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Periodic Korteweg de Vries equation with measures as initial data

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Abstract. The main result of the paper is that the periodic KdV equation $y_t + \partial_x^3 y + yy_x = 0$ has a unique global solution for initial data $y(0)$ given by a measure $\mu \in M(\mathbb{T})$ of sufficiently small norm $\|\mu\|$. There are two main ingredients in the proof. The first is the study of the local well-posedness problem in terms of the space-time Fourier-norms as exploited in [Bo] and also subsequent work such as [K-P-V$_2$]. At the end the estimates eventually depend on a uniform estimate in terms of the Fourier coefficients

$$\sup_{n\in\mathbb{Z},\, t\in\mathbb{R}} |\widehat{y}(n)(t)| < C.$$

Such a priori bound (in the space of pseudo-measures) on the solution may be derived from spectral theory and more precisely from the preservation of the periodic spectrum of a potential evolving according to KdV, which is the second ingredient. Thus the result at this stage depends heavily on integrability features of this particular equation. We also sketch an argument establishing almost periodicity properties of these solutions. This work is in spirit closely related to [Bo]. Natural questions suggested by these investigations is an extension of the result (at least for the IVP local in time) to a more general nonintegrable setting as well as to what extent the estimates on Fourier coefficients by spectral invariants and vice versa remains valid in distributional spaces.

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1. Introduction

The main result is the following

Theorem. *The periodic KdV equation*

$$y_t + \partial_x^3 y + yy_x = 0 \tag{1.1}$$

has a unique global solution for initial data $y(0)$ *given by a measure on the circle* $\mathbb{T}$ *of sufficiently small norm; thus* $y(0) = \mu \in M(\mathbb{T})$, $\|\mu\|$ *small enough.*

This smallness assumption is of a technical nature and may likely be removed by some additional work. As will be explained below, the method uses both the local in time analysis from [Bo] and certain a priori bounds provided by spectral theory. The content of the theorem may be reformulated as follows. Let $\{\phi_\alpha\}$ be a sequence of smooth data such that $\mu = \lim\limits_\alpha \phi_\alpha$ in the weak sense. Let $\{y_\alpha\}$ be the corresponding solutions of (1.1) satisfying $y_\alpha(0) = \phi_\alpha$. Then the sequence $\{y_\alpha(t)\}$ converges weakly for all time. Local well-posedness of he initial value problem

$$\begin{cases} y_t + \partial_x^3 y + y y_x = 0 \\ y(0) = \phi \end{cases} \tag{1.2}$$

in the period case, for $\phi \in H^s(\mathbb{T})$, $s \geq 0$, was established in [Bo]. In this situation, the result extends in fact to a global one because of the L^2-conservation

$$\|u(t)\|_2 = \|\phi\|_2. \tag{1.3}$$

It was observed in [KPV$_2$] that this argument in fact may be applied to establish local well-posedness for $\phi \in H^s(\mathbb{T})$, $s \geq -\frac{1}{2}$. In the periodic case, the exponent $-\frac{1}{2}$ turns out to be the boundary of the method. In fact, if one defines well-posedness as a smooth behavior of the time shifts acting on $H_0^s(\mathbb{T})$

$$\phi \longmapsto y_\phi(t), \qquad 0 \leq t \leq T, \tag{1.4}$$

where y_ϕ is the unique solution to (1.2), then the condition $s \geq -\frac{1}{2}$ is necessary. More comments on this will be given at the end of the paper. By $H_0^s(\mathbb{T})$ we mean the subpsace of $H^s(\mathbb{T})$ consisting of the functions of mean 0

$$\int_{\mathbb{T}} \phi(x) dx = 0. \tag{1.5}$$

Recall here that the mean is an invariant of motion under the KdV flow. Also when formulating the well-posedness results in the periodic case, one should specify the mean $\int \phi$, since otherwise the statement is incorrect. The procedure of mean zero reduction simply consists in replacing (1.2) by the equation

$$\begin{cases} y_t + \partial_x^3 y - \hat{\phi}(0) y_x + y y_x = 0 \\ y(0) = \phi - \hat{\phi}(0). \end{cases} \tag{1.6}$$

The additional $\hat{\phi}(0) y_x$-term plays otherwise a nonsignificant role in the [Bo]-analysis. This analysis requires however essentially that

$$\hat{y}(0)(t) = 0 \qquad \text{where} \qquad y(x,t) = \sum_{n \in \mathbb{Z}} \hat{y}(n)(t) e^{in.x}. \tag{1.7}$$

Starting from $\phi = y(0) = \mu$, one has

$$|\hat{\phi}(n)| \leq \|\mu\| \tag{1.8}$$

and hence ϕ as a distribution barely misses the $H^{-1/2}$-class.

However, the well-posedness limitation $s \geq -\frac{1}{2}$ brought up above turns out to be caused by data with Fourier coefficients of "extreme" size, such as

$$\phi(x) = N^{1/2} \sin 2\pi N x, \quad N \to \infty \tag{1.9}$$

and does not appear in the context of (1.8).

Thus the next issue to carry out the argument is how to obtain an a priori estimate on the Fourier coefficients of the form (1.8)

$$\sup_{n \in \mathbb{Z}} |\hat{y}(n)| < C. \tag{1.10}$$

Here we will rely on spectral theory (unfortunately depending strongly on the integrability aspects of KdV). Recall that if q is a real periodic potential, then the periodic spectrum

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \ldots < \lambda_{2n-1} \leq \lambda_{2n} < \ldots \tag{1.11}$$

of the Sturm-Liouville operator

$$Q = -\frac{d^2}{dx^2} + q(x) \tag{1.12}$$

is invariant if we let Q evolve according to the KdV flow. Recall that the periodic spectrum $\{\lambda_j\}$ is obtained as roots of the discriminant

$$\Delta^2(\lambda, q) - 4 = 0 \tag{1.13}$$

where

$$\Delta(\lambda, q) = y_1(1,\lambda) + y_2'(1,\lambda) \tag{1.14}$$

$$Qy_i = \lambda y_i \quad (i = 1, 2) \tag{1.15}$$

$$y_1(0,\lambda) = 1, \quad y_1'(0,\lambda) = 0 \tag{1.16}$$

and

$$y_2(0,\lambda) = 0, \quad y_2'(0,\lambda) = 1. \tag{1.17}$$

The functions $y_i = y_i(x,\lambda,q)$ $(i=1,2)$ satisfy equations

$$y_1(x,\lambda) = \cos\sqrt{\lambda}x \quad + \int_0^x \frac{\sin\sqrt{\lambda}(x-s)}{\sqrt{\lambda}} q(s) y_1(s,\lambda) ds \tag{1.18}$$

$$y_2(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \quad + \int_0^x \frac{\sin\sqrt{\lambda}(x-s)}{\sqrt{\lambda}} q(s) y_2(s,\lambda) ds. \tag{1.19}$$

Recall also the definition of the Dirichlet spectrum $\{\mu_n\} = \{\mu_n(q)\}$, given by the roots of the equation

$$y_2(1,\mu) = 0 \tag{1.20}$$

and interlacing the periodic spectrum

$$\lambda_0 < \lambda_1 \le \mu_1 \le \lambda_2 < \ldots < \lambda_{2n-1} \le \mu_n \le \lambda_{2n} < \ldots \tag{1.21}$$

If q is an $L^1(\mathbb{T})$-potential, then $\lambda_{2n-1}, \lambda_{2n} = \pi^2 n^2 + O(1)$. If q is sufficiently smooth, then $\{\lambda_{2n} - \lambda_{2n-1}\}$, the sequence of gaps, measures essentially the decay of the Fourier transform $\{|\hat{q}(n)|;\, n \in \mathbb{Z}_+\}$ (cf. [McK-T] and also [Bo] for inequalities in the context of nonsmooth potentials). Our aim here will be to derive from the preceding an a priori bound of the form (1.10).

Denote for given ϕ by $\mathcal{M}(\phi)$ the corresponding isospectral manifold of functions q with same periodic spectrum $\lambda_j(q) = \lambda_j(\phi)$. One may consider here smooth functions to avoid complications, but the conclusions below should only involve the $\|\phi\|_{L^1} = \|\phi\|_1$-norm. The scheme of the argument is then as follows

$$\|\phi\|_1 < c$$

$$\Downarrow$$

$$\lambda_{2n-1}(\phi), \lambda_{2n}(\phi) = \pi^2 n^2 + o(1)$$

$$\Downarrow$$

$$|\mu_n(q) - \pi^2 n^2| = o(1) \quad \text{for all} \quad q \in \mathcal{M}(\phi)$$

$$\Downarrow$$

$$\sup_n |\hat{q}(n)| < c'$$

and thus (1.10).

It is here that the smallness assumption is used, making the previous argument a bit easier. We remark that if $\phi \in L^2(\mathbb{T})$, then $\mathcal{M}(\phi)$ is a (compact) subset of L^2. However for $\phi \in L^1(\mathbb{T})$ (resp. $\mu \in M(\mathbb{T})$), the members of $\mathcal{M}(\phi)$ (resp. $\mathcal{M}(\mu)$) are distributions, not necessarily functions (resp. measures).

We now return to the local in time analysis along the line of [Bo]. The present work will be an elaboration of these arguments, with estimate (1.10) at our disposal. A crucial role is played by a system of space-time norms, defined as

$$\|y\|_{s,b} = \|y\|_{X_{s,b}[0,T]} = \left\{\sum_{n\in\mathbb{Z}} (1+|n|)^{2s} \int_{\mathbb{R}} d\lambda (1+|\lambda - n^3|)^{2b} |\hat{y}(n,\lambda)|^2\right\}^{1/2} \tag{1.22}$$

assuming ($T > 0$ bounded or small)

$$y(x,t) = \sum_{n\in\mathbb{Z}} \int d\lambda\, \hat{y}(n,\lambda) e^{i(n.x+\lambda.t)} \quad \text{on} \quad \mathbb{T} \times [0,T]. \tag{1.23}$$

Clearly $X_{s,b}[0,T] \subset C_{H^s(\mathbb{T})}[0,T]$ if $b > \frac{1}{2}$. For more details on the use of these and similar norms, the reader may consult [Bo] or the expository paper by J. Ginibre [Gi]. We will often use the notation $|\cdot|$ referring to $1+|\cdot|$ in the factors of (1.22). The use of the norm (1.22) in solving the Cauchy problem (1.2), (1.6) local in time is two-fold. First there are the Strichartz inequalities for the linear KdV-equation (cf. [Bo])

$$\begin{cases} y_t + \partial_x^3 y = 0 \\ y(0) = \phi. \end{cases} \tag{1.24}$$

In the periodic case, we have the inequality

$$\|y\|_{L^4(\mathbb{T} \times [0,T])} \leq C\|\phi\|_2 \tag{1.25}$$

where

$$y(x,t) = S(t)\phi(x) \equiv \sum_n \hat{\phi}(n) e^{i(nx+n^3t)} \tag{1.26}$$

solves (1.24). In fact, we will use following strengthening of (1.25) in the form of the general inequality

$$\|y\|_{L^4(\mathbb{T} \times [0,T])} \lesssim C\|y\|_{0,\frac{1}{3}} \tag{1.27}$$

(the exponent $\frac{1}{3}$ is sharp). We dispose also of an L^6-inequality ($a+$ denoting any number $> a$)

$$\|y\|_{L^6(\mathbb{T} \times [0,T])} \leq C\|y\|_{0+,\frac{1}{2}+}$$

which will not be used here however.

Secondly, the norms (1.22) permit us to capture certain smoothing phenomena in the nonlinear term of the (equivalent) integral equation

$$u(t) = S(t)\phi + \int_0^t S(t-\tau)(yy_x)(\tau) d\tau \tag{1.28}$$

(Duhamel's formula). These smoothing effects in the periodic case are more subtle than on the line $\mathbb{R}$ and result from certain arithmetic facts involving the $(\lambda - n^3)$-factors, namely the identities

$$(n_1 + n_2)^3 - n_1^3 - n_2^3 = 3n_1 n_2 (n_1 + n_2) \tag{1.29}$$

$$(n_1 + n_2 + n_3)^3 - n_1^3 - n_2^3 - n_3^3 = 3(n_1 + n_2)(n_2 + n_3)(n_3 + n_1). \tag{1.30}$$

For simplicity, we treat in the next sections the IVP (1.2), (1.6) with $\hat{\phi}(0) = 0$. Otherwise the $(\lambda - n^3)$-factor in (1.22) needs to be replaced by $\lambda - n^3 - \hat{\phi}(0)n$, making the formulas more cumbersome, although without essential differences.

It is not clear to me how the theorem above may be obtained by the sole use of spectral considerations. Explicit formulas such as given in [Tr] seem to require more smoothness of the data to converge. On the other hand, it is surely possible that the result here may be extended to larger classes of distributions as initial data than we are considering.

Most of the paper deals with the proof of the theorem stated in the beginning (except for section 6 where we make some comments on sharpness of known local well-posedness results in terms of required minimal regularity on data for the KdV equation and generalizations).

As explained above, the two main ingredients are

(i) The local well-posedness theory for the periodic KdV equation

The method developed in sections 2, 3 is much in the spirit of [Bo] and some later papers such as [KPV$_2$]. (Presently, there is in fact a rather vast literature of this approach to the Cauchy problem for various types of hyperbolic equations). Essentially, we reproduce the $H^{-1/2}$-well-posedness argument, but push the analysis to the point where it becomes clear what is the exact nature of the obstruction to go beyond $H^{-1/2}$ (namely the presence of "large" Fourier coefficients). The starting point is the use of Duhamel's formula (2.1) below and the estimate of the second (nonlinear term) in norms of the form (2.5). One significant difference with [Bo] is that here we will need to iterate (2.1) twice, leading to estimates on a trilinear expression in y (cf. (2.29)). Also norms $\| \ \|_{s,b}$ for different parameters values (s, b) are used yielding eventually in particular a bound on $\|y\|_{s,b}$ with $s < -\frac{1}{2}$, $b > \frac{1}{2}$. This analysis turns out to be conclusive provided there is an a priori uniform bound on the Fourier coefficients

$$\widehat{y}(n) = \widehat{y}(n)(t) = \int y(x,t)\, e^{-2\pi n x} dx.$$

Remark. It would be very interesting to develop a substitute for the norm (2.5) such that we would not have to rely on this additional assumption, since it would permit us to establish at least a local in time well-posedness result for nonintegrable KdV-variants.

(ii) A priori estimates from spectral theory

The a priori bound on $|\widehat{y}(n)|$ is derived in section 4 from the conservation of the periodic spectrum $\{\lambda_j(q)\}$ when the potential q evolves according to the KdV flow, and the general "principle" that the size of the Fourier coefficients $|\widehat{q}(n)|$ is essentially measured by the gap sequence $\{\lambda_{2n} - \lambda_{2n-1}\}$. If q is a smooth potential, such estimates appear in [McK-Tr]. For L^2-potentials, the case of the Dirichlet spectrum was studied in [P-Tr] and an L^2-extension of [McK-Tr] outlined in [Bo]. (Such results may also be found in some recent works of T. Kappeler and coauthors). The methods here are again similar and the main problem is to bound error terms in certain multilinear expansions involving the potential q (cf. (4.10)). The situation is a bit more delicate however, since at the start we dispose of a measure although the elements of the corresponding isospectral manifold are only expected to be pseudomeasures.

Solutions of periodic KdV are almost periodic in time. For finite gap potentials, this was proven by Lax and Novikov. The paper [McK-Tr] establishes the result in the smooth case. An easy extension of their method (see [Bo]) yields the result in L^2. In section 5 of this paper, almost periodicity in H^s, $s < -\frac{1}{2}$, of the previously constructed global solution, with initial data given by a measure, is established.

2. Estimates on the nonlinear term

We consider the integral equation

$$y(t) = \Phi + \int_0^t S(t-\tau)(\partial_x y^2)(\tau)d\tau \quad \text{with } \Phi = S(t)\phi \tag{2.1}$$

As mentioned above, we assume

$$\hat{y}(0) = \hat{\phi}(0) = 0. \tag{2.2}$$

There are essentially the following contributions to the nonlinear term (cf. [Bo])

$$\sum_{n\neq 0} \int_{|\lambda - n^3|>1} d\lambda \frac{n}{\lambda - n^3} \Bigg[\sum_{\substack{n=n_1+n_2 \\ \lambda=\lambda_1+\lambda_2}} \hat{y}(n_1,\lambda_1)\hat{y}(n_2,\lambda_2) \Bigg] e^{i(nx+\lambda t)} \tag{2.3}$$

and

$$\sum_{n\neq 0} n \Bigg[\int_{|\lambda - n^3|>1} d\lambda \frac{1}{\lambda - n^3} \Bigg[\sum_{\substack{n=n_1+n_2 \\ \lambda=\lambda_1+\lambda_2}} \int \hat{y}(n_1,\lambda_1)\hat{y}(n_2,\lambda_2) \Bigg] \Bigg] e^{i(nx+n^3 t)}. \tag{2.4}$$

Recall (1.22)

$$\|y\|_{s,b} = \left(\sum_n |n|^{2s} \int |\lambda - n^3|^{2b} |\hat{y}(n,\lambda)|^2 \right)^{1/2}. \tag{2.5}$$

Choose $\alpha < \frac{1}{2}$, close to $\frac{1}{2}$ (to be specified). We will consider the different norms $\| \ \|_{-(1-\alpha),\alpha}$, $\| \ \|_{-\alpha,\alpha}$, $\| \ \|_{-\alpha,(1-\alpha)}$ in what follows.
Write

$$\hat{y}(n,\lambda) = \frac{|n|^{1-\alpha} c(n,\lambda)}{|\lambda - n^3|^\alpha} \tag{2.6}$$

where

$$\|c\|_2 \equiv \|c\|_{\ell^2_n L^2_\lambda} = \|y\|_{-(1-\alpha),\alpha}. \tag{2.7}$$

Substituting both $\hat{y}$-factors in (2.3), (2.4) by (2.6), we get the different contributions

$$|\lambda - n^3| > |\lambda_1 - n_1^3|,\ |\lambda_2 - n_2^3| \tag{2.8}$$

and

$$|\lambda_1 - n_1^3| > |\lambda - n^3|,\ |\lambda_2 - n_2^3|, \tag{2.9}$$

similarly

$$|\lambda_2 - n_2^3| > |\lambda - n^3|,\ |\lambda_1 - n_1^3|. \tag{2.10}$$

(I.1) Estimation of $\|(2.3)\big|_{|\lambda - n^3| > |\lambda_1 - n_1^3|,\, |\lambda_2 - n_2^3|}\|_{-\alpha,\alpha}$

We get clearly

$$\left\{ \sum_n \int d\lambda \, \frac{|n|^{2(1-\alpha)}}{|\lambda - n^3|^{2(1-\alpha)}} \Bigg| \sum_{\substack{n=n_1+n_2 \\ \lambda=\lambda_1+\lambda_2 \\ (2.8)}} \int |\hat{y}(n_1,\lambda_1)|\, |\hat{y}(n_2,\lambda_2)| \Bigg|^2 \right\}^{1/2} \tag{2.11}$$

where by assumption, cf. (1.29)

$$|\lambda - n^3| = \max(|\lambda - n^3|,\ |\lambda_1 - n_1^3|,\ |\lambda_2 - n_2^3|) \geq |n^3 - n_1^3 - n_2^3| \geq |n\, n_1 n_2|. \tag{2.12}$$

By duality, (2.12) and (2.6), (2.11) is bounded by

$$\sum_{\substack{n=n_1+n_2\\ \lambda=\lambda_1+\lambda_2}}\int \frac{d(n,\lambda)}{|n_1n_2|^{2(1-\alpha)}}|\hat{y}(n_1,\lambda_1)|\ |\hat{y}(n_2,\lambda_2)| \tag{2.13}$$

$$(\|d\|_2\le 1)$$

$$\le \sum_{\substack{n=n_1+n_2\\ \lambda=\lambda_1+\lambda_2}}\int d(n,\lambda)\,\frac{c(n_1,\lambda_1)}{|\lambda_1-n_1^3|^\alpha}\,\frac{c(n_2,\lambda_2)}{|\lambda_2-n_2^3|^\alpha}\quad . \tag{2.14}$$

Assume

$$\alpha>\frac{1}{3}. \tag{2.15}$$

Invoking (1.27), (2.14) then admits clearly an $L^2_{xt}\times L^4_{xt}\times L^4_{xt}$-estimate, and we get the contribution

$$\|c\|_2^2=\|y\|^2_{-(1-\alpha),\alpha}. \tag{2.16}$$

In fact, from the denominator saving in (2.14), one may get for small time T an estimate in $X_{-(1-\alpha),\alpha}[0,T]$ by

$$T^{\varkappa}\|y\|^2_{-(1-\alpha),\alpha} \tag{2.17}$$

for some $\varkappa>0$.

Thus we reexpress (2.14) in terms of functions. Letting

$$F(x,\,t)=\sum_n\int d\lambda\Big\{d(n,\,\lambda)\,e^{i(nx+\lambda t)}\Big\}$$

$$G(x,\,t)=\sum_n\int d\lambda\Big\{\frac{c(n,\,\lambda)}{|\lambda-n^3|^\alpha}\,e^{i(nx+\lambda t)}\Big\}$$

(2.14) yields

$$\int\int F.G^2\,dxdt$$

(where the t-integration may be localized to the initial time interval $[0,\,T]$) which, by Hölder's inequality and (1.27), (2.15) is bounded by

$$\|F\|_2.\|G\|_4^2\le\|G\|_4^2\underset{(1.27)}{\le}C\|G\|^2_{0,\frac{1}{3}}\underset{\left(\alpha>\frac{1}{3}\right)}{\le}C.\|c\|_2^2. \tag{2.17$'$}$$

In order to get (2.17), refine using interpolation the last inequality in (2.17′) as follows

$$\|G\|_{0,\frac{1}{3}}\le\|G\|_{0,0}^{1-\frac{1}{3\alpha}}\|G\|_{0,\alpha}^{\frac{1}{3\alpha}}=\|G\|^{1-\frac{1}{3\alpha}}_{L^2(\mathbb{T}\times[0,\,T])}\cdot\|c\|_2^{\frac{1}{3\alpha}}.$$

and, from Hölder's inequality and again (1.27)

$$\|G\|_{L^2(\mathbb{T}\times[0,T])} \le T^{1/4}.\|G\|_4 \le CT^{1/4}\|G\|_{0,\frac{1}{3}}.$$

The last two inequalities yield

$$\|G\|_{0,\frac{1}{3}} \le CT^{1/4(3\alpha-1)}\|c\|_2. \tag{2.17''}$$

Substitution of (2.17″) in (2.17′) yields the improved estimate (2.17)

$$C\|G\|^2_{0,\frac{1}{3}} \le CT^{\frac{1}{2}(3\alpha-1)}\|c\|_2^2$$

with $\kappa = \frac{1}{2}(3\alpha-1)$.

Remark.

(i) The preceding is a typical example of an estimate with this approach. The problem is reexpressed using Fourier transform of the functions and multipliers. One then performs some "surgery" on those multipliers, often distinguishing several regions. The newly obtained expressions are again reformulated in a function setting and bounded for instance by a Strichartz type inequality such as (1.27). More details on this may be found in [Bo] or [Gi] for instance.

(ii) Similar improvement for small time interval of an initial bound (because of some denominator savings), leading to an extra factor T^κ, reappears repeatedly in later estimates below without explicit mention.

(I.2) Estimation of $\|(2.4)\big|_{|\lambda-n^3|>|\lambda_1-n_1^3|,\,|\lambda_2-n_2^3|}\|_{-\alpha},$

(From the form of (2.4), the exponent b in (2.5) may be chosen arbitrarily here.)

We get the bound

$$\left\{\sum_n |n|^{2(1-\alpha)}\left[\int \frac{d\lambda}{|\lambda-n^3|}\left[\sum_{\substack{n=n_1+n_2\\ \lambda=\lambda_1+\lambda_2}}\int |\hat{y}(n_1,\lambda_1)|\,|\hat{y}(n_2,\lambda_2)|\right]\right]^2\right\}^{1/2} \tag{2.18}$$

$$< \sum_{\substack{n=n_1+n_2\\ \lambda=\lambda_1+\lambda_2\\ (2.8)}}\int \frac{|n|^{1-\alpha}d(n)}{|\lambda-n^3|}\,\frac{|n_1|^{1-\alpha}c(n_1,\lambda_1)}{|\lambda_1-n_1^3|^\alpha}\,\frac{|n_2|^{1-\alpha}c(n_2,\lambda_2)}{|\lambda_2-n_2^3|^\alpha} \tag{2.19}$$

(from duality; $\|d\|_2 \le 1$).

Writing by (2.12)

$$|\lambda-n^3| > |\lambda_1-n_1^3|^{\frac{1}{2}-\alpha+}|\lambda_2-n_2^3|^{\frac{1}{2}-\alpha+}|nn_1n_2|^{2\alpha-} \tag{2.20}$$

we get thus

$$\sum_{n=n_1+n_2}\int d\lambda_1 d\lambda_2 |n|^{1-3\alpha+} d(n) \frac{|n_1|^{1-3\alpha+} c(n_1,\lambda_1)}{|\lambda_1 - n_1^3|^{\frac{1}{2}+}} \frac{|n_2|^{1-3\alpha+} c(n_2.\lambda_2)}{|\lambda_2 - n_2^3|^{\frac{1}{2}+}}. \tag{2.21}$$

Defining

$$\tilde{c}(n) = \int \frac{c(n,\lambda)}{|\lambda - n^3|^{\frac{1}{2}+}} d\lambda \tag{2.22}$$

it follows from Hölder's inequality that

$$\|\tilde{c}\|_2 \lesssim \|c\|_2 = \|y\|_{-(1-\alpha),\alpha}. \tag{2.23}$$

Thus

$$(2.21) = \left\langle \sum_n |n|^{1-3\alpha+} d(n) e^{inx}, \right.$$
$$\left. \left[\sum_{n_1} |n_1|^{1-3\alpha+} \tilde{c}(n_1) e^{in_1x}\right]\left[\sum_{n_2} |n_2|^{1-3\alpha+} \tilde{c}(n_2) e^{in_2x}\right]\right\rangle. \tag{2.24}$$

Assuming

$$1 - 3\alpha < -\frac{1}{6}, \quad \text{hence} \quad \alpha > \frac{7}{18} \tag{2.25}$$

one gets from the Hausdorff-Young inequality and an $L^3_x \times L^3_x \times L^3_x$-estimate, the bound

$$(2.24) \le \left\|\sum_n |n|^{1-3\alpha+} d(n) e^{inx}\right\|_3 \left\|\sum_n |n|^{1-3\alpha+} \tilde{c}(n) e^{inx}\right\|_3^2 \lesssim \|\tilde{c}\|_2^2 \le \|y\|^2_{-(1-\alpha),\alpha}. \tag{2.26}$$

(II) Estimation of $\|(2.3) + (2.4)\big|_{|\lambda_1 - n_1^3| > |\lambda - n^3|, |\lambda_2 - n_2^3|}\|_{-\alpha,1-\alpha}$ (other case is similar)

Denote for simplicity by $I_{-\alpha,1-\alpha}$ the quantity to be estimated here. Both contributions (2.3), (2.4) are clearly bounded by

$$\sum_{\substack{n=n_1+n_2\\ \lambda=\lambda_1+\lambda_2\\ (2.9)}}\int \frac{d(n,\lambda)|n|^{1-\alpha}}{|\lambda - n^3|^{\alpha}} \hat{y}(n_1,\lambda_1) \frac{|n_2|^{1-\alpha} c(n_2,\lambda_2)}{|\lambda_2 - n_2^3|^{\alpha}} \tag{2.27}$$

where

$$\|d\|_2 \le 1, \quad \|c\|_2 = \|u\|_{-(1-\alpha),\alpha}.$$

Since $|\lambda_1 - n_1^3| > |\lambda - n^3|$, $|\lambda_2 - n_2^3|$ by (2.9), the factor $\hat{y}(n_1, \lambda_1)$ in (2.27) is given by (2.3), thus replaced by

$$\frac{n_1}{\lambda_1 - n_1^3}\Bigg[\sum_{\substack{n_1=n_3+n_4\\ \lambda_1=\lambda_3+\lambda_4}} \int \hat{y}(n_3,\lambda_3)\hat{y}(n_4,\lambda_4)\Bigg]. \tag{2.28}$$

We get as estimate on $I_{-\alpha,1-\alpha}$

$$\sum_{\substack{n=n_1+n_2, n_1=n_3+n_4\\ \lambda=\lambda_1+\lambda_2, \lambda_1=\lambda_3+\lambda_4\\ (2.9)}} \int \frac{d(n,\lambda)|n|^{1-\alpha}}{|\lambda - n^3|^\alpha} \; \frac{n_1}{\lambda_1 - n_1^3} \; \frac{|n_2|^{1-\alpha} c(n_2,\lambda_2)}{|\lambda_2 - n_2^3|^\alpha} \; \hat{y}(n_3,\lambda_3)\hat{y}(n_4,\lambda_4). \tag{2.29}$$

Again we need to distinguish the cases

$$|\lambda_1 - n_1^3| > |\lambda_3 - n_3^3|,\ |\lambda_4 - n_4^3| \tag{2.30}$$

and

$$|\lambda_3 - n_3^3| > |\lambda_1 - n_1^3|,\ |\lambda_4 - n_4^3|. \tag{2.31}$$

Similarly

$$|\lambda_4 - n_4^3| > |\lambda_1 - n_1^3|,\ |\lambda_3 - n_3^3| \tag{2.32}$$

corresponding to (2.8)–(2.10).

(II.1) Contribution of $\|(2.29)\big|_{|\lambda_3 - n_3^3| > |\lambda_1 - n_1^3|, |\lambda_4 - n_4^3|}\|_{-\alpha,1-\alpha}$

One may clearly write

$$\frac{n_1}{\lambda_1 - n_1^3} \sum_{\substack{n_1=n_3+n_4\\ \lambda_1=\lambda_3+\lambda_4\\ (2.31)}} \int \hat{y}(n_3,\lambda_3)\hat{y}(n_4,\lambda_4) = \frac{|n_1|^\alpha b(n_1,\lambda_1)}{|\lambda_1 - n_1^3|^{1-\alpha}} \tag{2.33}$$

where

$$\|b\|_2 \le I_{-\alpha,1-\alpha}. \tag{2.34}$$

Substitution of (2.33) in (2.29) yields

$$\sum_{\substack{n=n_1+n_2\\ \lambda=\lambda_1+\lambda_2\\ (2.9)}}\int \frac{|n|^{1-\alpha}d(n,\lambda)}{|\lambda-n^3|^\alpha}\ \frac{|n_1|^\alpha b(n_1,\lambda_1)}{|\lambda_1-n_1^3|^{1-\alpha}}\ \frac{|n_2|^{1-\alpha}c(n_2,\lambda_2)}{|\lambda_2-n_2^3|^\alpha}. \tag{2.35}$$

Since by (2.9)

$$|\lambda_1-n_1^3|\geq|nn_1n_2| \tag{2.36}$$

$$(2.35) < \sum_{\substack{n=n_1+n_2\\ \lambda=\lambda_1+\lambda_2}}\int \frac{d(n,\lambda)}{|\lambda-n^3|^\alpha}\ \left[|n_1|^{2\alpha-1}|b(n_1,\lambda_1)|\right]\frac{c(n_2,\lambda_2)}{|\lambda_2-n_2^3|^\alpha} \tag{2.37}$$

$$< o(1)\|d\|_2.\|b\|_2\|c\|_2 \qquad (\text{by } L^4_{xt}\times L^2_{xt}\times L^4_{xt}\text{-estimate})$$

$$= o(1)\|y\|_{-(1-\alpha),\alpha}I_{-\alpha,1-\alpha} \tag{2.38}$$

for small time T.

(II.2) Contribution of $\|(2.29)\big|_{|\lambda_1-n_1^3|>|\lambda_3-n_3^3|,|\lambda_4-n_4^3|}\|_{-\alpha,(1-\alpha)}$

Substituting (2.6), replace (2.29) by

$$\sum_{\substack{n=n_2+n_3+n_4\\ \lambda=\lambda_2+\lambda_3+\lambda_4\\ (2.40)}}\int \frac{d(n,\lambda).|n|^{1-\alpha}}{|\lambda-n^3|^\alpha}\ \frac{|n_1|}{|\lambda_1-n_1^3|}\ \frac{|n_2|^{1-\alpha}c(n_2,\lambda_2)}{|\lambda_2-n_2^3|^\alpha}\ \cdot$$

$$\cdot\ \frac{|n_3|^{1-\alpha}c(n_3,\lambda_3)}{|\lambda_3-n_3^3|^\alpha}\ \frac{|n_4|^{1-\alpha}c(n_4,\lambda_4)}{|\lambda_4-n_4^3|^\alpha} \tag{2.39}$$

where (2.40) denotes the parameter restriction

$$|\lambda_1-n_1^3|>|\lambda-n^3|,\ |\lambda_2-n_2^3|,\ |\lambda_3-n_3^3|,\ |\lambda_4-n_4^3| \quad (n_1=n_3+n_4,\ \lambda_1=\lambda_3+\lambda_4) \tag{2.40}$$

One gets from (2.9), (2.30)

$$|\lambda_1-n_1^3|>|nn_1n_2|+|n_1n_3n_4|\geq|n_1|\,|nn_2|^\alpha|n_3n_4|^{1-\alpha}. \tag{2.41}$$

Hence there is the bound

$$(2.39) < \sum\int \frac{d(n,\lambda)|n|^{1-2\alpha}}{|\lambda-n^3|^\alpha}\ \frac{|n_2|^{1-2\alpha}|c(n_2,\lambda_2)|}{|\lambda_2-n_2^3|^\alpha}\ \frac{|c(n_3,\lambda_3)|}{|\lambda_3-n_3^3|^\alpha}\ \frac{|c(n_4,\lambda_4)|}{|\lambda_4-n_4^3|^\alpha}. \tag{2.42}$$

Fix a number $\gamma > 0$ satisfying

$$\gamma > \frac{2(1-2\alpha)}{\alpha - 1/3}. \tag{2.43}$$

In the parameter region

$$\max\big(|\lambda - n^3|,\ |\lambda_2 - n_2^3|,\ |\lambda_3 - n_3^3|,\ |\lambda_4 - n_4^3|\big) > (|n| + |n_2|)^\gamma \tag{2.44}$$

(2.42) may then clearly be estimated by

$$\sum \int \frac{d(n,\lambda)}{|\lambda - n^3|^{1/3}} \ \frac{|c(n_2,\lambda_2)|}{|\lambda_2 - n_2^3|^{1/3}} \ \frac{|c(n_3,\lambda_3)|}{|\lambda_3 - n_3^3|^{1/3}} \ \frac{|c(n_4,\lambda_4)|}{|\lambda_4 - n_4^3|^{1/3}} \cdot \atop (|n| + |n_2|)^{2(1-2\alpha) - \gamma\left(\alpha - \frac{1}{3}\right)} \tag{2.45}$$

$$< \sum \int \frac{d(n,\lambda)}{|\lambda - n^3|^{1/3}} \ \frac{|c(n_2,\lambda_2)|}{|\lambda_2 - n_2^3|^{1/3}} \ \frac{|c(n_3,\lambda_3)|}{|\lambda_3 - n_3^3|^{1/3}} \ \frac{|c(n_4,\lambda_4)|}{|\lambda_4 - n_4^3|^{1/3}}. \tag{2.46}$$

Again from an $(L^4_{xt} \times L^4_{xt} \times L^4_{xt} \times L^4_{xt})$-estimate and (1.27), this yields an $o(1).\|y\|^3_{-(1-\alpha),\alpha}$-bound. Thus we are restricted to the parameter region

$$\max\big(|\lambda_2 - n_2^3|,\ |\lambda_3 - n_3^3|,\ |\lambda_4 - n_4^3|,\ |\lambda - n^3|\big) \le (|n| + |n_2|)^\gamma. \tag{2.47}$$

If (2.47), observe that

$$\begin{aligned} |\lambda_1 - n_1^3| = |\lambda - \lambda_2 - (n-n_2)^3| &> |n^3 - n_2^3 - (n-n_2)^3| - 2(|n| + |n_2|)^\gamma \\ &> |n n_1 n_2| \\ &> |\lambda - n^3| + \max_{i=2,3,4} |\lambda_i - n_i^3| \\ &\text{(we may assume } n, n_1, n_2 \neq 0) \end{aligned} \tag{2.48}$$

and (2.9), (2.30) are thus automatically fulfilled.

Hence the (2.47)-contribution to (2.29) is thus

$$\sum_{\substack{n = n_2 + n_3 + n_4 \\ \lambda = \lambda_2 + \lambda_3 + \lambda_4}} \int \frac{|n|^{1-\alpha} d(n,\lambda)}{|\lambda - n^3|^\alpha} \ \frac{n - n_2}{\lambda - \lambda_2 - (n-n_2)^3} \ \hat{u}(n_2,\lambda_2)\, \hat{u}(n_3,\lambda_3)\, \hat{u}(n_4,\lambda_4). \tag{2.49}$$

Recall (1.30) implying that

$$\begin{aligned} \max\big(|\lambda - n^3|, |\lambda_2 - n_2^3|, |\lambda_3 - n_3^3|, |\lambda_4 - n_4^3|\big) &\ge |n^3 - n_2^3 - n_3^3 - n_4^3| \\ &\ge |n_2 + n_3|\, |n_3 + n_4|\, |n_4 + n_2|. \end{aligned} \tag{2.50}$$

If (2.47), we get thus clearly that

$$n_2 + n_3 = 0 \qquad \text{or} \qquad n_2 + n_4 = 0. \tag{2.51}$$

(Since $n_1 = n_3 + n_4$, we may assume $n_3 + n_4 \neq 0$.)

Assume $n_2 + n_3 = 0$. The case $n_2 + n_4 = 0$ is similar. Since

$$n_3 = -n_2,\ n_4 = n \tag{2.52}$$

(2.49) yields

$$\sum_{\substack{n, n_2 \neq n \\ \lambda = \lambda_2 + \lambda_3 + \lambda_4 \\ (2.54)}} \int \frac{|n|^{1-\alpha} d(n,\lambda)}{|\lambda - n^3|^\alpha} \frac{n - n_2}{\lambda - \lambda_2 - (n - n_2)^3} \hat{y}(n_2, \lambda_2) \hat{y}(-n_2, \lambda_3) \hat{y}(n, \lambda_4) \tag{2.53}$$

where (2.54) refers to the parameter restriction

$$|\lambda_2 - n_2^3|,\ |\lambda_3 + n_2^3|,\ |\lambda_4 - n^3|,\ |\lambda - n^3| < (|n| + |n_2|)^\gamma. \tag{2.54}$$

Write by (2.54)

$$\begin{aligned} \frac{n - n_2}{\lambda - \lambda_2 - (n - n_2)^3} &= \frac{n - n_2}{n^3 - n_2^3 - (n - n_2)^3 + [\lambda - n^3] - [\lambda_2 - n_2^3]} \\ &= \frac{1}{3nn_2 + \frac{[\lambda - n^3] - [\lambda_2 - n_2^3]}{n - n_2}} \\ &= \frac{1}{3nn_2} + O\left(\frac{(|n| + |n_2|)^\gamma}{|n|^2 |n_2|^2 |n - n_2|} \right). \end{aligned} \tag{2.55}$$

The contribution of the first term in (2.53) is given by

$$\sum_{\substack{n, n_2 \neq n \\ \lambda = \lambda_2 + \lambda_3 + \lambda_4 \\ (2.54)}} \frac{|n|^{1-\alpha} d(n,\lambda)}{n|\lambda - n^3|^\alpha} \frac{1}{n_2} \hat{y}(n_2, \lambda_2) \hat{y}(-n_2, \lambda_3) \hat{y}(n, \lambda_4). \tag{2.56}$$

To estimate it will require some care, taking into account certain cancellation in (2.56) as well as the a priori bound (1.10).

The second (error) term in (2.55) yields for (2.53) following straightforward estimate

$$\sum_{\substack{n;n_2\neq n\\ \lambda=\lambda_2+\lambda_3+\lambda_4\\ (2.54)}}\int \frac{|n|^{1-\alpha}d(n,\lambda)}{|\lambda-n^3|^\alpha}|\hat{y}(n_2,\lambda_2)|\,|\hat{y}(-n_2,\lambda_3)|\,|\hat{y}(n,\lambda_4)|\frac{(|n|+|n_2|)^\gamma}{n^2n_2^2} \tag{2.57}$$

$$\leq \sum\int \frac{d(n,\lambda)}{|\lambda-n^3|^\alpha}\,\frac{|c(n_2,\lambda_2)|}{|\lambda_2-n_2^3|^\alpha}\,\frac{|c(-n_2,\lambda_3)|}{|\lambda_3+n_2^3|^\alpha}\,\frac{|c(n,\lambda_4)|}{|\lambda_4-n^3|^\alpha}\,\frac{|n|^{\gamma+2(1-\alpha)}|n_2|^{\gamma+2(1-\alpha)}}{n^2n_2^2} \tag{2.58}$$

$$\leq \sum_{\substack{n=n_2+n_3+n_4\\ \lambda=\lambda_2+\lambda_3+\lambda_4}}\int \frac{d(n,\lambda)}{|\lambda-n^3|^\alpha}\,\frac{|c(n_2,\lambda_2)|}{|\lambda_2-n_2^3|^\alpha}\,\frac{|c(n_3,\lambda_3)|}{|\lambda_3-n_3^3|^\alpha}\,\frac{|c(n_4,\lambda_4)|}{|\lambda_4-n_4^3|^\alpha}$$

$$< o(1)\|y\|^3_{-(1-\alpha),\alpha} \tag{2.59}$$

provided

$$2(1-\alpha)+\gamma<2. \tag{2.60}$$

Next we analyze (2.56).

Rewrite (2.56) as

$$\sum_{\substack{n;n_2\neq n,-n\\ \lambda=\lambda_2+\lambda_3+\lambda_4\\ |\lambda-n^3|,|\lambda_4-n^3|<(|n|+|n_2|)^\gamma}}\int \frac{|n|^{1-\alpha}d(n,\lambda)}{n|\lambda-n^3|^\alpha}$$

$$\frac{1}{n_2}\Big[(\hat{y}(n_2,\lambda_2)\big|_{|\lambda_2-n_2^3|<(|n|+|n_2|)^\gamma})(\hat{y}(-n_2,\lambda_3)\big|_{|\lambda_3+n_2^3|<(|n|+|n_2|)^\gamma})\Big]\hat{y}(n,\lambda_4) \tag{2.61}$$

$$+0\Bigg\{\sum_{\substack{n;\lambda=\lambda_2+\lambda_3+\lambda_4\\ (2.54)}}\int \frac{|d(n,\lambda)|}{|n|^{1+\alpha}|\lambda-n^3|^\alpha}|\hat{y}(-n,\lambda_2)|\,|\hat{y}(n,\lambda_3)|\,|\hat{y}(n,\lambda_4)|\Bigg\}. \tag{2.62}$$

Estimation of (2.61)

Permuting λ_2 and λ_3, an antisymmetric expression in n_2 is obtained, so that (2.61) $=0$.

Estimation of (2.62)

Recalling (1.10), we have the a priori bound

$$|\hat{y}(n)(t)| < C \quad \text{for all} \quad t \tag{2.63}$$

and hence in particular

$$\|\hat{y}(n)\|_{L^2_\lambda} < C. \tag{2.64}$$

Thus

$$\begin{aligned}
(2.62) < \sum_n \int_{|\lambda_i \pm n^3| < 2|n|^\gamma} d\lambda_2 d\lambda_3 d\lambda_4 \times & \qquad (2.65)\\
\times \Big[|n|^{-2\alpha} |d(n, \lambda_2 + \lambda_3 + \lambda_4)| (|n|^{-(1-\alpha)} |\hat{y}(-n, \lambda_2)|) |\hat{y}(n, \lambda_3)| |\hat{y}(n, \lambda_4)| \Big] &\\
< o(1) \sum_n |n|^{-2\alpha+\gamma} \|d(n)\|_2 \Big(|n|^{-(1-\alpha)} \|\hat{y}(-n)\|_2 \Big) & \qquad (2.66)\\
< o(1) \|d\|_2 \, \|y\|_{-(1-\alpha),0} &\\
< o(1) \|y\|_{-(1-\alpha),\alpha}. & \qquad (2.67)
\end{aligned}$$

This completes our preliminary analysis which may be summarized as follows

$$\begin{aligned}
\|(2.3) + (2.4)\|_{-\alpha,\alpha} < & o(1) \|y\|^2_{-(1-\alpha),\alpha} \quad \text{(from (2.17), (2.26))}\\
& + I_{-\alpha,\alpha} \qquad (2.68)
\end{aligned}$$

and

$$\begin{aligned}
I_{-\alpha,1-\alpha} < & o(1) \|y\|_{-(1-\alpha),\alpha} I_{-\alpha,1-\alpha} && \text{(from (2.38))}\\
& + o(1) \|y\|^3_{-(1-\alpha),\alpha} && \text{(from (2.46), (2.59))}\\
& + o(1) \|y\|_{-(1-\alpha),\alpha} && \text{(from (2.67))} \qquad (2.69)
\end{aligned}$$

where $o(1)$ depends on small time T.

By (2.1) and the fact that $|\hat{\phi}(n)| < C$, it follows from the preceding that

$$\|y\|_{-(1-\alpha),\alpha} = o(1) + o(1) \|y\|^2_{-(1-\alpha),\alpha} + \frac{o(1) \|y\|_{-(1-\alpha),\alpha} + \|y\|^3_{-(1-\alpha),\alpha}}{1 - o(1) \|y\|_{-(1-\alpha),\alpha}}. \tag{2.70}$$

This enables us to deduce that

$$\|y\|_{-(1-\alpha),\alpha} = o(1) \tag{2.71}$$

$$I_{-\alpha,(1-\alpha)} = o(1) \tag{2.72}$$

$$\|(2.3) + (2.4)\|_{-\alpha,\alpha} = o(1). \tag{2.73}$$

The conditions on α, γ are

$$\alpha > \frac{7}{18} \qquad \text{(from (2.15), (2.25))}$$
$$\gamma > \frac{2(1-2\alpha)}{\alpha - 1/3} \tag{2.43}$$
$$2(1-\alpha) + \gamma < 2 \tag{2.60}$$

which are clearly compatible for α close enough to $\frac{1}{2}$.

Our next purpose is to derive an $\|y\|_{-s,b}$-estimate for some $s > \frac{1}{2}$, $b > \frac{1}{2}$. Considering the nonlinear term in (2.1) as a bilinear expression, denote for simplicity

$$(y, z) = \int_0^t S(t-\tau)\partial_x(y,z)(\tau)d\tau. \tag{2.74}$$

Thus

$$y = \Phi + (y, y) \tag{2.75}$$

where, by (2.73)

$$\|(y,y)\|_{-\alpha,\alpha} = o(1). \tag{2.76}$$

From the analysis of (2.4), (2.5), the contributions of

$$\max\big(|\lambda_1 - n_1^3|, |\lambda_2 - n_2^3|\big) \geq |\lambda - n^3| \tag{2.77}$$

are controlled in $\| \quad \|_{-\alpha,1-\alpha}$; we denoted this quantity by $I_{-\alpha,1-\alpha}$. Consider next the contribution to (y, y) of

$$\max\big(|\lambda_1 - n_1^3|,\ |\lambda_2 - n_2^3|\big) < |\lambda - n^3|. \tag{2.78}$$

Write

$$\begin{aligned}(y,y) &= (\Phi,\Phi) + 2(\Phi,(y,y)) + ((y,y),(y,y)) \\ &= (2.80) + (2.81) + (2.82).\end{aligned} \tag{2.79}$$

Fix $s > \frac{1}{2}$ and estimate the $\| \quad \|_{-s,\frac{1}{2}+}$ norm of $(2.79)\big|_{(2.78)}$

Contribution of (2.80)

We get

$$\left\{ \sum_{n,m\in\mathbb{Z}} \frac{|n|^{2(1-s)}}{|m-n^3|^{1-}} \Big| \sum_{\substack{n=n_1+n_2\\ m=n_1^3+n_2^3}} \hat{\phi}(n_1)\hat{\phi}(n_2)\Big|^2 \right\}^{1/2} \tag{2.83}$$

$$\leq C\bigg(\sum_{n=n_1+n_2} \frac{|n|^{2(1-s)}}{|nn_1n_2|^{1-}} |\hat{\phi}(n_1)|^2 |\hat{\phi}(n_2)|^2 \bigg)^{1/2} \tag{2.84}$$

since the system of equations in $n_1, n_2 \in \mathbb{Z}$

$$\begin{cases} n_1 + n_2 = n \\ n_1^3 + n_2^3 = m \end{cases} \tag{2.85}$$

has only a bounded number of solutions for given n, m. Thus

$$(2.84) \leq \bigg(\sum_{n_1,n_2} \frac{1}{|n_1|^{1-}|n_2|^{1-}|n_1+n_2|^{2s-1-}} \bigg)^{1/2} < C \tag{2.86}$$

since $s < \frac{1}{2}$.

Estimation of $\|(2.81)\big|_{(2.78)}\|_{-s,\frac{1}{2}+}$

From (2.76), we get the bound

$$\sum_{\substack{n=n_1+n_2\\ \lambda=\lambda_1+\lambda_2\\ (2.78)}} \int \frac{|n|^{1-s} d(n,\lambda)}{|\lambda-n^3|^{\frac{1}{2}-}} \frac{|n_1|^{\alpha}|b(n_1,\lambda_1)|}{|\lambda_1-n_1^3|^{\alpha}} |\hat{\Phi}(n_2,\lambda_2)| \tag{2.87}$$

where $\|d\|_2 \leq 1$, $\|b\|_2 = 0(1)$.
From (2.12)

$$(2.87) < \sum \int |n|^{\frac{1}{2}-s+} d(n,\lambda) \left[\frac{|n_1|^{\alpha-\frac{1}{2}+}|b(n_1,\lambda_1)|}{|\lambda_1-n_1^3|^{\alpha}} \right] \left[|n_2|^{-\frac{1}{2}+} |\hat{\Phi}(n_2,\lambda_2)| \right]. \tag{2.88}$$

Since $\alpha < \frac{1}{2}$, $s > \frac{1}{2}$ are fixed exponents and $|n_2| < |n| + |n_1|$, we have

$$(2.88) < \sum \int d(n,\lambda) \left[\frac{|b(n_1,\lambda_1)|}{|\lambda_1-n_1^3|^{\alpha}} \right] \left[|n_2|^{-\frac{1}{2}-} |\hat{\Phi}(n_2,\lambda_2)| \right] \tag{2.89}$$

and from (1.27), (1.25), (2.89) may be estimated in $L^2_{xt} \times L^4_{xt} \times L^4_{xt}$.

Estimation of $\|(2.82)\big|_{(2.78)}\|_{-s,\frac{1}{2}+}$

Again from (2.76), we get

$$\sum_{\substack{n=n_1+n_2\\ \lambda=\lambda_1+\lambda_2\\ (2.78)}}\int \frac{|n|^{1-s}d(n,\lambda)}{|\lambda-n^3|^{\frac{1}{2}-}}\,\frac{|n_1|^{\alpha}|b(n_1,\lambda_1)|}{|\lambda_1-n_1^3|^{\alpha}}\,\frac{|n_2|^{\alpha}|b(n_2,\lambda_2)|}{|\lambda_2-n_2^3|^{\alpha}} \tag{2.90}$$

$$< \sum\int \Big[|n|^{-s+\frac{1}{2}+}d(n,\lambda)\Big]\,\frac{|n_1|^{\alpha-\frac{1}{2}+}|b(n_1,\lambda_1)|}{|\lambda_1-n_1^3|^{\alpha}}\,\frac{|n_2|^{\alpha-\frac{1}{2}+}|b(n_2,\lambda_2)|}{|\lambda_2-n_2^3|^{\alpha}} \tag{2.91}$$

$$< \sum\int d(n,\lambda)\frac{|b(n_1,\lambda_1)|}{|\lambda_1-n_1^3|^{\alpha}}\,\frac{|b(n_2,\lambda_2)|}{|\lambda_2-n_2^3|^{\alpha}}$$

$$< C\|d\|_2\,\|b\|_2^2 = o(1). \tag{2.92}$$

Hence we get that also

$$\|(y,y)\big|_{(2.78)}\|_{-s,\frac{1}{2}+} = o(1) \tag{2.93}$$

and thus, since

$$\|(y,y)\big|_{(2.77)}\|_{-\alpha,1-\alpha} = I_{-\alpha,1-\alpha} = o(1) \tag{2.94}$$

we conclude that

$$\|(y,y)\|_{-s,\frac{1}{2}+} = o(1) \tag{2.95}$$

and, from (2.75)

$$\|y\|_{-s,\frac{1}{2}+} = o(1). \tag{2.96}$$

Here $s > \frac{1}{2}$ is any fixed number.

3. Convergence properties for the flow

Coming back to the Cauchy problem (1.2) with $\phi \in M_0(\mathbb{T})$ a measure of mean zero, inequality (2.96) yields a bound

$$\|y\|_{-s,\frac{1}{2}+} = o(1),\ \ s > \frac{1}{2} \tag{3.1}$$

on the solution for $t \in [0,T]$, T small enough (depending on the constant C in (1.10)and eventually on $\|\phi\|$ as will be shown in the next section).

Our purpose now is to show that if $\{\phi_\alpha\}$ is a sequence of (smooth) regularizations of ϕ and $\{y_\alpha\}$ the corresponding solutions to (1.2) with $y_\alpha(0) = \phi_\alpha$, then $\{y_\alpha(t)\}$ converges in $H^{-s}(\mathbb{T})$ for all $s > \frac{1}{2}$ and $|t| < T$. This statement will in particular result from the convergence of $\{y_\alpha\}$ in $\| \quad \|_{-s,\frac{1}{2}+}$ norm. Since we dispose already of the uniform bound (3.1), it will, by interpolation, suffice to establish convergence of $\{y_\alpha\}$ in the weaker norm $\| \quad \|_{-(1-\alpha),\alpha}$ considered in the previous section.

Let thus $y = y_\phi$, $z = y_\psi$ be the solutions of (1.2) corresponding to data $\phi, \psi \in M_0(\mathbb{T})$, respectively. We then repeat the estimates from the previous section to $y - z$. This is usually a straightforward modification of the argument establishing boundedness, replacing in the multilinear expression one of the y-factors by $y - z$. However, more care is needed here since the vanishing of (2.61) for instance depends on symmetry properties which do not remain valid for distinct factors. We will therefore indicate briefly the details. For the contribution (I), the modification is obvious and we get the estimate

$$\left(\|y\|_{-(1-\alpha),\alpha} + \|z\|_{-(1-\alpha),\alpha}\right) \|y - z\|_{-(1-\alpha),\alpha} = o(1)\|y - z\|_{-(1-\alpha),\alpha}. \tag{3.2}$$

Denote next

$$\widetilde{I}_{-\alpha,1-\alpha} = \Big\| [(y,y) - (z,z)] \big|_{|\lambda_1 - n_1^3| > |\lambda - n^3|, |\lambda_2 - n_2^3|} \Big\|_{-\alpha,(1-\alpha)} \tag{3.3}$$

the contribution of (II) in $\| \quad \|_{-\alpha,1-\alpha}$ norm. Substitute again the first factor y (resp. z) by the integral term (y,y) (resp. (z,z)).

For simplicity, we denote by $\underline{y}$ or $\underline{z}$ the component with largest denominator.

Case (II.1) $|\lambda_3 - n_3^3| > |\lambda_1 - n_1^3|, |\lambda_4 - n_4^3|$ (other case is again similar). Write

$$((\underline{y}, y), y) - ((\underline{z}, z), z) = ((\underline{y}, y) - (\underline{z}, z), y) + ((\underline{z}, z), y - z) \tag{3.4}$$

where

$$\big\|(\underline{y}, y) - (\underline{z}, z)\big\|_{-\alpha,1-\alpha} = \widetilde{I}_{-\alpha,1-\alpha} \tag{3.5}$$

and, by (2.71), (2.72)

$$\|y\|_{-(1-\alpha),\alpha} = o(1) \tag{3.6}$$

$$\|(\underline{z}, z)\|_{-\alpha,1-\alpha} = I_{-\alpha,1-\alpha} = o(1). \tag{3.7}$$

We then repeat the (II.1)-estimate from section 2 to get the bound

$$o(1)\widetilde{I}_{-\alpha,1-\alpha} + o(1)\|y - z\|_{-(1-\alpha),\alpha}. \tag{3.8}$$

Case (II.2): $|\lambda_1 - n_1^3| > |\lambda_2 - n_2^3|, |\lambda_3 - n_3^3|, |\lambda_4 - n_4^3|, |\lambda - n^3|$.

If (2.44), i.e.

$$\max\left(|\lambda_2 - n_2^3|,\ |\lambda_3 - n_3^3|,\ |\lambda_4 - n_4^3|,\ |\lambda - n^3|\right) > (|n| + |n_2|)^\gamma \tag{3.9}$$

we repeat the argument, cf. (2.39), (2.46), to get an $\|y - z\|_{-(1-\alpha),\alpha}$ estimate. Assume (2.47). Thus

$$\max\left(|\lambda_2 - n_2^3|,\ |\lambda_3 - n_3^3|,\ |\lambda_4 - n_4^3|,\ |\lambda - n^3|\right) < (|n| + |n_2|)^\gamma\,. \tag{3.10}$$

Write

$$[((\underline{u}, u), u) - ((\underline{v}, v), v)]\,\big|_{(3.10)} = [((\underline{u}, u), u) - ((\underline{v}, v), v)]\,\big|_{(3.10), n \neq n_3, n_4} \tag{3.11}$$

$$+ ((\underline{u}, u), u)\Big|_{\substack{(3.10)\\ n=n_3, n_2=-n_4 \neq n, -n}} - ((\underline{v}, v), v)\Big|_{\substack{(3.10)\\ n=n_3, n_2=-n_4 \neq n, -n}} \tag{3.12}$$

$$+ ((\underline{u}, u), u)\Big|_{\substack{(3.10)\\ n=n_4, n_2=-n_3 \neq n, -n}} - ((\underline{v}, v), v)\Big|_{\substack{(3.10)\\ n=n_4, n_2=-n_3 \neq n, -n}} \tag{3.13}$$

$$+\ [((\underline{u}, u), u) - ((\underline{v}, v), v)]\,\Big|_{\substack{(3.10)\\ n=n_3=n_4, n_2=-n}}\,. \tag{3.14}$$

Again by (2.50), (3.11) has no contribution. Each of the terms in (3.12), (3.13) are again decomposed in a vanishing part (2.61) and an "error term" obtained by replacement of the $\frac{n-n_2}{(\lambda-\lambda_2)-(n-n_2)^3}$-factor in (2.53) by $\frac{[\lambda-n^3]-[\lambda_2-n_2^3]}{(n-n_2)n_2^2n^2} + \cdots$. The collected contribution of these error terms in the differences (3.12), (3.13) may then again be estimated by $o(1)\|y - z\|_{-(1-\alpha),\alpha}$, cf. (2.59). Finally (3.14) admits a bound by $o(1)\|y-z\|_{-(1-\alpha),\alpha}$, passing to a trilinear expression and applying (2.64) to the two y, z factors with remaining $\|y - z\|_{-(1-\alpha),0}$ factor. Thus in conclusion

$$\|y - z\|_{-(1-\alpha),\alpha} \leq \|\phi - \psi\|_{-(1-\alpha)} + o(1)\|y - z\|_{-(1-\alpha),\alpha} + \widetilde{I}_{-\alpha,(1-\alpha)} \tag{3.15}$$

and

$$\widetilde{I}_{-\alpha,(1-\alpha)} \leq o(1)\widetilde{I}_{-\alpha,1-\alpha} + o(1)\|y - z\|_{-(1-\alpha),\alpha}. \tag{3.16}$$

Hence

$$\|y - z\|_{-(1-\alpha),\alpha} \leq \|\phi - \psi\|_{-(1-\alpha)} + o(1)\|y - z\|_{-(1-\alpha),\alpha}$$

thus

$$\|y - z\|_{-(1-\alpha),\alpha} \leq 2\|\phi - \psi\|_{-(1-\alpha)}. \tag{3.17}$$

Consequently, the map

$$\phi \longmapsto y_\phi \tag{3.18}$$

is Lipschitz from a ball in M_0 ($\mathbb{T}$) endowed with $H^{-(1-\alpha)}$ -topology into $X_{-(1-\alpha),\alpha}[0,T]$, T chosen sufficiently small. Hence, from the discussion in the beginning of this section, (3.18) is Hölder-continuous into $X_{-s,\frac{1}{2}+}[0,T]$ for all $s > \frac{1}{2}$ and the flow map for $|t| < T$

$$\phi \longmapsto y_\phi(t) \tag{3.19}$$

as well. Observe that the $M(\mathbb{T})$-boundedness of the data is only used to fulfill the estimate (1.10), (2.63). Since this is an a priori bound valid for all time, the map (3.19) will be (Hölder) continuous for all time, establishing in particular uniqueness of the weak solution.

4. Spectral estimates

The purpose of this section is to substantiate the bound (1.10) by showing that if

$$\|\phi\|_1 < c \tag{4.1}$$

then there are uniform estimates on the Fourier transform of all members q of the isospectral manifold $\mathcal{M}(\phi)$. Thus

$$|\hat{q}(n)| < c', \quad n \in \mathbb{Z}, \quad q \in \mathcal{M}(\phi). \tag{4.2}$$

As mentioned in the introduction, we deal here in fact only with the case in which the constant c in (4.1) is small, but this restriction is probably nonessential. At least part of the elements mentioned below are known and we recall them for completeness sake. Some of the calculations are borrowed from [Bo].

Recall that

$$Q = -\frac{d^2}{dx^2} + q(x) \tag{4.3}$$

where q is a real 1-periodic potential satisfying (4.1) with C small. There is no restriction in assuming $\int q = 0$. The periodic spectrum $\{\lambda_i = \lambda_{2n-1}, \lambda_{2n}\}$ is obtained as roots of the equation

$$\Delta(\lambda) = (-1)^n 2 \tag{4.4}$$

where

$$\Delta(\lambda) = y_1(1,\lambda) + y_2'(1,\lambda) \tag{4.5}$$

is the discriminant and

$$Qy = \lambda y \quad \begin{cases} y_1(0,\lambda) = y_2'(0,\lambda) = 1 \\ y_1'(0,\lambda) = y_2(0,\lambda) = 0. \end{cases} \tag{4.6}$$

The eigenfunctions y_1, y_2 satisfy the equations

$$y_1(x,\lambda) = \cos\sqrt{\lambda}\,x + \int_0^x \frac{\sin\sqrt{\lambda}\,(x-s)}{\sqrt{\lambda}}\, q(s) y_1(s,\lambda) ds \tag{4.7}$$

$$y_2(x,\lambda) = \frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}} + \int_0^x \frac{\sin\sqrt{\lambda}\,(x-s)}{\sqrt{\lambda}}\, q(s) y_2(s,\lambda) ds. \tag{4.8}$$

(i) Estimates on the periodic spectrum

Assume $q \in L^1(\pi)$,

$$\|q\|_1 = o(1), \qquad \int q = 0. \tag{4.9}$$

From (4.5) and iteration of (4.7), (4.8), we get

$$\begin{aligned} \Delta(\lambda) &= y_1(1,\lambda) + y_2'(1,\lambda) \\ &= 2\cos\sqrt{\lambda} + \int_0^1 \lambda^{-1/2} \left[\sin\sqrt{\lambda}\,(1-s).\cos\sqrt{\lambda}\,s + \cos\sqrt{\lambda}\,(1-s).\sin\sqrt{\lambda}\,s\right] q(s) ds \\ &\quad + \int_0^1 \int_0^s \lambda^{-1} \sin\sqrt{\lambda}\,(1-s).\sin\sqrt{\lambda}\,(s-s').q(s).q(s').y_1(s',\lambda) ds\, ds' \\ &\quad + \int_0^1 \int_0^s \lambda^{-1/2} \cos\sqrt{\lambda}\,(1-s).\sin\sqrt{\lambda}\,(s-s').q(s).q(s').y_2(s',\lambda) ds\, ds'. \end{aligned} \tag{4.10}$$

By (4.9), the second term in (4.10) vanishes and we get

$$\Delta(\lambda) = 2\cos\sqrt{\lambda} +$$

$$\lambda^{-1} \int_0^1 \int_0^s \sin\sqrt{\lambda}\,(1-s).\sin\sqrt{\lambda}\,(s-s').\cos\sqrt{\lambda}\,s'.q(s)q(s') ds\, ds' + \tag{4.11}$$

$$\lambda^{-1} \int_0^s \int_0^s \cos\sqrt{\lambda}\,(1-s).\sin\sqrt{\lambda}\,(s-s').\sin\sqrt{\lambda}\,s'.q(s)q(s') ds\, ds' \tag{4.12}$$

$$+\, 0(\lambda^{-3/2}) \tag{4.13}$$

where

$$\begin{aligned} (4.11) + (4.12) &= \lambda^{-1} \int_0^1 \int_0^s \sin\sqrt{\lambda}\,(1-s+s').\sin\sqrt{\lambda}\,(s-s') q(s) q(s') ds\, ds' \\ &= -\frac{\cos\sqrt{\lambda}}{\lambda} \int_0^1 \int_0^s \sin^2\sqrt{\lambda}\,(s-s') q(s) q(s') ds\, ds' \\ &\quad + 0\left(\lambda^{-1}|\sin\sqrt{\lambda}|\right). \end{aligned} \tag{4.14}$$

Since the integrand in the first term of (9.12) is symmetric in s, s', one may replace the double integration $\int_0^1 \int_0^s$ by $\frac{1}{2}\int_0^1 \int_0^1$. This gives, since $\int q = 0$

$$\begin{aligned} &- \frac{\cos\sqrt{\lambda}}{4\lambda} \int_0^1 \int_0^1 \left[1 - \cos 2\sqrt{\lambda}(s-s')\right] q(s)q(s')ds\,ds' \\ &= \frac{\cos\sqrt{\lambda}}{4\lambda} \left[\left(\int_0^1 \cos 2\sqrt{\lambda}s.q(s)ds\right)^2 + \left(\int_0^1 \sin 2\sqrt{\lambda}s.q(s)ds\right)^2\right]. \end{aligned} \tag{4.15}$$

From (4.4), (4.13), (4.14), (4.15), it follows that for $\lambda = \lambda_{2n-1}, \lambda_{2n}$

$$\begin{aligned} (-1)^n 2 = \Delta(\lambda) = 2\cos\sqrt{\lambda} + \frac{\cos\sqrt{\lambda}}{4\lambda}\left[\left\langle q, \cos 2\sqrt{\lambda}\,s\right\rangle^2 + \left\langle q, \sin 2\sqrt{\lambda}\,s\right\rangle^2\right] \\ + 0\left(\lambda^{-3/2} + \lambda^{-1}|\sin\sqrt{\lambda}\,|\right). \end{aligned} \tag{4.16}$$

Writing

$$(-1)^n - \cos\sqrt{\lambda} = (-1)^n\left[1 - \cos\left(\sqrt{\lambda} - \pi n\right)\right] \sim \left|\sqrt{\lambda} - \pi n\right|^2 \tag{4.17}$$

(4.16) yields

$$\begin{aligned} |\lambda - \pi^2 n^2|^2 &\sim \lambda \left|\sqrt{\lambda} - \pi n\right|^2 \\ &\sim \left(\left\langle q, \cos 2\sqrt{\lambda}\,s\right\rangle^2 + \left\langle q, \sin 2\sqrt{\lambda}\,s\right\rangle^2\right) + 0\left(n^{-1} + \left|\sqrt{\lambda} - \pi n\right|\right). \end{aligned} \tag{4.18}$$

In particular, it follows from (4.9) that

$$|\lambda_{2n-1} - \pi^2 n^2| = o(1) \quad \text{and} \quad |\lambda_{2n} - \pi^2 n^2| = o(1). \tag{4.19}$$

The property (4.19) is obviously preserved for all isospectral potentials. Denoting $\{\mu_n\}$ the Dirichlet spectrum, i.e. the roots of

$$y_1(1, \mu, q) = 0 \tag{4.20}$$

satisfying $\lambda_{2n-1} \le \mu_n \le \lambda_{2n}$, it follows thus that

$$|\mu_n(q) - \pi^2 n^2| = o(1) \tag{4.21}$$

for all potentials in the isospectral manifold. Our next goal will then be to show that conversely (4.19) implies (4.2). The main difficulty here when bounding the error terms in formulas as used above is the absence of the $\|q\|_1$-bound.

(ii) Eigenfunction estimates

Denote $\mu = \mu_n$ the n$^{\text{th}}$ Dirichlet eigenvalue and $y = y_2(x,\mu)$ the corresponding eigenfunction normalized in $L^2[0,1]$. Thus

$$y(0) = y(1) = 0 \qquad \text{and} \qquad \int_0^1 |y(x)|^2 = 1. \tag{4.22}$$

By (4.22) one may represent y on $[0,1]$ by a series

$$y = \sum c_k \sin \pi k x \qquad \text{for} \qquad x \in [0,1]. \tag{4.23}$$

Since

$$Qy = -y'' + qy = \mu y \tag{4.24}$$

it follows

$$-\int_0^1 y'' \sin \pi \ell x + \int_0^1 q(x)y(x) \sin \pi \ell x = \frac{\mu}{2}\, c_\ell. \tag{4.25}$$

By (4.22) and partial integration

$$\int_0^1 y'' \sin \pi \ell x = -\pi \ell \int_0^1 y'(x) \cos \pi \ell x = -\pi^2 \ell^2 \int_0^1 \sin \pi \ell x . y(x) = -\pi^2 \ell^2 \frac{c_\ell}{2}. \tag{4.26}$$

Hence, from (4.25), (4.26)

$$|c_\ell|\, |\mu_n - \pi^2 \ell^2| \lesssim \Big| \int q(x) y(x) \sin \pi \ell x \Big|. \tag{4.27}$$

Write from (4.23)

$$\int_0^1 q(x)y(x) \sin \pi \ell x = \frac{1}{2} \sum_{n,k} \hat{q}(n) c_k \int_0^1 e^{2\pi i n x} \left[\cos \pi (k+\ell) x - \cos \pi (k-\ell) x\right] \tag{4.28}$$

and thus

$$\Big| \int_0^1 q(x)y(x) \sin \pi \ell x \Big|$$

$$\leq C \|\hat{q}\|_\infty \sum_{n,k} |c_k| \frac{|\ell|}{(|2n \pm (k+\ell)| + 1)(|2n \pm (k-\ell)| + 1)} \tag{4.29}$$

(with sign correspondence)

$$\leq C \|\hat{q}\|_\infty (\log \ell) \Big(\sum_k |c_k| \Big). \tag{4.30}$$

From (4.27), (4.30) and the assumption $\mu_n = \pi^2 n^2 + o(1)$ (4.21)

$$|c_\ell| < C\|\hat{q}\|_\infty \frac{\log \ell}{|n^2 - \ell^2|}\Big(\sum_k |c_k|\Big) \quad \text{for} \quad \ell \neq n. \tag{4.31}$$

Consequently

$$\sum_{\ell \neq n} |c_\ell| < C\|\hat{q}\|_\infty \frac{(\log n)^2}{n}\Big(\sum_k |c_k|\Big). \tag{4.32}$$

Since by (4.22)

$$\sum c_k^2 = 1 \tag{4.33}$$

and one easily derives from (4.31) that

$$\sum |c_k| \lesssim \left(\frac{(\log M)^2}{M}\, \|\hat{q}\|_\infty\right)\left(\sum |c_k|\right) + M^{1/2} \tag{4.34}$$

for any positive integer M. Thus by appropriate choice of M in (4.34), we get

$$\sum |c_k| < (1 + \|\hat{q}\|_\infty)^{1/2} \log\left(2 + \|\hat{q}\|_\infty\right). \tag{4.35}$$

Consequently, from (4.31), (4.32), (4.35)

$$|c_\ell| \le C \frac{\log \ell}{|n^2 - \ell^2|} \|\hat{q}\|_\infty \left(1 + \|\hat{q}\|_\infty\right) \quad \text{for} \quad \ell \neq n \tag{4.36}$$

and

$$\sum_{\ell \neq n} |c_\ell| \le C \frac{(\log n)^2}{n} \|\hat{q}\|_\infty \left(1 + \|\hat{q}\|_\infty\right). \tag{4.37}$$

(iii) Estimate of $\hat{q}$

Let ϕ be a regularization of the measure in $M(\mathbb{T})$, $\int_{\mathbb{T}} \phi = 0$ and

$$\|\phi\|_1 < \eta \tag{4.38}$$

where η is taken sufficiently small. From the estimates on the periodic spectrum, we get then (cf. (4.19))

$$|\lambda_{2n-1}(q) - \pi^2 n^2| \lesssim \eta, \; |\lambda_{2n}(q) - \pi^2 n^2| \lesssim \eta, \; |\mu_n(q) - \pi^2 n^2| \lesssim \eta \tag{4.39}$$

for all q in the isospectral manifold $\mathcal{M}(\phi)$. We show here that if $q \in \mathcal{M}(\phi)$ and

$$\|\hat{q}\|_\infty < a_1 \tag{4.40}$$

(where a_1 is some fixed sufficiently small constant), then in fact

$$\|\hat{q}\|_\infty < a_1/10. \tag{4.41}$$

Since the map $q \mapsto \|\hat{q}\|_\infty$ is continuous on $\mathcal{M}(\phi)$ which is connected (in fact it is an infinite dimensional torus, cf. [McK-Tr]), it will then follow that (4.41) holds on all of $\mathcal{M}(\phi)$. Observe that the isospectral manifold is invariant under translation on $\mathbb{T}$. Hence, to derive (4.41), it clearly suffices to get

$$|\mathrm{Re}\,\hat{q}(n)| < a_1/10 \qquad \text{for all } n \in \mathbb{Z} \setminus \{0\} \tag{4.42}$$

assuming (4.40). To estimate $\mathrm{Re}\,\hat{q}(n)$, we use the Dirichlet spectrum for which the analysis is a bit simpler than for the periodic spectrum.

One has for $\mu = \mu_n$

$$0 = y_2(1, \mu_n) = \frac{\sin\sqrt{\mu}}{\sqrt{\mu}} + \int_0^1 \frac{\sin\sqrt{\mu}(1-s)}{\sqrt{\mu}} y_2(s,\mu) q(s) ds. \tag{4.43}$$

Denote $y = n y_2(\,\cdot\,, \mu)$, thus essentially L^2-normalized and satisfying thus estimates (4.36), (4.37). Thus

$$0 = n \sin\sqrt{\mu} + \int_0^1 \sin\sqrt{\mu}(1-s) y(s,\mu) q(s) ds$$

and by (4.39)

$$\eta \gtrsim |\mu_n - \pi^2 n^2| \sim n |\sin\sqrt{\mu}| =$$

$$c_n \int_0^1 \sin \pi n(1-s) . \sin \pi n s . q(s) ds + \tag{4.44}$$

$$2 \int_0^1 \sin \frac{\sqrt{\mu} - \pi n}{2} . \cos \frac{\sqrt{\mu} + \pi n}{2} (1-s) . y(s,\mu) . q(s) ds + \tag{4.45}$$

$$\int_0^1 \sin \pi n (1-s) . \left[y(s,\mu) - c_n \sin \pi n s \right] q(s) ds \tag{4.46}$$

where, by (4.36), (4.40)

$$y(s) = \sum c_\ell \sin \ell \pi s, \ c_n \sim 1 \quad \text{and} \quad |c_\ell| < \frac{\log \ell}{|\ell - n|\,|\ell + n|} \|\hat{q}\|_\infty \quad \text{for} \quad \ell \neq n. \tag{4.47}$$

Contribution of (4.44)

Since $\int q = 0$,

$$(4.44) \sim \int_0^1 \sin^2 \pi ns.q(s)ds \sim \int_0^1 \cos 2\pi ns.q(s)ds = \operatorname{Re} \hat{q}(n). \tag{4.48}$$

Estimation of (4.45)

Denote ψ the function

$$\psi(s) = \sin \frac{\sqrt{\mu} - \pi n}{2}(1-s) \cdot \cos \frac{\sqrt{\mu} + \pi n}{2}(1-s) \cdot y(s,\mu) \tag{4.49}$$

satisfying $\psi(0) = 0 = \psi(1)$. Thus

$$|(4.45)| \leq \|\hat{q}\|_\infty \sum_k |\hat{\psi}(k)|. \tag{4.50}$$

Estimate (4.50) distinguishing the cases $k < n$, $k > n$

Case $k < n$. One gets from (4.47), (4.49) the bound

$$\|\hat{q}\|_\infty \sum_{k<n} \sum_\ell |c_\ell| \Big| \int_0^1 e^{-2\pi iks} \cdot \sin \frac{\sqrt{\mu} - \pi n}{2}(1-s) \cdot$$

$$\cdot \cos \frac{\sqrt{\mu} + \pi n}{2}(1-s) \cdot \sin \ell\pi s \Big| \tag{4.51}$$

$$\leq \|\hat{q}\|_\infty \sum_{k<n} \sum_\ell |c_\ell| \min \Big\{ |\sqrt{\mu} - \pi n|,$$

$$\Bigg| \frac{e^{i\frac{\sqrt{\mu}-\pi n}{2}}}{-2k\pi - \frac{\sqrt{\mu}-\pi n}{2} + \sigma_1 \frac{\sqrt{\mu}+\pi n}{2} + \sigma_2 \ell\pi} -$$

$$- \frac{e^{-i\frac{\sqrt{\mu}-\pi n}{2}}}{-2k\pi + \frac{\sqrt{\mu}-\pi n}{2} + \sigma_1 \frac{\sqrt{\mu}+\pi n}{2} + \sigma_2 \ell\pi} \Bigg| \Big\} \tag{4.52}$$

(with $\sigma_1, \sigma_2 = \pm 1$)

by (4.39)

$$< \eta \|\hat{q}\|_\infty \frac{1}{n} \sum_{k<n} \sum_\ell |c_\ell| \frac{1}{1 + |-2k \pm n \pm \ell|} \tag{4.53}$$

by (4.37), (4.40)

$$< \eta \|\hat{q}\|_\infty \frac{\log n}{n} \Big(1 + \frac{(\log n)^2}{n} \|\hat{y}\|_\infty \Big) \tag{4.54}$$

Case $k > n$**.** Write

$$\int_0^1 \psi(s)e^{-2\pi iks}ds \sim \frac{1}{k}\int_0^1 e^{-2\pi iks}\psi'(s)ds \tag{4.55}$$

since $\psi(0) = \psi(1)$. One gets the contributions

$$\frac{1}{k}\,|\sqrt{\mu}-\pi n|\left|\int_0^1 e^{-2\pi iks}\cdot\cos\frac{\sqrt{\mu}-\pi n}{2}(1-s)\cdot\cos\frac{\sqrt{\mu}+\pi n}{2}(1-s)\cdot y(s,\mu)\right| \tag{4.56}$$

$$+\frac{n}{k}\left|\int_0^1 e^{-2\pi iks}\cdot\sin\frac{\sqrt{\mu}-\pi n}{2}(1-s)\cdot\sin\frac{\sqrt{\mu}+\pi n}{2}(1-s)\cdot y(s,\mu)\right| \tag{4.57}$$

$$+\frac{1}{k}\left|\int_0^1 e^{-2\pi iks}\cdot\sin\frac{\sqrt{\mu}-\pi n}{2}(1-s)\cdot\cos\frac{\sqrt{\mu}+\pi n}{2}(1-s)\cdot y'(s,\mu)\right| \tag{4.58}$$

The factors (4.56), (4.57) yield, by the estimates above

$$\eta\|\hat{q}\|_\infty\sum_{k>n}\frac{1}{k}\sum_\ell |c_\ell|\frac{1}{1+|-2k\pm n\pm\ell|} \tag{4.59}$$

$$<\eta\|\hat{q}\|_\infty\sum_\ell |c_\ell|\frac{\log(n+\ell)}{n+\ell} \tag{4.60}$$

and by (4.36), (4.40)

$$<\eta\|\hat{q}\|_\infty\frac{\log n}{n}+\eta\|\hat{q}\|_\infty^2\sum_{\ell\neq n}\frac{\log\ell.\log(n+\ell)}{|n+\ell|\,|n-\ell|.n} \tag{4.61}$$

$$<\eta\|\hat{q}\|_\infty\frac{\log\, n}{n}+\|\hat{q}\|_\infty^2\frac{(\log n)^3}{n^2}. \tag{4.62}$$

To estimate (4.58), replace the coefficient c_ℓ by $\frac{\ell}{k}c_\ell$ in (4.53). Thus we get

$$\eta\|\hat{q}\|_\infty\frac{1}{n}\sum_{k>n}\frac{1}{k}\sum_\ell \ell|c_\ell|\frac{1}{1+|2k\pm n\pm\ell|} \tag{4.63}$$

$$<\eta\|\hat{q}\|_\infty\frac{1}{n}\sum_\ell \ell|c_\ell|\frac{\log(n+\ell)}{n+\ell} \tag{4.64}$$

$$<\eta\|\hat{q}\|_\infty\frac{\log n}{n}+\eta\|\hat{q}\|_\infty^2\sum_{\ell\neq n}\frac{\ell.\log\ell.\log(n+\ell)}{(n+\ell)|n-\ell|n.(n+\ell)} \tag{4.65}$$

$$<\eta\|\hat{q}\|_\infty\frac{\log n}{n}+\|\hat{q}\|_\infty^2\frac{(\log n)^3}{n^2} \tag{4.66}$$

hence, collecting estimates (4.54), (4.62), (4.65) yields

$$(4.45) < \eta\|\hat{q}\|_\infty\frac{\log n}{n}+\|\hat{q}\|_\infty^2\frac{(\log n)^3}{n^2} \tag{4.67}$$

Estimation of (4.46)

Define now

$$\psi(s) = \sin \pi n(1-s)[y(s,\mu) - c_n \sin \pi ns] \tag{4.68}$$

again satisfying $\psi(0) = \psi(1) = 0$. For $k < n$, we get the estimate

$$\begin{aligned} &\|\hat{q}\|_\infty \sum_{k<n} |\hat{\psi}(k)| \\ &< \|\hat{q}\|_\infty \sum_{k<n} \sum_{\ell \neq n} |c_\ell| \frac{1}{1 + |-2k \pm n \pm \ell|} \\ &< \|\hat{q}\|_\infty \frac{\log n}{n} \sum_{\ell \neq n} |c_\ell| \\ &< \|\hat{q}\|_\infty^2 \frac{(\log n)^3}{n} \quad \text{by (4.37).} \end{aligned} \tag{4.69}$$

For $k > n$, we estimate again by (4.36)

$$\begin{aligned} &\|\hat{q}\|_\infty \sum_{k>n} \frac{1}{k} \left| \int_0^1 e^{-2\pi iks} \psi'(s) ds \right| < \\ &\|\hat{q}\|_\infty \sum_{k>n} \frac{1}{k} \sum_{\ell \neq n} (\ell + n)|c_\ell| \frac{1}{1 + |-2k \pm n \pm \ell|} \\ &< \|\hat{q}\|_\infty \sum_{\ell \neq n} (\ell + n)|c_\ell| \frac{\log(n+\ell)}{n+\ell} \\ &< \|\hat{q}\|_\infty^2 \sum_{\ell \neq n} \frac{(\log \ell)\log(n+\ell)}{(n+\ell)|n-\ell|} \\ &< \|\hat{q}\|_\infty^2 \frac{(\log n)^3}{n}. \end{aligned} \tag{4.70}$$

Thus

$$|(4.46)| \lesssim \|\hat{q}\|_\infty^2 \frac{(\log n)^3}{n}. \tag{4.71}$$

In summary, from (4.44) – (4.46), (4.48), (4.67), (4.71)

$$|\mathrm{Re}\, \hat{q}(n)| \lesssim \eta + \eta \|\hat{q}\|_\infty \frac{\log n}{n} + \|\hat{q}\|_\infty^2 \frac{(\log n)^3}{n} \lesssim \eta + \eta a_1 + a_1^2 \tag{4.72}$$

invoking (4.40). Hence (4.42) holds, completing the proof. □

5. Almost periodicity properties

Let $y = y(t)$ be the solution of the IVP

$$\begin{cases} \partial_t y + \partial_x^3 y + y y_x = 0 \\ y(0) = \mu \end{cases} \tag{5.1}$$

with $\mu \in M(\mathbb{T})$, $\|\mu\|$ small, obtained above. Then $y \in \mathcal{C}_{H^s(\mathbb{T})}(\mathbb{R})$ for all $s < -\frac{1}{2}$ and we claim that y is an almost periodic H^s-valued function of time. The corresponding result was obtained in [Bo] for L^2-data using essentially the analysis from [McK-Tr] for smooth data.

The proof of the result will be a rather simple adaptation of the argument in [Bo], see Appendix 1. Recall that almost periodicity means that for given $\varepsilon > 0$ there is a number $\ell(\varepsilon)$ such that any t-interval of length at least $\ell(\varepsilon)$ contains an "almost period" T, i.e. satisfying

$$\|y(\,.\,, t+T) - y(\,.\,, t)\|_{H^s} < \varepsilon \quad \text{for all } t \in \mathbb{R}. \tag{5.2}$$

We fix here some $s < -\frac{1}{2}$.

As in [Bo], in order to use the formulas from [McK-Tr] for smooth potentials without having to extend explicitly this theory to the case of non-smooth functions or distributions, we regularize the data μ, considering a sequence of smooth functions $\{\phi_\alpha\}$ such that

$$\lim_{\alpha\to\infty} \phi_\alpha = \mu \quad \text{in} \quad H^{-\frac{1}{2}-}, \ |\hat{\phi}_\alpha| \le |\hat{\mu}|, \ \|\phi_\alpha\|_1 \le \|\mu\| \quad \text{and} \quad \int_{\mathbb{T}} \phi_\alpha(x)dx = \int d\mu. \tag{5.3}$$

Let $\{y_\alpha\}$ be the corresponding sequence of solutions of the IVP

$$\begin{cases} \partial_t y_\alpha + \partial_x^3 y_\alpha + y_\alpha (y_\alpha)_x = 0 \\ y_\alpha(0) = \phi_\alpha. \end{cases} \tag{5.4}$$

From the preceding, we know that

$$y_\alpha(t) \longrightarrow y(t) \quad \text{in} \quad H^{-\frac{1}{2}-} \tag{5.5}$$

for all time $t \in \mathbb{R}$. Assume the almost periodicity statement valid for all y_α, with $\ell(\varepsilon)$ independent of α. Then, passing to a subsequence, we may assume the corresponding sequence (T_α) of almost periods to converge

$$\lim_{\alpha\to\infty} T_\alpha = T. \tag{5.6}$$

Thus we get for all time t

$$\begin{aligned} &\|y(\,\cdot\,,t+T)-y(\,\cdot\,,t)\|_{H^s} = \\ &\lim_{\alpha\to\infty}\|y(\,\cdot\,,t+T_\alpha)-y(\,\cdot\,,t)\|_{H^s} = &&\text{(since } y\in C_{H^s}(\mathbb{R})\text{)} \\ &\lim_{\alpha\to\infty}\|y_\alpha(\,\cdot\,,t+T_\alpha)-y_\alpha(\,\cdot\,,t)\|_{H^s} &&\text{(since } y=\lim y_\alpha \text{ in } H^{-\frac12-} \text{ uniformly on finite time intervals)} \\ &< \varepsilon. \end{aligned} \tag{5.7}$$

Hence it suffices to consider smooth data $y(0)=\phi$, assuming a uniform bound

$$\|\phi\|_1 < c \tag{5.8}$$

provided $\ell(s)$ remains uniformly bounded.

At this stage, we briefly recall the main points from the [McK-Tr] analysis. This analysis is rather involved and we refer the reader to [McK-Tr], [Bo], especially [McK-Tr], for details. Rewrite (by appropriate rescaling) the KdV equation as

$$\partial_t y = V_2 y \tag{5.9}$$

where V_2 stands for the differential operator

$$V_2(q) = 3qq' - \frac{1}{2}q'''. \tag{5.10}$$

We consider the KdV-flow acting on the isospectral manifold $\mathcal{M}=\mathcal{M}(\phi)$ of potentials q with same periodic spectrum $\{\lambda_i\}$ as ϕ, thus

$$\Delta^2(\lambda_i, q) = 4 \qquad i = 0,1,2,\dots \tag{5.11}$$

We assume (cf. [Bo]) the spectrum simple. There is a uniform estimate on the gaps, cf. (4.19),

$$|\lambda_{2n-1} - n^2\pi^2| < C, \quad |\lambda_{2n} - n^2\pi^2| < C. \tag{5.12}$$

Recall the product formula

$$4 - \Delta^2(\lambda) = 4(\lambda - \lambda_0)\prod_{n\geq 1}\frac{(\lambda_{2n-1}-\lambda)(\lambda_{2n}-\lambda)}{n^4\pi^4}. \tag{5.13}$$

Points $q\in\mathcal{M}$ may be identified in a $1-1$ way with the sequence $\overline{p}=(p_n)$, where

$$p_n = \left(\mu_n, \sqrt{\Delta^2(\mu_n)-4}\right) \tag{5.14}$$

with $\mu_n = \mu_n(q)$ the n^{th} Dirichlet eigenvalue of q and where the square root carries a $+$ or $-$ sign (identifying $\mathcal{M}$ with the infinite dimensional torus constructed on the intervals of instability).

On $\mathcal{M}$, one considers a sequence of vector fields $\{X_n\}$, where at a given $q \in \mathcal{M}$, $X_n(q)$ is defined by

$$X_n(q) = D\,\frac{\partial\Delta(\lambda_{2n}(q))}{\partial q} = -\dot{\Delta}(\lambda_{2n}) D(f_{2n}^2) \tag{5.15}$$

where f_{2n} stands for the L^2-normalized eigenfunction corresponding to λ_{2n} and D is the differentiation operator. (In the case of simple spectrum, periodic eigenvalues and eigenfunctions depend on q in a real analytic way.) For each q, the vectors $X_n(q)$ form a basis for the tangent space of $\mathcal{M}$ at q. The corresponding flows e^{tX_n} are defined for all t and commute. They also commute with the KdV flow e^{tV_2} since, for smooth potentials V_2 may be written as a convergent expansion in the X_n's.

Let $q_0 \in \mathcal{M}$ be the point of $\mathcal{M}$ with Dirichlet spectrum $(\lambda_1, \lambda_3, \dots, \lambda_{2n-1}, \dots)$ (which is an even function). Let $X = \sum x_n X_n$ be a linear combination of the X_n-fields. Let $\overline{p} = (p_1, p_2, \dots)$ be the point of the torus corresponding to $q = e^X q_0$ according to the identification made above, cf. (5.14). Then the relation between $\{x_n\}$ and $\overline{p}$ is expressed as follows

$$2\sum_{n=1}^{\infty} \int_{0_n}^{p_n} \phi(\mu)\,\frac{d\mu}{\sqrt{\Delta^2(\mu)-4}} = \sum_{n=1}^{\infty} x_n \phi(\lambda_{2n}) \quad (0_n \equiv \lambda_{2n-1}) \tag{5.16}$$

for appropriately chosen paths of integration for $\int_0^{p_n}$. The identity (5.16) is valid for any $\phi \in I^{3/2}$, the class of integral functions of order $1/2$ and type 1, satisfying $\int_0^\infty |\phi(\mu)|^2 \mu^{3/2} d\mu < \infty$. Thus the way e^X acts on M is determined by an element of the dual space $(I^{3/2})^*$, defined by the right member of (5.16), modulo the lattice $L \subset (I^{3/2})^*$ generated by the elements

$$2\int_{\lambda_{2n-1}}^{\lambda_{2n}} \phi(\mu)\,\frac{d\mu}{\sqrt{\Delta^2(\mu)-4}}. \tag{5.17}$$

As a consequence of this discussion, it follows that, given $X = \sum x_n X_n$, there is $Y = \sum y_n Y_n$, such that

$$e^X q_0 = e^Y q_0 \qquad \text{and thus} \qquad e^X = e^Y \tag{5.18}$$

(using the fact that the maps e^X are commuting and act transitively) where the coefficients y_n are bounded by

$$|y_n| < 4\sum_{i=1}^{\infty} \int_{\lambda_{2i-1}}^{\lambda_{2i}} \frac{\mathbb{1}_n(\mu)}{\sqrt{\Delta^2(\mu)-4}}\,d\mu \tag{5.19}$$

with

$$\mathbb{1}_n(\mu) = \frac{\prod_{k\neq n}\left(1 - \frac{\mu}{\lambda_{2n}}\right)}{\prod_{k\neq n}\left(1 - \frac{\lambda_{2n}}{\lambda_{2k}}\right)} \tag{5.20}$$

(by Lagrange interpolation).

Calculating (5.19), using (5.12), (5.13), yields that

$$|y_n| \lesssim n + \sum_{i\neq n} \frac{n^2}{i|i^2 - n^2|} \lesssim n. \tag{5.21}$$

Let now

$$X(q) = \sum_{n>n_0} x_n X_n(q), \quad |x_n| \lesssim n \tag{5.22}$$

for some $q \in \mathcal{M}$, where the $X_n(q)$ are defined by (5.15). We need to estimate the Fourier coefficients $\widehat{X(q)}(\ell)$, $\ell \in \mathbb{Z}$. Thus

$$\left|\widehat{X(q)}(\ell)\right| \lesssim \sum_{n>n_0} n|\dot{\Delta}(\lambda_{2n})||\ell| \left|\widehat{f_{2n}^2}(\ell)\right| \lesssim |\ell| \sum_{n>n_0} \frac{1}{n} \left|\widehat{f_{2n}^2}(\ell)\right| \tag{5.23}$$

since

$$|\dot{\Delta}(\lambda_{2n}| \lesssim \frac{\lambda_{2n} - \lambda_{2n-1}}{n^2} \lesssim \frac{1}{n^2}. \tag{5.24}$$

The eigenfunction $y = f_{2n}$ is periodic or antiperiodic, hence has an expansion

$$y = \sum_{k\in\mathbb{Z}} c_k e^{i\pi kx}, \quad c_{-k} = \overline{c}_k \quad \text{for} \quad x \in \mathbb{R}. \tag{5.25}$$

Since

$$-y'' + qy = \lambda_{2n} y \tag{5.26}$$

we get

$$|c_\ell||\lambda_{2n} - \pi^2\ell^2| \leq \sum_k |\hat{q}(\ell - k)||c_k| < a_1 \sum |c_k| \tag{5.27}$$

(cf. (4.31)).

It follows that

$$|c_\ell| \lesssim \frac{a_1}{|n^2 - \ell^2|} \quad \text{for} \quad |n| \neq |\ell|. \tag{5.28}$$

Coming back to (5.23), estimate by

$$\begin{aligned}|\ell| \sum_{n>n_0,k} \frac{1}{n}|\hat{f}_{2n}(\ell-k)||\hat{f}_{2n}(k)| &\lesssim |\ell| \sum_{n>n_0,k} \frac{1}{n}\frac{1}{1+|n^2-k^2|}\frac{1}{1+|n^2-(k-\ell)^2|} \\ &\lesssim |\ell| \sum_{n>n_0,k} \frac{1}{n}\frac{1}{1+|n^2-k^2|}\frac{1}{[|\ell|\,|2k-\ell|+1]} \\ &\lesssim |\ell| \sum_{n>n_0} \frac{1}{n}\frac{1}{\left|n^2-\left[\frac{\ell}{2}\right]^2\right|+1} \\ &\quad + \sum_{n>n_0,k\neq\frac{\ell}{2}} \frac{1}{n}\frac{1}{1+|n^2-k^2|}\frac{1}{|2k-\ell|}. \end{aligned} \tag{5.29}$$

Distinguishing the cases $n^2=\left[\frac{\ell}{2}\right]^2$, $n^2\neq\left[\frac{\ell}{2}\right]^2$ in the first sum and $n^2=k^2$, $n^2\neq k^2$ in the second, we conclude that

$$\left|\widehat{X(q)}(\ell)\right| \lesssim 1 \qquad \text{for all } \ell$$

and

$$\left|\widehat{X(q)}(\ell)\right| < \frac{\log n_0}{n_0} \qquad \text{if } |\ell|<n_0. \tag{5.30}$$

As in [Bo], it is our aim to prove that the family

$$\{e^X \mid X=\sum x_n X_n\}$$

acts equicontinuously as a subset of $\mathcal{C}_s(\mathcal{M};\mathcal{M})$, the space of continuous maps from $\mathcal{M}$ to itself, where $\mathcal{M}$ is endowed with H^s-topology, $s<-\frac{1}{2}$. This properly needs moreover to hold uniformly for $\mathcal{M}(\phi)$, ϕ subject to (5.8).

From (5.21), we may assume $|x_n|\leq n$. We first establish a tale estimate, letting

$$X=\sum_{n>n_0} x_n X_n\ , \qquad |n_0|\lesssim n. \tag{5.31}$$

Writing

$$e^X q - q = \int_0^1 \frac{d}{dt}(e^{tX}q)dt = \int_0^1 X(e^{tX}q)dt$$

it follows

$$\begin{aligned}
\|e^X q - q\|_{H^s} &\leq \sup_{q_1 \in \mathcal{M}} \|X(q_1)\|_{H^s} \\
&= \left(\sum (1+|\ell|)^{2s} \left| \widehat{X(q_1)}(\ell) \right|^2 \right)^{1/2} \\
\text{by (5.30)} \qquad &\lesssim \left(\sum_{|\ell|<n_0} (1+|\ell|)^{2s} \right)^{1/2} \frac{\log n_0}{n_0} + \left(\sum_{|\ell|>n_0} (1+|\ell|)^{2s} \right)^{1/2} \\
&\leq \frac{\log n_0}{n_0} + n_0^{2s+1} \longrightarrow 0 \quad \text{for} \quad n_0 \to \infty. \qquad (5.32)
\end{aligned}$$

Observe that this estimate is in particular uniform under the assumption (5.8). To complete the argument of equicontinuity we have to check the continuity of individual maps e^{X_n}, n fixed, where, by (5.15), (5.24)

$$X_n(q) \sim \frac{\lambda_{2n} - \lambda_{2n-1}}{n^2} \cdot D\, f_{2n}^2(q). \tag{5.33}$$

Thus, for this issue, we may assume some separation of λ_{2n-1}, λ_{2n}, since in particular

$$\|X_n(q)\|_2 \lesssim \frac{1}{n} |\lambda_{2n} - \lambda_{2n-1}| \to 0 \quad \text{if} \quad |\lambda_{2n} - \lambda_{2n-1}| \to 0. \tag{5.34}$$

Let, as in (5.25)

$$f_{2n}(q) = y = \sum_k c_k e^{i\pi kx}\,, \qquad x \in \mathbb{R}. \tag{5.35}$$

Expressing then the equation

$$-y'' + qy = \lambda y\,; \qquad \lambda = \lambda_{2n} = \pi^2 n^2 + o(1) \tag{5.36}$$

in the Fourier modes, yields

$$(k^2 - \lambda) c_k + \widehat{qy}(k) = 0 \tag{5.37}$$

or

$$Tc = (D+S)c = 0 \tag{5.38}$$

where the linear operators D, S are defined by

$$D \text{ is the diagonal operator with } D_k = -\lambda + k^2 \tag{5.39}$$

and S in the Toeplitz operator S_q,

$$S_q(k,\ell) = \hat{q}(k-\ell). \tag{5.40}$$

Replace the operator T by

$$T_1 = \frac{\mathbb{1}}{(|k^2-n^2|+1)^{1/2}} \circ T \circ \frac{\mathbb{1}}{(|\ell^2-n^2|+1)^{1/2}} = D_1 + S_1 \tag{5.41}$$

where

$$\begin{cases} D_1(n) = D_1(-n) = n^2 - \lambda \\ D_1(k) = \frac{k^2-\lambda}{|k^2-n^2|+1}, \quad \text{hence } |D_1(k)| \sim 1 \text{ for } |k| \neq n \end{cases} \tag{5.42}$$

and

$$S_1(k,\ell) = \frac{\hat{q}(k-\ell)}{(|k^2-n^2|+1)^{1/2}(|\ell^2-n^2|+1)^{1/2}}. \tag{5.43}$$

Thus for $q \in \mathcal{M}$, we clearly get that

$$\|S_1\| \le \|S_1\|_{H^s} = \left[\sum_{k,\ell} \frac{|\hat{q}(k-\ell)|^2}{(1+|k^2-n^2|)(1+|\ell^2-n^2|)}\right]^{1/2} = o(1) \tag{5.44}$$

since $\|\hat{q}\|_\infty < a_1 = o(1)$.

We partition the index set $\mathbb{Z}$ as

$$\mathbb{Z} = \wedge \cup \{-n, n\} \tag{5.45}$$

From (5.42), (5.44), the restricted operator $T_\wedge = P_\wedge T_1 P_\wedge$ is clearly invertible by a Neumann series. The equation

$$T_1 c' = 0\,, \quad c'(k) = (1+|k^2-n^2|)^{1/2} c(k) \tag{5.46}$$

reduces therefore to following equation in c_n, c_{-n}

$$U \begin{pmatrix} c_n \\ c_{-n} \end{pmatrix} = 0 \tag{5.47}$$

where

$$U = \begin{pmatrix} n^2-\lambda & \hat{q}(n) \\ \overline{\hat{q}(n)} & n^2-\lambda \end{pmatrix} - \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad \text{where} \tag{5.48}$$

$$B_{11} = \sum_{k,\ell\in\Lambda} \frac{\hat{q}(k-n)\overline{\hat{q}(\ell-n)}}{(1+|k^2-n^2|)^{1/2}(1+|\ell^2-n^2|)^{1/2}} T_\Lambda^{-1}(k,\ell)$$

$$B_{12} = \sum_{k,\ell\in\Lambda} \frac{\hat{q}(k+n)\overline{\hat{q}(\ell-n)}}{(1+|k^2-n^2|)^{1/2}(1+|\ell^2-n^2|)^{1/2}} T_\Lambda^{-1}(k,\ell)$$

$$B_{21} = \sum_{k,\ell\in\Lambda} \frac{\hat{q}(k-n)\overline{\hat{q}(\ell+n)}}{(1+|k^2-n^2|)^{1/2}(1+|\ell^2-n^2|)^{1/2}} T_\Lambda^{-1}(k,\ell)$$

$$B_{22} = \sum_{k,\ell\in\Lambda} \frac{\hat{q}(k+n)\overline{\hat{q}(\ell+n)}}{(1+|k^2-n^2|)^{1/2}(1+|\ell^2-n^2|)^{1/2}} T_\Lambda^{-1}(k,\ell)$$

This matrix depends continuously on $q \in \mathcal{M}(\phi)$ endowed with H^s-topology for any $s < -\frac{1}{2}$. Since $\lambda = \lambda_{2n}$ is in the periodic spectrum,

$$\det U = 0. \tag{5.49}$$

On the other hand, λ is simple, and, more precisely, a separation of λ_{2n} and λ_{2n-1} is assumed. This implies that

$$\Big| n^2 - \lambda - \sum_{k,\ell\in\Lambda} \frac{\hat{q}(k-n)\hat{q}(\ell-n)}{(1+|k^2-n^2|)^{1/2}(1+|\ell^2-n^2|)^{1/2}} T_\Lambda^{-1}(k,\ell)\Big|^2 + \\ + \Big|\hat{q}(n) - \sum_{k,\ell\in\Lambda} \frac{\hat{q}(k+n)\overline{\hat{q}(\ell-n)}}{(1+|k^2-n^2|)^{1/2}(1+|\ell^2-n^2|)^{1/2}} T_\Lambda^{-1}(k,\ell)\Big|^2 \tag{5.50}$$

does not vanish, allowing to obtain an eigenvector $y = y(q)$ with

$$f_{2n}(q)^2 = \frac{y(q)^2}{\int y(q)^2} \tag{5.51}$$

depending continuously on q.

From the preceding discussion, the equicontinuity of the family$\{e^X | X = \sum x_n X_n\}$, acting on $\mathcal{M}$, follows.

We may now complete the proof of the almost periodicity of the KdV flow e^{tV_2} acting on $\mathcal{M}$, following the argument [McK-Tr]. First, from the equicontinuity property established above, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|e^X q_1 - e^X q_2\|_s < \varepsilon \tag{5.52}$$

whenever $X = \sum x_n X_n$ and $q_1, q_2 \in \mathcal{M}$, $\|q_1 - q_2\|_s < \delta$. We let here $s < -\frac{1}{2}$ be a fixed exponent. Since $\mathcal{M}(\phi)$ is obviously compact in H^s, so is the set

$$A = \{e^{tV_2}\phi \mid t \in \mathbb{R}\}. \tag{5.53}$$

Hence, there is T_0 such that

$$\inf_{q_2 \in A_{T_0}} \|q_1 - q_2\|_s < \delta \qquad \text{for all} \qquad q_1 \in A \tag{5.54}$$

denoting

$$A_{T_0} = \Big\{e^{tV_2}\phi \mid |t| < T\Big\}. \tag{5.55}$$

Consider an interval $I = [t', t'']$ in $\mathbb{R}$,

$$t'' - t' > 10T_0. \tag{5.56}$$

It follows from (5.54) that there is $t_0 \in [-T_0, T_0]$ such that

$$\Big\| e^{\frac{t'+t''}{2}V_2}\phi - e^{t_0 V_2}\phi \Big\|_s < \delta. \tag{5.57}$$

Hence, from the commutation property and (5.52)

$$\left\| e^{\left(\frac{t'+t''}{2} - t_0\right)V_2} e^X \left(e^{t_0 V_2}\phi\right) - e^X \left(e^{t_0 V_2}\phi\right) \right\|_s < \varepsilon \tag{5.58}$$

for any map e^X. Since they act transitively on $\mathcal{M}$, (5.58) implies

$$\|e^{TV_2}q - q\|_s < \varepsilon \qquad \text{for all} \qquad q \in \mathcal{M} \tag{5.59}$$

where, by (5.56)

$$T = \frac{t'+t''}{2} - t_0 \in I. \tag{5.60}$$

In particular

$$\|e^{(t+T)V_2}\phi - e^{tV_2}\phi\|_s < \varepsilon \tag{5.61}$$

which is (5.2).

6. Remarks on wellposedness

If one strengthens the usual notion of well-posedness, requiring that the flow maps

$$\phi \longmapsto y_\phi(t) \qquad |t| < T \tag{6.1}$$

should act smoothly on the H^s-space under consideration (instead of just continuously) it turns out that the known results on KdV and MKdV both on the line and in the periodic case are essentially optimal. These results are

	$\mathbb{R}$	$\mathbb{T}$
KdV	$s > -3/4$	$s \geq -1/2$
MKdV	$s > 1/4$	$s \geq \frac{1}{2}$

(see [Bo], [KPV$_1$], [KPV$_2$]), in which cases the flow map is in fact real analytic.

Fixing $\phi \in H^s$, consider the solution $y = y(\delta)$ on the IVP

$$\begin{cases} y_t + \partial_x^3 y + \partial_x(y^2) = 0 \\ y(0) = \delta\phi. \end{cases} \tag{6.2}$$

Then clearly

$$\frac{\partial y}{\partial \delta}\Big|_{\delta=0} = S(t)\phi \equiv \Phi \tag{6.3}$$

$$\frac{\partial^2 y}{\partial \delta^2}\Big|_{\delta=0} \sim \int_0^t S(t-\tau)(\partial_x \Phi^2)(\tau) d\tau \equiv \Psi_2 \tag{6.4}$$

$$\frac{\partial^3 y}{\partial \delta^3}\Big|_{\delta=0} \sim \int_0^t S(t-\tau)(\partial_x(\Phi.\Psi_2))(\tau d\tau \equiv \Psi_3 \tag{6.5}$$

etc.

and well-posedness in the strong sense mentioned above implies in particular for $|t|$ bounded

$$\|\psi_2(t)\|_{H^s} \leq C\|\phi\|_{H^s}^2 \tag{6.6}$$

$$\|\psi_3(t)\|_{H^s} \leq C\|\phi\|_{H^s}^3 \tag{6.7}$$

etc.

Consider first the case of periodic KdV. Take

$$\phi(x) = N^{-s} \cos Nx, \ N \to \infty \tag{6.8}$$

as H^s-normalized initial data (s will be negative). Thus

$$\Phi(x,t) \sim N^{-s}\cos(Nx+N^3t) \tag{6.9}$$

$$\Psi_2(x,t) \sim N^{-2s+1}\int_0^t S(t-\tau)\Big[\sin(2Nx+2N^3\tau)\Big]d\tau$$
$$\sim N^{-2s-2}[\sin(2Nx+2N^3t)-\sin(2Nx+8N^3t)] \tag{6.10}$$

and

$$\begin{aligned}\Psi_3(x,t)&\\ &\sim N^{-3s-2}\int_0^1 S(t-\tau)\partial_x\Big[\cos(Nx+N^3\tau)\sin(2Nx+2N^3\tau)\\ &\quad -\cos(Nx+N^3\tau)\sin(2Nx+8N^3\tau)\Big]d\tau\\ &\sim N^{-3s-1}\int_0^t S(t-\tau)\\ &\quad\Big[3\cos(3Nx+3N^3\tau)+\cos(Nx+N^3\tau)-3\cos(3Nx+9N^3\tau)\\ &\quad -\cos(Nx+7N^3\tau)\Big]d\tau\end{aligned} \tag{6.11}$$

with the main contribution

$$\sim N^{-3s-1}t\,\cos(Nx+N^3t). \tag{6.12}$$

Thus

$$\|\Psi_3(t)\|_{H^s}\sim N^{-2s-1} \tag{6.13}$$

and inequality (6.7) requires

$$2s+1\geq 0, \qquad \text{i.e.}\quad s\geq -\frac{1}{2}. \tag{6.14}$$

To treat the $\mathbb{R}$-case, the preceding needs to be modified a bit. Take

$$\phi(x)\sim\gamma^{-1/2}N^{-s}\cos\,Nx\cdot\left(\int_{-\gamma}^{\gamma}e^{i\lambda x}d\lambda\right) \tag{6.15}$$

where

$$\gamma=\gamma_N\sim\frac{1}{\sqrt{N}}. \tag{6.16}$$

Thus

$$\|\phi\|_{H^s}=1. \tag{6.17}$$

From (6.15), (6.3), (6.4), (6.5)

$$\Phi(x,t) \sim \gamma^{-1/2} N^{-s} \int\limits_{|\lambda \pm N| < \gamma} e(\lambda x + \lambda^3 t) d\lambda \tag{6.18}$$

$$\Psi_2(x,t) \sim \gamma^{-1} N^{-2s} \int\limits_{\substack{|\lambda_1 \pm N| < \gamma \\ |\lambda_2 \pm N| < \gamma}} \frac{1}{\lambda_1 \lambda_2} e\Big[(\lambda_1 + \lambda_2)x + (\lambda_1^3 + \lambda_2^3)t\Big] d\lambda_1\, d\lambda_2 \tag{6.19}$$

$$-\gamma^{-1} N^{-2s} \int\limits_{\substack{|\lambda_1 \pm N| < \gamma \\ |\lambda_2 \pm N| < \gamma}} \frac{1}{\lambda_1 \lambda_2} e\Big[(\lambda_1 + \lambda_2)x + (\lambda_1 + \lambda_2)^3 t\Big] d\lambda_1\, d\lambda_2. \tag{6.20}$$

The contribution of (6.19) to Ψ_3 is thus

$$\gamma^{-3/2} N^{-3s} \int\limits_{\substack{|\lambda_1 \pm N| < \gamma \\ |\lambda_2 \pm N| < \gamma \\ |\lambda_3 \pm N| < \gamma}} \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1 \lambda_2} e\Big[(\lambda_1 + \lambda_2 + \lambda_3)x + (\lambda_1 + \lambda_2 + \lambda_3)^3 t\Big] \cdot$$

$$\frac{e^{i[\lambda_1^3 + \lambda_2^3 + \lambda_3^3 - (\lambda_1 + \lambda_2 + \lambda_3)^3]t} - 1}{\lambda_1^3 + \lambda_2^3 + \lambda_3^3 - (\lambda_1 + \lambda_2 + \lambda_3)^3} \tag{6.21}$$

where

$$(\lambda_1 + \lambda_2 + \lambda_3)^3 - \lambda_1^3 - \lambda_2^3 - \lambda_3^3 = 3(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1). \tag{6.22}$$

Thus by (6.16) either $|(6.22)| \sim N^3$ or $|(6.22)| < \gamma^2 N = o(1)$. The main contribution to (6.21) is obviously gotten from the second alternative, in which case we get

$$\gamma^{-3/2} N^{-3s} \int\limits_{\begin{cases} \lambda = \lambda_1 + \lambda_2 + \lambda_3 \\ |\lambda_\alpha \pm N| < \gamma \\ |(5.22)| = 0(1) \end{cases}} \frac{\lambda}{\lambda_1 \lambda_2} e(\lambda x + \lambda^3 t) d\lambda_1 d\lambda_2 d\lambda_3 \tag{6.23}$$

with H^s-norm

$$\sim \gamma^{-1} N^{-2s+1} \left[\frac{\gamma^2}{N^2} - \frac{\gamma^2}{N^2} - \frac{\gamma^2}{N^2}\right] \sim \gamma N^{-2s-1} \sim N^{-2s-3/2}. \tag{6.24}$$

The contribution of (6.20) to Ψ_3 yields

$$\gamma^{-3/2} N^{-3s} \int\limits_{|\lambda_\alpha \pm N| < \gamma} \frac{1}{\lambda_2 \lambda_2} e\Big[(\lambda_1 + \lambda_2 + \lambda_3)x + (\lambda_1 + \lambda_2 + \lambda_3)^3 t\Big]$$

$$\frac{e^{i[(\lambda_1 + \lambda_2)^3 + \lambda_3^3 - (\lambda_1 + \lambda_2 + \lambda_3)^3]t} - 1}{(\lambda_1 + \lambda_2)\lambda_3} \tag{6.25}$$

for which the H^s-norm is bounded by

$$\gamma^{-3/2}N^{-2s-3}\Big\| \int_{\substack{\lambda=\lambda_1+\lambda_2+\lambda_3\\ |\lambda_\alpha\pm N|<\gamma}} e^{i\lambda x}\frac{1}{|\lambda_1+\lambda_2|+\frac{1}{N^2}}d\lambda_1 d\lambda_2 d\lambda_3\Big\|_2 \tag{6.26}$$

$$\sim \gamma^{-3/2}\gamma^{1/2}N^{-2s-3}\Bigg(\int_{|\lambda_\alpha\pm N|<\gamma}\frac{1}{|\lambda_1+\lambda_2|+\frac{1}{N^2}}d\lambda_1 d\lambda_2\Bigg)$$

$$< N^{-2s-3}\log N. \tag{6.27}$$

Thus (6.24) yields the main contribution to $\|\Psi_3\|_{H^s}$ and we find the condition

$$2s+3/2\geq 0, \qquad \text{i.e.} \qquad s\geq -3/4. \tag{6.28}$$

The calculations for MKdV are similar, in fact included in preceding analysis. The results need to be multiplied by N^2 and we get

$$N^{-2s+1} \Longrightarrow s\geq \frac{1}{2} \qquad \text{in the periodic case}$$

$$N^{-2s+\frac{1}{2}} \Longrightarrow s\geq \frac{1}{4} \qquad \text{on the line.}^{(*)} \tag{6.29}$$

It is interesting to notice that the exponents in the table above differ from the scaling exponents for instance for KdV on the line, this exponent is $-\frac{3}{2}$. In the periodic case, the failure of well-posedness may be due to the need for certain renormalizations; in fact, the results in the table above for the periodic case require to specify the mean $\int\phi$, for the KdV equation and the L^2-norm, $\int\phi^2$, for the MKdV-equation. The renormalization consists here in changing slightly linear and nonlinear part of the equation as follows (see [Bo])

$$y_t+\partial_x^3 y+2\left(\int\phi\right)y \;+\left[y^2-2\left(\int y\right)y\right]=0 \tag{6.30}$$

$$y_t+\partial_x^3 y+3\left(\int\phi^2\right)y+\left[y^3-3\left(\int y^2\right)y\right]=0. \tag{6.31}$$

Such normalizations are, however, usually only useful in the context of discrete modes and not likely to be helpful in the $\mathbb{R}$-case.

Finally, we observe that when expressing the KdV-equation in Darboux coordinates, i.e. in the standard Hamiltonian format $\dot p=-\frac{\partial H}{\partial q}$, $\dot q=\frac{\partial H}{\partial p}$, the (symplectic phase space normalization immerging, is amazingly $H^{-1/2}$, cf. [Kuk].

(*) Examples, as described above, were in fact investigated by Kenig, Ponce and Vega in closely related context and the optimality of the $\frac{1}{4}$-exponent for MKdV observed by these authors.

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