}$$

where $P_z(u)$ is the Poisson kernel, and $\widehat{f}$ the Fourier transform of f. □

One should compare this lemma with Corollary 2 of Theorem III.1. In the latter only the critical zeros were coming into play and with a multiplicity controlled by δ.

In the above lemma, all zeros do appear and with their full multiplicity, but while the critical zeros appear perse, the noncritical ones play the role of resonances as in the Fermi theory.

Let us now explain how the above results extend to number fields k. We first need to analyze, as above, the relative position of the projections P_Λ and $\widehat{P}_\Lambda$. Let us first remind the reader of the well known geometry of pairs of projectors. Recall that a pair of orthogonal projections P_i in Hilbert space is the same as a unitary representation of the dihedral group $\Gamma = \mathbb{Z}/2 * \mathbb{Z}/2$. To the generators U_i of Γ correspond the operators $2P_i - 1$. The group Γ is the semidirect product of the subgroup generated by $U = U_1 U_2$ by the group $\mathbb{Z}/2$, acting by $U \mapsto U^{-1}$. Its irreducible unitary representations are parametrized by an angle $\theta \in [0, \frac{\pi}{2}]$, the corresponding orthogonal projections P_i being associated to the one dimensional subspaces $y = 0$ and $y = x\, tg(\theta)$ in the Euclidean x, y plane. In particular these representations are at most two dimensional. A general unitary representation is characterized by the operator Θ whose value is the above angle θ in the irreducible case. It is uniquely defined by $0 \leq \Theta \leq \frac{\pi}{2}$ and the equality

$$\mathrm{Sin}\,(\Theta) = |P_1 - P_2|, \tag{30}$$

and commutes with P_i.

The first obvious difficulty is that when v is an Archimedean place there exists no nonzero function on k_v which vanishes as well as its Fourier transform for $|x| > \Lambda$. This would be a difficult obstacle were it not for the work of Landau, Pollak and Slepian ([LPS]) in the early sixties, motivated by problems of electrical engineering, which allows us to overcome it by showing that though the projections P_Λ and $\widehat{P}_\Lambda$ do not commute exactly even for large Λ, their angle is sufficiently well behaved so that the subspace B_Λ makes good sense.

For simplicity we shall take $k = \mathbb{Q}$, so that the only infinite place is real. Let P_Λ be the orthogonal projection onto the subspace

$$P_\Lambda = \{\xi \in L^2(\mathbb{R})\,;\ \xi(x) = 0,\ \forall x,\ |x| > \Lambda\} \tag{31}$$

and $\widehat{P}_\Lambda = F P_\Lambda F^{-1}$, where F is the Fourier transform associated to the basic character $\alpha(x) = e^{-2\pi i x}$. What the above authors have done is to analyze the relative position of the projections P_Λ, $\widehat{P}_\Lambda$ for $\Lambda \to \infty$ in order to account for the obvious existence of signals (a recorded music piece for instance) which for all practical purposes have finite support both in the time variable and the dual frequency variable.

The key observation of ([LPS]) is that the following second order differential operator on $\mathbb{R}$ actually commutes with the projections P_Λ, $\widehat{P}_\Lambda$,

$$H_\Lambda \psi(x) = -\partial((\Lambda^2 - x^2)\,\partial)\psi(x) + (2\pi\Lambda x)^2\,\psi(x), \tag{32}$$

where ∂ is ordinary differentiation in one variable. To be more precise the above equality defines a symmetric operator with natural domain the Schwartz space. One can show that this operator has both deficiency indices equal to 4 and admits a unique selfadjoint extension which commutes with the dihedral group Γ associated to the projections P_Λ, $\widehat{P}_\Lambda$. We let H_Λ be this selfadjoint operator. It commutes with Fourier transform F.

If one restricts to $[-\Lambda,\Lambda]$, the operator H_Λ has discrete simple spectrum, and was studied long before the work of [LPS]. It appears from the factorization of the Helmoltz equation $\Delta\,\psi + k^2\,\psi = 0$ in one of the few separable coordinate systems in Euclidean 3-space, called the prolate spheroidal coordinates. Its eigenvalues $\chi_n(\Lambda), n \geq 0$ are simple, positive. The corresponding eigenfunctions ψ_n are called the prolate spheroidal wave functions and since $P_\Lambda\,\widehat{P}_\Lambda\,P_\Lambda$ commutes with H_Λ, they are the eigenfunctions of $P_\Lambda\,\widehat{P}_\Lambda\,P_\Lambda$. A lot is known about them; in particular one can take them to be real valued, they are even for n even and odd for n odd and they have exactly n zeros in the interval $[-\Lambda,\Lambda]$. When $\Lambda \to \infty$ the function ψ_n converges to the Weber-Hermite function of order n (cf. [Si]).

The key result of [LPS] is that the eigenvalues λ_n of the operator $P_\Lambda\,\widehat{P}_\Lambda\,P_\Lambda$ are simple and if we let $\lambda_0 > \lambda_1 > \cdots > \lambda_n > \ldots$ be their list in decreasing order, one has

$$P_\Lambda\,\widehat{P}_\Lambda\,P_\Lambda\,\psi_n = \lambda_n \psi_n \tag{33}$$

They behave qualitatively in the following manner. They stay very close to 1, $\lambda_n \simeq 1$ until n falls in an interval of size $\simeq \log(\Lambda)$ around the value $4\Lambda^2$. Their behaviour in this interval is governed by the relation ([Sl])

$$\lambda_n = (1 + e^{\pi\delta})^{-1} \tag{34}$$

where δ is the solution of smallest absolute value of the equation

$$(n + 1/2)\pi = 4\pi\Lambda^2 + \delta\ \mathrm{Log}\,(8\pi\Lambda^2) - \delta\,(\,\mathrm{Log}\,(|\delta/2|) - 1). \tag{35}$$

Beyond this interval the λ_n tend very rapidly to zero. Of course this gives the eigenvalues of Θ. One can show that for n in the above interval and $h \in C_c^\infty(\mathbb{R}_+^*)$ one has

$$\|U(h)\psi_n\| = O(\Lambda^{-a}) \tag{36}$$

for some $a > 0$. This allows to define the analogue of the subspace B_Λ of Lemma 1, as the linear span of the ψ_n, $n \leq 4\Lambda^2$, the exact value of the upper bound being irrelevant by (36). It also gives the justification of the semiclassical counting of the number of quantum mechanical states which are localized in the interval $[-\Lambda,\Lambda]$ as well as their Fourier transform as the area of the corresponding square in phase space.

We now know what is the subspace B_Λ for the single place ∞, and to obtain it for an arbitrary set of places (containing the infinite one), we just use the same rule as in the case of function fields, i.e., we consider the map,

$$\psi \mapsto \psi \otimes 1_R, \tag{37}$$

which suffices when we deal with the Riemann zeta function. Note also that in that case we restrict ourselves to even functions on $\mathbb{R}$. This gives the analogue of Lemma 1, Theorem 5, and Lemma 3.

To end this section we shall come back to our original motivation of Section I and show how the formula for the number of zeros

$$N(E) \sim (E/2\pi)(\log(E/2\pi) - 1) + 7/8 + o(1) + N_{\rm osc}(E) \tag{38}$$

appears from our spectral interpretation.

Let us first do a semiclassical computation for the number of quantum mechanical states in one degree of freedom which fulfill the conditions

$$|q| \leq \Lambda, \quad |p| \leq \Lambda, \quad |H| \leq E\,, \tag{39}$$

where $H = qp$ is the Hamiltonian which generates the group of scaling transformations

$$(U(\lambda)\xi)(x) = \xi(\lambda^{-1}x) \qquad \lambda \in \mathbb{R}_+^*, \quad x \in \mathbb{R}, \quad \xi \in L^2(\mathbb{R})\,, \tag{40}$$

as in our general framework.

To comply with our analysis of Section III, we have to restrict ourselves to even functions so that we exclude the region $pq \leq 0$ of the semiclassical (p,q) plane.

Now the region given by the above condition is equal to $D = D_+ \cup (-D_+)$ where

$$D_+ = \{(p,q) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad p \leq \Lambda, \quad q \leq \Lambda, \quad pq \leq E\}\,. \tag{41}$$

Let us compute the area of D_+ for the canonical symplectic form

$$\omega = \frac{1}{2\pi}\, dp \wedge dq\,. \tag{42}$$

By construction D_+ is the union of a rectangle with sides E/Λ, Λ with the subgraph, from $q = E/\Lambda$ to $q = \Lambda$, of the hyperbola $pq = E$. Thus,

$$\int_{D_+} \omega = \frac{1}{2\pi}\, E/\Lambda \times \Lambda + \frac{1}{2\pi} \int_{E/\Lambda}^{\Lambda} \frac{E\,dq}{q} = \frac{E}{2\pi} + \frac{2E}{2\pi}\, \log \Lambda - \frac{E}{2\pi}\, \log E\,. \tag{43}$$

Now the above computation corresponds to the standard normalization of the Fourier transform with basic character of $\mathbb{R}$ given by

$$\alpha(x) = \exp(ix)\,. \tag{44}$$

But we need to comply with the natural normalization at the infinite place,

$$\alpha_0(x) = \exp(-2\pi i x)\,. \tag{45}$$

We thus need to perform the transformation

$$P = p/2\pi, \quad Q = q\,. \tag{46}$$

The symplectic form is now $dP \wedge dQ$ and the domain

$$D' = \{(P,Q); |Q| \leq \Lambda, \quad |P| \leq \Lambda, \quad |PQ| \leq E/2\pi\}. \tag{47}$$

The computation is similar and yields the result

$$\int_{D'_+} \omega = \frac{2E}{2\pi}\log\Lambda - \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right). \tag{48}$$

In this formula we thus see the overall term $\langle N(E)\rangle$ which appears with a *minus* sign which shows that the number of quantum mechanical states corresponding to D' is less than $\frac{4E}{2\pi}\log\Lambda$ by the first approximation to the number of zeros of zeta whose imaginary part is less than E in absolute value (one just multiplies by 2 the equality (43) since $D' = D'_+ \cup (-D'_+)$. Now $\frac{1}{2\pi}(2E)(2\log\Lambda)$ is the number of quantum states in the Hilbert space $L^2(\mathbb{R}^*_+, d^*x)$ which are localized in $\mathbb{R}^*_+$ between Λ^{-1} and Λ and are localized in the dual group $\mathbb{R}$ (for the pairing $\langle\lambda, t\rangle = \lambda^{it}$) between $-E$ and E. Thus we see clearly that the first approximation to $N(E)$ appears as the lack of surjectivity of the map which associates to quantum states ξ belonging to D' the function on $\mathbb{R}^*_+$,

$$E(\xi)(x) = |x|^{1/2}\sum_{n\in\mathbb{Z}} \xi(nx) \tag{49}$$

where we assume the additional conditions $\xi(0) = \int \xi(x)dx = 0$.

A finer analysis, which is just what the trace formula is doing, does yield the additional terms $7/8 + o(1) + N_{\rm osc}(E)$. The above discussion yields an explicit construction of a large matrix whose spectrum approaches the zeros of zeta as $\Lambda \to \infty$.

It is quite remarkable that the eigenvalues of the angle operator Θ discussed above also play a key role in the theory of random unitary matrices. To be more specific, let $E(n,s)$ be the large N limit of the probability that there are exactly n eigenvalues of a random Hermitian $N \times N$ matrix in the interval $[-\frac{\pi}{\sqrt{2N}}t, \frac{\pi}{\sqrt{2N}}t]$, $t = s/2$. Clearly $\sum_n E(n,s) = 1$. Let P_t be as above the operator of multiplication by $1_{[-t,t]}$ — characteristic function of the interval $[-t,t]$ in the Hilbert space $L^2(\mathbb{R})$. In general (cf. [Me]), $E(n,s)$ is $(-1)^n$ times the n-th coefficient of the Taylor expansion at $z=1$ of $\zeta_s(z) = \prod_1^\infty (1 - z\lambda_j(s))$, where $\lambda_j(s)$ are the eigenvalues of the operator $\widehat{P_\pi}P_t$. (Here we denote by $\widehat{P_\lambda} = \mathcal{F}P_\lambda\mathcal{F}^{-1}$, and $\mathcal{F}$ denotes the Fourier transform, $\mathcal{F}\xi(u) = \int e^{ixu}\xi(x)dx$. Note also that the eigenvalues of $\widehat{P_a}P_b$ only depend upon the product ab so that the relation with the eigenvalues of Θ should be clear.)

General remarks

a) There is a close analogy between the construction of the Hilbert space $L^2(X)$ in Section III, and the construction of the physical Hilbert space ([S] Theorem 2.1) in constructive quantum field theory, in the case of gauge theories. In both cases the action of the invariance group (the group $k^* = GL_1(k)$ in our case, the gauge group in the case of gauge theories) is wiped out by the very definition of the inner product. Compare the comment after III(9) with ([S]) top of page 17.

b) For global fields of zero characteristic, the Idele class group has a non trivial connected component of the identity and this connected group has so far received no interpretation from Galois theory (cf. [W4]). The occurrence of type III factors in [BC] indicates that the classification of hyperfinite type III factors [C] should be viewed as a refinement of local class field theory for Archimedean places, and provide the missing interpretation of the connected component of the identity in the Idele class group. In particular hyperfinite type III factors are classified by closed (virtual) subgroups of $\mathbb{R}_+^*$ (cf. [C]) and they all appear as "unramified" extensions of the hyperfinite factor of type III_1.

c) Our construction of the Pólya-Hilbert space bears some resemblance to [Z] and in fact one should clarify their relation, as well as the relation of the space X of Adele classes with the hypothetical arithmetic site of Deninger [D]. Note that the division of A by k^* eliminates the linear structure of A and that it transforms drastically the formulas for dimensions of function spaces, replacing products by sums (cf. Theorem 4 of Section VII). It should be clear to the reader that the action of the Idele class group on the space of Adele classes is the analogue (through the usual dictionnary of class field theory) of the action of the Frobenius on the curve. (To be more specific one needs to divide first the space X of Adele classes by the action of the maximal compact subgroup of the Idele class group).

d) There is a superficial resemblance between the way the function $N(E)$ appears in the last computation of section VIII and the discussion in [BK], directly inspired from [Co]. It is amusing to note that the computation of [BK] is actually coincidental, the two rectangles are eliminated for no reason, which changes appropriately the sign of the term in E. What [BK] had not taken into account is that the spectral interpretation of [Co] is as an *absorption spectrum* rather than an emission spectrum.

e) There is an even more superficial resemblance with the work of D. Goldfeld [G]; in the latter the Weil distribution is used to define a corresponding inner product on a function space on the Idele class group. The positivity of the inner product is of course equivalent to the positivity of the Weil distribution (and by the result of A. Weil to RH) but this does not give any clue as to how to prove this positivity, nor does it give any explanation (except for a nice observation at the Archimedean place) for what the Weil distribution is, since it is introduced by hands in the formula for the inner product.

f) The above framework extends naturally from the case of $GL(1)$ to the case of $GL(n)$ where the Adele class space is replaced by the quotient of $M_n(A)$ by the action on the right of $GL_n(k)$. Preliminary work of C. Soule shows that the analogue of Theorem III.1 remains valid; the next step is to work out the analogue of the Lefchetz trace formula in this context.

g) I have been told by P. Sarnak and E. Bombieri that Paul Cohen has considered the space of Adele classes in connection with RH, but never got any detail of his unpublished ideas.

All the results of the present paper have been announced in the September 98 conference on the Riemann hypothesis held in the Schrodinger Institute in Vienna and have been published as a preprint of the Schrodinger Institute. We are grateful to the American Institute of Mathematics for its sponsoring of the conference.

Appendix I. Proof of Theorem 1

In this appendix we give the proof of Theorem 1. Let us first recall as a preliminary the results of Tate an Iwasawa as interpreted in [W2]

L functions and homogeneous distributions on A

In general for a non archimedean local field K we use the notations R for the maximal compact subring, P for the maximal ideal of R, π for a generator of the ideal P (i.e., $P = \pi R$).

Let k be a global field and A the ring of Adeles of k. It is the restricted product of the local fields k_v indexed by the set of places v of k, with respect to the maximal compact subrings R_v. Similarly, the Bruhat-Schwartz space $\mathcal{S}(A)$ is the restricted tensor product of the local Bruhat-Schwartz spaces $\mathcal{S}(k_v)$, with respect to the vectors 1_{R_v}.

L functions on k are associated to Grössencharakters, i.e., to characters of the Idele class group

$$C_k = J_k/k^* \,. \tag{1}$$

Let $\mathcal{X}$ be a character of the idele class group, we consider $\mathcal{X}$ as a character of J_k which is 1 on k^*. As such it can be written as a product

$$\mathcal{X}(j) = \Pi\, \mathcal{X}_v(j_v) \qquad j = (j_v) \in J_k \,. \tag{2}$$

By considering the restriction of $\mathcal{X}$ to the compact subgroup

$$G_0 = \Pi\, R_v^* \times 1 \subset J_k \,, \tag{3}$$

it follows that for all finite v but a finite number, one has

$$\mathcal{X}_v/R_v^* = 1\,. \tag{4}$$

One says that $\mathcal{X}$ is unramified at v when this holds.

Then $\mathcal{X}_v(x)$ only depends upon the module $|x|$, since

$$k_v^*/R_v^* = \bmod\,(k_v)\,. \tag{5}$$

Thus $\mathcal{X}_v$ is determined by

$$\mathcal{X}_v(\pi_v) \tag{6}$$

which does not depend upon the choice of π_v (mod R_v^*).

Let $\mathcal{X}$ be a quasi-character of C_k, of the form

$$\mathcal{X}(x) = \mathcal{X}_0(x)\,|x|^s \tag{7}$$

where $s \in \mathbb{C}$ and $\mathcal{X}_0$ is a character of C_k. The real part σ of s is uniquely determined by

$$|\mathcal{X}(x)| = |x|^\sigma\,. \tag{8}$$

Let P be the finite set of finite places where $\mathcal{X}_0$ is ramified. The L function $L(\mathcal{X}_0, s)$ is defined for $\sigma = \mathrm{Re}\,(s) > 1$ as

$$L(\mathcal{X}_0, s) = \left(\prod_{\substack{v \text{ finite} \\ v \notin P}} (1 - \mathcal{X}_{0,v}(\pi_v)\, q_v^{-s})^{-1}\right) = \left(\prod_{\substack{v \text{ finite} \\ v \notin P}} (1 - \mathcal{X}_v(\pi_v))^{-1}\right) \tag{9}$$

where

$$|\pi_v| = q_v^{-1}. \tag{10}$$

Let us now recall from [W2] how $L(\mathcal{X}_0, s)$ appears as a normalization factor for homogeneous distributions on A.

First let K be a local field and $\mathcal{X}$ a quasi-character of K^*,

$$\mathcal{X}(x) = \mathcal{X}_0(x)\,|x|^s \quad , \quad \mathcal{X}_0 : K^* \to U(1)\,. \tag{11}$$

A distribution D on K is homogeneous of weight $\mathcal{X}$ iff one has

$$\langle f^a, D\rangle = \mathcal{X}(a)^{-1}\,\langle f, D\rangle \tag{12}$$

for all test functions f and all a in K^*, where by definition

$$f^a(x) = f(ax)\,. \tag{13}$$

When $\sigma = \text{Re}\,(s) > 0$, there exists up to normalization only one homogeneous distribution of weight $\mathcal{X}$ on K,(cf. [W2]). It is given by the absolutely convergent integral

$$\int_{K^*} f(x)\,\mathcal{X}(x) d^*x = \Delta_{\mathcal{X}}(f)\,. \tag{14}$$

In particular, let K be non archimedean. Then, for any compactly supported locally constant function f on K, one has

$$f(x) - f(\pi^{-1}x) = 0 \quad \forall x, \quad |x| \le \delta\,. \tag{15}$$

Thus, for any $s \in \mathbb{C}$ the integral

$$\int_{K^*} (f(x) - f(\pi^{-1}x))\,|x|^s\, d^*x = \Delta_s'(f) \tag{16}$$

with the multiplicative Haar measure d^*x normalized by

$$\langle 1_{R^*}, d^*x\rangle = 1\,. \tag{17}$$

defines a distribution on K with the properties

$$\langle 1_R, \Delta_s'\rangle = 1 \tag{18}$$

$$\langle f^a, \Delta_s'\rangle = |a|^{-s}\,\langle f, \Delta_s'\rangle \tag{19}$$

and

$$\Delta_s' = (1 - q^{-s})\,\Delta_s, \tag{20}$$

where $|\pi| = q^{-1}$. (Let us check (18)–(20). With $f = 1_R$ one has $f(\pi^{-1}x) = 1$ iff $\pi^{-1}x \in R$, i.e., $x \in \pi R = P$. Thus $\Delta_s'(1_R) = \int_{R^*} d^*x = 1$. Let us check (20). One has $\int f(\pi^{-1}x)\,|x|^s\, d^*x = \int f(y)\,|\pi|^s\,|y|^s\, d^*y = |\pi|^s\,\Delta_s(f)$. But $|\pi| < 1$, $|\pi| = \frac{1}{q}$. Note also that for $s = 1$ and $f = 1_R$, one gets $\int_{R^*} dx = \left(1 - \frac{1}{q}\right) \int_R dx$).

Let then $\mathcal{X}$ be a quasi-character of C_k and write as above

$$\mathcal{X} = \Pi\,\mathcal{X}_v\,, \qquad \mathcal{X}(x) = \mathcal{X}_0(x)\,|x|^s \tag{21}$$

where $s \in \mathbb{C}$ and $\mathcal{X}_0$ is a character. Let P be the finite set of finite places where it is ramified. For any finite place $v \notin P$, let $\Delta_v'(s)$ be the unique homogeneous distribution of weight $\mathcal{X}_v$ normalized by

$$\langle \Delta_v'(s), 1_{R_v}\rangle = 1\,. \tag{22}$$

For any $v \in P$ or any infinite place, let, for $\sigma = \text{Re}\,(s) > 0$, Δ_v' be given by (14) which is homogeneous of weight $\mathcal{X}_v$ but unnormalized. Then the infinite tensor product

$$\Delta_s' = \Pi\,\Delta_v'(s) \tag{23}$$

makes sense as a continuous linear form on $\mathcal{S}(A)$ and is homogeneous of weight $\mathcal{X}$.

This solution is not equal to 0 since $\Delta_v' \ne 0$ for any $v \in P$ and any infinite place as well. It is finite by construction of the space $\mathcal{S}(A)$ of test functions as an infinite tensor product

$$\mathcal{S}(A) = \otimes\,(\mathcal{S}(k_v), 1_{R_v})\,. \tag{24}$$

Lemma 1 (cf. [W2]). *For $\sigma = \operatorname{Re}(s) > 1$, the following integral converges absolutely*

$$\int f(x)\, \mathcal{X}_0(x)\, |x|^s\, d^*x = \Delta_s(f) \qquad \forall\, f \in \mathcal{S}(A)$$

and $\Delta_s(f) = L(\mathcal{X}_0, s)\, \Delta'_s(f)$.

Proof. To get the absolute convergence, one can assume that $f = 1_R$ and $\mathcal{X}_0 = 1$. Then one has to control an infinite product of local terms, given locally for the Haar measure d^*x on k_v^* such that $\int_{R_v^*} d^*x = 1$, by

$$\int_{R \cap k_v^*} |x|^s\, d^*x \qquad (s \text{ real}) \tag{25}$$

which is $1 + q_v^{-s} + q_v^{-2s} + \ldots = (1 - q_v^{-s})^{-1}$. Thus the convergence for $\sigma > 1$ is the same as for the zeta function.

To prove the second equality, one only needs to consider the infinite tensor product for finite places $v \notin P$. Then by (20) one has $\Delta'_v = (1 - q_v^{-\alpha_v})\, \Delta_v$ where

$$q_v^{-\alpha_v} = \mathcal{X}_v(\pi) = \mathcal{X}_{0,v}(\pi)\, q_v^{-s} \tag{26}$$

with $|\pi| = q_v^{-1}$.

Thus one gets

$$\Delta_s = \left(\prod_{\substack{v \text{ finite} \\ v \notin P}} (1 - \mathcal{X}_{0,v}(\pi)\, q_v^{-s})^{-1} \right) \Delta'_s = L(\mathcal{X}_0, s)\, \Delta'_s .$$

□

By construction Δ'_s makes sense whenever $\sigma > 0$ and is a holomorphic function of s (for fixed f). Let us review briefly (cf. [W2]) how to extend the definition of Δ_s.

We let as above k be a global field, we fix a nontrivial additive character α of A, trivial on k,

$$\alpha(x + y) = \alpha(x)\, \alpha(y) \in U(1), \quad \alpha(q) = 1 \quad \forall\, q \in k\,. \tag{27}$$

We then identify the dual of the locally compact additive group A with A itself by the pairing

$$\langle x, y \rangle = \alpha(xy)\,. \tag{28}$$

One shows (cf. [W1]) that the lattice $k \subset A$, i.e., the discrete and cocompact additive subgroup k, is its own dual

$$\langle x, q \rangle = 1 \qquad \forall\, q \in k \qquad \Leftrightarrow \qquad x \in k\,. \tag{29}$$

Since A is the restricted product of the local fields k_v, one can write α as an infinite product

$$\alpha = \Pi\, \alpha_v \tag{30}$$

where for almost all v, one has $\alpha_v = 1$ on R_v. Let us recall the definition of the space $\mathcal{S}(A)_0$

$$\mathcal{S}(A)_0 = \{f \in \mathcal{S}(A)\ ;\ f(0) = 0, \quad \int f\, dx = 0\}\,. \tag{31}$$

Lemma 2. *Let $f \in \mathcal{S}(A)_0$. Then the series*

$$E(f)\,(g) = |g|^{1/2} \sum_{q \in k^*} f(qg) \qquad \forall\, g \in C_k$$

converges absolutely and one has

$$\forall\, n\,,\ \exists\, c, \qquad |E(f)(g)| \leq c\, e^{-n|\log|g||} \qquad \forall\, g \in C_k$$

and $E(\widehat{f})\,(g) = E(f)\,(g^{-1})$.

Proof. Let us first recall the formal definition ([Br]) of the Bruhat-Schwartz space $\mathcal{S}(G)$ for an arbitrary locally compact abelian group G. One considers all pairs of subgroups G_1, G_2 of G such that G_1 is generated by a compact neighborhood of 0 in G, while G_2 is a compact subgroup of G_1 such that the quotient group is elementary, i.e., is of the form $\mathbb{R}^a\, \mathbb{T}^b\, \mathbb{Z}^c\, F$ for F a finite group. By definition the Bruhat-Schwartz space $\mathcal{S}(G)$ is the inductive limit of the Schwartz spaces $\mathcal{S}(G_1/G_2)$ where the latter have the usual definition in terms of rapid decay of all derivatives. Since G_1 is open in G, any element of $\mathcal{S}(G_1/G_2)$ extended by 0 outside G_1 defines a continuous function on G. By construction $\mathcal{S}(G)$ is the union of the subspaces $\mathcal{S}(G_1/G_2)$ and it is endowed with the inductive limit topology.

Let $\widehat{G}$ be the Pontrjagin dual of G. Then the Fourier transform, which depends upon the normalization of the Haar measure on G, gives an isomorphism of $\mathcal{S}(G)$ with $\mathcal{S}(\widehat{G})$.

Let Γ be a lattice in the locally compact abelian group G. Then any function $f \in \mathcal{S}(G)$ is admissible for the pair G, Γ in the sense of [W1], and the Poisson summation formula (cf. [W1]) is the equality

$$\text{Covol}\ (\Gamma) \sum_{\gamma \in \Gamma} f(\gamma) = \sum_{\beta \in \Gamma^{\perp}} \widehat{f}(\beta)\,, \tag{32}$$

where $\Gamma^{\perp}$ is the dual of the lattice Γ, and

$$\widehat{f}(\beta) = \int f(a)\, \beta(a)\, da\,. \tag{33}$$

Both sides of (32) depend upon the normalization of the Haar measure on G.

In our case we let A be as above the additive group of Adeles on k. We normalize the additive Haar measure dx on A by

$$\mathrm{Covol}\,(k) = 1\,. \tag{34}$$

We then take $\Gamma = xk$, for some $x \in A^{-1}$. One has

$$\mathrm{Covol}\ (xk) = |x|\,. \tag{35}$$

The dual $\Gamma^{\perp}$ of the lattice xk, for x invertible in A, is the lattice $\Gamma^{\perp} = x^{-1}k$. Thus the Poisson formula (32) reads, for any $f \in \mathcal{S}(A)$,

$$|x| \sum_{q \in k} f(xq) = \sum_{q \in k} \widehat{f}(x^{-1}q)\,. \tag{36}$$

Which we can rewrite as

$$|x| \sum_{k^*} f(xq) = \sum_{k^*} \widehat{f}(x^{-1}q) + \delta \tag{37}$$

$\delta = -|x|\, f(0) + \int f(y)\, dy$.

We can then rewrite (37) as the equality, valid for all $f \in \mathcal{S}(A)_0$

$$E(f)(x) = E(\widehat{f}) \left(\frac{1}{x}\right) \qquad f \in \mathcal{S}(A)_0. \tag{38}$$

It remains to control the growth of $E(f)(x)$ on C_k, but by (38), it is enough to understand what happens for $|x|$ large.

We only treat the case of number fields, the general case is similar. Let $A = A_f \times A_\infty$ be the decomposition of the ring of Adeles corresponding to finite and infinite places. Thus $A_\infty = \prod_{S_\infty} k_v$ where S_∞ is the set of infinite places.

Any element of $\mathcal{S}(A)$ is a finite linear combination of test functions of the form

$$f = f_0 \otimes f_1 \tag{39}$$

where $f_0 \in \mathcal{S}(A_f)$, $\quad f_1 \in \mathcal{S}(A_\infty)$ (cf. [W5] 39); thus it is enough to control the growth of $E(f)(x)$ for such f and $|x|$ large.

Let $J_{k,1} = \{x \in J_k; |x| = 1\}$ be the group of Ideles of module one. Since $J_{k,1}/k^*$ is compact (cf. [W1]), we shall fix a compact subset K_1 of $J_{k,1}$ whose image in $J_{k,1}/k^*$ is this compact group.

Let μ be the diagonal embedding

$$\lambda \in \mathbb{R}_+^* \xrightarrow{\ \mu\ } (\lambda, \ldots, \lambda) \in \prod_{S_\infty} k_v^* \tag{40}$$

which yields an isomorphism

$$J_k = J_{k,1} \times \text{Im } \mu \,. \tag{41}$$

One has $f_0 \in \mathcal{S}(A_f)$, hence (cf. [W5]), $f_0 \in C_c(A_f)$ and we let $K_0 = \text{Support } f_0$. Since K_0 is compact, one can find a finite subset P of the set of finite places, and $C < \infty$ such that

$$y \in K = (K_f)^{-1} K_0 \Rightarrow |y_v| \leq 1 \quad \forall\, v \notin P, \qquad |y_v| \leq C \quad \forall\, v \,, \tag{42}$$

where K_f is the projection of K_1 on A_f.

We let Ω be the compact open subgroup of A_f determined by

$$|a_v| \leq 1 \quad \forall\, v \notin P, \qquad |a_v| \leq C \quad \forall\, v \,. \tag{43}$$

By construction $E(f)(x)$ only depends upon the class of x in J_k/k^*. Thus, to control the behaviour of $E(f)(x)$ for $|x| \to \infty$, we can take $x = (x_f, x_\infty) \in K_1$ and consider $E(f)(\lambda\, x)$ for $\lambda \in \mathbb{R}_+^*\,, \lambda \to \infty$. Now let $q = (q_f, q_\infty) \in k$. Then,

$$f(q\, \lambda\, x) = f_0(q_f\, x_f)\, f_1(q_\infty\, \lambda\, x_\infty) \tag{44}$$

and this vanishes unless $q_f\, x_f \in K_0$, i.e., unless $q_f \in K$. But then by (42) one has $q_f \in \Omega$. Let Γ be the lattice in $\prod_{S_\infty} k_v$ determined by

$$\Gamma = \{q_\infty;\, q \in k,\, q_f \in \Omega, \} \,. \tag{45}$$

The size of $E(f)(\lambda\, x)$ is thus controlled (up to the square root of $|\lambda\, x|$) by

$$C \sum_{n \in \Gamma^*} |f_1(\lambda\, x_\infty\, n)| \,, \tag{46}$$

where x_∞ varies in the projection K_∞ of K_1 on $\prod_{S_\infty} k_v^*$.

Since $f_1 \in \mathcal{S}(A_\infty)$, this shows that $E(f)(x)$ decays faster than any power of $|x|$ for $|x| \to \infty$.

We have shown that $E(f)$ has rapid decay in terms of $|x|$, for $|x| \to \infty$. Using (38) and the stability of $\mathcal{S}(A)_0$ under Fourier, we see that it also has exponential decay in terms of $|\log |x||$ when $|\log |x|| \to \infty$.

We then get

Lemma 3 (cf. [W2]). *For $\sigma = \text{Re}\,(s) > 0$, and any character $\mathcal{X}_0$ of C_k, one has*

$$\int E(f)(x)\, \mathcal{X}_0(x)\, |x|^{s-1/2}\, d^*x = cL(\mathcal{X}_0, s)\, \Delta'_s(f) \qquad \forall\, f \in \mathcal{S}(A)_0$$

where the nonzero constant c depends upon the normalization of the Haar measure d^*x on C_k.

Proof. For $\sigma = \text{Re}\,(s) > 1$, the equality follows from Lemma 1, but since both sides are analytic in s it holds in general.

As in Lemma 1, we shall continue to use the notation $\Delta_s(f)$ for $\sigma = \text{Re}\,(s) > 0$.

Approximate units in the Sobolev spaces $L^2_\delta(C_k)$

We first consider, for $\delta > 1$, the Hilbert space $L^2_\delta(\mathbb{R})$ of functions $\xi(u)$, $u \in \mathbb{R}$ with square norm given by

$$\int_{\mathbb{R}} |\xi(u)|^2 \, (1+u^2)^{\delta/2} \, du \,. \tag{1}$$

We let $\rho(u) = (1+u^2)^{\delta/2}$. It is comparable to $(1+|u|)^\delta$ and in particular,

$$\frac{\rho(u+a)}{\rho(u)} \leq c\,\rho(a) \qquad \forall\, u \in \mathbb{R}\,,\ a \in \mathbb{R} \tag{2}$$

with $c = 2^{\delta/2}$.

We then let $V(v)$ be the translation operator

$$(V(v)\xi)(u) = \xi(u-v) \qquad \forall\, u, v \in \mathbb{R}\,. \tag{3}$$

One has $\int_{\mathbb{R}} |\xi(u-v)|^2 \, \rho(u) \, du = \int_{\mathbb{R}} |\xi(u)|^2 \, \rho(u+v) \, du$ so that by (2) it is less than $c \int_{\mathbb{R}} |\xi(u)|^2 \, \rho(u) \, \rho(v) \, du = c\,\rho(v) \, \|\xi\|^2$,

$$\|V(v)\| \leq (c\,\rho(v))^{1/2} \,. \tag{4}$$

This shows that $V(f) = \int f(v) \, V(v) \, dv$ makes sense as soon as

$$\int |f(v)| \, \rho(v)^{1/2} \, dv < \infty \,. \tag{5}$$

This holds for all $f \in \mathcal{S}(\mathbb{R})$.

Lemma 4. *There exists an approximate unit $f_n \in \mathcal{S}(\mathbb{R})$, such that $\widehat{f_n}$ has compact support, $\|V(f_n)\| \leq C \quad \forall\, n$, and*

$$V(f_n) \to 1 \ \textit{strongly in} \ L^2_\delta(\mathbb{R}) \,.$$

Proof. Let f be a function, $f \in \mathcal{S}(\mathbb{R})$, whose Fourier transform $\widehat{f}$ has compact support, and such that $\int f \, dx = 1$ (i.e., $\widehat{f}(0) = 1$). Let then

$$f_n(v) = n\, f(nv) \qquad n = 1, 2, \ldots \tag{6}$$

One has $\int |f_n(v)| \ \ \rho(v)^{1/2} \, dv = \int |f(u)| \ \ \rho\left(\frac{u}{n}\right)^{1/2} \, du \leq \int |f(u)| \ \ \rho(u)^{1/2} \, du$. Thus $\|V(f_n)\|$ is uniformly bounded.

We can assume that $\widehat{f}$ is equal to 1 on $[-1,1]$. Then $\widehat{f_n}$ is equal to 1 on $[-n,n]$ and $V(f_n)\xi = \xi$ for any ξ with $\operatorname{Supp} \widehat{\xi} \subset [-n,n]$. By uniformity one gets that $V(f_n) \to 1$ strongly. $\square$

Let us now identify the dual $(L^2_\delta)^*$ of the Hilbert space L^2_δ with $L^2_{-\delta}$ by means of the pairing

$$\langle \xi, \eta \rangle_0 = \int_{\mathbb{R}} \xi(u)\,\eta(u)\,du\,. \tag{7}$$

Since L^2_δ is a Hilbert space, it is its own dual using the pairing

$$\langle \xi, \eta_1 \rangle = \int_{\mathbb{R}} \xi(u)\,\eta_1(u)(1+u^2)^{\delta/2}\,du\,. \tag{8}$$

If we let $\eta(u) = \eta_1(u)(1+u^2)^{\delta/2}$, then

$$\int |\eta_1(u)|^2\,(1+u^2)^{\delta/2}\,du = \int |\eta(u)|^2\,(1+u^2)^{-\delta/2}\,du$$

which is the natural norm square for $L^2_{-\delta}$.

Given a quasicompact group such as C_k with module

$$|\ | : C_k \to \mathbb{R}^*_+\,, \tag{9}$$

we let d^*g be the Haar measure on C_k normalized by

$$\int_{|g|\in[1,\Lambda]} d^*g \sim \log \Lambda \qquad \Lambda \to \infty \tag{10}$$

and we let $L^2_\delta(C_k)$ be defined by the norm

$$\int_{C_k} |\xi(g)|^2\,(1+\log|g|^2)^{\delta/2}\,d^*g\,. \tag{11}$$

It is, when the module of k is $\mathbb{R}^*_+$, a direct sum of spaces (1), labelled by the characters $\mathcal{X}_0$ of the compact group

$$C_{k,1} = \text{Ker mod}\,. \tag{12}$$

The pairing between $L^2_\delta(C_k)$ and $L^2_{-\delta}(C_k)$ is given by

$$\langle \xi, \eta \rangle = \int \xi(g)\,\eta(g)\,d^*g\,. \tag{13}$$

The natural representation V of C_k by translations is given by

$$(V(a)\xi)(g) = \xi(a^{-1}g) \qquad \forall\, g, a \in C_k\,. \tag{14}$$

It is not unitary but by (4) one has

$$\|V(g)\| = 0\,|\log|g||^{\delta/2}\,,\ |\log|g|| \to \infty\,. \tag{15}$$

Finally, one has, using Lemma 4 and the decomposition $C_k = C_{k,1} \times N$.

Lemma 5. *There exists an approximate unit $f_n \in \mathcal{S}(C_k)$, such that $\widehat{f_n}$ has compact support, $\|V(f_n)\| \le C \quad \forall\, n$, and*

$$V(f_n) \to 1 \text{ strongly in } L^2_\delta(C_k)\,.$$

Proof of Theorem III 1

We first consider the subspace of codimension 2 of $\mathcal{S}(A)$ given by

$$f(0) = 0\,, \quad \int f\,dx = 0\,. \tag{1}$$

On this subspace $\mathcal{S}(A)_0$ we put the inner product

$$\int_{C_k} |E(f)(x)|^2\,(1+\log|x|^2)^{\delta/2}\,d^*x\,. \tag{2}$$

We let U be the representation of C_k on $\mathcal{S}(A)$ given by

$$(U(a)\xi)(x) = \xi(a^{-1}x) \qquad \forall\,a \in C_k\,,\ x \in A\,. \tag{3}$$

We let $L^2_\delta(X)_0$ be the separated completion of $\mathcal{S}(A)_0$ for the inner product given by (2). The linear map $E : \mathcal{S}(A)_0 \to L^2_\delta(C_k)$ satisfies

$$\|E(f)\|^2_\delta = \|f\|^2_\delta \tag{4}$$

by construction. Thus it extends to an isometry, still noted E,

$$E : L^2_\delta(X)_0 \hookrightarrow L^2_\delta(C_k)\,. \tag{5}$$

One has

$$\begin{aligned} E(U(a)f)(g) = {} & |g|^{1/2}\sum_{k^*}(U(a)f)(qg) = |g|^{1/2}\sum_{k^*} f(a^{-1}qg) \\ = {} & |g|^{1/2}\sum_{k^*} f(qa^{-1}g) = |a|^{1/2}\,|a^{-1}g|^{1/2}\sum_{k^*} f(qa^{-1}g) \\ = {} & |a|^{1/2}(V(a)\,E(f))(g) \end{aligned}$$

$$E\,U(a) = |a|^{1/2}\,V(a)\,E\,. \tag{6}$$

The equality (6) shows that the natural representation U of C_k on $L^2_\delta(X)_0$ corresponds by the isometry E to the restriction of $|a|^{1/2}\,V(a)$ to the invariant subspace given by the range of E.

In order to understand $\operatorname{Im} E$ we consider its orthogonal in the dual space $L^2_{-\delta}(C_k)$. The compact subgroup

$$C_{k,1} = \{g \in C_k\,;\ |g| = 1\} \tag{7}$$

acts by the representation V which is unitary when restricted to $C_{k,1}$. Thus one can decompose $L^2_\delta(C_k)$ and its dual $L^2_{-\delta}(C_k)$, in the direct sum of the subspaces,

$$L^2_{\delta,\mathcal{X}_0} = \{\xi \in L^2_\delta(C_k)\,;\, \xi(a^{-1}g) = \mathcal{X}_0(a)\,\xi(g) \qquad \forall\, g \in C_k\,,\ a \in C_{k,1}\} \tag{8}$$

and

$$L^2_{-\delta,\mathcal{X}_0} = \{\xi \in L^2_{-\delta}(C_k)\,;\, \xi(a\,g) = \mathcal{X}_0(a)\,\xi(g) \qquad \forall\, g \in C_k\,,\ a \in C_{k,1}\} \tag{9}$$

which corresponds to the projections $P_{\mathcal{X}_0} = \int \overline{\mathcal{X}_0}(a)\,V(a)\,d_1\,a$ for L^2_δ and $P^t_{\mathcal{X}_0} = \int \overline{\mathcal{X}_0}(a)\,V(a)^t\,d_1\,a$ for the dual space $L^2_{-\delta}$.

In (9) we used the formula

$$(V(g)^t\,\eta)(x) = \eta(gx) \tag{10}$$

which follows from the definition of the transpose, $\langle V(g)\xi,\eta\rangle = \langle \xi, V(g)^t\eta\rangle$ using

$$\int \xi(g^{-1}x)\,\eta(x)\,d^*x = \int \xi(y)\,\eta(gy)\,d^*y\,.$$

In these formulas one only uses the character $\mathcal{X}_0$ as a character of the compact subgroup $C_{k,1}$ of C_k. One now chooses, noncanonically, an extension $\widetilde{\mathcal{X}}_0$ of $\mathcal{X}_0$ as a character of C_k

$$\widetilde{\mathcal{X}}_0(g) = \mathcal{X}_0(g) \qquad \forall g \in C_{k,1}\,. \tag{11}$$

This choice is not unique but any two such extensions differ by a character which is principal, i.e., of the form: $g \to |g|^{is_0}$, $s_0 \in \mathbb{R}$.

Let us fix a factorization $C_k = C_{k,1} \times \mathbb{R}^*_+$, and fix $\widetilde{\mathcal{X}}_0$ as being equal to 1 on $\mathbb{R}^*_+$.

We then write any element of $L^2_{-\delta,\mathcal{X}_0}(C_k)$ in the form

$$g \in C_k \to \eta(g) = \widetilde{\mathcal{X}}_0(g)\,\psi(|g|) \tag{12}$$

where

$$\int |\psi(|g|)|^2\,(1 + (\log|g|)^2)^{-\delta/2}\,d^*g < \infty\,. \tag{13}$$

This vector is in the orthogonal of Im E iff

$$\int E(f)(x)\,\widetilde{\mathcal{X}}_0(x)\,\psi(|x|)\,d^*x = 0 \qquad \forall\, f \in \mathcal{S}(A)_0\,. \tag{14}$$

We first proceed formally and write $\psi(|x|) = \int \widehat{\psi}(t)\,|x|^{it}\,dt$ so that the left hand side of (14) becomes

$$\int\!\!\int E(f)(x)\,\widetilde{\mathcal{X}}_0(x)\,|x|^{it}\,\widehat{\psi}(t)\,d^*x\,dt = \int \Delta_{1/2+it}(f)\,\widehat{\psi}(t)\,dt \tag{15}$$

(using the notations of Lemmas 1 and 3).

Let us justify this formal manipulation; since we deal with the orthogonal of an invariant subspace, we can assume that

$$V^t(h)\,\eta = \eta, \tag{16}$$

for some h such that $\widehat{h}$ has compact support. Indeed we can use Lemma 5 to only consider vectors which belong to the range of

$$V^t(h) = \int h(g)\,V(g)^t\,d^*g\,,\ \widehat{h} \text{ with compact support.}$$

Then, using (16), the Fourier transform of the tempered distribution ψ on $\mathbb{R}_+^*$ has compact support in $\mathbb{R}$. Thus, since $E(f)(x)$ has rapid decay, the equality between (14) and (15) follows from the definition of the Fourier transform of the tempered distribution ψ on $\mathbb{R}_+^*$.

Let us now describe suitable test functions $f \in \mathcal{S}(A)_0$ in order to test the distribution

$$\int \Delta_{\frac{1}{2}+it}\,\widehat{\psi}(t)\,dt\,. \tag{17}$$

We treat the case of characteristic zero; the general case is similar. For the finite places we take

$$f_0 = \underset{v\notin P}{\otimes} 1_{R_v} \otimes f_{\mathcal{X}_0} \tag{18}$$

where $f_{\mathcal{X}_0}$ is the tensor product over ramified places of the functions equal to 0 outside R_v^* and to $\overline{\mathcal{X}}_{0,v}$ on R_v^*. It follows then by the definition of Δ'_s that

$$\langle \Delta'_s, f_0 \otimes f\rangle = \int f(x)\,\mathcal{X}_{0,\infty}(x)\,|x|^s\,d^*x \tag{19}$$

for any $f \in \mathcal{S}(A_\infty)$. Moreover if the set P of finite ramified places is not empty, one has

$$f_0(0) = 0\,,\ \int_{A_f} f_0(x)\,dx = 0 \tag{20}$$

so that $f_0 \otimes f \in \mathcal{S}(A)_0 \qquad \forall\, f \in \mathcal{S}(A_\infty)$.

Now let ℓ be the number of infinite places of k and consider the map $\rho : (\mathbb{R}_+^*)^\ell \to \mathbb{R}_+^*$ given by

$$\rho(\lambda_1, \dots, \lambda_\ell) = \lambda_1 \dots \lambda_\ell\,. \tag{21}$$

As soon as $\ell > 1$ this map is not proper. Given a smooth function with compact support, $b \in C_c^\infty(\mathbb{R}_+^*)$, we need to find $a \in C_c^\infty((\mathbb{R}_+^*)^\ell)$ such that the direct image of the measure $a(x)\,d^*x$ is $b(y)\,d^*y$ where $d^*x = \Pi\,d^*x_i$ is the product of the multiplicative Haar measures.

Equivalently, one is dealing with a finite dimensional vector space E and a linear form $L : E \to \mathbb{R}$. One is given $b \in C_c^\infty(\mathbb{R})$ and asked to lift it. One can write $E = \mathbb{R} \times E_1$ and the lift can be taken as $a = b \otimes b_1$ where $b_1 \in C_c^\infty(E_1)$, $\int b_1 \, dx = 1$.

Thus we can in (19) take a function f of the form

$$f(x) = g(x) \, \overline{\mathcal{X}}_{0,\infty}(x) \,, \tag{22}$$

where the function $g \in C_c^\infty(A_\infty)$ only depends upon $(|x|_v)$, $v \in S_\infty$ and is smooth with compact support, disjoint from the closed set

$$\left\{ x \in \prod_{v \in S_\infty} k_v \, ; \, \exists \, v \, , \, x_v = 0 \right\} .$$

Thus, to any function $b \in C_c^\infty(\mathbb{R}_+^*)$ we can assign a test function $f = f_b$ such that for any s (Re $s > 0$)

$$\langle \Delta'_s, f_0 \otimes f_b \rangle = \int_{\mathbb{R}_+^*} b(x) \, |x|^s \, d^*x \, . \tag{23}$$

By Lemma 3, we get

$$\begin{aligned} \left\langle \int \Delta_{\frac{1}{2}+it} \, \widehat{\psi}(t) \, dt \, , \, f_0 \otimes f_b \right\rangle &= \left\langle \int L\,(\mathcal{X}_0, \frac{1}{2} + it) \, \Delta'_{\frac{1}{2}+it} \, \widehat{\psi}(t) \, dt \, , \, f_0 \otimes f_b \right\rangle \\ &= \int \int L\,(\mathcal{X}_0, \frac{1}{2} + it) \, \widehat{\psi}(t) \, b(x) \, |x|^{\frac{1}{2}+it} \, d^*x \, dt \, . \end{aligned}$$

Thus, from (14) and (15) we conclude, using arbitrary test functions b that the Fourier transform of the distribution $L\,(\mathcal{X}_0, 1/2 + it) \, \widehat{\psi}(t)$ actually vanishes,

$$L\,(\mathcal{X}_0, \frac{1}{2} + it) \, \widehat{\psi}(t) = 0 \, . \tag{24}$$

To justify the above equality, we need to control the growth of the L function in the variable t. One has

$$\left| L\Big(\frac{1}{2} + it\Big) \right| = 0\,(|t|^N) \, . \tag{25}$$

In particular, since $L\left(\frac{1}{2} + it\right)$ is an analytic function of t, we see that it is a multiplier of the algebra $\mathcal{S}(\mathbb{R})$ of Schwartz functions in the variable t. Thus the product $L\,(\frac{1}{2} + it) \, \widehat{\psi}(t)$ is still a tempered distribution, and so is its Fourier transform. To say that the latter vanishes when tested on arbitrary functions which are smooth with compact support implies that it vanishes.

The above argument uses the hypothesis $\mathcal{X}_0 / C_{k,1} \neq 1$.

In the case $\mathcal{X}_0/C_{k,1} = 1$ we need to impose to the test function f used in (22) the condition $\int f\,dx = 0$ which means

$$\int b(x)\,|x|\,d^*x = 0\,. \tag{26}$$

But the space of functions $b(x)\,|x|^{1/2} \in C_c^\infty(\mathbb{R}_+^*)$ such that (26) holds is still dense in the Schwartz space $\mathcal{S}(\mathbb{R}_+^*)$.

To understand the equation (24), let us consider an equation for distributions $\alpha(t)$ of the form

$$\varphi(t)\,\alpha(t) = 0\,, \tag{27}$$

where we first work with distributions α on S^1 and we assume that $\varphi \in C^\infty(S^1)$ has finitely many zeros $x_i \in Z(\varphi)$, of finite order n_i. Let J be the ideal of $C^\infty(S^1)$ generated by φ. One has $\psi \in J \Leftrightarrow$ order of ψ at x_i is $\geq n_i$.

Thus the distributions $\delta_{x_i}, \delta'_{x_i}, \ldots, \delta_{x_i}^{(n_i-1)}$ form a basis of the space of solutions of (27).

Now $\widehat{\psi}(t)$ is, for η orthogonal to $\mathrm{Im}\,(E)$ and satisfying (16), a distribution with compact support, and $L\left(\mathcal{X}_0, \frac{1}{2} + it\right)\,\widehat{\psi}(t) = 0$. Thus by the above argument we get that $\widehat{\psi}$ is a finite linear combination of the distributions

$$\delta_t^{(k)}, \quad L\left(\mathcal{X}_0, \frac{1}{2} + it\right) = 0, \quad k < \text{order of the zero},\ k < \frac{\delta - 1}{2}\,. \tag{28}$$

The condition $k <$ order of the zero is necessary and sufficient to get the vanishing on the range of E. The condition $k < \frac{\delta-1}{2}$ is necessary and sufficient to ensure that ψ belongs to $L^2_{-\delta}$, i.e., that

$$\int (\log|x|)^{2k}\,(1 + |\log|x||^2)^{-\delta/2}\,d^*x < \infty \tag{29}$$

which is $2k + \delta < -1$, i.e., $k < \frac{\delta-1}{2}$.

Conversely, let s be a zero of $L(\mathcal{X}_0, s)$ and $k > 0$ its order. By Lemma 3 and the finiteness and analyticity of Δ'_s (for $\mathrm{Re}\ s > 0$) we get

$$\left(\frac{\partial}{\partial s}\right)^a \Delta_s(f) = 0 \qquad \forall\, f \in \mathcal{S}(A)_0\,,\ a = 0, 1, \ldots, k-1\,. \tag{30}$$

We can differentiate the equality of Lemma 3 and get

$$\left(\frac{\partial}{\partial s}\right)^a \Delta_s(f) = \int_{C_k} E(f)(x)\,\mathcal{X}_0(x)\,|x|^{s-1/2}\,(\log|x|)^a\,d^*x\,. \tag{31}$$

Thus η belongs to the orthogonal of $\mathrm{Im}\,(E)$ and satisfies (16) iff it is a finite linear combination of functions of the form

$$\eta_{t,a}(x) = \mathcal{X}_0(x)\,|x|^{it}\,(\log|x|)^a, \tag{32}$$

where

$$L\left(\mathcal{X}_0, \frac{1}{2} + it\right) = 0, \qquad a < \text{order of the zero},\ a < \frac{\delta - 1}{2}\,. \tag{33}$$

The restriction to the subgroup $\mathbb{R}_+^*$ of C_k of the transposed of W is thus given in the above basis by

$$W(\lambda)^t\,\eta_{t,a} = \sum_{b=0}^{a} C_a^b\,\lambda^{it}\,(\,\mathrm{Log}\,(\lambda))^b\,\eta_{t,a-b}. \tag{34}$$

The multiplication operator by a function with bounded derivatives is a bounded operator in any Sobolev space one checks directly using the density in the orthogonal of $\mathrm{Im}\,(E)$ of vectors satisfying (16), that is, if $L\left(\mathcal{X}_0, \frac{1}{2} + is\right) \neq 0$, then is does not belong to the spectrum of $D^t_{\mathcal{X}_0}$.

This determines the spectrum of the operator $D^t_{\mathcal{X}_0}$ and hence of its transpose $D_{\mathcal{X}_0}$ as indicated in Theorem 1 and ends the proof of Theorem 1.

Let us now prove the corollary. Let us fix $h_0 \in \mathcal{S}(C_k)$ such that $\widehat{h}_0$ has compact support contained in $\{\mathcal{X}_0\} \times \mathbb{R}$ and $\widehat{h}_0(\mathcal{X}_0, s) = 1$ for s small.

Let then h_s be given by $h_s(g) = h_0(g)\,|g|^{is}$. The Fourier transform $\widehat{h}_s$ is then the translate of $\widehat{h}_0$, and one can choose h_0 such that

$$\sum_{n\in\mathbb{Z}} \widehat{h}_n(\mathcal{X}_0\,, u) = 1 \qquad u \in \mathbb{R}\,. \tag{35}$$

When $|s| \to \infty$, the dimension of the range of $W^t(h_s)$ is of the order of $\mathrm{Log}\,|s|$ as is the number of zeros of the L function in the translates of a fixed interval (cf. [W3]).

Let $h \in \mathcal{S}(C_k)$. One has $W^t(h) = \sum_{n\in\mathbb{Z}} W^t(h * h_n)$.

It follows then from the polynomial growth of the norm of $W^t(g)$ that the operator

$$\int h(g)\,W(g)^t\,d^*g \tag{36}$$

is of trace class for any $h \in \mathcal{S}(C_k)$.

Moreover using the triangular form given by (34) we get its trace, and hence the trace of its transpose $W(h)$ as

$$\mathrm{Trace}\;W(h) = \sum_{\substack{L\left(\mathcal{X}, \frac{1}{2}+\rho\right)=0 \\ \rho\in i\,\mathbb{R}}} \widehat{h}(\mathcal{X}, \rho) \tag{37}$$

where the multiplicity is counted as in Theorem 1 and where the Fourier transform $\widehat{h}$ of h is defined by

$$\widehat{h}(\mathcal{X},\rho) = \int_{C_k} h(u)\,\widetilde{\mathcal{X}}(u)\,|u|^\rho\,d^*\,u\,. \tag{38}$$

Appendix II. Explicit formulas

We check in detail that the rewriting of the Weil explicit formulas which is predicted by the global trace formula of Theorem VII 4 is correct. Our computation is straightforward but can be better understood conceptually using [H]. Let us first recall the Weil explicit formulas ([W3]). One lets k be a global field. One identifies the quotient $C_k/C_{k,1}$ with the range of the module

$$N = \{|g|\,;\, g \in C_k\} \subset \mathbb{R}_+^*\,. \tag{1}$$

One endows N with its normalized Haar measure d^*x. Given a function F on N such that, for some $b > \frac{1}{2}$,

$$|F(\nu)| = 0(\nu^b) \quad \nu \to 0, \quad |F(\nu)| = 0(\nu^{-b}), \quad \nu \to \infty\,, \tag{2}$$

one lets,

$$\Phi(s) = \int_N F(\nu)\,\nu^{1/2-s}\,d^*\nu\,. \tag{3}$$

Given a Grössencharakter $\mathcal{X}$, i.e., a character of C_k and any ρ in the strip $0 < \mathrm{Re}\,(\rho) < 1$, one lets $N(\mathcal{X},\rho)$ be the order of $L(\mathcal{X},s)$ at $s=\rho$. One lets

$$S(\mathcal{X},F) = \sum_\rho N(\mathcal{X},\rho)\,\Phi(\rho) \tag{4}$$

where the sum takes place over ρ's in the above open strip. One then defines a distribution Δ on C_k by

$$\Delta = \log|d^{-1}|\,\delta_1 + D - \sum_v D_v\,, \tag{5}$$

where δ_1 is the Dirac mass at $1 \in C_k$, where d is a differental idele of k so that $|d|^{-1}$ is up to sign the discriminant of k when char $(k)=0$, and is q^{2g-2} when k is a function field over a curve of genus g with coefficients in the finite field $\mathbb{F}_q$.

The distribution D is given by

$$D(f) = \int_{C_k} f(w)\,(|w|^{1/2} + |w|^{-1/2})\,d^*w\,, \tag{6}$$

where the Haar measure d^*w is normalized (cf. IIb). The distributions D_v are parametrized by the places v of k and are obtained as follows. For each v one considers the natural proper homomorphism

$$k_v^* \to C_k, \quad x \to \text{class of } (1, \dots, x, 1 \dots) \tag{7}$$

of the multiplicative group of the local field k_v in the idele class group C_k.

One then has

$$D_v(f) = Pfw \int_{k_v^*} \frac{f(u)}{|1-u|} |u|^{1/2} \, d^*u \,, \tag{8}$$

where the Haar measure d^*u is normalized (cf. IIb), and where the Weil principal value Pfw of the integral is obtained as follows, for a local field $K = k_v$,

$$Pfw \int_{k_v^*} 1_{R_v^*} \frac{1}{|1-u|} \, d^*u = 0 \,, \tag{9}$$

if the local field k_v is non Archimedean, and otherwise

$$Pfw \int_{k_v^*} \varphi(u) \, d^*u = PF_0 \int_{\mathbb{R}_+^*} \psi(\nu) \, d^*\nu \,, \tag{10}$$

where $\psi(\nu) = \int_{|u|=\nu} \varphi(u) \, d_\nu u$ is obtained by integrating φ over the fibers, while

$$PF_0 \int \psi(\nu) d^*\nu = 2\log(2\pi)\, c + \lim_{t\to\infty} \left(\int (1 - f_0^{2t}) \, \psi(\nu) \, d^*\nu - 2c \log t \right) \,, \tag{11}$$

where one assumes that $\psi - c \, f_1^{-1}$ is integrable on $\mathbb{R}_+^*$, and

$$f_0(\nu) = \inf(\nu^{1/2}, \nu^{-1/2}) \qquad \forall \, \nu \in \mathbb{R}_+^*, \qquad f_1 = f_0^{-1} - f_0 \,.$$

The Weil explicit formula is

Theorem 1 ([W])**.** *With the above notations one has* $S(\mathcal{X}, F) = \Delta(F(|w|) \, \mathcal{X}(w))$.

We shall now elaborate on this formula and in particular compare the principal values Pfw with those of Theorem V.3.

Let us make the following change of variables,

$$|g|^{-1/2} \, h(g^{-1}) = F(|g|) \, \mathcal{X}_0(g) \,, \tag{12}$$

and rewrite the above equality in terms of h.

By (3) one has

$$\Phi\left(\frac{1}{2} + is\right) = \int_{C_k} F(|g|) \, |g|^{-is} \, d^*g \,. \tag{13}$$

Thus, in terms of h,

$$\int h(g)\, \mathcal{X}_1(g)\, |g|^{1/2+is}\, d^*g = \int F(|g^{-1}|)\, \mathcal{X}_0(g^{-1})\, \mathcal{X}_1(g)\, |g|^{is}\, d^*g\,, \tag{14}$$

which is equal to 0 if $\mathcal{X}_1/C_{k,1} \neq \mathcal{X}_0/C_{k,1}$ and for $\mathcal{X}_1 = \mathcal{X}_0$,

$$\int h(g)\, \mathcal{X}_0(g)\, |g|^{1/2+is}\, d^*g = \Phi\left(\frac{1}{2} + is\right). \tag{15}$$

Thus, with our notations we see that

$$\mathrm{Supp}\, \widehat{h} \subset \mathcal{X}_0 \times \mathbb{R}, \quad \widehat{h}(\mathcal{X}_0, \rho) = \Phi(\rho)\,. \tag{16}$$

Thus we can write,

$$S(\mathcal{X}_0, F) = \sum_{\substack{L(\mathcal{X},\rho)=0, \mathcal{X}\in\widehat{C}_{k,1} \\ 0<\mathrm{Re}\,\rho<1}} \widehat{h}(\mathcal{X}, \rho) \tag{17}$$

using a fixed decomposition $C_k = C_{k,1} \times N$.

Let us now evaluate each term in (5). The first gives $(\log|d^{-1}|)\, h(1)$. One has, using (6) and (12),

$$\begin{aligned} \langle D, F(|g|)\, \mathcal{X}_0(g)\rangle &= \int_{C_k} |g|^{-1/2}\, h(g^{-1})\, (|g|^{1/2} + |g|^{-1/2})\, d^*g \\ &= \int_{C_k} h(u)\, (1+|u|)\, d^*u = \widehat{h}(0) + \widehat{h}(1)\,, \end{aligned}$$

where for the trivial character of $C_{k,1}$ one uses the notation

$$\widehat{h}(z) = \widehat{h}(1, z) \qquad \forall\, z \in \mathbb{C}\,. \tag{18}$$

Thus the first two terms of (5) give

$$(\log|d^{-1}|)\, h(1) + \widehat{h}(0) + \widehat{h}(1)\,. \tag{19}$$

Let then v be a place of k. One has by (8) and (12),

$$\langle D_v, F(|g|)\, \mathcal{X}_0(g)\rangle = Pfw \int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u\,.$$

We can thus write the contribution of the last term of (5) as

$$-\sum_v Pfw \int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u\,. \tag{20}$$

Thus the equality of Weil can be rewritten as

$$\begin{aligned}\widehat{h}(0)+\widehat{h}(1)-\sum_{\substack{L(\mathcal{X},\rho)=0,\mathcal{X}\in\widehat{C}_{k,1}\\ 0<\text{Re }\rho<1}}\widehat{h}(\mathcal{X},\rho)&=(\log|d|)\,h(1)\\ &+\sum_v Pfw\int_{k_v^*}\frac{h(u^{-1})}{|1-u|}\,d^*u\,,\end{aligned}\tag{21}$$

which now holds for finite linear combinations of functions h of the form (12).

This is enough to conclude when $h(1)=0$.

Let us now compare the Weil principal values, with those dictated by Theorem V.3. We first work with a local field K and compare (9), (10) with our prescription. Let first K be non Archimedean. Let α be a character of K such that

$$\alpha/R=1,\quad \alpha/\pi^{-1}R\neq 1\,.\tag{22}$$

Then, for the Fourier transform given by

$$(Ff)(x)=\int f(y)\,\alpha(xy)\,dy\,,\tag{23}$$

with dy the selfdual Haar measure, one has

$$F(1_R)=1_R\,.\tag{24}$$

Lemma 2. *With the above choice of* α *one has*

$$\int'\frac{h(u^{-1})}{|1-u|}\,d^*u=Pfw\int\frac{h(u^{-1})}{|1-u|}\,d^*u$$

with the notations of Theorem 3.

Proof. By construction the two sides can only differ by a multiple of $h(1)$. Let us recall from Theorem 3 that the left hand side is given by

$$\left\langle L,\,\frac{h(u^{-1})}{|u|}\right\rangle\,,\tag{25}$$

where L is the unique extension of $\rho^{-1}\frac{du}{|1-u|}$ whose Fourier transform vanishes at 1, $\widehat{L}(1)=0$. Thus from (9) we just need to check that (25) vanishes for $h=1_{R^*}$, i.e., that

$$\langle L,1_{R^*}\rangle=0\,.\tag{26}$$

Equivalently, if we let $Y = \{y \in K\,;\, |y-1| = 1\}$ we just need to show, using Parseval, that

$$\langle \log|u|\,,\, \widehat{1}_Y\rangle = 0\,. \tag{27}$$

One has $\widehat{1}_Y(x) = \int_Y \alpha(xy)\,dy = \alpha(x)\,\widehat{1}_{R^*}(x)$, and $1_{R^*} = 1_R - 1_P$, $\widehat{1}_{R^*} = 1_R - |\pi|\,1_{\pi^{-1}R}$, thus, with $q^{-1} = |\pi|$,

$$\widehat{1}_Y(x) = \alpha(x)\left(1_R - \frac{1}{q}\,1_{\pi^{-1}R}\right)(x)\,. \tag{28}$$

We now need to compute $\int \log|x|\,\widehat{1}_Y(x)\,dx = A + B$,

$$A = -\frac{1}{q}\int_{\pi^{-1}R^*} \alpha(x)\,(\log q)\,dx, \quad B = \left(1-\frac{1}{q}\right)\int_R \log|x|\,dx\,. \tag{29}$$

Let us show that $A+B=0$. One has $\int_R dx = 1$, and

$$\begin{aligned} A = & -\int_{R^*} \alpha(\pi^{-1}y)\,(\log q)\,dy = -\log q\left(\int_R \alpha(\pi^{-1}y)\,dy - \int_P dy\right) \\ = & \;\frac{1}{q}\,\log q\ ,\ \text{since}\ \int_R \alpha(\pi^{-1}y)\,dy = 0 \text{ as } \alpha/\pi^{-1}R \neq 1\,. \end{aligned}$$

To compute B, note that $\int_{\pi^n R^*} dy = q^{-n}\left(1-\frac{1}{q}\right)$ so that

$$B = \left(1-\frac{1}{q}\right)^2 \sum_{n=0}^{\infty}(-n\log q)\,q^{-n} = -q^{-1}\log q\,.$$

and $A+B=0$. □

Let us now treat the case of Archimedean fields. We take $K = \mathbb{R}$ first, and we normalize the Fourier transform as

$$(Ff)(x) = \int f(y)\,e^{-2\pi i xy}\,dy \tag{30}$$

so that the Haar measure dx is selfdual.

With the notations of (10) one has

$$Pfw\int_{\mathbb{R}^*} f_0^3(|u|)\,\frac{|u|^{1/2}}{|1-u|}\,d^*u = \log\pi + \gamma \tag{31}$$

where γ is Euler's constant, $\gamma = -\Gamma^{'}(1)$. Indeed integrating over the fibers gives $f_0^4 \times (1-f_0^4)^{-1}$, and one gets

$$PF_0 \int_{\mathbb{R}_+^*} f_0^4 \times (1-f_0^4)^{-1}\, d^*u = \Bigg(\log(2\pi) + \lim_{t\to\infty} \left(\int_{\mathbb{R}_+^*} (1-f_0^{2t})\, f_0^4 (1-f_0^4)^{-1}\, d^*u - \log t\right)\Bigg) = \log 2\pi + \gamma - \log 2\,.$$

Now let $\varphi(u) = -\log|u|$. It is a tempered distribution on $\mathbb{R}$ and one has

$$\langle \varphi, e^{-\pi u^2}\rangle = \frac{1}{2}\,\log \pi + \frac{\gamma}{2} + \log 2\,, \tag{32}$$

as one obtains from $\frac{\partial}{\partial s} \int |u|^{-s}\, e^{-\pi u^2}\, du = \frac{\partial}{\partial s}\left(\pi^{\frac{s-1}{2}}\, \Gamma\left(\frac{1-s}{2}\right)\right)$ evaluated at $s=0$, using $\frac{\Gamma^{'}(\frac{1}{2})}{\Gamma(\frac{1}{2})} = -\gamma - 2\log 2$.

Thus by the Parseval formula one has

$$\langle \widehat{\varphi}, e^{-\pi x^2}\rangle = \frac{1}{2}\log \pi + \frac{\gamma}{2} + \log 2\,, \tag{33}$$

which gives, for any test function f,

$$\langle \widehat{\varphi}, f\rangle = \lim_{\varepsilon\to 0}\left(\int_{|x|\geq\varepsilon} f(x)\, d^*x + (\log\varepsilon)\, f(0)\right) + \lambda\, f(0) \tag{34}$$

where $\lambda = \log(2\pi) + \gamma$. In order to get (34), one uses the equality

$$\lim_{\varepsilon\to 0}\left(\int_{|x|\geq\varepsilon} f(x)\, d^*x + (\log\varepsilon)\, f(0)\right) = \lim_{\varepsilon\to 0}\left(\int f(x)\, |x|^{\varepsilon}\, d^*x - \frac{1}{\varepsilon}\, f(0)\right), \tag{35}$$

which holds since both sides vanish for $f(x) = 1$ if $|x| \leq 1$, $f(x) = 0$ otherwise.

Thus from (34) one gets

$$\int_{\mathbb{R}}' f(u)\, \frac{1}{|1-u|}\, d^*u = \lambda\, f(1) + \lim_{\varepsilon\to 0}\left(\int_{|1-u|\geq\varepsilon} \frac{f(u)}{|1-u|}\, d^*u + (\log\varepsilon)\, f(1)\right). \tag{36}$$

Taking $f(u) = |u|^{1/2}\, f_0^3(|u|)$, the right hand side of (36) gives $\lambda - \log 2 = \log\pi + \gamma$; thus we conclude using (31) that for any test function f,

$$\int_{\mathbb{R}}' f(u)\, \frac{1}{|1-u|}\, d^*u = Pfw \int_{\mathbb{R}} f(u)\, \frac{1}{|1-u|}\, d^*u\,. \tag{37}$$

Let us finally consider the case $K = \mathbb{C}$. We choose the basic character α as

$$\alpha(z) = \exp 2\pi i(z + \overline{z}), \tag{38}$$

the selfdual Haar measure is $dz\, d\overline{z} = |dz \wedge d\overline{z}|$, and the function $f(z) = \exp -2\pi|z|^2$ is selfdual.

The normalized multiplicative Haar measure is

$$d^*z = \frac{|dz \wedge d\overline{z}|}{2\pi|z|^2}. \tag{39}$$

Let us compute the Fourier transform of the distribution

$$\varphi(z) = -\log|z|_{\mathbb{C}} = -2\log|z|. \tag{40}$$

One has

$$\langle \varphi, \exp -2\pi|z|^2 \rangle = \log 2\pi + \gamma, \tag{41}$$

as is seen using $\frac{\partial}{\partial\varepsilon}\left(\int e^{-2\pi|z|^2}\,|z|^{-2\varepsilon}\,|dz \wedge d\overline{z}|\right) = \frac{\partial}{\partial\varepsilon}\left((2\pi)^{\varepsilon}\,\Gamma(1-\varepsilon)\right)$.

Thus $\langle \widehat{\varphi}, \exp -2\pi|u|^2 \rangle = \log 2\pi + \gamma$ and one gets

$$\langle \widehat{\varphi}, f \rangle = \lim_{\varepsilon \to 0}\left(\int_{|u|_{\mathbb{C}} \geq \varepsilon} f(u)\, d^*u + \log\varepsilon\, f(0)\right) + \lambda'\, f(0) \tag{42}$$

where $\lambda' = 2(\log 2\pi + \gamma)$.

To see this one uses the analogue of (35) for $K = \mathbb{C}$ to compute the right hand side of (42) for $f(z) = \exp -2\pi|z|^2$.

Thus, for any test function f, one has

$$\int_{\mathbb{C}}' f(u)\, \frac{1}{|1-u|_{\mathbb{C}}}\, d^*u = \lambda'\, f(1) + \lim_{\varepsilon \to 0}\left(\int_{|1-u|_{\mathbb{C}} \geq \varepsilon} \frac{f(u)}{|1-u|_{\mathbb{C}}}\, d^*u + (\log\varepsilon) f(1)\right). \tag{43}$$

Let us compare it with Pfw. When one integrates over the fibers of $\mathbb{C}^* \xrightarrow{|\ |_{\mathbb{C}}} \mathbb{R}_+^*$ the function $|1-z|_{\mathbb{C}}^{-1}$, one gets

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-e^{i\theta}z|^2}\, d\theta = \frac{1}{1-|z|^2} \text{ if } |z| < 1, \text{ and } \frac{1}{|z|^2 - 1} \text{ if } |z| > 1. \tag{44}$$

Thus for any test function f on $\mathbb{R}_+^*$ one has, by (10),

$$Pfw \int f(|u|_{\mathbb{C}})\, \frac{1}{|1-u|_{\mathbb{C}}}\, d^*u = PF_0 \int f(\nu)\, \frac{1}{|1-\nu|}\, d^*\nu \tag{45}$$

with the notations of (11). With $f_2(\nu) = \nu^{\frac{1}{2}} f_0(\nu)$ we thus get, using (11),

$$Pfw \int f_2(|u|_{\mathbb{C}}) \frac{1}{|1-u|_{\mathbb{C}}} d^*u = PF_0 \int f_0 f_1^{-1} d^*\nu = 2(\log 2\pi + \gamma). \tag{46}$$

We shall now show that

$$\lim_{\varepsilon \to 0} \left(\int_{|1-u|_{\mathbb{C}} \geq \varepsilon} \frac{f_2(|u|_{\mathbb{C}})}{|1-u|_{\mathbb{C}}} d^*u + \log \varepsilon \right) = 0. \tag{47}$$

It will then follow that, using (43),

$$\int_{\mathbb{C}}' f(u) \frac{1}{|1-u|_{\mathbb{C}}} d^*u = Pfw \int f(u) \frac{1}{|1-u|_{\mathbb{C}}} d^*u. \tag{48}$$

To prove (47) it is enough to investigate the integral

$$\int_{|z|\leq 1, |1-z|\geq \varepsilon} ((1-z)(1-\overline{z}))^{-1} |dz \wedge d\overline{z}| = j(\varepsilon), \tag{49}$$

and show that $j(\varepsilon) = \alpha \log \varepsilon + o(1)$ for $\varepsilon \to 0$. A similar statement then holds for

$$\int_{|z|\leq 1, |1-z^{-1}|\geq \varepsilon} ((1-z)(1-\overline{z}))^{-1} |dz \wedge d\overline{z}|.$$

One has $j(\varepsilon) = \int_D |dZ \wedge d\overline{Z}|$, where $Z = \log(1-z)$ and the domain D is contained in the rectangle

$$\{Z = (x+iy)\,;\, \log \varepsilon \leq x \leq \log 2\,,\, -\frac{\pi}{2} \leq y \leq \pi/2\} = R_\varepsilon\,, \tag{50}$$

and bounded by the curve $x = \log(2\cos y)$ which comes from the equation of the circle $|z| = 1$ in polar coordinates centered at $z = 1$. One thus gets

$$j(\varepsilon) = 4 \int_{\log \varepsilon}^{\log 2} \text{Arc}\cos(e^x/2)\, dx\,; \tag{51}$$

when $\varepsilon \to 0$ one has $j(\varepsilon) \sim 2\pi \log(1/\varepsilon)$, which is the area of the following rectangle (in the measure $|dz \wedge d\overline{z}|$),

$$\{Z = (x+iy)\,;\, \log \varepsilon \leq x \leq 0\,,\, -\pi/2 \leq y \leq \pi/2\}. \tag{52}$$

One has $|R_\varepsilon| - 2\pi \log 2 = 2\pi \log(1/\varepsilon)$. When $\varepsilon \to 0$ the area of $R_\varepsilon \backslash D$ converges to

$$4 \int_{-\infty}^{\log 2} \text{Arc}\sin(e^x/2)\, dx = -4 \int_0^{\pi/2} \log(\sin u)\, du = 2\pi \log 2\,, \tag{53}$$

so that $j(\varepsilon) = 2\pi \log(1/\varepsilon) + o(1)$ when $\varepsilon \to 0$.

Thus we can assert that with the above choice of basic characters for local fields one has, for any test function f,

$$\int_K' f(u) \frac{1}{|1-u|} d^*u = Pfw \int f(u) \frac{1}{|1-u|} d^*u. \tag{54}$$

Lemma 3. *Let K be a local field, α_0 a normalized character as above and α, $\alpha(x) = \alpha_0(\lambda x)$ an arbitrary character of K. Let $\int'$ be defined as in Theorem* V.3 *relative to α. Then, for any test function f,*

$$\int_K' f(u) \frac{1}{|1-u|} d^*u = \log|\lambda| f(1) + Pfw \int f(u) \frac{1}{|1-u|} d^*u .$$

Proof. The new selfdual Haar measure is

$$da = |\lambda|^{1/2} d_0 a \tag{55}$$

with $d_0 a$ selfdual for α_0. Similarly, the new Fourier transform is given by

$$\widehat{f}(x) = \int \alpha(xy) f(y) dy = \int \alpha_0(\lambda xy) f(y) |\lambda|^{1/2} d_0 y .$$

Thus

$$\widehat{f}(x) = |\lambda|^{1/2} \widehat{f}^0(\lambda x) . \tag{56}$$

Let then $\varphi(u) = -\log|u|$. Its Fourier transform as a distribution is given by

$$\langle \widehat{\varphi}, f \rangle = \int (-\log|u|) \widehat{f}(u) du . \tag{57}$$

One has

$$\begin{aligned} \int (-\log|u|) \widehat{f}(u) du &= \int (-\log|u|) \widehat{f}^0(\lambda u) |\lambda| d_0 u \\ &= \int (-\log|v|) \widehat{f}^0(v) d_0 v + \int \log|\lambda| \widehat{f}^0(v) d_0 v \\ &= \int (-\log|v|) \widehat{f}^0(v) d_0 v + \log|\lambda| f(0) . \end{aligned}$$

Thus the lemma follows from (54). □

Let us now pass to the global case. Recall that if α, $\alpha \neq 1$, is a character of A such that $\alpha/k = 1$, there exists a differental idele $d = (d_v)$ such that, (cf. [W1])

$$\alpha_v(x) = \alpha_{0,v}(d_v x) \tag{58}$$

where $\alpha = \Pi \alpha_v$ and each local character $\alpha_{0,v}$ is normalized as above.

We can thus rewrite the Weil formula (Theorem 1) as

Theorem 6. *Let k be a global field, α a nontrivial character of A/k and $\alpha = \Pi\, \alpha_v$ its local factors.*

Let $h \in \mathcal{S}(C_k)$ have compact support. Then

$$\widehat{h}(0) + \widehat{h}(1) - \sum_{\substack{L(\mathcal{X},\rho)=0 \\ 0<\,\mathrm{Re}\,\rho<1}} \widehat{h}(\mathcal{X},\rho) = \sum_v \int_{k_v^*}' \frac{h(u^{-1})}{|1-u|}\, d^*u$$

where the normalization of $\int'$ is given by α_v as in Theorem V.3, and $\widehat{h}(\mathcal{X},z) = \int h(u)\, \mathcal{X}(u)\, |u|^z\, d^*u$.

Proof. This follows from formula (21), Lemma 3 and the equality $\log |d| = \sum_v \log |d_v|$. □

Normalization of Haar measure on modulated group

We let G be a locally compact abelian group with a proper morphism

$$g \to |g|\,,\ G \to \mathbb{R}_+^* \tag{1}$$

whose range is cocompact in $\mathbb{R}_+^*$.

There exists a unique Haar measure d^*g on G such that

$$\int_{|g|\in[1,\Lambda]} d^*g \sim \log \Lambda \quad \text{when} \quad \Lambda \to +\infty\,. \tag{2}$$

Let $G_0 = \mathrm{Ker\ mod} = \{g \in G\,;\, |g| = 1\}$. It is a compact group by hypothesis, and one can identify G/G_0 with the range N of the module. Let us determine the measure d^*n on $N \subset \mathbb{R}_+^*$ such that (2) holds for

$$\int f\, d^*g = \int \left(\int f(ng_0)\, dg_0 \right) d^*n \tag{3}$$

where the Haar measure dg_0 is normalized by

$$\int_{G_0} dg_0 = 1\,. \tag{4}$$

We let ρ_Λ be the function on G given by

$$\rho_\Lambda(g) = 0 \quad \text{if } |g| \notin [1,\Lambda]\,,\ \rho_\Lambda(g) = \frac{1}{\log \Lambda} \quad \text{if } g \in [1,\Lambda]\,. \tag{5}$$

The normalization (2) means that $\int \rho_\Lambda \, d^*g \to 1$ when $\Lambda \to \infty$.

Let first $N = \mathbb{R}_+^*$. Then the unique measure satisfying (2) is

$$d^*\lambda = \frac{d\lambda}{\lambda}\,. \tag{6}$$

Let then $N = \mu^{\mathbb{Z}}$ for some $\mu > 1$. Let us consider the measure

$$\int f \, d^*g = \alpha \sum f(\mu^n)\,. \tag{7}$$

We take $f = \rho_\Lambda$. Then the right hand side is $\alpha \, \frac{N}{\log \Lambda}$ where N is the number of $\mu^n \in [1, \Lambda]$, i.e., $\sim \frac{\log \Lambda}{\log \mu}$. This shows that (2) holds iff

$$\alpha = \log \mu\,. \tag{8}$$

Let us show more generally that if $H \subset G$ is a compact subgroup of G and if both d^*g and d^*h are normalized by (2), one has

$$\int \left(\int f(hy) \, d^*h \right) d_0\, y = \int f \, d^*g \tag{9}$$

where $d_0\, y$ is the Haar measure of integral 1 on G/H,

$$\int_{G/H} d_0\, y = 1\,. \tag{10}$$

The left hand side of (9) defines a Haar measure on G and we just need to show that it satisfies (2).

One has $\|\rho_\Lambda(\cdot\, y) - \rho_\Lambda\|_1 \to 0 \quad$ when $\Lambda \to \infty$, and

$$\int \rho_\Lambda(hy) \, d^*h \to 1 \quad \text{when} \quad \Lambda \to \infty \tag{11}$$

uniformly on compact sets of $y \in G$. Thus

$$\int \left(\int \rho_\Lambda(hy) \, d^*h \right) d_0\, y \to 1 \quad \text{when} \quad \Lambda \to \infty\,. \tag{12}$$

Appendix III. Distribution trace formulas

In this appendix we recall for the convenience of the reader the coordinate free treatment of distributions of [GS] and give the details of the transversality conditions.

Given a vector space E over $\mathbb{R}$, $\dim E = n$, a density is a map, $\rho \in |E^*|$,

$$\rho : \wedge^n E \to \mathbb{C} \tag{1}$$

such that $\rho(\lambda v) = |\lambda|\, \rho(v) \quad \forall\, \lambda \in \mathbb{R}, \quad \forall\, v \in \wedge^n E$. Given a linear map $T : E \to F$ we let $|T^*| : |F^*| \to |E^*|$ be the corresponding linear map; it depends contravariantly on T.

A smooth compactly supported density $\rho \in C_c^\infty(M, |T^* M|)$ on an arbitrary manifold M has a canonical integral

$$\int \rho \in \mathbb{C}\,. \tag{2}$$

One defines the generalized sections of a vector bundle L on M as the dual space of $C_c^\infty(M, L^* \otimes |T^* M|)$

$$C^{-\infty}(M, L) = \text{ dual of } C_c^\infty(M, L^* \otimes |T^* M|) \tag{3}$$

where L^* is the dual bundle. One has a natural inclusion

$$C^\infty(M, L) \subset C^{-\infty}(M, L) \tag{4}$$

given by the pairing

$$\sigma \in C^\infty(M, L), \quad s \in C_c^\infty(M, L^* \otimes |T^* M|) \to \int \langle s, \sigma \rangle \tag{5}$$

where $\langle s, \sigma \rangle$ is viewed as a density, $\langle s, \sigma \rangle \in C_c^\infty(M, |T^* M|)$.

One has a similar notion of generalized section with compact support. Given a smooth map $\varphi : X \to Y$, then if φ is *proper*, it gives a (contravariantly) associated map

$$\varphi^* : C_c^\infty(Y, L) \to C_c^\infty(X, \varphi^*(L)), \quad (\varphi^*\, \xi)(x) = \xi(\varphi(x)) \tag{6}$$

where $\varphi^*(L)$ is the pullback of the vector bundle L. Thus, given a linear form on $C_c^\infty(X, \varphi^*(L))$ one has a (covariantly) associated linear form on $C_c^\infty(Y, L)$. In particular with L trivial, we see that generalized densities $\rho \in C^{-\infty}(X, |T^* X|)$ pushforward,

$$\varphi_*(\rho) \in C^{-\infty}(Y, |T^* Y|) \tag{7}$$

with $\langle \varphi_*(\rho), \xi \rangle = \langle \rho, \varphi^*\, \xi \rangle \quad \forall\, \xi \in C_c^\infty(X)$.

This gives the natural functoriality of generalized sections, they pushforward under proper maps. However under suitable transversality conditions which are automatic for submersions, generalized sections also pull back. For instance, if φ is a fibration and $\rho \in C_c^\infty(X, |T^* X|)$ is a density then one can integrate ρ along the fibers, the obtained density on Y, $\varphi_*(\rho)$ is given as in (7) by

$$\langle \varphi_*(\rho), f \rangle = \langle \rho, \varphi^*\, f \rangle \qquad \forall\, f \in C^\infty(Y). \tag{8}$$

The point is that the result is not only a generalized section but a smooth section $\varphi_*(\rho) \in C_c^\infty(Y, |T^*Y|)$.

It follows that if $f \in C^{-\infty}(Y)$ is a generalized function, then one obtains a generalized function $\varphi^*(f)$ on X by

$$\langle \varphi^*(f), \rho\rangle = \langle f, \varphi_*(\rho)\rangle \qquad \forall\, \rho \in C_c^\infty(X, |T^*X|)\,. \tag{9}$$

In general, the pullback $\varphi^*(f)$ of a generalized function f, continues to make sense provided the following transversality condition holds,

$$d(\varphi^*(l)) \neq 0 \qquad \forall l \in WF(f)\,. \tag{10}$$

where $WF(f)$ is the wave front set of f ([GS]).

Next, let us recall the construction ([GS]) of the generalized section of a vector bundle L on a manifold X associated to a submanifold $Z \subset X$ and a symbol

$$\sigma \in C^\infty(Z, L \otimes |N_Z|)\,, \tag{11}$$

where N_Z is the normal bundle of Z. The construction is the same as that of the current of integration on a cycle. Given $\xi \in C_c^\infty(X, L^* \otimes |T^*X|)$, the product $\sigma\,\xi/Z$ is a density on Z, since it is a section of $|T_Z^*| = |T_X^*| \otimes |N_Z|$. One can thus integrate it over Z. When $Z = X$, one has $N_Z = \{0\}$ and $|N_Z|$ has a canonical section, so that the current associated to σ is just given by (5).

Now let $\varphi : X \to Y$ with Z a submanifold of Y and σ as in (11). Let us assume that φ is transverse to Z, so that for each $x \in X$ with $y = \varphi(x) \in Z$ one has

$$\varphi_*(T_x) + T_{\varphi(x)}(Z) = T_y\, Y\,. \tag{12}$$

Let

$$\tau_x = \{X \in T_x, \quad \varphi_*(X) \in T_y(Z)\}\,. \tag{13}$$

Then φ_* gives a canonical isomorphism

$$\varphi_* : T_x(X)/\tau_x \simeq T_y(Y)/T_y(Z) = N_y(Z)\,. \tag{14}$$

And $\varphi^{-1}(Z)$ is a submanifold of X of the same codimension as Z with a natural isomorphism of normal bundles

$$\varphi_* :\ N_{\varphi^{-1}(Z)} \simeq \varphi^*\, N_Z\,. \tag{15}$$

In particular, given a (generalized) δ-section of a bundle L with support Z and symbol $\sigma \in C^\infty(Z, L \otimes |N_Z|)$, one has a corresponding symbol on $\varphi^{-1}(Z)$ given by

$$\varphi^*\, \sigma(x) = |(\varphi_*)^{-1}|\sigma(\varphi(x)) \in (\varphi^*\, L)_x \otimes |N_x| \tag{16}$$

using the inverse of the isomorphism (15), which requires the transversality condition.

For any δ-section associated to Z, σ, the wave front set is contained in the conormal bundle of the submanifold Z which shows that if φ is transverse to Z the pull back $\varphi^* \, \delta_{Z,\sigma}$ of the distribution on Y associated to Z, σ makes sense, it is equal to $\delta_{\varphi^{-1}(Z), \varphi^*(\sigma)}$.

Let us now recall the formulation ([GS]) of the Schwartz kernel theorem. One considers a continuous linear map

$$T : C_c^\infty(Y) \to C^{-\infty}(X) \, . \tag{17}$$

The statement is that one can write it as

$$(T\,\xi)\,(x) = \int k(x,y)\,\xi(y)\,dy \tag{18}$$

where $k(x,y)\,dy$ is a generalized section

$$k \in C^{-\infty}(X \times Y, \quad \mathrm{pr}_Y^*(|T^*Y|)) \, . \tag{19}$$

Let $f : X \to Y$ be a smooth map, and $T = f^*$ the operator

$$(T\,\xi)\,(x) = \xi\,(f(x)) \qquad \forall\, \xi \in C_c^\infty(Y) \, . \tag{20}$$

The corresponding k is the δ-section associated to the submanifold of $X \times Y$ given by

$$\mathrm{Graph}\,(f) = \{(x, f(x)) \; ; \; x \in X\} = Z \tag{21}$$

and its symbol, $\sigma \in C^\infty(Z, \, \mathrm{pr}_Y^*(|T^*Y|) \otimes |N_Z|)$ is obtained as follows.

Given $\xi \in T_x^*(X)$, $\eta \in T_y^*(Y)$, one has $(\xi, \eta) \in N_Z^*$ iff it is orthogonal to $(v, f_*\,v)$ for any $v \in T_x(X)$, i.e., $\langle v, \xi \rangle + \langle f_*\,v, \eta \rangle = 0$ so that

$$\xi = -f_*^t\,\eta \, . \tag{22}$$

Thus one has a canonical isomorphism $j : T_y^*(Y) \simeq N_Z^*$, $\eta \overset{j}{\to} (-f_*^t\,\eta, \eta)$. The transposed $(j^{-1})^t$ is given by $(j^{-1})^t(Y) =$ class of $(0, Y)$ in N_Z, $\forall\, Y \in T_y(Y)$. One has,

$$\sigma = |j^{-1}| \in C^\infty(Z, \, \mathrm{pr}_Y^*(|T^*Y|) \otimes |N_Z|) \, . \tag{23}$$

We denote the corresponding δ-distribution by

$$k(x,y)\,dy = \delta(y - f(x))\,dy \, . \tag{24}$$

One then checks the formula

$$\int \delta(y - f(x))\,\xi(y)\,dy = \xi(f(x)) \qquad \forall\, \xi \in C_c^\infty(Y) \, . \tag{25}$$

Let us now consider a manifold M with a flow F_t

$$F_t(x) = \exp(t\,v)\,x \qquad v \in C^\infty(M, T_M) \tag{26}$$

and the corresponding map f,

$$f : M \times \mathbb{R} \to M, \quad f(x,t) = F_t(x)\,. \tag{27}$$

We apply the above discussion with $X = M \times \mathbb{R}$, $Y = M$. The graph of f is the submanifold Z of $X \times Y$

$$Z = \{(x,t,y)\ ;\ y = F_t(x)\}\,. \tag{28}$$

One lets φ be the diagonal map

$$\varphi(x,t) = (x,t,x), \quad \varphi : M \times \mathbb{R} \to X \times Y \tag{29}$$

and the first issue is the transversality $\varphi \pitchfork Z$.

We thus need to consider (12) for each (x,t) such that $\varphi(x,t) \in Z$, i.e., such that $x = F_t(x)$. One looks at the image by φ_* of the tangent space $T_x\,M \times \mathbb{R}$ to $M \times \mathbb{R}$ at (x,t). One lets ∂_t be the natural vector field on $\mathbb{R}$. The image of $(X, \lambda\,\partial_t)$ is $(X, \lambda\,\partial_t, X)$ for $X \in T_x\,M, \lambda \in \mathbb{R}$. Dividing the tangent space of $M \times \mathbb{R} \times M$ by the image of φ_*, one gets an isomorphism

$$(X, \lambda\,\partial_t, Y) \to Y - X \tag{30}$$

with $T_x\,M$. The tangent space to Z is $\{(X', \mu\,\partial_t, (F_t)_*\,X' + \mu\,v_{F_t(x)});\ X' \in T_x\,M, \mu \in \mathbb{R}\}$. Thus the transversality condition means that every element of $T_x\,M$ is of the form

$$(F_t)_*\,X - X + \mu\,v_x \qquad X \in T_x\,M, \quad \mu \in \mathbb{R}\,. \tag{31}$$

One has

$$(F_t)_*\,\mu\,v_x = \mu\,v_x \tag{32}$$

so that $(F_t)_*$ defines a quotient map, the Poincaré return map

$$P : T_x/\mathbb{R}\,v_x \to T_x/\mathbb{R}\,v_x = N_x \tag{33}$$

and the transversality condition (31) means exactly that

$$1 - P \qquad \text{is invertible.} \tag{34}$$

Let us make this hypothesis and compute the symbol σ of the distribution

$$\tau = \varphi^*(\delta(y - F_t(x))\,dy)\,. \tag{35}$$

First, as above, let $W = \varphi^{-1}(Z) = \{(x,t)\ ;\ F_t(x) = x\}$. The codimension of $\varphi^{-1}(Z)$ in $M \times \mathbb{R}$ is the same as the codimension of Z in $M \times \mathbb{R} \times M$, so it is $\dim M$ which shows that $\varphi^{-1}(Z)$ is 1-dimensional. If $(x,t) \in \varphi^{-1}(Z)$, then $(F_s(x),t) \in \varphi^{-1}(Z)$. Thus, if we assume that v does not vanish at x, the map

$$(x,t) \xrightarrow{q} t \tag{36}$$

is locally constant on the connected component of $\varphi^{-1}(Z)$ containing (x,t).

This allows us to identify the transverse space to $W = \varphi^{-1}(Z)$ as the product

$$N_{x,t}^W \simeq N_x \times \mathbb{R}\,, \tag{37}$$

where to $(X, \lambda\,\partial_t) \in T_{x,t}(M \times \mathbb{R})$, we associate the pair $(\widetilde{X}, \lambda)$ given by the class of X in $N_x = T_x/\mathbb{R}\,v_x$ and $\lambda \in \mathbb{R}$.

The symbol σ of the distribution (35) is a smooth section of $|N^W|$ tensored by the pull back $\varphi^*(L)$ where $L = \mathrm{pr}_Y^*\,|T_M^*|$, and one has

$$\varphi^*(L) \simeq |p^*\,T_M^*| \tag{38}$$

where

$$p(x,t) = x \qquad \forall\,(x,t) \in M \times \mathbb{R}\,. \tag{39}$$

To compute σ one needs the isomorphism

$$N_{(x,t)}^W \xrightarrow{\varphi_*} T_{\varphi(x,t)}(M \times \mathbb{R} \times M)/T_{\varphi(x,t)}(Z) = N^Z\,. \tag{40}$$

The map $\varphi_* : N_{x,t}^W \to N^Z$ is given by

$$\varphi_*(X, \lambda\,\partial_t) = (1 - (F_t)_*)\,X - \lambda\,v \qquad X \in N_x, \quad \lambda \in \mathbb{R} \tag{41}$$

and the symbol σ is just

$$\sigma = |\varphi_*^{-1}| \in |p^*\,T_M^*| \otimes |N^W|\,. \tag{42}$$

Let us now consider the second projection

$$q(x,t) = t \in \mathbb{R}\,, \tag{43}$$

and compute the pushforward $q_*(\tau)$ of the distribution τ. By construction $q_*(\tau)$ is a generalized function. We first look at the contribution of a periodic orbit. The corresponding part of $\varphi^{-1}(Z)$ is of the form

$$\varphi^{-1}(Z) = V \times \Gamma \subset M \times \mathbb{R}\,, \tag{44}$$

where Γ is a discrete cocompact subgroup of $\mathbb{R}$, while $V \subset M$ is a one dimensional compact submanifold of M.

To compute $q_*(\tau)$, we let $h(t)\,|dt|$ be a 1-density on $\mathbb{R}$ and pull it back by q as the section on $M \times \mathbb{R}$ of the bundle $q^*\,|T^*|$,

$$\xi(x,t) = h(t)\,|dt|\,. \tag{45}$$

We now need to compute $\int_{\varphi-1(Z)} \xi\,\sigma$. We can look at the contribution of each component $V \times \{T\}$, $T \in \Gamma$.

One gets ([GS]),

$$T^{\#} \frac{1}{|1 - P_T|}\; h(T)\,, \tag{46}$$

where $T^{\#}$ is the length of the primitive orbit or equivalently the covolume of Γ in $\mathbb{R}$ for the Haar measure $|dt|$. We can thus write the contributions of the periodic orbits as

$$\sum_{\gamma_p} \sum_{\Gamma} \operatorname{Covol}(\Gamma)\, \frac{1}{|1 - P_T|}\; h(T)\,, \tag{47}$$

where the test function h vanishes at 0.

The next case to consider is when the vector field v_x has an isolated 0, $v_{x_0} = 0$. In this case, the transversality condition (31) becomes

$$1 - (F_t)_* \quad \text{invertible (at } x_0)\,. \tag{48}$$

One has $F_t(x_0) = x_0$ for all $t \in \mathbb{R}$ and now the relevant component of $\varphi^{-1}(Z)$ is $\{x_0\} \times \mathbb{R}$. The transverse space N^W is identified with T_x and the map $\varphi_* : N^W \simeq N^Z$ is given by

$$\varphi_* = 1 - (F_t)_*\,. \tag{49}$$

Thus the symbol σ is the scalar function $|1 - (F_t)_*|^{-1}$. The generalized section $q_*\,\varphi^*(\delta(y - F_t(x))\,dy)$ is the function, $t \to |1 - (F_t)_*|^{-1}$. We can thus write the contribution of the zeros of the flow as ([GS])

$$\sum_{\text{zeros}} \int \frac{h(t)}{|1 - (F_t)_*|}\; dt \tag{50}$$

where h is a test function vanishing at 0.

We can thus collect the contributions 47 and 50 as

$$\sum_{\gamma} \int_{I_\gamma} \frac{h(u)}{|1 - (F_u)_*|}\; d^*u\,, \tag{51}$$

where h is as above, I_γ is the isotropy group of the periodic orbit γ, the Haar measure d^*u on I_γ is normalized so that the covolume of I_γ is equal to one, and we still write $(F_u)_*$ for its restriction to the transverse space of γ.

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