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Geometry of conservation laws for a class of parabolic PDE's, II: Normal forms for equations with conservation laws

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Abstract. We consider conservation laws for second-order parabolic partial differential equations for one function of three independent variables. An explicit normal form is given for such equations having a nontrivial conservation law. It is shown that any such equation whose space of conservation laws has dimension at least four is locally contact equivalent to a quasi-linear equation. Examples are given of nonlinear equations that have an infinite-dimensional space of conservation laws parameterized (in the sense of Cartan-Kähler) by two arbitrary functions of one variable. Furthermore, it is shown that any equation whose space of conservation laws is larger than this is locally contact equivalent to a linear equation.

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1. Introduction

In [4], the author studied conservation laws for second-order, parabolic partial differential equations for one function of three independent variables. These are equations of the form

$$u_t = G\big(x,\, y,\, t,\, u,\, u_x,\, u_y,\, u_{xx},\, u_{xy},\, u_{yy}\big) \tag{1.1}$$

where the right-hand side of (1.1) is an elliptic operator, i.e.,

$$\begin{vmatrix} G_{u_{xx}} & \frac{1}{2}G_{u_{xy}} \\ \frac{1}{2}G_{u_{xy}} & G_{u_{yy}} \end{vmatrix} > 0.$$

Such an equation may be formulated as an exterior differential system $\mathcal{I}$ on a 12-dimensional manifold M, and its space of conservation laws is identified with the vector space of closed 3-forms in the infinite prolongation of $\mathcal{I}$ modulo the so-called "trivial" conservation laws.

The principal result of [4] is the following theorem.

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Theorem 1. *Any conservation law for a second-order, parabolic PDE for one function of three independent variables can be represented by a closed* 3*-form in the differential ideal* $\mathcal{I}$ *on the original* 12*-manifold* M.

In other words, passing to the infinite prolongation of $\mathcal{I}$ is not necessary. This means that every conservation law has an expression that depends on at most second derivatives of the unknown function u.

The author showed that if an equation of this type has a nontrivial conservation law, then the differential system $\mathcal{I}$ is locally equivalent to a system $\mathcal{J}$ of a certain type, called a *parabolic Monge-Ampère system*, on a 7-manifold N, and any conservation law for the original system can be represented by a closed 3-form in $\mathcal{J}$. For equations in the real analytic category, the following normal form result was obtained.

Theorem 2. *Let* $\mathcal{J}$ *be a real analytic, parabolic Monge-Ampère system on a* 7*-manifold* N. *Then* $\mathcal{J}$ *is locally contact equivalent to a Monge-Ampère system that corresponds to a (parabolic) equation of the form*

$$A\left(u_{xx}\,u_{yy}-u_{xy}^2\right)+B\,u_{xx}+2C\,u_{xy}+D\,u_{yy}+E=0 \tag{1.2}$$

where A, B, C, D, E *are functions of the variables* x, y, t, u, u_x, u_y, u_t *such that equation* (1.2) *can locally be solved for* u_t *and* $BD-AE-C^2>0$. *Conversely, for any functions* A, B,C, D, E *of* x, y, t, u, u_x, u_y, u_t *that satisfy these conditions, the equation* (1.2) *corresponds to a parabolic Monge-Ampère system.*

In particular, any equation of this type in the real analytic category that has a nontrivial conservation law is locally contact equivalent to an equation of the form (1.2).

In this paper we refine this result by obtaining an explicit normal form for equations of this type having a nontrivial conservation law. We also consider equations whose space of conservation laws has dimension greater than one, and we obtain the following results:

Theorem (cf. Theorem 3.1)**.** *Any parabolic PDE of the form* (2.2) *whose space of conservation laws has dimension at least* 2 *is locally contact equivalent to an equation of the form*

$$u_t=A'\,(u_{xx}u_{yy}-u_{xy}^2)+B'\,u_{xx}+2C'\,u_{xy}+D'\,u_{yy}+E'$$

where the coefficients A', B', C', D', E' *are functions of* x, y, t, u, u_x, u_y.

Theorem (cf. Theorem 4.1)**.** *Any parabolic PDE of the form* (2.2) *whose space of conservation laws has dimension at least* 4 *is locally contact equivalent to an equation of the form*

$$u_t=B'\,u_{xx}+2C'\,u_{xy}+D'\,u_{yy}+E'$$

where the coefficients B', C', D', E' *are functions of* x, y, t, u, u_x, u_y. *In particular, such an equation is locally contact equivalent to a quasi-linear equation.*

Theorem (cf. Theorem 5.3)**.** *Let $\mathcal{I}$ be a parabolic Monge-Ampère system whose space of conservation laws has parameter space (in the sense of Cartan-Kähler) strictly larger than the space of two arbitrary functions of one variable. Then $\mathcal{I}$ is locally contact equivalent to a system that arises from a linear parabolic partial differential equation, and the space of conservation laws is parameterized by two arbitrary functions of two variables.*

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2. A normal form for equations that have a conservation law

A few words about notation are in order. We will work with a parabolic Monge-Ampère system $\mathcal{I}$ on a 7-dimensional manifold M with local coordinates $\{x^1, x^2, x^3, u, p_1, p_2, p_3\}$. The system $\mathcal{I}$ that corresponds to the PDE (1.2) is generated locally by the forms

$$\begin{aligned}
\theta &= du - p_1\,dx^1 - p_2\,dx^2 - p_3\,dx^3 \\
d\theta &= -dp_1 \wedge dx^1 - dp_2 \wedge dx^2 - dp_3 \wedge dx^3 \\
\Upsilon &= \Big[A\,dp_1 \wedge dp_2 + B\,dp_1 \wedge dx^2 + C\left(dx^1 \wedge dp_1 + dp_2 \wedge dx^2\right) \\
&\qquad + D\,dx^1 \wedge dp_2 + E\,dx^1 \wedge dx^2\Big] \wedge dx^3.
\end{aligned}$$

Define vector fields X_i on M as follows:

$$\begin{aligned}
X_0 &= \frac{\partial}{\partial u} \\
X_1 &= \frac{\partial}{\partial x^1} + p_1 \frac{\partial}{\partial u} \\
X_2 &= \frac{\partial}{\partial x^2} + p_2 \frac{\partial}{\partial u} \\
X_3 &= \frac{\partial}{\partial x^3} + p_3 \frac{\partial}{\partial u} \\
X_4 &= \frac{\partial}{\partial p_1} \\
X_5 &= \frac{\partial}{\partial p_2} \\
X_6 &= \frac{\partial}{\partial p_3}.
\end{aligned}$$

At each point $p \in M$, the vectors $\{X_0, X_1, X_2, X_3, X_4, X_5, X_6\}$ form a basis of T_pM which is dual to the basis $\{\theta,\ dx^1,\ dx^2,\ dx^3,\ dp_1,\ dp_2,\ dp_3\}$ of T_p^*M. For any function f on M, let f_i denote $X_i(f)$. Then

$$df = f_0\,\theta + f_1\,dx^1 + f_2\,dx^2 + f_3\,dx^3 + f_4\,dp_1 + f_5\,dp_2 + f_6\,dp_3.$$

For second and higher-order derivatives, it is not difficult to show that mixed partials *almost* commute: $f_{ij} = f_{ji}$ unless $\{i,j\}$ is one of the pairs $\{1,4\}$, $\{2,5\}$, $\{3,6\}$. These mixed partials satisfy the equations

$$\begin{aligned} f_{14} &= f_{41} + f_0 \\ f_{25} &= f_{52} + f_0 \\ f_{36} &= f_{63} + f_0. \end{aligned}$$

2.1. A refinement of Theorem 2

Suppose that the parabolic equation

$$A(u_{xx}u_{yy} - u_{xy}^2) + B\,u_{xx} + 2C\,u_{xy} + D\,u_{yy} + E = 0 \tag{2.1}$$

has a nontrivial conservation law Φ. It is shown in [4] that Φ has a representative of the form

$$\begin{aligned} \Phi = Q\Big[&A\,dp_1 \wedge dp_2 + B\,dp_1 \wedge dx^2 + C\left(dx^1 \wedge dp_1 + dp_2 \wedge dx^2\right) \\ &+ D\,dx^1 \wedge dp_2 + E\,dx^1 \wedge dx^2\Big] \wedge dx^3 + \theta \wedge \psi \end{aligned}$$

for some function Q and some 2-form ψ. By scaling the coefficients A, B, C, D, E appropriately, we can assume that $Q \equiv 1$. (Note that this choice of scaling depends on the *particular* conservation law Φ.) So suppose that

$$\begin{aligned} \Phi = \Big[&A\,dp_1 \wedge dp_2 + B\,dp_1 \wedge dx^2 + C\left(dx^1 \wedge dp_1 + dp_2 \wedge dx^2\right) \\ &+ D\,dx^1 \wedge dp_2 + E\,dx^1 \wedge dx^2\Big] \wedge dx^3 + \theta \wedge \psi \end{aligned}$$

is a conservation law. Computing $d\Phi \equiv 0 \mod \theta$ shows that

$$\begin{aligned} \psi = {} & A_6\,dp_1 \wedge dp_2 + B_6\,dp_1 \wedge dx^2 + C_6\left(dx^1 \wedge dp_1 + dp_2 \wedge dx^2\right) + D_6\,dx^1 \wedge dp_2 \\ & + E_6\,dx^1 \wedge dx^2 + \left(E_4 - B_1 - C_2\right) dx^2 \wedge dx^3 + \left(E_5 - C_1 - D_2\right) dx^3 \wedge dx^1 \\ & + \left(A_2 - B_5 + C_4\right) dp_1 \wedge dx^3 + \left(A_1 + C_5 - D_4\right) dx^3 \wedge dp_2. \end{aligned}$$

Now computing $d\Phi \equiv 0$ modulo various combinations of forms shows that

$$\begin{aligned} A_{66} &= 0 \\ B_{66} &= 0 \\ C_{66} &= 0 \\ D_{66} &= 0 \\ E_{66} &= 0. \end{aligned}$$

Therefore, A, B, C, D, E have the form

$$\begin{aligned} A &= A' - A''p_3 \\ B &= B' - B''p_3 \\ C &= C' - C''p_3 \\ D &= D' - D''p_3 \\ E &= E' - E''p_3, \end{aligned}$$

where $A', B', C', D', E', A'', B'', C'', D'', E''$ are functions of the variables x^1, x^2, x^3, u, p_1, p_2. The condition that (2.1) can locally be solved for u_t implies that at least one of A'', B'', C'', D'', E'' is nonzero; therefore equation (2.1) can be written as

$$u_t = \frac{A'(u_{xx}u_{yy} - u_{xy}^2) + B'u_{xx} + 2C'u_{xy} + D'u_{yy} + E'}{A''(u_{xx}u_{yy} - u_{xy}^2) + B''u_{xx} + 2C''u_{xy} + D''u_{yy} + E''}, \tag{2.2}$$

where the coefficients are functions of x, y, t, u, u_x, u_y and the right-hand side of the equation is elliptic.

2.2. An equivalent system on a 6-manifold

Let Φ be a conservation law as above. Set

$$\begin{aligned} \Phi' = A'dp_1 \wedge dp_2 + B'dp_1 \wedge dx^2 + C'\big(dx^1 \wedge dp_1 + dp_2 \wedge dx^2\big) \\ + D'dx^1 \wedge dp_2 + E'dx^1 \wedge dx^2 \end{aligned}$$

$$\begin{aligned} \Phi'' = A''dp_1 \wedge dp_2 + B''dp_1 \wedge dx^2 + C''\big(dx^1 \wedge dp_1 + dp_2 \wedge dx^2\big) \\ + D''dx^1 \wedge dp_2 + E''dx^1 \wedge dx^2. \end{aligned}$$

From the computation of ψ above (note that $A_6 = -A''$, etc.), it follows that Φ may be written as

$$\begin{aligned} \Phi &= \big(\Phi' - p_3\Phi''\big) \wedge dx^3 - \theta \wedge \big(\Phi'' + \kappa \wedge dx^3\big) \\ &= \Phi' \wedge dx^3 - \Phi'' \wedge \big(du - p_1\, dx^1 - p_2\, dx^2\big) + \kappa \wedge \theta \wedge dx^3 \end{aligned}$$

where κ is a 1-form whose exact expression is not relevant, except to note that it contains no dp_3 terms.

Since Φ contains no dp_3 terms and $d\Phi = 0$, Φ is well-defined on an open subset $U \subset \mathbb{R}^6$ with coordinates $\{x^1, x^2, x^3, u, p_1, p_2\}$. Set

$$\overline{\theta} = du - p_1\, dx^1 - p_2\, dx^2$$

and let $\mathcal{I}'$ be the ideal on U defined by

$$\mathcal{I}' = \{\overline{\theta} \wedge dx^3, \Phi\}.$$

Integral manifolds of $\mathcal{I}'$ are locally in one-to-one correspondence with integral manifolds of $\mathcal{I}$, and any conservation law for $\mathcal{I}$ may be expressed as a closed 3-form in $\mathcal{I}'$.

Define vector fields X_i on U as on M, with the exceptions that there is no vector field X_6 and

$$X_3 = \frac{\partial}{\partial x^3}.$$

The noncommuting mixed partial derivatives on U are

$$\begin{aligned} f_{14} &= f_{41} + f_0 \\ f_{25} &= f_{52} + f_0. \end{aligned}$$

2.3. A normal form

Since the conservation law Φ is a closed form on U, locally there exists a 2-form Ψ on U such that

$$\Phi = d\Psi.$$

Such a form Ψ must have the property that

$$\begin{aligned} d\Psi \wedge \overline{\theta} \wedge dx^3 &= 0 \\ d\Psi \wedge d\overline{\theta} \wedge dx^3 &= 0. \end{aligned} \tag{2.3}$$

Furthermore, the coefficients of $dx^1 \wedge dp_1 \wedge dx^3$ and $dp_2 \wedge dx^2 \wedge dx^3$ in $d\Psi$ must be equal.

Conversely, for any 2-form Ψ on U which satisfies these conditions and the nondegeneracy condition that the right-hand side of the equation (2.2) determined by $\Phi = d\Psi$ is elliptic (note that this condition may depend on the particular solution u as well as on Ψ), the 3-form $\Phi = d\Psi$ uniquely determines a parabolic PDE up to contact equivalence, together with a conservation law for the PDE.

For notational convenience, set

$$\begin{aligned} \eta_0 &= \overline{\theta} \\ \eta_1 &= dx^1 \\ \eta_2 &= dx^2 \\ \eta_3 &= dx^3 \\ \eta_4 &= dp_1 \\ \eta_5 &= dp_2 \end{aligned}$$

and set

$$\Psi = F^{ij}\,\eta_i \wedge \eta_j$$

with $F^{ij} = -F^{ji}$. It is not difficult to show that the conditions (2.3) together with the equality of the coefficients of $dx^1 \wedge dp_1 \wedge dx^3$ and $dp_2 \wedge dx^2 \wedge dx^3$ in $d\Psi$ are equivalent to the first-order PDEs

$$\begin{aligned}
F^{01} &= F^{15}_2 - F^{25}_1 - F^{12}_5\\
F^{02} &= F^{24}_1 + F^{12}_4 - F^{14}_2\\
F^{03} &= \frac{1}{2}\big(F^{23}_5 + F^{35}_2 - F^{25}_3 - F^{14}_3 + F^{13}_4 + F^{34}_1\big)\\
F^{04} &= F^{45}_2 + F^{24}_5 - F^{25}_4\\
F^{05} &= F^{15}_4 - F^{14}_5 - F^{45}_1.
\end{aligned} \tag{2.4}$$

Note that Ψ is not uniquely determined by Φ; Ψ can be modified by an arbitrary exact form without affecting Φ. Given $\Psi = F^{ij}\,\eta_i \wedge \eta_j$, define a 1-form $\gamma = G^j\,\eta_j$ by setting $G^3 = 0$ and choosing G^j, $j \neq 3$, so that

$$G^j_3 = F^{j3}.$$

Replacing Ψ by $\Psi + d\gamma$, we can assume that $F^{j3} = 0$ for all j. The third equation in (2.4) then becomes

$$F^{14}_3 + F^{25}_3 = 0,$$

which implies that

$$F^{25} = -F^{14} + f\big(u, x^1, x^2, p_1, p_2\big)$$

for some function f.

The general parabolic PDE which admits a conservation law can now be described as follows: let $F^{12}, F^{14}, F^{15}, F^{24}, F^{45}$ be essentially arbitrary functions of the variables $x^1, x^2, x^3, u, p_1, p_2$ (subject only to the nondegeneracy condition mentioned above), and let f be an arbitrary function of the variables x^1, x^2, u, p_1, p_2. (The function f will actually turn out to be unnecessary.) With $F^{3j} = 0$ and $F^{25} = -F^{14} + f$, define functions F^{0j} by the equations (2.4). A straightforward computation shows that the PDE which arises from the 3-form $\Phi = d\Psi$ has the form

$$u_t = \frac{A'(u_{xx}u_{yy} - u_{xy}^2) + B'u_{xx} + 2C'u_{xy} + D'u_{yy} + E'}{A''(u_{xx}u_{yy} - u_{xy}^2) + B''u_{xx} + 2C''u_{xy} + D''u_{yy} + E''}$$

where

$$\begin{aligned}
A' &= F_3^{45}\\
B' &= -F_3^{24}\\
C' &= F_3^{14}\\
D' &= F_3^{15}\\
E' &= F_3^{12}\\
A'' &= F_4^{05} - F_5^{04} - F_0^{45}\\
&= -F_0^{45} - F_{14}^{45} - F_{25}^{45} + F_{44}^{15} - F_{55}^{24} - 2F_{45}^{14} + f_{45}\\
B'' &= F_0^{24} + F_4^{02} - F_2^{04}\\
&= 2F_0^{24} + F_{14}^{24} - F_{25}^{24} + F_{44}^{12} - F_{22}^{45} - 2F_{24}^{14} + f_{24}\\
C'' &= -F_0^{14} - F_4^{01} + F_1^{04}\\
&= -2F_0^{14} + F_{45}^{12} + F_{12}^{45} - F_{24}^{15} + F_{15}^{24} + f_0\\
D'' &= -F_0^{15} + F_1^{05} - F_5^{01}\\
&= -2F_0^{15} + F_{14}^{15} - F_{25}^{15} + F_{55}^{12} - F_{11}^{45} - 2F_{15}^{14} + f_{15}\\
E'' &= -F_0^{12} + F_1^{02} = F_2^{01}\\
&= -3F_0^{12} + F_{14}^{12} + F_{25}^{12} + F_{11}^{24} - F_{22}^{15} - 2F_{12}^{14} + f_{12}.
\end{aligned}$$

Replacing F^{14} by $F^{14} + \frac{1}{2}f$, we can assume that $f \equiv 0$.

3. Multiple conservation laws: first reduction

From now on we will assume that the PDE (2.2) has a nontrivial conservation law. We now consider the case where the space of conservation laws has dimension greater than one. The principal result of this section is the following theorem.

Theorem 3.1. *Any parabolic PDE of the form* (2.2) *whose space of conservation laws has dimension at least* 2 *is locally contact equivalent to an equation of the form*

$$u_t = A'\left(u_{xx}u_{yy} - u_{xy}^2\right) + B'\,u_{xx} + 2C'\,u_{xy} + D'\,u_{yy} + E'$$

where the coefficients A', B', C', D', E' are functions of x, y, t, u, u_x, u_y.

Suppose that

$$\Phi_0 = \Phi_0' \wedge dx^3 - \Phi_0'' \wedge \overline{\theta} + \kappa_0 \wedge \overline{\theta} \wedge dx^3$$

is a conservation law for (2.2). Any other conservation law Φ has the form

$$\Phi = Q\left(\Phi_0' \wedge dx^3 - \Phi_0'' \wedge \overline{\theta}\right) + \kappa \wedge \overline{\theta} \wedge dx^3$$

where $Q = Q(x^1, x^2, x^3, u, p_1, p_2)$. Computing $d\Phi \equiv 0 \mod \overline{\theta}$ shows that

$$\begin{aligned}\kappa = &\big[(E'Q)_5 - (C'Q)_1 - (D'Q)_2\big]\, dx^1 + \big[(B'Q)_1 + (C'Q)_2 - (E'Q)_4\big]\, dx^2 \\ &+ \big[(B'Q)_5 - (C'Q)_4 - (A'Q)_2\big]\, dp_1 + \big[(A'Q)_1 + (C'Q)_5 - (D'Q)_4\big]\, dp_2.\end{aligned}$$

Then computing $d\Phi = 0$, using the fact that $d\Phi_0 = 0$, yields the following equations for Q:

$$B''\, Q_1 + C''\, Q_2 - E''\, Q_4 = 0 \tag{3.1}$$
$$C''\, Q_1 + D''\, Q_2 - E''\, Q_5 = 0 \tag{3.2}$$
$$A''\, Q_1 - D''\, Q_4 + C''\, Q_5 = 0 \tag{3.3}$$
$$A''\, Q_2 + C''\, Q_4 - B''\, Q_5 = 0 \tag{3.4}$$
$$\begin{aligned}A''\, Q_3 + 3A'\, Q_0 + A'_4\, Q_1 + A'_5\, Q_2 + (A'_1 + 2C'_5 - 2D'_4)\, Q_4 + (A'_2 - 2B'_5 + 2C'_4)\, Q_5 \\ +A'\,(Q_{41} + Q_{52}) - D'\, Q_{44} + 2C'\, Q_{45} - B'\, Q_{55} = 0\end{aligned} \tag{3.5}$$
$$\begin{aligned}B''\, Q_3 + 2B'\, Q_0 + B'_4\, Q_1 + (2A'_2 - B'_5 + 2C'_4)\, Q_2 + (B'_1 + 2C'_2 - 2E'_4)\, Q_4 - B'_2\, Q_5 \\ +B'\,(Q_{41} - Q_{52}) + A'\, Q_{22} + 2C'\, Q_{24} - E'\, Q_{44} = 0\end{aligned} \tag{3.6}$$
$$\begin{aligned}C''\, Q_3 + 2C'\, Q_0 + (B'_5 - A'_2)\, Q_1 + (D'_4 - A'_1)\, Q_2 + (D'_2 - E'_5)\, Q_4 + (B'_1 - E'_4)\, Q_5 \\ +B'\, Q_{15} + D'\, Q_{24} - A'\, Q_{12} - E'\, Q_{45} = 0\end{aligned} \tag{3.7}$$
$$\begin{aligned}D''\, Q_3 + 2D'\, Q_0 + (2A'_1 + 2C'_5 - D'_4)\, Q_1 + D'_5\, Q_2 - D'_1\, Q_4 + (2C'_1 + D'_2 - 2E'_5)\, Q_5 \\ +D'\,(Q_{52} - Q_{41}) + A'\, Q_{11} + 2C'\, Q_{15} - E'\, Q_{55} = 0\end{aligned} \tag{3.8}$$
$$\begin{aligned}E''\, Q_3 + E'\, Q_0 + (2B'_1 + 2C'_2 - E'_4)\, Q_1 + (2C'_1 + 2D'_2 - E'_5)\, Q_2 - E'_1\, Q_4 - E'_2\, Q_5 \\ -E'\,(Q_{41} + Q_{52}) + B'\, Q_{11} + 2C'\, Q_{12} + D'\, Q_{22} = 0.\end{aligned} \tag{3.9}$$

In general, this is an overdetermined system of linear equations for Q. The fact that $Q =$ constant is always a solution reflects the assumption that Φ_0 is a conservation law. Additional conservation laws, if they exist, are given by nonconstant solutions of equations (3.1)–(3.9).

Lemma 3.2. *Equations* (3.1)–(3.9) *have no nonconstant solutions unless*

$$B''D'' - A''E'' - (C'')^2 = 0.$$

Proof. Suppose that $B''D'' - A''E'' - (C'')^2 \neq 0$. Equations (3.1)–(3.4) can be written in matrix form as

$$\begin{bmatrix} B'' & C'' & -E'' & 0 \\ C'' & D'' & 0 & -E'' \\ A'' & 0 & -D'' & C'' \\ 0 & A'' & C'' & -B'' \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_4 \\ Q_5 \end{bmatrix} = 0.$$

This matrix has determinant $(B''D'' - A''E'' - (C'')^2)^2$, which by hypothesis is nonzero. Therefore $Q_1 = Q_2 = Q_4 = Q_5 = 0$, and

$$dQ = Q_0\,\overline{\theta} + Q_3\,dx^3.$$

Since dQ is exact, it must lie in the last derived system of $\{\overline{\theta}, dx^3\}$. But

$$\{\overline{\theta}, dx^3\}^{(1)} = \{dx^3\},$$

and so

$$dQ = Q_3\,dx^3.$$

Since at least one of A'', B'', C'', D'', E'' must be nonzero, equations (3.5)–(3.9) now imply that $Q_3 = 0$, and hence $dQ = 0$. Therefore, there are no nonconstant solutions Q. $\square$

With this lemma in hand, we can now prove Theorem 3.1.

Proof. Suppose that the space of conservation laws for equation (2.2) has dimension at least 2. By Lemma 3.2,

$$B''D'' - A''E'' - (C'')^2 = 0.$$

This implies that the 2-form

$$\begin{aligned}\Phi_0'' = A''dp_1 \wedge dp_2 + B''dp_1 \wedge dx^2 + C''(dx^1 \wedge dp_1 + dp_2 \wedge dx^2)\\ + D''dx^1 \wedge dp_2 + E''dx^1 \wedge dx^2\end{aligned}$$

is decomposable. Therefore, there exist 1-forms ω^1, ω^2 such that

$$\Phi_0'' = \omega^1 \wedge \omega^2.$$

We will now show that the system $\mathcal{L} = \{\overline{\theta}, \omega^1, \omega^2, dx^3\}$ is integrable. Since Φ_0 is a conservation law and

$$\Phi_0 \equiv -\Phi_0'' \wedge \overline{\theta} \equiv -\omega^1 \wedge \omega^2 \wedge \overline{\theta} \mod dx^3,$$

it follows that

$$\begin{aligned}0 &\equiv d\Phi_0 \mod \overline{\theta},\, dx^3\\ &\equiv -\omega^1 \wedge \omega^2 \wedge d\overline{\theta} \mod \overline{\theta},\, dx^3.\end{aligned}$$

Therefore,

$$d\overline{\theta} \equiv 0 \mod \overline{\theta},\, \omega^1,\, \omega^2,\, dx^3.$$

Let α be a 1-form such that

$$d\overline{\theta} \equiv \alpha \wedge \overline{\theta} \mod \omega^1,\, \omega^2,\, dx^3.$$

Then

$$\begin{aligned} 0 &\equiv d\Phi_0 \mod dx^3 \\ &\equiv -d\omega^1 \wedge \omega^2 \wedge \overline{\theta} + \omega^1 \wedge d\omega^2 \wedge \overline{\theta} - \omega^1 \wedge \omega^2 \wedge \alpha \wedge \overline{\theta}. \end{aligned}$$

Reducing this equation modulo ω^1 shows that

$$d\omega^1 \wedge \omega^2 \equiv 0 \mod \overline{\theta},\, \omega^1,\, dx^3$$

which implies that

$$d\omega^1 \equiv 0 \mod \overline{\theta},\, \omega^1,\, \omega^2,\, dx^3.$$

Similarly,

$$d\omega^2 \equiv 0 \mod \overline{\theta},\, \omega^1,\, \omega^2,\, dx^3.$$

Therefore $\mathcal{L}$ is integrable, as claimed. By the Frobenius theorem, locally there exist functions $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{u}$, $\tilde{p}_1$, $\tilde{p}_2$ such that

$$\mathcal{L} = \left\{d\tilde{u},\, d\tilde{x}^1,\, d\tilde{x}^2,\, dx^3\right\}$$

and

$$\overline{\theta} \equiv \lambda\big(d\tilde{u} - \tilde{p}_1\, d\tilde{x}^1 - \tilde{p}_2\, \tilde{x}^2\big) \mod dx^3$$

for some nonzero function λ. In this coordinate system,

$$\Phi_0 \equiv -\tilde{E}'' d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \tilde{\overline{\theta}} \mod dx^3$$

and the PDE takes the form

$$u_t = \frac{1}{\tilde{E}''}\Big[\tilde{A}'\left(u_{xx}u_{yy} - u_{xy}^2\right) + \tilde{B}'\, u_{xx} + 2\tilde{C}'\, u_{xy} + \tilde{D}'\, u_{yy} + \tilde{E}'\Big].$$

Setting $A' = \frac{\tilde{A}'}{\tilde{E}''}$, etc., yields the desired form. $\square$

In fact, we can arrange via a contact change of coordinates that $E'' \equiv 1$, as follows. Suppose that

$$\Phi_0 \equiv -E'' dx^1 \wedge dx^2 \wedge \overline{\theta} \mod dx^3.$$

A straightforward computation shows that

$$0 \equiv d\Phi_0 \equiv -E''_4\, dp_1 \wedge dx^1 \wedge dx^2 \wedge \overline{\theta} - E''_5\, dp_2 \wedge dx^1 \wedge dx^2 \wedge \overline{\theta} \mod dx^3,$$

which implies that $E''_4 = E''_5 = 0$. Therefore E'' can be expressed in the form

$$E'' = f(x^1, x^2, x^3, u),$$

and

$$\Phi_0 \equiv -f(x^1, x^2, x^3, u)\, du \wedge dx^1 \wedge dx^2 \mod dx^3.$$

Set $F(x^1, x^2, x^3, u) = \int_0^u f(x^1, x^2, x^3, \tau)\, d\tau$, and make the following change of coordinates:

$$\begin{aligned}
\tilde{x}^1 &= x^1 \\
\tilde{x}^2 &= x^2 \\
\tilde{x}^3 &= x^3 \\
\tilde{u} &= F(x^1, x^2, x^3, u) \\
\tilde{p}_1 &= f(x^1, x^2, x^3, u)\, p_1 + \frac{\partial F}{\partial x^1} \\
\tilde{p}_2 &= f(x^1, x^2, x^3, u)\, p_2 + \frac{\partial F}{\partial x^2}.
\end{aligned}$$

It is not difficult to check that this is a contact change of coordinates and that

$$\Phi_0 \equiv -d\tilde{u} \wedge d\tilde{x}^1 \wedge d\tilde{x}^2 \mod dx^3.$$

Therefore $E'' \equiv 1$ in this coordinate system.

The condition that Φ''_0 be decomposable is a significant restriction, even among equations that have one conservation law. For example, the parabolic equation

$$u_t = \frac{u_{xx}u_{yy} - u_{xy}^2 - 1}{2u_{xx} + 2u_{yy} + 1}$$

has one conservation law, which is represented by the 3-form

$$\Phi_0 = (dp_1 \wedge dp_2 - dx^1 \wedge dx^2) \wedge dx^3 - (2dp_1 \wedge dx^2 + 2dx^1 \wedge dp_2 + dx^1 \wedge dx^2) \wedge \overline{\theta}.$$

But

$$\Phi''_0 = 2dp_1 \wedge dx^2 + 2dx^1 \wedge dp_2 + dx^1 \wedge dx^2$$

is not decomposable; therefore the space of conservation laws for this equation is one-dimensional and is spanned by Φ_0. (This can be easily verified by a direct computation.)

Applying Theorem 3.1 and assuming that $E'' \equiv 1$, equations (3.1)–(3.4) imply that $Q_4 = Q_5 = 0$. Equations (3.5)–(3.9) now take the form

$$3A'\, Q_0 + A'_4\, Q_1 + A'_5\, Q_2 = 0 \tag{3.5$'$}$$

$$2B'\, Q_0 + B'_4\, Q_1 + \left(2A'_2 - B'_5 + 2C'_4\right) Q_2 + A'\, Q_{22} = 0 \tag{3.6$'$}$$

$$2C'\, Q_0 + \left(B'_5 - A'_2\right) Q_1 + \left(D'_4 - A'_1\right) Q_2 - A'\, Q_{12} = 0 \tag{3.7$'$}$$

$$2D'\, Q_0 + \left(2A'_1 + 2C'_5 - D'_4\right) Q_1 + D'_5\, Q_2 + A'\, Q_{11} = 0 \tag{3.8$'$}$$

$$\begin{aligned} Q_3 + E'\, Q_0 + \left(2B'_1 + 2C'_2 - E'_4\right) Q_1 + \left(2C'_1 + 2D'_2 - E'_5\right) Q_2 \\ + B'\, Q_{11} + 2C'\, Q_{12} + D'\, Q_{22} = 0. \end{aligned} \tag{3.9$'$}$$

We will consider separately the cases $A' \neq 0,\ A' = 0$.

4. Case 1: $A' \neq 0$

Equation (3.5′) can be used to express Q_0 as a linear combination of Q_1, Q_2. Then equations (3.6′)–(3.8′) can be used to express Q_{11}, Q_{12}, Q_{22} as linear combinations of Q_1, Q_2. Finally, equation (3.9′) can be used to express Q_3 as a linear combination of Q_1, Q_2. As a result of this computation, dQ may be expressed as

$$dQ = Q_1\,\omega^1 + Q_2\,\omega^2$$

for certain 1-forms ω^1, ω^2. Therefore, dQ must lie in the last derived system of $\mathcal{L} = \{\omega^1,\,\omega^2\}$.

4.1. Case 1.1: rank $\mathcal{L}^{(\infty)} = 0$

Then there are no nonconstant solutions Q, and the space of conservation laws is 1-dimensional.

4.2. Case 1.2: rank $\mathcal{L}^{(\infty)} = 1$

Then locally there exists a function y such that $\mathcal{L}^{(\infty)} = \{dy\}$. Q must be a function of y alone, and

$$dQ = Q_y\,dy.$$

Using the chain rule, the derivatives Q_i, Q_{ij} can be expressed in terms of Q_y and Q_{yy}, and at least one of equations (3.6′)–(3.8′) can be written in the form

$$Q_{yy} = \lambda\,Q_y.$$

This equation has at most a 2-dimensional space of solutions, so the space of conservation laws in this case is at most 2-dimensional.

4.3. Case 1.3: rank $\mathcal{L} = 2$

Then locally there exist functions y, z such that $\mathcal{L}^{(\infty)} = \{dy,\,dz\}$. Q must be a function of y, z alone, and

$$dQ = Q_y\,dy + Q_z\,dz.$$

Equations (3.6′)–(3.8′) yield equations of the form

$$\begin{aligned} Q_{yy} &= a_1\,Q_y + a_2\,Q_z \\ Q_{yz} &= b_1\,Q_y + b_2\,Q_z \\ Q_{zz} &= c_1\,Q_y + c_2\,Q_z. \end{aligned}$$

If the functions a_i, b_i, c_i depend nontrivially on any variables other than y, z, then differentiating with respect to these variables yields a linear relation between Q_y and Q_z, and the argument of the previous case shows that the space of conservation laws is at most 2-dimensional. So assume that a_i, b_i, c_i are functions of y, z alone.

On the manifold $\mathbb{R}^5 = \{(y, z, Q, Q_1, Q_2)\}$, let

$$\begin{aligned} \eta &= dQ - Q_1\,dy - Q_2\,dz \\ \eta_1 &= dQ_1 - \big(a_1\,Q_1 + a_2\,Q_2\big)\,dy - \big(b_1\,Q_1 + b_2\,Q_2\big)\,dz \\ \eta_2 &= dQ_2 - \big(b_1\,Q_1 + b_2\,Q_2\big)\,dy - \big(c_1\,Q_1 + c_2\,Q_2\big)\,dz. \end{aligned}$$

Let $\mathcal{J}$ be the differential system generated by $\{\eta,\ \eta_1,\ \eta_2\}$. The space of conservation laws may be identified with the space of 2-dimensional integral manifolds of $\mathcal{J}$ that satisfy the independence condition $dy \wedge dz \neq 0$. This space is at most 3-dimensional, with dimension equal to 3 if and only if $\mathcal{J}$ is Frobenius.

From this discussion, it is clear that if $A' \neq 0$, then the space of conservation laws is at most 3-dimensional. We state this result as the following theorem.

Theorem 4.1. *Any parabolic PDE of the form* (2.2) *whose space of conservation laws has dimension at least* 4 *is locally contact equivalent to an equation of the form*

$$u_t = B'\,u_{xx} + 2C'\,u_{xy} + D'\,u_{yy} + E'$$

where the coefficients B', C', D', E' are functions of x, y, t, u, u_x, u_y. In particular, such an equation is locally contact equivalent to a quasi-linear equation. □

5. Case 2: $A' = 0$

Since the right-hand side of equation (2.2) is an elliptic operator, it must be true that $B'D' - (C')^2 > 0$. In particular, both B' and D' must be nonzero.

Equation (3.5′) is now an identity, and equations (3.6′)–(3.9′) take the form

$$2B'\,Q_0 + B'_4\,Q_1 + \big(2C'_4 - B'_5\big)\,Q_2 = 0 \tag{3.6″}$$

$$2C'\,Q_0 + B'_5\,Q_1 + D'_4\,Q_2 = 0 \tag{3.7″}$$

$$2D'\,Q_0 + \big(2C'_5 - D'_4\big)\,Q_1 + D'_5\,Q_2 = 0 \tag{3.8″}$$

$$\begin{aligned} Q_3 + E'\,Q_0 + \big(2B'_1 + 2C'_2 - E'_4\big)\,Q_1 + \big(2C'_1 + 2D'_2 - E'_5\big)\,Q_2& \\ +B'\,Q_{11} + 2C'\,Q_{12} + D'\,Q_{22} = 0.& \end{aligned} \tag{3.9″}$$

Moreover, computing $d\Phi_0 = 0$ shows that the following equations must hold:

$$B'_{55} - 2C'_{45} + D'_{44} = 0 \tag{5.1}$$
$$E'_{44} = 2B'_0 + B'_{41} - B'_{52} + 2C'_{24} \tag{5.2}$$
$$E'_{45} = 2C'_0 + B'_{15} + D'_{24} \tag{5.3}$$
$$E'_{55} = 2D'_0 + D'_{52} - D'_{41} + 2C'_{15} \tag{5.4}$$
$$E'_0 - E'_{41} - E'_{52} + B'_{11} + 2C'_{12} + D'_{22} = 0. \tag{5.5}$$

Let

$$S = \begin{bmatrix} 2B' & B'_4 & 2C'_4 - B'_5 \\ 2C' & B'_5 & D'_4 \\ 2D' & 2C'_5 - D'_4 & D'_5 \end{bmatrix}.$$

Since $B'D' - (C')^2 > 0$, S has rank at least 1. Equations (3.6″)–(3.8″) are equivalent to the statement that the vector ${}^t[\,Q_0 \quad Q_1 \quad Q_2\,]$ lies in the kernel of S.

Let $\mathcal{L}$ be the linear span of the forms

$$\bigl\{Q_0\,\overline{\theta} + Q_1\,dx^1 + Q_2\,dx^2 \mid {}^t[\,Q_0 \quad Q_1 \quad Q_2\,] \in \ker(S)\bigr\} \cup \bigl\{dx^3\bigr\},$$

and let $\mathcal{L}^{(\infty)}$ be the last derived system of $\mathcal{L}$. Note that $\mathcal{L}^{(\infty)}$ has rank at most 3. We will consider separately the possibilities for the rank of $\mathcal{L}^{(\infty)}$.

In the course of this discussion, we will use the following theorem, which is easily proved using standard techniques.

Theorem 5.1. *Let θ be a contact form on a $(2n+1)$-dimensional manifold M. Let $\{\theta, \omega^1, \dots, \omega^p\}$, $p \le n$, be a rank $p+1$ system such that*

1. *the system $\{\omega^1, \dots, \omega^p\}$ is integrable*
2. $(d\theta)^{n-p+1} \equiv 0 \mod \{\theta, \omega^1, \dots, \omega^p\}$
3. $(d\theta)^{n-p} \not\equiv 0 \mod \{\theta, \omega^1, \dots, \omega^p\}$.

Then there exist local coordinates x^i, u, p_i, $1 \le i \le n$, such that

$$\theta = \lambda\bigl(du - p_i\,dx^i\bigr)$$

for some $\lambda \ne 0$ and

$$\bigl\{\omega^1, \dots, \omega^p\bigr\} = \bigl\{dx^1, \dots, dx^p\bigr\}.$$

5.1. Case 2.1: rank $\mathcal{L}^{(\infty)} = 1$

Then $\mathcal{L}^{(\infty)} = \{dx^3\}$, and

$$dQ = Q_3\,dx^3.$$

Equation (3.9″) implies that $Q_3 = 0$; therefore there are no nonconstant solutions Q, and the space of conservation laws is 1-dimensional.

5.2. Case 2.2: rank $\mathcal{L}^{(\infty)} = 2$

Then $\mathcal{L}^{(\infty)}$ has the form

$$\mathcal{L}^{(\infty)} = \left\{a_0\,\overline{\theta} + a_1\,dx^1 + a_2\,dx^2,\ dx^3\right\}$$

with at least one of a_1, a_2 nonzero. Applying Theorem 5.1 to the system

$$\left\{\theta = \overline{\theta} - p_3\,dx^3\right\} \cup \mathcal{L}^{(\infty)}$$

on the manifold $U \times \mathbb{R}$ with coordinate p_3 on the second factor shows that there exist local contact coordinates $\tilde{x}^i$, $\tilde{u}$, $\tilde{p}_i$ with $d\tilde{x}^3 = dx^3$ and

$$\mathcal{L}^{(\infty)} = \left\{d\tilde{x}^1,\ d\tilde{x}^3\right\}.$$

In this coordinate system,

$$dQ = Q_1\,dx^1 + Q_3\,dx^3.$$

To avoid the previous case, we assume that $Q_1 \neq 0$. Equations (3.6″)–(3.8″) imply that

$$B'_4 = B'_5 = 2C'_5 - D'_4 = 0 \tag{5.6}$$

and equation (3.9″) now takes the form

$$Q_3 + \left(2B'_1 + 2C'_2 - E'_4\right) Q_1 + B'\,Q_{11} = 0. \tag{3.9‴}$$

Using equations (5.2), (5.3), and (5.6), it can be shown that the coefficients of Q_1 and Q_{11} in equation (3.9‴) are independent of p_1 and p_2. If either of these coefficients depends nontrivially on u or x^2, then the space of solutions of equation (3.9‴) (and hence the space of conservation laws for equation (2.2)) has dimension at most 2. Otherwise, equation (3.9‴) is a linear parabolic equation for one function of two independent variables, and the space of solutions of equation (3.9‴) (and hence the space of conservation laws for equation (2.2)) is parameterized (in the sense of Cartan-Kähler) by two arbitrary functions of one variable.

Example. $u_t = u_{xx} + u_y\,u_{yy}$. This equation is parabolic whenever $u_y > 0$, and it is represented by the closed 3-form

$$\Phi_0 = \left(dp_1 \wedge dx^2 + p_2\,dx^1 \wedge dp_2\right) \wedge dx^3 - dx^1 \wedge dx^2 \wedge \overline{\theta}.$$

Any other conservation law has the form

$$\Phi = Q\,\Phi_0 - Q_1\,\overline{\theta} \wedge dx^2 \wedge dx^3$$

where $Q(x^1, x^3)$ satisfies the backward heat equation

$$Q_3 + Q_{11} = 0.$$

In classical terms, $\Phi = d\phi$, where

$$\phi = Q\left(p_1\, dx^2 \wedge dx^3 + \frac{1}{2} p_2^2\, dx^3 \wedge dx^1 - u\, dx^1 \wedge dx^2\right) - Q_1\, u\, dx^2 \wedge dx^3.$$

This corresponds to the fact that for any solution $u(x, y, t)$ with asymptotically zero boundary conditions,

$$\frac{d}{dt} \iint_{\mathbb{R}^2} Q(x,t)\, u(x,y,t)\, dx\, dy = 0.$$

5.3. Case 2.3: rank $\mathcal{L}^{(\infty)} = 3$

In this case the matrix S necessarily has rank 1. Using equations (3.6″)–(3.8″), Q_0 can be expressed as a linear combination of Q_1 and Q_2. $\mathcal{L}$ can then be written in the form

$$\mathcal{L} = \left\{\omega^1 = dx^1 - \lambda_1\, \overline{\theta}, \quad \omega^2 = dx^2 - \lambda_2\, \overline{\theta}, \quad dx^3\right\}.$$

Applying Theorem 5.1 to the system $\mathcal{L}$ shows that there exist local contact coordinates $\tilde{x}^i, \tilde{u}, \tilde{p}_i$ with $d\tilde{x}^3 = dx^3$ and

$$\mathcal{L} = \left\{dx^1,\, dx^2,\, dx^3\right\}.$$

In this coordinate system, $\ker(S)$ is spanned by vectors of the form

$${}^t\left[0 \quad Q_1 \quad Q_2\right].$$

This implies that the second and third columns of S vanish, i.e.,

$$B'_4 = B'_5 = C'_4 = C'_5 = D'_4 = D'_5 = 0. \tag{5.7}$$

Equations (3.6″)–(3.8″) imply that $Q_0 = 0$, and equation (3.9″) now takes the form

$$\begin{aligned} Q_3 + \left(2B'_1 + 2C'_2 - E'_4\right) Q_1 + \left(2C'_1 + 2D'_2 - E'_5\right) Q_2 \\ + B'\, Q_{11} + 2C'\, Q_{12} + D'\, Q_{22} = 0. \end{aligned} \tag{3.9''''}$$

Using equations (5.2), (5.3), (5.4), and (5.7), it can be shown that the coefficients of Q_1, Q_2, Q_{11}, Q_{12}, and Q_{22} in equation (3.9⁗) are independent of p_1 and p_2. The solution space of equation (3.9⁗) depends on how these coefficients depend on the variable u. If any of these coefficients depend nontrivially on u, then equation (3.9⁗) actually splits into two or more equations, and the space of conservation laws is parameterized by at most two functions of one variable, as in the previous case.

Example. $u_t = \frac{1}{2}e^{-u}(u_{xx} + u_{yy})$. This equation arises naturally in a geometric problem and is discussed in more detail in [4]. The equation is parabolic and is represented by the closed 3-form

$$\Phi_0 = \big(dp_1 \wedge dx^2 + dx^1 \wedge dp_2\big) \wedge dx^3 - 2e^u\, dx^1 \wedge dx^2 \wedge \overline{\theta}.$$

Any other conservation law has the form

$$\Phi = Q\,\Phi_0 + \big(Q_1\, dx^2 - Q_2\, dx^1\big) \wedge \overline{\theta} \wedge dx^3$$

where $Q(x^1, x^2)$ satisfies Laplace's equation

$$Q_{11} + Q_{22} = 0.$$

In classical terms, $\Phi = -d\phi$, where

$$\begin{aligned}\phi = 2\,Q\,e^u\, dx^1 \wedge dx^2 - tQ\,\big(dp_1 \wedge dx^2 + dx^1 \wedge dp_2\big)\\ + t\,\big(du - p_1\, dx^1 - p_2\, dx^2\big) \wedge \big(Q_1\, dx^2 - Q_2\, dx^1\big).\end{aligned}$$

This corresponds to the fact that for any solution $u(x, y, t)$ for which the integral converges,

$$\frac{d}{dt}\iint_{\mathbb{R}^2} Q(x, y)\,\big(2e^{u(x,y,t)} - t(u_{xx} + u_{yy})\big)\, dx\, dy = 0.$$

Proposition 5.2. *If all of the coefficients in equation* (3.9'''') *are independent of u, then the PDE* (2.2) *is linear.*

Proof. By hypothesis,

$$B'_0 = C'_0 = D'_0$$

so B', C', D' are functions of x^1, x^2, x^3 alone. Equations (5.2), (5.3), (5.4) imply that

$$\begin{aligned}E'_{44} &= 2B'_0 = 0\\ E'_{45} &= 2C'_0 = 0\\ E'_{55} &= 2D'_0 = 0.\end{aligned}$$

The fact that the coefficients of Q_1, Q_2 in (3.9'''') are independent of u implies that

$$\begin{aligned}E'_{04} &= 2B'_{01} + 2C'_{02} = 0\\ E'_{05} &= 2C'_{01} + 2D'_{02} = 0.\end{aligned}$$

Finally, differentiating (5.5) with respect to u yields

$$E'_{00} = E'_{041} + E'_{052} + B'_{011} + 2C'_{012} + D'_{022} = 0.$$

Therefore,

$$E' = f(x^i)\,u + g(x^i)\,p_1 + h(x^i)\,p_2 + k(x^i)$$

and the PDE (2.2) is linear. □

It is well-known that the space of conservation laws for a linear equation is isomorphic to the space of solutions of its adjoint equation. The result of this discussion is the following theorem.

Theorem 5.3. *Let $\mathcal{I}$ be a parabolic Monge-Ampère system whose space of conservation laws has parameter space (in the sense of Cartan-Kähler) strictly larger than the space of two arbitrary functions of one variable. Then $\mathcal{I}$ is locally contact equivalent to a system that arises from a linear parabolic partial differential equation, and the space of conservation laws is parameterized by two arbitrary functions of two variables.* □

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