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GL-algebras in positive characteristic I: the exterior algebra

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Abstract
We study the category of **GL**-equivariant modules over the infinite exterior algebra in positive characteristic. Our main structural result is a shift theorem à la Nagpal. Using this, we obtain a Church–Ellenberg type bound for the Castelnuovo–Mumford regularity. We also prove finiteness results for local cohomology.

Keywords Exterior algebra · Infinite general linear group · Castelnuovo–Mumford regularity

Mathematics Subject Classification Primary 13A50 · 13C05; Secondary 13D02 · 15A75 · 20G15

List of symbols

k	The base field, always algebraically closed of characteristic $p > 0$
$\mathbf{V}$	A fixed infinite dimensional k-vector space with basis $\{e_i\}_{i\geq 1}$
$\mathbf{GL}$	The group of automorphisms of $\mathbf{V}$ fixing all but finitely many of the basis vectors e_i
Vec	The category of k-vector spaces
$\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$	The category of polynomial representations of $\mathbf{GL}$
$\mathbf{Pol}$	The category of strict polynomial functors Vec $\to$ Vec
R	The exterior algebra of $\mathbf{V}$
L_λ	The irreducible polynomial representation of $\mathbf{GL}$ with highest weight λ
$\mathbf{\Sigma}$	The Schur derivative functor, which we also call the shift functor
$\mathbf{\Delta}$	The difference functor
$-^{<n}$	The submodule generated by all elements of degree less than n
$-^{(r)}$	The r-th Frobenius twist of a $\mathbf{GL}$-representation
$t_i(-)$	The generation degree of $\mathrm{Tor}_i^R(-, k)$

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1 Introduction

Over the last decade, multiple authors have proved interesting asymptotic and uniformity results in fields ranging from topology [1, 6, 29] and number theory [28, 39] to algebraic statistics [4, 9, 10, 27] and commutative algebra [11, 23, 46]. A unifying feature of these seemingly unrelated works is the prevalence of highly symmetric infinite-dimensional objects. The class of **GL**-*algebras*, i.e., rings which admit an (appropriate) action of the infinite general linear group **GL** plays a central role in this endeavour. Snowden [47, 48], Sam–Snowden [42, 44], and other authors [3, 34] have extensively studied **GL**-algebras over a field of characteristic zero; their work has already found a diverse array of applications.

On the contrary, **GL**-algebras in positive characteristic have hardly been studied despite being intimately connected to other ubiquitous algebraic structures like twisted commutative algebras. We fill this gap in the literature and initiate the systematic study of **GL**-algebras in positive characteristic. In this paper, we study the infinite exterior algebra in detail; in the sequel [17], we investigate the infinite polynomial ring. In positive characteristic, there are many complications in the polynomial ring (like Frobenius-twisted ideals) that disappear in the exterior algebra, making the latter easier to analyze.

Throughout the paper, we fix k an algebraically closed field of characteristic $p > 0$. Let $\mathbf{GL} = \bigcup \mathbf{GL}_n(k)$ be the infinite general linear group, and let $\mathbf{V} = \bigcup k^n$ be the defining representation of **GL**. Let R be the exterior algebra of $\mathbf{V}$ regarded as an algebra object in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$, and similarly, let S be the symmetric algebra of $\mathbf{V}$. The category of R-modules (resp. S-modules) in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ will be denoted by Mod_R (resp. Mod_S). In this paper, we provide a detailed description of Mod_R. Over a field of characteristic zero, Sam–Snowden [42] have studied the algebraic and homological properties of Mod_R and Mod_S. We make many of their results characteristic independent. For instance, we prove:

Theorem 1.1 *The Castelnuovo–Mumford regularity of a finitely generated R-module is finite.*

We emphasize that in characteristic zero, the analogue of Theorem 1.1 is true for both R and S. In characteristic p, this result fails for the polynomial ring S: the ideal generated by the p^{th} power of the variables is a **GL**-stable ideal with infinite regularity.

Theorem 1.1 is almost a formal consequence of our main structural result (see Sect. 2.3 for the definition of the Schur derivative):

Theorem 1.2 (Shift Theorem) *Let M be a finitely generated R-module and $\mathbf{\Sigma}$ be the Schur derivative. The R-module $\mathbf{\Sigma}^n(M)$ is flat for $n \gg 0$.*

The shift theorem is known for R-modules in characteristic zero by work of Nagpal [32].

Remark 1.3 Nagpal's version of the shift theorem is for FI-modules in all characteristics. An FI-module is a functor from FI, the category of finite sets with injections, to the category of k-vector spaces. In characteristic zero, the category of FI-modules is equivalent to the category Mod_S (by Schur–Weyl duality) which is equivalent to the

category Mod_R (using the "transpose functor" [41, Section 7]). All these equivalences fail in positive characteristic.

We now explain the key technical novelty of our paper. Our proof of the shift theorem is motivated by Li–Yu's simplification [26] of Nagpal's work. Crucial to Li–Yu's proof is the fact that for an FI-module M, the generation degree of $\mathbf{\Delta}(M) = \mathrm{coker}(M \to \mathbf{\Sigma}(M))$ is exactly $t_0(M) - 1$. This property fails for R-modules: in Corollary 4.4, we show that for a Frobenius twisted $\mathbf{GL}$-representation W, we have $\mathbf{\Delta}(R \otimes W) = 0$. Our main idea is to classify the R-modules for which $\mathbf{\Delta}$ vanishes. We achieve this in Sect. 4.2 by controlling the $\mathbf{GL}$-representations that occur in such modules using the Steinberg tensor product theorem. In fact, we show that if $\mathbf{\Delta}$ vanishes for a torsion-free R-module, then the module is (up to extensions) of the form $R \otimes W$ with W Frobenius twisted. We also use these ideas in our forthcoming paper on Mod_S [17].

The theory of $\mathbf{GL}$-algebras is more difficult in positive characteristic, but not merely due to the lack of semisimplicity. Instead, as the previous paragraph shows, the presence of Frobenius twisted representations is the primary obstacle to extending many results from characteristic zero to positive characteristic. Furthermore, recent work [3, 47] about $\mathbf{GL}$-algebras in characteristic zero heavily exploit the presence of weight vectors of weight (1^n) in an arbitrary $\mathbf{GL}$-representation; all the weights occurring in a Frobenius twisted representation are divisible by p so we cannot use these methods. The Frobenius powers of an ideal also cause significant pathology. We [17] show that in the infinite polynomial ring S, the Frobenius powers of the maximal ideal behave like prime ideals in a suitable sense. In characteristic zero, the appropriate spectrum of S only has two points (the zero ideal and the maximal ideal); in positive characteristic, it has infinitely many points. We hope to address these issues more systematically in the future.

Remark 1.4 A VI-module is a functor from VI, the category of finite dimensional vector spaces over $\mathbf{F}_q$ with injections, to the category of vector spaces over a field L. Nagpal [33] also proved a shift theorem for VI-modules in non-describing characteristic; i.e., when $\mathrm{char}(\mathbf{F}_q) \neq \mathrm{char}(L)$. The aforementioned property about the shift functor also fails for VI-modules. As a workaround, Nagpal proved that in non-describing characteristic, a VI-module is flat, in a suitable sense, if and only if its local cohomology modules vanish. We prove a similar equivalence for R-modules (Proposition 5.11), but obtain it as a corollary of the shift theorem. It would be interesting to see if the strategy we employ here can be used to reprove Nagpal's theorem.

1.1 Additional results on *R*-modules

Apart from the shift theorem, we extend a number of other results about the exterior algebra to all characteristics.

- We prove Mod_R is locally noetherian using Gröbner theory (Theorem 3.2).
- We show that the Krull–Gabriel dimension of Mod_R is one (Proposition 3.8).
- We obtain finiteness results for the local cohomology functors $\mathrm{R}\Gamma$; i.e., prove that for a finitely generated R-module M, the R-module $\mathrm{R}^i\Gamma(M)$ is finite length for all i, and vanishes for sufficiently large i (Theorem 5.17).

Table 1 Other examples from equivariant commutative algebra

Group	Ring	char(k)	Results
GL	$\mathrm{Sym}(\mathbf{V})$ & $\bigwedge(\mathbf{V})$	char 0	[6, 32, 42]
GL	$\mathrm{Sym}(\mathbf{V}^{\oplus n})$ & $\bigwedge(\mathbf{V}^{\oplus n})$	char 0	[40, 44]
$\mathfrak{S}_\infty$	$\mathrm{Sym}(\mathbf{V})$	any char	[37, 38]
Sp	$\mathrm{Sym}(\mathbf{V})$	char 0	[45]
GL	$\mathrm{Sym}(\mathbf{V})$	char $p > 0$	[17]
GL	$\bigwedge(\bigwedge^2(\mathbf{V}))$ & $\bigwedge(\mathrm{Sym}^2(\mathbf{V}))$	char 0	[36]
GL	$\mathrm{Sym}(\mathrm{Sym}^2(\mathbf{V})$ & $\mathrm{Sym}(\bigwedge^2(\mathbf{V}))$	char 0	[34]
GL	$\mathrm{Sym}(\mathbf{V} \oplus \bigwedge^2(\mathbf{V}))$	char 0	[45]

- We show that the derived category $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$ is generated by the simple R-modules L_λ and the flat R-modules $R \otimes L_\lambda$ for arbitrary λ (Theorem 5.4).
- We explain how to make Theorem 1.1 effective using work of Gan–Li [16] (Remark 5.7). This mirrors a theorem of Church–Ellenberg [5] on FI-modules.

Not everything generalizes, however. In characteristic zero, Sam–Snowden [42] proved that $\mathrm{Mod}_R^{\mathrm{gen}}$ and $\mathrm{Mod}_R^{\mathrm{tors}}$ are equivalent, and showed that every finitely generated R-module has finite injective dimension. Both these properties fail in our setting (Proposition 5.15), though remarkably the functor $\mathrm{R\,Hom}(M, -)$ still preserves the derived category $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$ (Theorem 5.16).

There are many more results known in characteristic zero; we are unaware if they continue to hold in positive characteristic. For example, Nagpal–Sam–Snowden [35] showed that the regularity of an R-module can also be computed using the local cohomology functors. It would be interesting if one can prove this in positive characteristic as well (the analogous fact for $\mathbf{Z}$-graded Gorenstein rings is classical).

1.2 Relation to other work

This work fits in with the broader goal of equivariant commutative algebra wherein one studies G-equivariant modules over a k-algebra A. We collect some of the relevant examples from the literature in Table 1. Our paper suggests that the exterior algebra is a much more tractable object in positive characteristic compared to the polynomial ring.

1.2.1 GL-algebras

Fix a positive integer r, and let $A = \bigwedge(k^r \otimes \mathbf{V})$ and $B = \mathrm{Sym}(k^r \otimes \mathbf{V})$. In characteristic zero, it is easy to see that the two categories Mod_A and Mod_B are equivalent; Sam–Snowden [44] have studied the structural properties of Mod_B. The results (and techniques) of our paper is strong evidence that in positive characteristic, at least Mod_A should parallel the characteristic zero story.

1.2.2 Invariant theory

Gandini [18, 19] has studied equivariant resolutions over exterior algebras in characteristic zero by transferring well-established results about the polynomial ring to the exterior algebra using the transpose functor (see [41, Section 7.4]). In positive characteristic, the transpose functor does not exist, and moreover these results usually fail for the polynomial ring. It would be interesting to see whether one can directly prove analogues of Gandini's results in positive characteristic.

1.2.3 The infinite symmetric group

Nagpal–Snowden [37, 38] have classified the $\mathfrak{S}_\infty$-stable ideals in the infinite polynomial ring. There are many remarkable results and open problems [24, 25, 30] about these ideals and their resolutions. The exterior algebra should be a good testing ground for these problems, being somewhat less complicated structurally.

1.2.4 Twisted commutative algebras

A twisted commutative algebra (tca) is a commutative algebra object in $\mathrm{Rep}(\mathfrak{S}_\star)$, the category of sequences of representations of $\mathfrak{S}_n$. We have a symmetric monoidal functor $\mathcal{F}\colon \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}) \to \mathrm{Rep}(\mathfrak{S}_\star)$ often referred to as the Schur functor. Given an object W in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$, the functor $\mathcal{F}$ induces a functor $\mathcal{F}_W\colon \mathrm{Mod}_{\mathrm{Sym}(W)} \to \mathrm{Mod}_{\mathrm{Sym}(\mathcal{F}(W))}$. In characteristic zero, the functor $\mathcal{F}$ is an equivalence, so the theory of tca's and **GL**-algebras in characteristic zero agree. In positive characteristic, the Schur functor is not an equivalence. However, we are hopeful that the theory of **GL**-algebras in positive characteristic will enable us to prove new results about tca's in positive characteristic.

1.2.5 Categories with shift-like functors

Gan–Li [15] have described an inductive machinery to prove results (like Noetherianity and finite regularity) about a module category equipped with a shift-like functor (see Subsection 1.9 of loc. cit.). In their work, they assume $t_0(\mathbf{\Delta}(M)) = t_0(M) - 1$ so it does not apply in our case. Nevertheless, developing an inductive machinery without this assumption on $\mathbf{\Delta}(M)$ deserves serious consideration as more and more shift functors do not satisfy this condition (see also Remark 1.4).

1.2.6 The infinite polynomial ring

As we remarked earlier, the infinite polynomial ring S is more complicated in positive characteristic. In a forthcoming paper [17], we provide a fairly comprehensive picture of Mod_S. The structural results of this paper will be used to prove finiteness properties for resolutions of S-modules.

1.3 Outline

In Sect. 2, we recall the necessary background. In particular, we recall the notion of a polynomial representation in Sect. 2.1, and define **GL**-algebras and related concepts in Sect. 2.2. The Schur derivative and some of its properties are discussed in Sect. 2.3. We prove the noetherianity result (Theorem 3.2) and facts about torsion R-modules in Sect. 3. Section 4 contains the proof of the shift theorem with the key technical results appearing in Sect. 4.2. In Sect. 5, we use the shift theorem to obtain structural results about the derived category $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$.

For the expert, we suggest first reading Sect. 2.3, Proposition 2.4, and then directly jumping to Sect. 4. Most of the other results in Sect. 2 and Sect. 3 are as one would expect.

2 Background

2.1 Polynomial representations

We let $\mathbf{GL} = \bigcup_{n\geq 1} \mathbf{GL}_n(k)$, and $\mathbf{V} = \bigcup_{n\geq 1} k^n$ be the standard representation of **GL**. A representation of **GL** is *polynomial* if it appears as a subquotient of a (possibly infinite) direct sum of tensor powers of **V**. We denote the category of all polynomial representations by $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$. It is a Grothendieck abelian category closed under tensor products whose objects are locally finite length. For a quick review of polynomial representations, we refer the interested reader to [49, Section 1].

Examples

The irreducible polynomial representations of **GL** are indexed by partitions of arbitrary size. For a partition λ, we let L_λ be the irreducible representation with highest weight vector of weight λ, and we let $\mathbf{S}_\lambda(\mathbf{V})$ be the Schur module corresponding to λ. The representation L_λ is the socle of $\mathbf{S}_\lambda(\mathbf{V})$. We note that $\mathbf{S}_{(d)}(\mathbf{V}) = \mathrm{Sym}^d(\mathbf{V})$, and $\mathbf{S}_{(1^d)}(\mathbf{V}) = \bigwedge^d(\mathbf{V})$. Furthermore, we let $\mathrm{Div}^d(\mathbf{V})$ be the d-th divided power of **V**. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we let $\mathrm{Div}^\lambda(\mathbf{V}) = \mathrm{Div}^{\lambda_1}(\mathbf{V}) \otimes \mathrm{Div}^{\lambda_2}(\mathbf{V}) \otimes \ldots \mathrm{Div}^{\lambda_n}(\mathbf{V})$, and define $\mathrm{Sym}^\lambda(\mathbf{V})$ similarly. The modules $\mathrm{Div}^\lambda(\mathbf{V})$ are projective objects in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$, and any polynomial representation is a quotient of direct sums of $\mathrm{Div}^\lambda(\mathbf{V})$ (as λ varies). Similarly, the representations $\mathrm{Sym}^\lambda(\mathbf{V})$ are injective objects in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$.

Polynomial functors

Let **Pol** denote the category of strict polynomial endofunctors of Vec in the sense of Friedlander–Suslin [13]. The category $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ is equivalent to **Pol** [13, Lemma 3.4]. The equivalence is given by evaluating a polynomial functor F on **V**. We let $\Phi\colon \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}) \to \mathbf{Pol}$ denote the inverse. Under this equivalence, the Schur functor $\mathbf{S}_\lambda$ in **Pol** is mapped to the polynomial representation $\mathbf{S}_\lambda(\mathbf{V})$, and if W is a

polynomial representation of **GL**, then $W \cong \Phi(W)(\mathbf{V})$. Every concept we define for polynomial representations can be transferred to **Pol** using this equivalence.

Frobenius twist

Let Frob: $\mathbf{GL} \to \mathbf{GL}$ denote the Frobenius map, which raises each entry of a matrix to its p-th power. The Frobenius map is a group homomorphism. Given a representation W of **GL**, the *Frobenius twist* of W, denoted $W^{(1)}$, is the **GL**-representation obtained by pulling W back along this map. If W is a polynomial representation, then $W^{(1)}$ is also a polynomial representation. For $r > 1$, we recursively define $W^{(r)} = (W^{(r-1)})^{(1)}$.

Grading

A *homogeneous* polynomial representation of *degree* d is a polynomial representation of **GL** which has a (possibly infinite) filtration where the successive quotients are the simple objects L_λ with $|\lambda| = d$. The degree is also compatible with the tensor product and Frobenius twist: if V is homogeneous of degree d and W is homogeneous of degree e, then $V \otimes W$ is homogeneous of degree $d + e$, and $V^{(1)}$ is homogeneous of degree pd. Given an arbitrary polynomial representation W of **GL**, we can decompose it as $\bigoplus_{i \in \mathbf{N}} W_i$ where W_d is homogeneous of degree d [13, Lemma 2.5]. Therefore, every polynomial representation has a canonical **N**-grading. We let $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})_d$ be the full subcategory of $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ on the homogeneous polynomial representation of degree d.

Steinberg tensor product theorem

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is *p-restricted* if $\lambda_i - \lambda_{i+1} < p$ for all $1 \le i \le n$ (here $\lambda_{n+1} = 0$). We state an important result that relates p-restricted partitions, Frobenius twists, tensor products, and irreducible representations, see e.g., [22, Theorem 1.1].

Theorem 2.1 *Let λ be a partition and write $\lambda = \sum_{i=0}^{n} p^i \lambda^i$ for p-restricted partitions $\lambda^0, \lambda^1, \ldots, \lambda^n$. Then, we have an isomorphism $L_\lambda \cong L_{\lambda^0} \otimes L_{\lambda^1}^{(1)} \otimes L_{\lambda^2}^{(2)} \ldots \otimes L_{\lambda^n}^{(n)}$.*

2.2 GL-algebras

A **GL**-*algebra* is a commutative, associative, unital k-algebra equipped with an action of **GL** via algebra homomorphisms, under which it forms a polynomial representation. A *skew* **GL**-*algebra* is a skew-commutative associative, unital $\mathbf{Z}/2$-graded k-algebra equipped with an action of **GL** via algebra homomorphisms under which it forms a polynomial representation. We write $|A|$, when we want to forget the **GL**-action on A and think of it as an ordinary k-algebra. The **GL**-action endows $|A|$ with a canonical **N**-grading. In a skew **GL**-algebra, this canonical **N**-grading is unrelated to the $\mathbf{Z}/2$-grading. Given a polynomial representation W, the algebra $\mathrm{Sym}(W)$ is a **GL**-algebra and $\bigwedge(W)$ is canonically a skew **GL**-algebra.

A-modules

Let A be a (skew) $\mathbf{GL}$-algebra. We let Mod_A denote the category of A-modules in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$. Concretely, an *$A$-module* M is an $|A|$-module, equipped with an action of $\mathbf{GL}$ that is compatible with the action on A (i.e., $g(am) = g(a)g(m)$ for all $a \in A, m \in M$ and $g \in \mathbf{GL}$) such that M is a polynomial representation. The action of $\mathbf{GL}$ endows M with an $\mathbf{N}$-grading, making M an $\mathbf{N}$-graded $|A|$-module. A *$\mathbf{GL}$-ideal* of A is a $\mathbf{GL}$-stable ideal of $|A|$. A *submodule* of an A-module is a $\mathbf{GL}$-stable $|A|$-submodule. We denote by A_+ the ideal of positive degree elements of A.

Finite generation

A $\mathbf{GL}$-algebra A is *finitely generated* if there exists a finite length subrepresentation $V \subset A$ such that the canonical map $\mathrm{Sym}(V) \to A$ is surjective. A skew $\mathbf{GL}$-algebra A is *finitely generated* if there exists finite length subrepresentations $V \subset A_{\bar{0}}$ and $W \subset A_{\bar{1}}$ such that the canonical map $\mathrm{Sym}(V) \otimes \bigwedge(W) \to A$ is surjective. Equivalently, the k-algebra $|A|$ is generated by the $\mathbf{GL}$ orbits of finitely many elements. An A-module M is *finitely generated* if there exists a finite length subrepresentation $V \subset M$ such that the canonical map $A \otimes V \to M$ is surjective. Equivalently, we can find finitely many elements in M such that their $\mathbf{GL}$-orbits generate $|M|$ as an $|A|$-module.

Let M be an A-module. The *generation degree* of M is the smallest integer $n \geq -1$ such that M is generated by elements of degree $\leq n$ (or ∞ if no such integer exists). We say M is *generated in degree n* if M is generated by its degree n elements (in particular, we have $M_i = 0$ for $i < n$).

Noetherianity

A (skew) $\mathbf{GL}$-algebra is *noetherian* if every submodule of a finitely generated A-module is also finitely generated. It is *weakly noetherian* if every $\mathbf{GL}$-ideal of A is finitely generated.

Nakayama's Lemma

We have the following version of Nakayama's lemma. The usual proof of the (graded version of the) lemma applies.

Lemma 2.2 *Let A be a $\mathbf{GL}$-algebra and M be an A-module. Let $V \to M$ be a map of $\mathbf{GL}$-representations such that the composition $V \to M \to M/A_+M$ is surjective. The map $A \otimes V \to M$, obtained by adjunction, is also surjective.*

Projective and injective modules

For a projective polynomial representation V, the A-module $A \otimes V$ is projective. Since there are enough (finite length) projectives in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$, the category Mod_A has enough (finitely generated) projectives. In fact, every A-module is a quotient of direct sums of the projective modules $\{A \otimes \mathrm{Div}^{\lambda}(\mathbf{V})\}_{\lambda}$. The category Mod_A has enough

injectives as well since it is a Grothendieck abelian category (though the injectives may not be finitely generated).

2.3 The Schur derivative

In this subsection, we define an endofunctor of $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ called the *Schur derivative*. We first define it for **Pol** and then define it for $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ using the equivalence Φ. Some properties of the Schur derivative are outlined in [41, Section 6.4] (under char 0 assumptions). The Schur derivative and related notions also appear in [2, Section 3] under the name "linear approximation".

Schur derivative of polynomial functors

Let $F\colon \mathrm{Vec} \to \mathrm{Vec}$ be a polynomial functor. We define the Schur derivative of F, denoted $\mathbf{\Sigma}(F)$, to be the polynomial functor which on objects is defined by

$$\mathbf{\Sigma}(F)(V) = F(k \oplus V)^{[1]},$$

where the superscript here denotes the subspace of $F(k \oplus V)$ on which $k^\star$ acts with weight 1. For a linear map $f\colon V \to W$, the induced map $\mathbf{\Sigma}(F)(f)\colon \mathbf{\Sigma}(F)(V) \to \mathbf{\Sigma}(F)(W)$ is given by restricting the map $F(\mathrm{id}_k \oplus f)\colon F(k \oplus V) \to F(k \oplus W)$ to the 1 weight-space of the $k^\star$ action. It is easy to see that $\mathbf{\Sigma}(F)$ is a polynomial functor, and that the assignment $F \to \mathbf{\Sigma}(F)$ is functorial.

Basic properties of the Schur derivative

The Schur derivative is an exact functor (because $k^\star$ is semisimple). It preserves finite length objects of **Pol**. Furthermore, the Schur derivative is a categorification of the ordinary derivative (of polynomials) as it satisfies:

- Additivity: $\mathbf{\Sigma}(F \oplus G) = \mathbf{\Sigma}(F) \oplus \mathbf{\Sigma}(G)$
- Leibniz rule: $\mathbf{\Sigma}(F \otimes G) = (\mathbf{\Sigma}(F) \otimes G) \oplus (F \otimes \mathbf{\Sigma}(G))$
- Chain rule: $\mathbf{\Sigma}(F \circ G) = (\mathbf{\Sigma}(F) \circ G) \otimes \mathbf{\Sigma}(G)$
- Kills "p-th powers": $\mathbf{\Sigma}(F^{(1)}) = 0$

For the second property, we use the fact that for $k^\star$ representations V and W, we have $(V \otimes W)^{[1]} = (V^{[1]} \otimes W) \oplus (V \otimes W^{[1]})$, and for the last property, we use the fact that all the weights occurring in a Frobenius twisted representation are divisible by p.

Schur derivative of a polynomial representation

Since **Pol** and $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ are equivalent, we obtain an endofunctor of $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$, which we again denote by $\mathbf{\Sigma}$ and call the Schur derivative. For a polynomial representation W, we identify $\mathbf{\Sigma}(W)$ as a subspace of $\Phi(W)(k \oplus \mathbf{V})$, where $k^\star$ acts with weight 1. We also identify W is as a subspace of $\Phi(W)(k \oplus \mathbf{V})$ (where $k^\star$ acts trivially). The newly introduced basis vector of $k \oplus \mathbf{V}$ will usually be denoted by f_1, or y_1 depending

on the situation. The proof of the next lemma is easy, but demonstrates how we will work with the Schur derivative in the rest of the paper.

Lemma 2.3 *For all $n \geq 1$, we have isomorphisms $\mathbf{\Sigma}(\mathrm{Sym}^n(\mathbf{V})) \cong \mathrm{Sym}^{n-1}(\mathbf{V})$, and $\mathbf{\Sigma}(\bigwedge^n(\mathbf{V})) \cong \bigwedge^{n-1}(\mathbf{V})$.*

Proof The vector space $\mathrm{Sym}^n(\mathbf{V})$ has basis indexed by degree n monomials in $x_1, x_2, \ldots$. We shall identify $\mathbf{\Sigma}(\mathrm{Sym}^n(\mathbf{V}))$ as a subspace of $\mathrm{Sym}^n(k \oplus \mathbf{V})$, where the newly introduced basis vector will be y_1.

We define a map $\mathrm{Sym}^{n-1}(\mathbf{V}) \to \mathbf{\Sigma}(\mathrm{Sym}^n)(\mathbf{V}) = \mathrm{Sym}^n(k \oplus \mathbf{V})^{[1]}$, where

$$x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m} \mapsto y_1 x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m}.$$

This is $\mathbf{GL}(\mathbf{V})$-equivariant ($\mathbf{GL}(\mathbf{V})$ acts trivially on y_1), and easily checked to be both injective and surjective. The proof for $\bigwedge^n(\mathbf{V})$ is similar. □

Schur derivative of an *A*-module

Let A be a $\mathbf{GL}$-algebra, and M be an A-module. The $\mathbf{GL}$-representation $\mathbf{\Sigma}(M)$ is canonically an A-module, as we now explain. Firstly, we have a canonical $\mathbf{GL}$-equivariant map $A \to \Phi(A)(k \oplus \mathbf{V})$ induced by the canonical inclusion $\mathbf{V} \to k \oplus \mathbf{V}$. Now, the $\mathbf{GL}$-representation $\Phi(M)(k \oplus \mathbf{V})$ is a $\mathbf{GL}(k \oplus \mathbf{V})$-equivariant module over $\Phi(A)(k \oplus \mathbf{V})$. By restriction of scalars, it is a $\mathbf{GL}$-equivariant module over A. The subspace $\mathbf{\Sigma}(M)$ of $\Phi(M)(k \oplus V)$ is stable under the action of $\mathbf{GL}$ as well as $|A|$, so $\mathbf{\Sigma}(M)$ is an A-module. To summarize, the Schur derivative induces a functor $\mathbf{\Sigma}\colon \mathrm{Mod}_A \to \mathrm{Mod}_A$. For example, given a polynomial representation V, we have $\mathbf{\Sigma}(A \otimes V) = (A \otimes \mathbf{\Sigma}(V)) \oplus (\mathbf{\Sigma}(A) \otimes V)$.

2.4 Semi-induced modules

Throughout this subsection, we fix a (skew) $\mathbf{GL}$-algebra A such that $A_0 = k$.

An A-module is *induced* if it is isomorphic to $A \otimes W$ for some polynomial representation W. An A-module is *semi-induced* if it has a finite filtration where the successive quotients are induced modules. The terminology is motivated by similarly defined objects for FI-modules (and related algebraic structures) [32].

Given an A-module M, we let $\mathrm{Tor}_i^A(M, -)$ denote the left derived functors of the right exact functor $M \otimes_A -$. An A-module M is *flat* if the functor $M \otimes_A -$ is exact. An A-module M is flat if and only if the functors $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$ and all A-modules N. Finally, it is easy to see that the $|A|$-modules $|\mathrm{Tor}_i^A(M, N)|$ and $\mathrm{Tor}_i^{|A|}(|M|, |N|)$ are isomorphic.

The main result of this subsection is a characterization of semi-induced modules using Tor.

Proposition 2.4 *Let M be a finitely generated A-module. The following are equivalent:*

(a) M is semi-induced,
(b) M is flat,

(c) $\operatorname{Tor}_i^A(M,k) = 0$ *for all* $i > 0$, *and*
(d) $\operatorname{Tor}_1^A(M,k) = 0$.

We let $t_i(M) = \max \deg \operatorname{Tor}_i^A(M,k)$ as a polynomial representation with the convention that $\max \deg(0) = -1$. By Nakayama's lemma (Lemma 2.2), the module M is generated in degrees $\leq n$ if and only if $t_0(M) \leq n$. We denote by $M^{<n}$ the A-submodule of M generated by all elements of degree $< n$. If $t_0(M) = n$, the quotient $M/M^{<n}$ will be generated in degree n; we implicitly use this throughout the paper.

Lemma 2.5 *Let M be a semi-induced A-module and N be any A-module. Then $\operatorname{Tor}_i^A(M,N) = 0$ for $i > 0$.*

Proof The induced module $A \otimes V$ is flat since the functor $(A \otimes V) \otimes_A -$ is isomorphic to $V \otimes_k -$ which is exact. By dévissage, we see that semi-induced modules are also flat. □

Lemma 2.6 *Let M be an A-module generated in degree n such that $t_1(M) \leq n$. The natural map $A \otimes M_n \to M$ is an isomorphism.*

Proof The map $A \otimes M_n \to M$ is surjective since M is generated in degree n. The kernel K of this map must be supported in degrees $> n$. By Lemma 2.5, the module $\operatorname{Tor}_1^A(A \otimes M_n, k) = 0$. Therefore, the long exact sequence of $\operatorname{Tor}_\bullet^A(-,k)$ gives us the exact sequence

$$0 \to \operatorname{Tor}_1^A(M,k) \to \operatorname{Tor}_0^A(K,k) \to \operatorname{Tor}_0^A(A \otimes M_n, k) \to \operatorname{Tor}_0^A(M,k).$$

The rightmost map is an isomorphism, so the first map is also an isomorphism. Now, since $t_1(M) \leq n$, we have $t_0(K) \leq n$ which implies K is zero since $K_i = 0$ for $i \leq n$. Therefore the map $A \otimes M_n \to M$ is an isomorphism, as required. □

Lemma 2.7 *Let M be a finitely generated A-module such that $\operatorname{Tor}_1^A(M,k) = 0$ and $t_0(M) = n$. The module $M/M^{<n}$ is induced, and $\operatorname{Tor}_1^A(M^{<n},k) = 0$.*

Proof The module $M/M^{<n}$ is generated in degree n. We have the long exact sequence

$$\begin{aligned}\operatorname{Tor}_2^A(M/M^{<n},k) &\to \operatorname{Tor}_1^A(M^{<n},k) \to \operatorname{Tor}_1^A(M,k) \to \operatorname{Tor}_1^A(M/M^{<n},k)\\ &\to \operatorname{Tor}_0^A(M^{<n},k),\end{aligned}$$

from which we see that $t_1(M/M^{<n}) < n$. So by Lemma 2.6, the module $M/M^{<n}$ is an induced module. In particular, the module $\operatorname{Tor}_2^A(M/M^{<n},k) = 0$ by Lemma 2.5, which in turn implies that $\operatorname{Tor}_1^A(M^{<n},k)$ vanishes (from the long exact sequence above), as required. □

Proof of Proposition 2.4 (a) implies (b) is Lemma 2.5. (b) implies (c), and (c) implies (d) is trivial. We now prove (d) implies (a). Using lemma 2.7, we see that the module $M/M^{<t_0(M)}$ is induced, and $\operatorname{Tor}_1^A(M^{<t_0(M)},k) = 0$. So by induction on generation degree, we have that $M^{<t_0(M)}$ is semi-induced, and therefore M is also semi-induced, as required. □

Corollary 2.8 *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of finitely generated A-modules.*

(a) If M_1 *and* M_3 *are semi-induced, then so is* M_2.
(b) If M_2 *and* M_3 *are semi-induced, then so is* M_1.

Proof The analogous results for flat modules can be obtained using the long exact sequence of Tor. We obtain the corollary for semi-induced modules using Proposition 2.4. □

2.5 Serre subcategories and Serre quotients

We now review Serre quotients; see [14] or [12, Chapter 15] for general background.

A *Serre subcategory* $\mathcal{B}$ of an abelian category $\mathcal{A}$ is a full abelian subcategory of $\mathcal{A}$ that is closed under extensions, and taking subquotients. Given a Serre subcategory $\mathcal{B} \subset \mathcal{A}$, we let $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ denote the exact functor to the Serre quotient $\mathcal{A}/\mathcal{B}$ and $S\colon \mathcal{A}/\mathcal{B} \to \mathcal{A}$ denote the right adjoint to T (the section functor), if it exists.

If $\mathcal{A}$ and $\mathcal{B}$ are Grothendieck abelian categories, then the section functor exists (see e.g., [12, Theorem 15.11]). Furthermore, under these assumptions, we also have a functor $\Gamma\colon \mathcal{A} \to \mathcal{B}$ which takes an object M to the maximal subobject $\Gamma(M)$ of M contained in $\mathcal{B}$. The functor Γ is right adjoint to the inclusion functor $i\colon \mathcal{B} \to \mathcal{A}$, so left exact; its right derived functors are the local cohomology functors.

All the results we use about local cohomology will be from [44, Section 4]. Many of the results from loc. cit. are stated under the assumption that the $\mathcal{B}$ satisfies Property (Inj), i.e., injective objects in $\mathcal{B}$ remain injective in $\mathcal{A}$. We will verify this in cases of interest.

3 Basic results on *R*-modules

For the rest of the paper, we let $R = \bigwedge(\mathbf{V})$ be the exterior algebra. This is a skew $\mathbf{GL}$-algebra. We let $\mathfrak{m}$ denote the ideal of all positive degree elements of R. In degree i, the $\mathbf{GL}$-representation R_i is the i-th exterior power $\bigwedge^i(\mathbf{V})$ which is irreducible of highest weight (1^i). The irreducibility of R_i implies that the only nonzero proper $\mathbf{GL}$-ideals of R are the ideals $\mathfrak{m}^r$ for $r \geq 1$. Clearly, the ascending chain condition holds for $\mathbf{GL}$-ideals of R, i.e., the skew $\mathbf{GL}$-algebra R is weakly noetherian. In Sect. 3.1, we show that R is noetherian. In Sect. 3.2, we define the notion of a torsion R-module (which makes sense only in the infinite variable setting).

3.1 Noetherianity

In this subsection, we show Mod_R is locally noetherian. We will prove a stronger statement that given a finitely generated $\mathrm{Inc}(\mathbf{N})$-equivariant R-module M, every $\mathrm{Inc}(\mathbf{N})$-stable R-submodule of M is also finitely generated. Our proof uses the following theorem of Cohen [7] (see also [31, Corollary 6.16]):

Theorem 3.1 *Let M be a finitely generated* $\mathrm{Inc}(\mathbf{N})$*-equivariant module over the infinite polynomial ring $S = k[x_1, x_2, \dots,]$. Every* $\mathrm{Inc}(\mathbf{N})$*-stable S-submodule of M is finitely generated.*

For the rest of the subsection (except the proof of Theorem 3.2), we forget the action of $\mathbf{GL}$ on R and S.

Notation

Let $\mathrm{Inc}(\mathbf{N})$ denote the *increasing monoid*, i.e., the monoid of increasing injective functions from $\mathbf{N}$ to itself. Let B denote the k-algebra $S/(x_1^2, x_2^2, \dots, x_n^2, \dots)$. We let P_n denote the R-module $R \otimes \mathbf{V}^{\otimes n}$, and let Q_n denote the B-module $B \otimes \mathbf{V}^{\otimes n}$ for all $n \geq 0$.

Monomial submodules

An element of P_n is *monomial* if it is of the form

$$m = (x_{i_1} \wedge x_{i_2} \dots \wedge x_{i_r}) \otimes (e_{j_1} \otimes e_{j_2} \dots \otimes e_{j_n})$$

with $i_1 > i_2 > \cdots > i_r$. We also have a similar notion of monomials for the B-module Q_n.

A submodule M of the R-module P_n (resp. the B-module Q_n) is *monomial* if it is generated by all the monomials it contains.

Bijection between monomial submodules of P_n and Q_n

We define a map Ψ from the set of monomials of P_n and the set of monomials of Q_n. Given a monomial

$$m = (x_{i_1} \wedge x_{i_2} \dots \wedge x_{i_r}) \otimes (e_{j_1} \otimes e_{j_2} \dots \otimes e_{j_n}) \in P_n$$

we define

$$\Psi(m) = (x_{i_1} x_{i_2} \dots x_{i_r}) \otimes (e_{j_1} \otimes e_{j_2} \dots \otimes e_{j_n}) \in Q_n.$$

Clearly, the map Ψ is a bijection. The map Ψ also induces a bijection $\widetilde{\Psi}$ between the set of monomial submodules of P_n and Q_n in the following way: given a monomial submodule M of P_n, the submodule $\widetilde{\Psi}(M) \subset Q_n$ is generated by the monomials $\Psi(m)$ with m varying over all the monomials in M.

Action of the increasing monoid

We define an action of $\mathrm{Inc}(\mathbf{N})$ on P_n. If $\sigma \in \mathrm{Inc}(\mathbf{N})$ and

$$m = (x_{i_1} \wedge x_{i_2} \dots \wedge x_{i_p}) \otimes (e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_n}) \in P_n,$$

we define

$$\sigma m := (x_{\sigma(i_1)} \wedge x_{\sigma(i_2)} \ldots \wedge x_{\sigma(i_p)}) \otimes (e_{\sigma(j_1)} \otimes e_{\sigma(j_2)} \otimes \ldots \otimes e_{\sigma(j_n)})$$

and linearly extend this to all of P_n. We similarly define an action of $\mathrm{Inc}(\mathbf{N})$ on Q_n as well.

The action we have just defined endows P_n and Q_n with the structure of an $\mathrm{Inc}(\mathbf{N})$-equivariant module over R and B respectively. Furthermore, the map $\widetilde{\Psi}$ restricts to an *inclusion-preserving* bijection:

$$\{\mathrm{Inc}(\mathbf{N})\text{-stable monomial submodules of } P_n\} \xrightarrow{\widetilde{\Psi}} \{\mathrm{Inc}(\mathbf{N})\text{-stable monomial submodules of } Q_n\}$$

Total order on monomials of P_n

Fix an $n \in \mathbf{N}$. We endow a total ordering (denoted $\leq$) on the set of monomials of P_n. Given monomials $m = (x_{i_1} \wedge x_{i_2} \ldots \wedge x_{i_r}) \otimes (e_{j_1} \otimes e_{j_2} \ldots \otimes e_{j_n})$, we associate to it the word $(I, J) = ((i_1, i_2, \ldots, i_r), (j_1, j_2, \ldots, j_n))$ in $\mathbf{N}^* \times \mathbf{N}^n$ (here, $\mathbf{N}^*$ is the set of words of arbitrary length on the alphabet $\mathbf{N}$). The lexicographic order on $\mathbf{N}^* \times \mathbf{N}^n$ restricts to a total order on the set of monomials. For example, in P_3, the monomial $(x_3 \wedge x_1) \otimes (e_1 \otimes e_4 \otimes e_1)$ will be bigger than $(x_2 \wedge x_1) \otimes (e_7 \otimes e_2 \otimes e_4)$ but smaller than $x_5 \otimes (e_1 \otimes e_1 \otimes e_1)$.

The total order we have defined is also compatible with multiplication by a monomial of R in the following sense: for a monomial $n \in R$, and monomials $m, m' \in P_n$ with $m > m'$, we have $nm > nm'$. Here, we are implicitly identifying a monomial with its negative, as it is possible that nm or nm' is not a monomial as we have defined, in which case $-nm$ or $-nm'$ will be a monomial.

Initial submodules

Given an element m in P_n, we can write it as a sum of monomials $m = \sum a_i m_i$, with $a_i \in k^\star$, and m_i being monomials. We define the *initial term* of m, denoted $\mathrm{init}(m)$, to be $a_n m_n$, where m_n is the largest monomial (under the total order just defined) among all the m_i. It is easy to see that $\mathrm{init}(\sigma(m)) = \sigma\,\mathrm{init}(m)$, for $\sigma \in \mathrm{Inc}(\mathbf{N})$ and $m \in P_n$. Given an arbitrary $\mathrm{Inc}(\mathbf{N})$-stable submodule M of P_n, we define its *initial submodule*, denoted $\mathrm{init}(M)$, to be the submodule generated by all elements of the form $\mathrm{init}(m)$, with $m \in M$. By definition, the submodule $\mathrm{init}(M)$ is an $\mathrm{Inc}(\mathbf{N})$-stable monomial submodule of P_n.

Theorem 3.2 Mod_R *is locally noetherian.*

Proof For each λ, let $P_\lambda = R \otimes \mathrm{Div}^\lambda(\mathbf{V})$. Every finitely generated R-module is a quotient of a finite direct sum of P_λ (as λ varies). Furthermore, the module P_λ is a submodule of P_n for $n = |\lambda|$ (as $\mathrm{Div}^\lambda(\mathbf{V})$ is a subrepresentation of $\mathbf{V}^{\otimes|\lambda|}$). Therefore, it suffices to show that P_n is $\mathbf{GL}$-noetherian for all n, because finite direct sums, submodules and quotients of $\mathbf{GL}$-noetherian modules are $\mathbf{GL}$-noetherian. We will

prove that P_n is $\mathrm{Inc}(\mathbf{N})$-noetherian; this suffices as every $\mathbf{GL}$-stable submodule of P_n is easily seen to be $\mathrm{Inc}(\mathbf{N})$-stable.

Fix an n. For $\mathrm{Inc}(\mathbf{N})$-stable submodules $M \subset N \subset P_n$, clearly $\mathrm{init}(M) \subset \mathrm{init}(N)$. The key point here is that the init operation detects strict inclusions, i.e., for $M \subset N \subset P_n$, we have $\mathrm{init}(M) = \mathrm{init}(N)$ if and only if $M = N$ (see proof of [43, Proposition 4.2.2]). Therefore, if ACC holds for $\mathrm{Inc}(\mathbf{N})$-stable monomial submodules of P_n, then ACC holds for all $\mathrm{Inc}(\mathbf{N})$-stable submodule.

Using the bijection $\widetilde{\Psi}$, it suffices to prove that ACC holds for $\mathrm{Inc}(\mathbf{N})$-stable monomial submodules of Q_n, which follows from Theorem 3.1. □

3.2 Torsion and generic *R*-modules

An element m in an R-module is *torsion* if $Im = 0$ for a nonzero $\mathbf{GL}$-ideal of R. Since the only nonzero proper $\mathbf{GL}$-ideals of R are of the form $\mathfrak{m}^n$ for $n \geq 1$, an element m is torsion if and only if $\mathfrak{m}^n m = 0$ for some $n \in \mathbf{N}$. A module M is *torsion* if all elements of M are torsion.

The subcategory $\mathrm{Mod}_R^{\mathrm{tors}}$ of torsion modules is a Serre subcategory of Mod_R. We let $\mathrm{Mod}_R^{\mathrm{gen}} = \mathrm{Mod}_R / \mathrm{Mod}_R^{\mathrm{tors}}$. A *generic module* is an object of $\mathrm{Mod}_R^{\mathrm{gen}}$. All the functors from Sect. 2.5 exist in this case. In particular, the functor $\Gamma : \mathrm{Mod}_R \to \mathrm{Mod}_R^{\mathrm{tors}}$ takes a module M to its submodule $\Gamma(M)$ of torsion elements. A nonzero module M is *torsion-free* if $\Gamma(M) = 0$.

We now give some examples of torsion and torsion-free R-modules. Given a nonzero module M, let $M^{>n}$ denote the submodule of M containing all elements of degree $> n$. The module $M/M^{>n}$ is torsion (provided it is nonzero). Any finite length R-module is clearly torsion. The module R is torsion-free: given a nonzero element $r \in R$, it is clear that $\mathfrak{m}^n r \neq 0$ for all $n \geq 0$ (since $\mathfrak{m}^n \mathfrak{m}^m = \mathfrak{m}^{n+m} \neq 0$). We have crucially used the fact that we have infinitely many variables here, for otherwise, $\mathfrak{m}^n = 0$ for $n \gg 0$. More generally, the induced module $R \otimes W$ are all torsion-free, and so are the semi-induced R-modules.

Lemma 3.3 *The submodule $\Gamma(M)$ of a finitely generated R-module M is finite length.*

Proof First, assume M is a finitely generated torsion R-module generated by homogeneous elements $m_1, m_2, \ldots, m_n$. Assume $\mathfrak{m}^r m_i = 0$ for all i. Then M_i is a finite length $\mathbf{GL}$-representation for all i (this holds all finitely generated R-modules), and vanishes for all $i > r + \max \deg(m_i)$. So M has finite length as a $\mathbf{GL}$-representation. If M is a finitely generated R-module, then the submodule $\Gamma(M)$ is a finitely generated torsion R-module by Theorem 3.2, and hence $\Gamma(M)$ is finite length, as required. □

We now verify property (Inj) for $\mathrm{Mod}_R^{\mathrm{tors}}$, i.e., injectives in the Serre subcategory $\mathrm{Mod}_R^{\mathrm{tors}}$ remain injective in the ambient category Mod_R.

Lemma 3.4 $\mathrm{Mod}_R^{\mathrm{tors}}$ *satisfies property (Inj).*

Proof Let M be a finitely generated R-module. The submodule $\Gamma(M)$ is supported in finitely many degrees by Lemma 3.3; assume $\Gamma(M)_i = 0$ for $i > N$. Let $M^{>N}$ be the submodule of M consisting of all elements of degrees $> N$. The submodule $M^{>N}$

is torsion-free and the quotient $M/M^{>N}$ is torsion. The result now follows from [44, Proposition 4.18]. □

This implies:

Corollary 3.5 *Let M be a torsion R-module. For $i > 0$, the local cohomology functors $\mathrm{R}^i\Gamma(M)$ vanish.*

Proof This is [44, Proposition 4.2]. □

Even if $rm = 0$ for an element $r \in \mathfrak{m}$ and $m \in M$, the element m need not be torsion. For example, we have $x_1 \wedge x_1 = 0$, but R is torsion-free. The next lemma shows that if an element m is killed by $r \in R$ which is *disjoint* from m (i.e., the basis vectors of $\mathbf{V}$ which are used in m are disjoint from the ones used in r), then m is a torsion element.

Lemma 3.6 *Let M be a finitely generated R-module, and $m \in M$ be an element such that $m \in M(k^n)$, i.e., m only uses the first n basis vectors. If $x_{n+1}x_{n+2}\ldots x_{n+r}m = 0$, then m is a torsion element killed by $\mathfrak{m}^r$.*

Proof We give a proof when $r = 1$, leaving the easy generalization to the reader. Given $i \neq n+1$, let g be the element in $\mathbf{GL}$ that maps e_{n+1} to $e_{n+1} + e_i$, and fixes e_j for $j \neq n+1$. We have

$$0 = g(x_{n+1}m) = (gx_{n+1})(gm) = (x_i + x_{n+1})m = x_i m + x_{n+1}m = x_i m,$$

where for the third equality, we use the fact that m is fixed by g since m only uses the first n basis vectors. So the $\mathbf{GL}$-ideal $\mathfrak{m}$ kills m, as required. □

For an abelian category $\mathcal{C}$, the *Krull–Gabriel dimension* measures how far $\mathcal{C}$ is from being locally of finite length (see [14, § 4.1] for a precise definition). We have $\mathrm{KG.dim}(\mathcal{C}) = 0$ if and only if all objects in $\mathcal{C}$ are locally finite length. It is easy to see using Lemma 3.3 that $\mathrm{KG.dim}(\mathrm{Mod}_R^{\mathrm{tors}}) = 0$. We prove that $\mathrm{Mod}_R^{\mathrm{gen}}$ is also zero-dimensional.

Lemma 3.7 *Fix an $n \in \mathbf{N}$. There exists an $F(n) \in \mathbf{N}$ such that the length of $\bigwedge^i(\mathbf{V}) \otimes \mathbf{V}^{\otimes n}$ as a $\mathbf{GL}$-representation is less than $F(n)$ for all $i \geq 0$.*

Proof By [21, Theorem 2.8], the lengths of $\bigwedge^i(\mathbf{V}) \otimes \mathbf{V}^{\otimes n}$ is eventually periodic, i.e., there exists q (a power of p) and N such that for $r, s > N$ with $q|(r-s)$, we have

$$\mathrm{len}_{\mathbf{GL}}(\textstyle\bigwedge^r(\mathbf{V}) \otimes \mathbf{V}^{\otimes n}) = \mathrm{len}_{\mathbf{GL}}(\bigwedge^s \mathbf{V} \otimes \mathbf{V}^{\otimes n}).$$

In particular, the lengths are bounded above by some $F(n) \in \mathbf{N}$, as required. □

Proposition 3.8 *The category $\mathrm{Mod}_R^{\mathrm{gen}}$ has Krull–Gabriel dimension zero. Consequently, the Krull–Gabriel dimension of Mod_R is one.*

Proof We first show that for a finitely generated R-module M, we have

$$\mathrm{len}_{\mathrm{Mod}_R^{\mathrm{gen}}}(T(M)) < \liminf_i \mathrm{len}_{\mathbf{GL}}(M_i).$$

Indeed, given a submodule $N \subset M$, the equality $T(N) = T(M)$ holds if and only if M/N is finite length, which holds if and only $N_i = M_i$ for $i \gg 0$. Therefore, if $T(N) \subsetneq T(M)$, then we have $\mathrm{len}_{\mathbf{GL}}(N_i) < \mathrm{len}_{\mathbf{GL}}(M_i)$ for sufficiently large i, whence we get the above inequality.

Now, every finitely generated R-module is a subquotient of direct sums of $P_n = R \otimes \mathbf{V}^{\otimes n}$. So it suffices to show that $T(P_n)$ has finite length. By Lemma 3.7 and the discussion in the previous paragraph, the length of $T(P_n)$ is at most $F(n)$, as required. □

4 The shift theorem

In the last section, we showed that Mod_R splits into two pieces, the Serre subcategory of torsion R-modules, and the Serre quotient of generic R-modules. In this section, we prove the shift theorem which, in some sense, lets us glue these two pieces together. We implicitly use the equivalence Φ between **Pol** and $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ throughout. In particular, we suppress the functor Φ. So if M is a polynomial representation, and we write $M(W)$ for some vector space W, we mean the $\mathbf{GL}(W)$-representation $\Phi(M)(W)$.

4.1 The shift functor

As we explained in Sect. 2.3, the Schur derivative restricts to an endofunctor of Mod_R. We now define a natural map $i_M \colon M \to \mathbf{\Sigma}(M)$ for an R-module M. We have the R-module map $m \mapsto y_1 m$ from M to $M(k \oplus \mathbf{V})$ (where y_1 is the new variable in $R(k \oplus \mathbf{V})$). Since m uses only the basis vectors e_i, the $k^\star \subset \mathbf{GL}(k \oplus \mathbf{V})$-weight of $y_1 m$ is 1. Therefore, the map factors $M \to \mathbf{\Sigma}(M) \to M(k \oplus \mathbf{V})$. We let i_M be the first map. It is easy to see that i is natural. We let $\mathbf{\Delta}(M) = \mathrm{coker}(i_M)$ be the *difference functor*. Since $\mathbf{\Sigma}$ is an exact functor, it follows from the snake lemma that $\mathbf{\Delta}$ is right exact. We also let $\mathbf{\Sigma}^r$ be the r-fold composition of $\mathbf{\Sigma}$ for $r > 0$ and let $\mathbf{\Delta}_r(M) = \mathrm{coker}(M \to \mathbf{\Sigma}^r(M))$.

We first prove some basic properties of the Schur derivative functor (which hereafter we refer to as the *shift functor*). See [26, Section 2] for analogous results for FI-modules.

Proposition 4.1 *Let M be an R-module.*

(a) The kernel of i_M is torsion.
(b) The map i_M is injective if and only if M is torsion-free.
(c) If M is torsion-free then so is $\mathbf{\Sigma}(M)$.
(d) If M is finitely generated, then $\mathbf{\Sigma}^r(M)$ is torsion-free for sufficiently large r.
(e) If M is nonzero, the module $\mathbf{\Delta}_r(M)$ is generated in degrees $\leq t_0(M) - 1$ for all $r \geq 1$.
(f) If M is a finitely generated semi-induced module, then so are $\mathbf{\Sigma}(M)$ and $\mathbf{\Delta}(M)$.
(g) If M is finitely generated, then so are $\mathbf{\Sigma}(M)$ and $\mathbf{\Delta}(M)$.
(h) For all $r \geq 1$, the functors $\mathbf{\Sigma}\mathbf{\Delta}_r$ and $\mathbf{\Delta}_r\mathbf{\Sigma}$ are naturally isomorphic (i.e., the functors $\mathbf{\Sigma}$ and $\mathbf{\Delta}_r$ commute).

(i) The functors $\mathbf{\Sigma}\Gamma$ and $\Gamma\mathbf{\Sigma}$ are naturally isomorphic (i.e., the functors $\mathbf{\Sigma}$ and Γ commute).

The proposition will be proved at the end of this subsection, after we prove a few preliminary results.

Lemma 4.2 *Let $i : \mathrm{id} \to \mathbf{\Sigma}$ be the natural map of functors. We have two natural maps from $\mathbf{\Sigma}$ to $\mathbf{\Sigma}^2$: one given by $\mathbf{\Sigma}(i)$, and the other given by $i_{\mathbf{\Sigma}}$. There exists an involution τ of $\mathbf{\Sigma}^2$ such that $\mathbf{\Sigma}(i) = \tau \circ i_{\mathbf{\Sigma}}$.*

Proof The module $\mathbf{\Sigma}^2(M)$ can be identified as a submodule of $M(k^2 \oplus \mathbf{V})$. Under this identification, the map $\mathbf{\Sigma}(i_M)$ is given by multiplying with the variable introduced first (call it y_1), and $i_{\mathbf{\Sigma}(M)}$ is given by multiplying with the variable introduced second (call it y_2). We have an isomorphism of $M(k^2 \oplus \mathbf{V})$ induced by the isomorphism of $k^2 \oplus \mathbf{V}$ which swaps the two basis vectors of the first factor and is identity on $\mathbf{V}$. This isomorphism clearly restricts to a natural isomorphism of $\mathbf{\Sigma}^2(M)$; call this natural map τ. We have

$$(\tau_M \circ \mathbf{\Sigma}(i_M))(m) = \tau_M(y_1 m) = y_2 m = i_{\mathbf{\Sigma}(M)}(m),$$

as required. □

Lemma 4.3 *Let V be a polynomial representation of $\mathbf{GL}$. We have isomorphisms*

$$\mathbf{\Sigma}(R \otimes V) \cong (\mathbf{\Sigma}(R) \otimes V) \oplus (R \otimes \mathbf{\Sigma}(V)) \cong (R \otimes V) \oplus (R \otimes \mathbf{\Sigma}(V)).$$

Furthermore, the natural map $i_{R \otimes V}$ is the identity map onto the first summand. Therefore, the module $\mathbf{\Delta}(R \otimes V)$ is isomorphic to $R \otimes \mathbf{\Sigma}(V)$.

Proof Firstly, it is easy to check that i_R is an isomorphism (see Lemma 2.3). Therefore $R \cong \mathbf{\Sigma}(R)$, giving us the second isomorphism in the statement of the lemma. For the first isomorphism, we use the fact that $\mathbf{\Sigma}$ is a categorical derivation. The claim about $i_{R \otimes V}$ follows from the definition of i.

□

Corollary 4.4 *Let W be a polynomial representation of $\mathbf{GL}$. Then $\mathbf{\Delta}(R \otimes W^{(1)}) = 0$.*

Proof All the weights of $W^{(1)}$ are divisible by p, so $\mathbf{\Sigma}(W^{(1)}) = 0$. The result now follows from Lemma 4.3. □

Proof of Proposition 4.1

(a) Assume m is in the kernel of i_M, i.e., $y_1 m = 0$ in $\mathbf{\Sigma}(M)$. Viewing $\mathbf{\Sigma}(M)$ as a submodule of $M(k \oplus \mathbf{V})$, we see that the element $m \in M(\mathbf{V}) \subset M(k \oplus \mathbf{V})$ is annihilated by $y_1 \in R(k \oplus \mathbf{V})$. Since y_1 and m are disjoint, the element m is torsion by Lemma 3.6.

(b) If M is torsion-free, then by part (a), the map i_M is injective. Now, suppose M is not torsion-free; let m be a torsion element and assume $m \in M(k^i)$ (i.e., it only uses the basis vectors $e_1, e_2, \ldots, e_i$). Let n be the minimal integer for which $\mathfrak{m}^n m = 0$. Pick a nonzero element $\widetilde{m}$ in the submodule $\mathfrak{m}^{n-1}m$. Since $\mathfrak{m}^n m = 0$, we have $\mathfrak{m}\widetilde{m} = 0$. So, in $M(k \oplus \mathbf{V})$, we also have that $y_1\widetilde{m} = 0$, which implies that $\widetilde{m}$ is in the kernel of i_M, as required.

(c) If M is torsion-free, then by part (a), the map i_M is injective. Applying the exact functor $\mathbf{\Sigma}$, we see that the map $\mathbf{\Sigma}(i_M)\colon \mathbf{\Sigma}(M) \to \mathbf{\Sigma}^2(M)$ is also injective. By Lemma 4.2, we have $i_{\mathbf{\Sigma}(M)} = \tau_M \circ \mathbf{\Sigma}(i_M)$, where τ_M is an involution of $\mathbf{\Sigma}^2(M)$. Therefore, the map $i_{\mathbf{\Sigma}(M)}$ is also injective, and so by part (b), the module $\mathbf{\Sigma}(M)$ is torsion-free.

(d) The submodule $\Gamma(M)$ of torsion elements is finitely generated, and so supported in degrees $< n$ for some $n \in \mathbf{N}$. We have the short exact sequence

$$0 \to \Gamma(M) \to M \to M/\Gamma(M) \to 0.$$

Applying $\mathbf{\Sigma}^n$, we obtain an isomorphism $\mathbf{\Sigma}^n(M) \cong \mathbf{\Sigma}^n(M/\Gamma(M))$ as $\mathbf{\Sigma}^n(\Gamma(M)) = 0$. The module $\mathbf{\Sigma}^n(M/\Gamma(M))$ is torsion-free by part (c), and so $\mathbf{\Sigma}^n(M)$ is also torsion-free, as required.

(e) We prove the $r = 1$ case which can be easily extended to all r by the reader. For a polynomial representation W of degree $\leq n$, the representation $\mathbf{\Sigma}(W)$ has degree $\leq n - 1$. Therefore, by Lemma 4.3, applying $\mathbf{\Delta}$ to an induced module generated in degree $\leq n$ results in an induced module generated in degree $\leq n - 1$. Now, for an R-module M, we have a surjection $R \otimes W \to M$, with W a polynomial representation of degree $\leq t_0(M)$. Since $\mathbf{\Delta}$ is right exact, we have a surjection $R \otimes \mathbf{\Delta}(W) \to \mathbf{\Delta}(M)$, which implies that $t_0(\mathbf{\Delta}(M)) \leq t_0(M) - 1$, as required.

(f) This follows from Lemma 4.3, and induction on the length of the filtration of M.

(g) This follows from Lemma 4.3 and the fact that the Schur derivative of a finite length polynomial representation is also finite length.

(h) We again prove the $r = 1$ case with the easy generalization for all $r > 1$ left to the reader. The module $\mathbf{\Sigma}(\mathbf{\Delta}(M))$ is the cokernel of $\mathbf{\Sigma}(i_M)$, and $\mathbf{\Delta}(\mathbf{\Sigma}(M))$ is the cokernel of $i_{\mathbf{\Sigma}(M)}$. These two maps only differ by an automorphism of the target, and so their cokernels are isomorphic.

(i) We have the short exact sequence $0 \to \Gamma(M) \to M \to M/\Gamma(M) \to 0$. Applying the functor $\mathbf{\Sigma}$ to this short exact sequence, we get

$$0 \to \mathbf{\Sigma}(\Gamma(M)) \to \mathbf{\Sigma}(M) \to \mathbf{\Sigma}(M/\Gamma(M)) \to 0.$$

By part (c), the module $\mathbf{\Sigma}(M/\Gamma(M))$ is torsion-free, and therefore, the submodule $\Gamma(\mathbf{\Sigma}(M))$ is contained in the image of $\mathbf{\Sigma}(\Gamma(M))$. For the reverse containment, the image of $\mathbf{\Sigma}(\Gamma(M))$ is a torsion submodule of $\mathbf{\Sigma}(M)$, and so is contained in $\Gamma(\mathbf{\Sigma}(M))$.

□

Lemma 4.5 *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. We have a six-term exact sequence*

$$0 \to \mathbf{K}(L) \to \mathbf{K}(M) \to \mathbf{K}(N) \to \mathbf{\Delta}(L) \to \mathbf{\Delta}(M) \to \mathbf{\Delta}(N) \to 0$$

where $\mathbf{K} = \ker(\mathrm{id} \to \mathbf{\Sigma})$.

Proof This is just the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_L} & & \downarrow{\scriptstyle i_M} & & \downarrow{\scriptstyle i_N} & & \\ 0 & \longrightarrow & \mathbf{\Sigma}(L) & \longrightarrow & \mathbf{\Sigma}(M) & \longrightarrow & \mathbf{\Sigma}(N) & \longrightarrow & 0. \end{array}$$

□

Remark 4.6 It is not hard to see that the functor $\mathbf{K}$ is isomorphic to $\mathbf{L}^1\mathbf{\Delta}$, and $\mathbf{L}^i\mathbf{\Delta}$ vanishes for $i > 1$ (for FI-modules, the analogous result can be found in [5, Lemma 4.7]).

Corollary 4.7 *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules such that N is torsion-free. We have a short exact sequence*

$$0 \to \mathbf{\Delta}(L) \to \mathbf{\Delta}(M) \to \mathbf{\Delta}(N) \to 0$$

Proof The module $\mathbf{K}(N) = 0$ by Proposition 4.1(a). The result now follows from Lemma 4.5. □

4.2 Vanishing of Δ

This subsection contains the main new technical result of this paper. We classify the torsion-free R-modules for which $\mathbf{\Delta}(M)$ vanishes. In characteristic zero, it is not hard to prove that for a nonzero R-module M, we have $t_0(\mathbf{\Delta}(M)) = t_0(M) - 1$ [26, Proposition 2.4]. So if $\mathbf{\Delta}(M) = 0$, then M must be generated in degree zero and therefore induced. In positive characteristic we only have the inequality $t_0(\mathbf{\Delta}(M)) \leq t_0(M) - 1$. In this subsection, we prove that if $\mathbf{\Delta}$ vanishes for a torsion-free module, then it is an extension of modules of the form $R \otimes W^{(1)}$. In particular, they are all semi-induced. We first prove this when the module M is generated in one degree (Proposition 4.10), and then use the short exact sequence $0 \to M^{<n} \to M \to M/M^{<n} \to 0$ to deduce the result in general (Proposition 4.12).

The key lemma is:

Lemma 4.8 *Let W be a polynomial representation of $\mathbf{GL}$ that is not Frobenius twisted. There exists a weight vector $w \in W$ of weight μ with $\mu_1 = 1$.*

Proof Since W is not Frobenius twisted, at least one of the irreducible constituent of W will not be Frobenius twisted. So we may assume that W is an *irreducible* polynomial representation that is not Frobenius twisted. By the Steinberg tensor product theorem, we have an isomorphism $W \cong L_{\lambda^0} \otimes L_{\lambda^1}^{(1)} \otimes L_{\lambda^2}^{(2)} \ldots \otimes L_{\lambda^r}^{(r)}$ for p-restricted partitions

$\lambda^0, \lambda^1, \ldots, \lambda^r$ with $\lambda^0 \neq \varnothing$. In L_{λ^0}, we can find a nonzero weight vector v_0 of weight α with $\alpha_1 = 1$ and we can find a weight vector v_1 of weight β with $\beta_1 = 0$ in $L^{(1)}_{\lambda^1} \otimes L^{(2)}_{\lambda^2} \ldots \otimes L^{(r)}_{\lambda^r}$. The vector $v_0 \otimes v_1$ is a weight vector in W, and the first component of its weight is 1, as claimed. □

Lemma 4.9 *Assume M is a finitely generated R-module with $M_n = 0$ and M_{n+1} not Frobenius twisted. Then $\mathbf{\Delta}(M)_n \neq 0$. In particular, if M is a finitely generated R-module generated in degree $n+1$ with $\mathbf{\Delta}(M) = 0$, then M_{n+1} is Frobenius twisted.*

Proof Since the representation M_{n+1} is not Frobenius twisted, by Lemma 4.8, we can find a vector v of weight $(1, \lambda)$ in M_n. In $M(k \oplus \mathbf{V})$, let $w = g \cdot v$ where g swaps the newly introduced basis vector, say f, with e_1 (and fixes the other e_i). The vector w lies in $\mathbf{\Sigma}(M)$ as it has weight $(1, 0, \lambda)$ for the action of $\mathbf{GL}(k \oplus \mathbf{V})$. Furthermore, the element w has degree n so it is not in the image of the natural map $M \to \mathbf{\Sigma}(M)$ (as $M_n = 0$), which implies that the image of w is nonzero in $\mathbf{\Delta}(M)$, as required. The second part of the statement follows easily since for a module generated in degree $n+1$, we have $M_n = 0$. □

Proposition 4.10 *Let M be a torsion-free R-module generated in degree n such that $\mathbf{\Delta}(M) = 0$. The natural map $R \otimes M_n \to M$ is an isomorphism, i.e., M is an induced module.*

Proof The irreducible components of R in positive degree are p-restricted representations of $\mathbf{GL}$ (in degree i, it is the i-th exterior power $\bigwedge^i(\mathbf{V})$ which is irreducible with highest weight (1^i)). Therefore, if W is an irreducible representation of $\mathbf{GL}$, then $R_n \otimes W^{(1)}$ is also irreducible by the Steinberg tensor product theorem (Theorem 2.1), and in particular, if $n > 0$, it is not Frobenius twisted. It follows that for a finite length polynomial representation W, the irreducible components of $R \otimes W^{(1)}$ are not Frobenius twisted in degrees $> \deg(W^{(1)})$.

By the assumptions on M, we see that M_n is Frobenius twisted by Lemma 4.9. Consider the natural map $\varphi\colon R \otimes M_n \to M$. The map φ is surjective, and $K = \ker(\varphi)$ is zero in degrees $\leq n$. We have to prove that $K = 0$. First, we claim that $\mathbf{\Delta}(K) = 0$. Indeed, since M is torsion-free, by Corollary 4.7, we have a short exact sequence

$$0 \to \mathbf{\Delta}(K) \to \mathbf{\Delta}(R \otimes M_n) \to \mathbf{\Delta}(M) \to 0.$$

By Corollary 4.4, we have $\mathbf{\Delta}(R \otimes M_n) = 0$ and so $\mathbf{\Delta}(K) = 0$, as claimed. Now, suppose K is nonzero, and let r be the minimal degree in which K is nonzero (i.e., $K_r \neq 0$ and $K_i = 0$ for $i < r$). Note $r > n$ and so K_r is not Frobenius twisted by the first paragraph. Therefore, by Lemma 4.9, we see that $\mathbf{\Delta}(K)$ must be nonzero, which is a contradiction. So the map φ is an isomorphism, as required. □

Lemma 4.11 *Let M be a torsion-free R-module with $t_0(M) = n$ such that $\mathbf{\Delta}(M)$ is semi-induced and $\mathbf{\Delta}(M/M^{<n}) = 0$. Then $M/M^{<n}$ is torsion-free.*

Proof As in Lemma 4.5, we have the short exact sequence

$$0 \to \mathbf{K}(M/M^{<n}) \to \mathbf{\Delta}(M^{<n}) \to \mathbf{\Delta}(M) \to 0.$$

Applying Tor to this gives us the exact sequence

$$\mathrm{Tor}_1^R(\mathbf{\Delta}(M), k) \to \mathrm{Tor}_0^R(\mathbf{K}(M/M^{<n}), k) \to \mathrm{Tor}_0^R(M^{<n}, k).$$

However, since $\mathbf{\Delta}(M)$ is semi-induced, the module $\mathrm{Tor}_1^R(\mathbf{\Delta}(M), k) = 0$, which implies that

$$t_0(\mathbf{K}(M/M^{<n})) \leq t_0(M^{<n}) < n.$$

But $\mathbf{K}(M/M^{<n})$ is supported only in degrees $\geq n$ (as it is a submodule of $M/M^{<n}$), and so must be zero. Therefore, the natural map $M/M^{<n} \to \mathbf{\Sigma}(M/M^{<n})$ is injective, which implies that the module $M/M^{<n}$ is torsion-free by Proposition 4.1(b). □

We can now prove our main result.

Proposition 4.12 *Let M be a finitely generated torsion-free R-module such that $\mathbf{\Delta}(M) = 0$. Then M is semi-induced.*

Proof We proceed by induction on the generation degree of M. When $t_0(M) = 0$, this follows from Proposition 4.10. Now, assume that $t_0(M) = n > 0$. Since $\mathbf{\Delta}$ is right exact, we see that $\mathbf{\Delta}(M/M^{<n}) = 0$. Therefore, by Lemma 4.11, the module $M/M^{<n}$ is torsion-free, and so by Proposition 4.10, the module $M/M^{<n}$ is semi-induced. By Corollary 4.7, we have a short exact sequence

$$0 \to \mathbf{\Delta}(M^{<n}) \to \mathbf{\Delta}(M) \to \mathbf{\Delta}(M/M^{<n}) \to 0.$$

As $\mathbf{\Delta}(M) = 0$, we have that $\mathbf{\Delta}(M^{<n}) = 0$, which implies that $M^{<n}$ is semi-induced (by induction) and therefore, so is M. □

We finally note a corollary (of Lemma 4.11) that we will use in the proof of the shift theorem.

Corollary 4.13 *Let M be a finitely generated torsion-free R-module with $t_0(M) = n$ such that $\mathbf{\Delta}(M)$ is semi-induced with $t_0(\mathbf{\Delta}(M)) \leq \max(-1, n-2)$. Then $M/M^{<n}$ is semi-induced.*

Proof Since the functor $\mathbf{\Delta}$ is right exact, we have

$$t_0(\mathbf{\Delta}(M/M^{<n})) \leq t_0(\mathbf{\Delta}(M)) < n - 1.$$

Since $M/M^{<n}$ is supported in degrees $\geq n$, the module $\mathbf{\Sigma}(M/M^{<n})$ is supported in degrees $\geq n-1$, and so $\mathbf{\Delta}(M/M^{<n})$ is also supported in degrees $\geq n-1$. Therefore, the above inequality implies that $\mathbf{\Delta}(M/M^{<n}) = 0$. Now, by Lemma 4.11, the module $M/M^{<n}$ is torsion-free and so by Proposition 4.10, the module $M/M^{<n}$ is semi-induced. □

4.3 Semi-induced subquotients of semi-induced modules

We prove some lemmas bounding the generation degree of semi-induced subquotients of semi-induced modules. The only idea used in this subsection is that $R(k^n)$, the exterior algebra on k^n, is supported in degrees 0 through n.

Lemma 4.14 *Let M be an R-module. Then for all n, we have $M(k^n)_i = 0$ for $i > n + t_0(M)$. Furthermore, if M is semi-induced, then $M(k^n)_{n+t_0(M)}$ is nonzero for all sufficiently large n.*

Proof This is clear since R evaluated at k^n is supported in degrees $\leq n$. □

Corollary 4.15 *Let F be a semi-induced module and Z be a submodule of F such that $Z/Z^{<t_0(Z)}$ is semi-induced. Then $t_0(Z) \leq t_0(F)$.*

Proof For sufficiently large n, the module $Z(k^n)$ is nonzero in degree $n + t_0(Z)$ as this is true for $Z/Z^{<t_0(Z)}$ by the previous lemma. Since Z is a submodule of F, we have $n + t_0(Z) \leq n + t_0(F)$ for large n by the previous lemma, giving us the required inequality. □

4.4 Proof of the shift theorem

Lemma 4.16 *Let M be a torsion-free R-module such that $\mathbf{\Delta}(M)$ is semi-induced. Then $t_1(M) \leq t_0(M)$.*

Proof Let F be a semi-induced module surjecting onto M such that $t_0(F) = t_0(M)$. We have an exact sequence

$$0 \to Z \to F \to M \to 0.$$

The long exact sequence of $\mathrm{Tor}_\bullet^R(-, k)$ yields:

$$0 \to \mathrm{Tor}_1^R(M, k) \to \mathrm{Tor}_0^R(Z, k) \to \mathrm{Tor}_0^R(F, k) \to \mathrm{Tor}_0^R(M, k) \to 0.$$

Therefore, we have the inequality $t_1(M) \leq t_0(Z)$. So it suffices to show that $t_0(Z) \leq t_0(M)$.

Since M is torsion-free, we also obtain a short exact sequence

$$0 \to \mathbf{\Delta}(Z) \to \mathbf{\Delta}(F) \to \mathbf{\Delta}(M) \to 0$$

by Corollary 4.7. The module $\mathbf{\Delta}(F)$ is semi-induced by Proposition 4.1(f), and $\mathbf{\Delta}(M)$ is semi-induced by assumption. So by Corollary 2.8, the module $\mathbf{\Delta}(Z)$ is also semi-induced.

We know $t_0(\mathbf{\Delta}(Z)) \leq t_0(Z) - 1$ by Proposition 4.1(e). If $t_0(\mathbf{\Delta}(Z)) < t_0(Z) - 1$, then $Z/Z^{<t_0(Z)}$ is semi-induced by Corollary 4.13 (note that Z is torsion-free being a submodule of F) and therefore, by Corollary 4.15, we have $t_0(Z) \leq t_0(F) = t_0(M)$.

Instead, if $t_0(\mathbf{\Delta} Z) = t_0(Z) - 1$, then from the short exact sequence above, we have the inequality

$$t_0(\mathbf{\Delta}(Z)) = t_0(Z) - 1 \leq t_0(\mathbf{\Delta}(F)) \leq t_0(F) - 1 = t_0(M) - 1,$$

or $t_0(Z) \leq t_0(M)$. □

Proposition 4.17 *Let M be a finitely generated torsion-free R-module such that $\mathbf{\Delta}(M)$ is semi-induced. Then M is also semi-induced.*

Proof We induct on $t_0(M)$. When $t_0(M) = 0$, the module $\mathbf{\Delta}(M) = 0$, and so M is semi-induced by Proposition 4.12. Now, assume $t_0(M) = n > 0$. We have the short exact sequence

$$0 \to M^{<n} \to M \to M/M^{<n} \to 0.$$

So it suffices to prove that $M^{<n}$ and $M/M^{<n}$ are semi-induced.

We first show that $M/M^{<n}$ is semi-induced. The long exact sequence of $\mathrm{Tor}^R_\bullet(-, k)$ associated to the above short exact sequence gives us

$$\mathrm{Tor}^R_1(M, k) \to \mathrm{Tor}^R_1(M/M^{<n}, k) \to \mathrm{Tor}^R_0(M^{<n}, k)$$

from which we see that $t_1(M/M^{<n}) \leq \max(t_1(M), t_0(M^{<n})) = \max(t_1(M), n-1) \leq n$. For the last step, we use Lemma 4.16 to get $t_1(M) \leq t_0(M) = n$. So by Lemma 2.6, the module $M/M^{<n}$ is semi-induced.

We proceed to show that $M^{<n}$ is also semi-induced. Since $M/M^{<n}$ is semi-induced, it is torsion-free, and therefore, we have a short exact sequence

$$0 \to \mathbf{\Delta}(M^{<n}) \to \mathbf{\Delta}(M) \to \mathbf{\Delta}(M/M^{<n}) \to 0$$

by Corollary 4.7. By Proposition 4.1(f), we have that $\mathbf{\Delta}(M/M^{<n})$ is semi-induced, and so from the short exact sequence above, we see that $\mathbf{\Delta}(M^{<n})$ is also semi-induced by Corollary 2.8. Since $t_0(M^{<n}) < n$, by the induction hypothesis, it now follows that $M^{<n}$ is semi-induced, as required. □

Theorem 4.18 (Shift Theorem) *Let M be a finitely generated R-module. We can find an $L \in \mathbf{N}$ such that for all $l > L$, the module $\mathbf{\Sigma}^l(M)$ is semi-induced.*

Proof We follow the proof of [26, Theorem 3.13] and proceed by induction on $t_0(M)$. It suffices to prove the result for $\mathbf{\Sigma}^r(M)$ for $r \gg 0$, and so we may additionally assume M is torsion-free by Proposition 4.1(d). When $t_0(M) = 0$, the module $\mathbf{\Delta}(M) = 0$ and so M is semi-induced by Proposition 4.17. Now, assume $t_0(M) = n > 0$. We have a short exact sequence $0 \to M \to \mathbf{\Sigma}(M) \to \mathbf{\Delta}(M) \to 0$ with $t_0(\mathbf{\Delta}(M)) < n$. By induction, for $l \gg 0$ the module $\mathbf{\Sigma}^l(\mathbf{\Delta}(M))$ is semi-induced. By Proposition 4.1(h) we have $\mathbf{\Sigma}^l(\mathbf{\Delta}(M)) \cong \mathbf{\Delta}(\mathbf{\Sigma}^l(M))$, and so by Proposition 4.17 we have that $\mathbf{\Sigma}^l(M)$ is semi-induced, as required. □

5 Structure of $\mathbf{Mod}_R^{\mathbf{fg}}$

In this section, we use the shift theorem to study the structure of finitely generated R-modules.

5.1 Generators of the derived category

In this subsection, we show that the derived category is generated by the torsion and flat R-modules. Using this, we prove that all finitely generated R-modules have finite regularity. We first note a useful corollary of the shift theorem. Throughout this section, we let $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$ denote the bounded derived category of Mod_R with finitely generated cohomology.

Theorem 5.1 *(Resolution Theorem) Let M be a finitely generated R-module. We have a chain complex of R-modules*

$$0 \to M \to P^0 \to P^1 \to \ldots \to P^r \to 0$$

satisfying the following properties:

- *each P^i is a finitely generated semi-induced module with $t_0(P^i) \leq t_0(M) - i$,*
- *$r \leq t_0(M)$, and*
- *the cohomology of this complex is torsion (and so supported in finitely many degrees).*

Furthermore, given a map of R-modules $f : M \to N$, we can find complexes $M \to P^\bullet$ and $N \to Q^\bullet$ satisfying the above properties, and a map of complexes $\tilde{f}$ extending f.

Proof We prove the result by induction on $n = t_0(M)$. If $n = 0$, then by the shift theorem, for $l \gg 0$, the module $\mathbf{\Sigma}^l(M)$ is semi-induced. Set $P^0 = \mathbf{\Sigma}^l(M)$. The kernel of the natural map $M \to P^0$ is torsion by Proposition 4.1(a), and this map must be surjective as the cokernel is generated in degrees $< t_0(M) = 0$ by Proposition 4.1(e). Therefore, the complex $0 \to M \to P^0 \to 0$ is the required complex for M. Now assume $n > 0$. Again, by the shift theorem, the module $\mathbf{\Sigma}^l(M)$ is semi-induced for $l \gg 0$ (and $t_0(\mathbf{\Sigma}^l(M)) = t_0(M)$). Set $P^0 = \mathbf{\Sigma}^l(M)$. As before, the kernel of the natural map from $M \to P^0$ is torsion. Furthermore, the cokernel of this map is generated in degrees $\leq n-1$ by Proposition 4.1(e). By induction, we have a complex $0 \to N \to P^1 \to P^2 \ldots P^n \to 0$ satisfying all the conditions of the theorem for N. It is easy to check that the complex $0 \to M \to P^0 \to P^1 \to \ldots \to P^n \to 0$ satisfies all the desired properties. The last statement is clear upon noting that if $M \to P^\bullet$ satisfies all the required properties, then so does the complex $M \to \mathbf{\Sigma}(P^\bullet)$. □

Lemma 5.2 *Given a bounded complex C of finitely generated R-modules with finite length cohomology, there exists a complex of finite length R-modules D that is quasi-isomorphic to C.*

Proof Given an R-module M, we let $M^{>n}$ be the submodule of M consisting of all elements of degree $> n$, and given a complex of R-modules C, we let $C^{>n}$ be the canonical subcomplex of C with $(C^{>n})^i = (C^i)^{>n}$. Now, assume C is a bounded complex of finitely generated R-modules and its cohomology is supported in degrees $< N$. We have a short exact sequence

$$0 \to C^{>N} \to C \to C/C^{>N} \to 0$$

with $C/C^{>N}$ a bounded complex of finite length modules. By assumption on the cohomology of C, the complex $C^{>N}$ is acyclic so the the complex C is quasi-isomorphic to $C/C^{>N}$, as required. □

Proposition 5.3 *Given a finitely generated R-module M, there exists a distinguished triangle $T \to M \to F \to$ in $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$, with T quasi-isomorphic to a bounded complex of finite length R-modules, and F quasi-isomorphic to a bounded complex of finitely generated semi-induced R-modules.*

Proof Let $M \to P^\bullet$ be a complex satisfying Theorem 5.1. We consider M to be a complex supported in degree 0, and let j denote the map of complexes $M \to P^\bullet$. We obtain a distinguished triangle

$$M \to P^\bullet \to \mathrm{Cone}(j) \to$$

in $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$. The complex $\mathrm{Cone}(j)$ has finite length cohomology, so is quasi-isomorphic to a finite complex of finite length torsion R-modules T by Lemma 5.2. The rotated triangle $T[-1] \to M \to P^\bullet \to$ satisfies the requirements. □

Theorem 5.4 *The category $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$ is generated as a triangulated category by the modules L_λ and $R \otimes L_\lambda$ for arbitrary λ.*

Proof By Proposition 5.3, the smallest triangulated subcategory of $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$ containing the torsion and semi-induced R-modules is all of $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$. Any torsion R-module (resp. semi-induced R-module) has a filtration where the successive quotients are L_λ (resp. $R \otimes L_\lambda$). So the modules L_λ and $R \otimes L_\lambda$ generate $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$, as required. □

We can now prove the theorem on finite regularity.

Definition 5.5 For an R-module M, we define the *Castelnuovo–Mumford regularity* of M to be the minimal integer ρ such that $t_i(M) \leq \rho + i$ for all i, or ∞ if no such integer exists.

Theorem 5.6 *The Castelnuovo–Mumford regularity of a finitely generated R-module is finite.*

Proof Let λ be a partition. The induced module $R \otimes L_\lambda$ is flat, and so has regularity $|\lambda|$. The module $k \cong R/\mathfrak{m}$ has regularity 0 as witnessed by the Koszul resolution (which is a flat resolution of k). Applying the functor $- \otimes_k L_\lambda$ to the Koszul resolution of k,

we obtain a flat resolution of L_λ. Therefore, the module L_λ has regularity $|\lambda|$. Since these two classes of modules generate the bounded derived category, every bounded complex has finite regularity, and in particular, so does every finitely generated module. □

Remark 5.7 Using Nagpal's shift theorem [33], Gan–Li showed that the regularity of a finitely generated VI-module M is at most $t_0(M) + t_1(M)$ [16, Theorem 3.2]. The results (and proofs) of Section 2 and Theorem 3.2 of loc. cit. also hold for R-modules. The only additional ingredient required for their proof is that for a finite length R-module M, we have $\operatorname{reg}(M) \leq \max \deg(M)$. This can be proved using the Koszul complex. Therefore, by mimicking their approach, we obtain the bound $\operatorname{reg}(M) \leq t_0(M) + t_1(M)$ for every finitely generated R-module.

5.2 The shift functor commutes with local cohomology

In Proposition 4.1(i), we showed that the shift functor commutes with the functor Γ. In this subsection, we show that the shift functor also commutes with the derived functors of Γ. This technical result will be used to prove several finiteness results for local cohomology. We first explain the setup.

Let M be an R-module, and let $M \to I^\bullet$ and $\mathbf{\Sigma}(M) \to J^\bullet$ be injective resolution of M and $\mathbf{\Sigma}(M)$ respectively. The identity map of $\mathbf{\Sigma}(M)$ induces a map of complexes $\mathbf{\Sigma}(I^\bullet) \to J^\bullet$ as $J^\bullet$ is an injective resolution. Applying Γ and taking cohomology, we obtain a map $H^i(\Gamma(\mathbf{\Sigma}(I^\bullet))) \to H^i(\Gamma(J^\bullet)) = \mathrm{R}^i\Gamma(\mathbf{\Sigma}(M))$. We also have natural isomorphisms,

$$H^i(\Gamma(\mathbf{\Sigma}(I^\bullet))) \cong H^i(\mathbf{\Sigma}(\Gamma(I^\bullet))) \cong \mathbf{\Sigma}(H^i(\Gamma(I^\bullet))) = \mathbf{\Sigma}(\mathrm{R}^i\Gamma(M)),$$

where for the first isomorphism, we use that $\mathbf{\Sigma}$ commutes with Γ by Proposition 4.1(i), and for the second isomorphism, we use that $\mathbf{\Sigma}$ is an exact functor, and therefore commutes with taking cohomology. So we get a natural map $F_{i,M} \colon \mathbf{\Sigma}(\mathrm{R}^i\Gamma(M)) \to \mathrm{R}^i\Gamma(\mathbf{\Sigma}(M))$.

Proposition 5.8 *The map F_i is an isomorphism for all $i \geq 0$.*

Proof Our proof is an expanded version of the proof of [33, Lemma 4.20] and [8, Proposition A.3]. We proceed by induction on i. For $i = 0$, this is just Proposition 4.1(h). Now, assume $i > 0$, and we know that F_{i-1} is an isomorphism. We have to show that F_i is an isomorphism.

Before we prove the result, we make two reductions. First, we may assume that M is torsion-free. Indeed, given an arbitrary module N, we have the short exact sequence $0 \to \Gamma(N) \to N \to N/\Gamma(N) \to 0$. Using the long exact sequence of local cohomology and Corollary 3.5, we see that for $i > 0$, the map $F_{i,N}$ is an isomorphism if and only if $F_{i,N/\Gamma(N)}$ is an isomorphism.

Second, we may assume that M is finitely generated. It is easy to see that the functors Γ and $\mathbf{\Sigma}$ commute with filtered colimits. In a locally noetherian Grothendieck category, a filtered colimit of injective objects remains injective. Therefore, by [20,

Proposition A.4], the right derived functors of Γ also commute with filtered colimits. So, for an arbitrary module N, we may write $F_{i,N}$ as the colimit of the maps $\{F_{i,N_\alpha}\}_\alpha$ where $\{N_\alpha\}_\alpha$ is the directed system of finitely generated submodules of N. Therefore, if we show $F_{i,M}$ is an isomorphism for finitely generated M, then $F_{i,N}$ will also be an isomorphism. So for the rest of the proof, we assume that M is a finitely generated torsion-free R-module.

We first show that $F_{i,M}$ is injective. Consider a short exact sequence

$$0 \to M \to I \to N \to 0$$

where I is an injective R-module. We obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{\Sigma}(\mathrm{R}^{i-1}\Gamma(I)) & \longrightarrow & \mathbf{\Sigma}(\mathrm{R}^{i-1}\Gamma(N)) & \longrightarrow & \mathbf{\Sigma}(\mathrm{R}^{i}\Gamma(M)) & \longrightarrow & \mathbf{\Sigma}(\mathrm{R}^{i}\Gamma(I)) = 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{R}^{i-1}\Gamma(\mathbf{\Sigma}(I)) & \longrightarrow & \mathrm{R}^{i-1}\Gamma(\mathbf{\Sigma}(N)) & \longrightarrow & \mathrm{R}^{i}\Gamma(\mathbf{\Sigma}(M)) & \longrightarrow & \mathrm{R}^{i}\Gamma(\mathbf{\Sigma}(I))
\end{array}$$

where the rows are exact. The first two vertical arrows are isomorphisms (by induction), and the fourth vertical map is injective, so by the four lemma, the third vertical map, which is $F_{i,M}$ is also injective. This implies that the canonical map $\mathbf{\Sigma}^l \mathrm{R}^i\Gamma(M) \to \mathrm{R}^i\Gamma(\mathbf{\Sigma}^l(M))$ is also injective for all $l \in \mathbf{N}$.

Let $\kappa_l(M) = \ker[\mathrm{R}^i\Gamma(M) \to \mathbf{\Sigma}^l(\mathrm{R}^i\Gamma(M))]$. We will now show that the modules $\mathbf{\Sigma}\kappa_l(M)$ and $\kappa_l(\mathbf{\Sigma}(M))$ are isomorphic for $l \gg 0$. For all l, we have a short exact sequence

$$0 \to M \to \mathbf{\Sigma}^l(M) \to \mathbf{\Delta}_l(M) \to 0.$$

Therefore, we get the exact sequence

$$\mathrm{R}^{i-1}\Gamma(\mathbf{\Sigma}^l(M)) \to \mathrm{R}^{i-1}\Gamma(\mathbf{\Delta}_l(M)) \to \mathrm{R}^i\Gamma(M) \to \mathrm{R}^i\Gamma(\mathbf{\Sigma}^l(M)).$$

Let $I^\bullet$ and $J^\bullet$ be injective resolutions of M and $\mathbf{\Sigma}^l(M)$ respectively. The induced map $I^\bullet \to J^\bullet$ is easily seen to factor via $I^\bullet \to \mathbf{\Sigma}(I^\bullet) \to J^\bullet$. Therefore, the last map in the above exact sequence factors

$$\mathrm{R}^i\Gamma(M) \to \mathbf{\Sigma}^l\mathrm{R}^i\Gamma(M) \to \mathrm{R}^i\Gamma(\mathbf{\Sigma}^l(M)).$$

We have already shown that $\mathbf{\Sigma}^l\mathrm{R}^i\Gamma(M) \to \mathrm{R}^i\Gamma(\mathbf{\Sigma}^l(M))$ is injective, so we get that $\kappa_l(M)$ can also be written as $\ker[\mathrm{R}^i\Gamma(M) \to \mathrm{R}^i\Gamma(\mathbf{\Sigma}^l(M))]$ and from the exact sequence above, we also get

$$\kappa_l(M) = \ker[\mathrm{R}^i\Gamma(M) \to \mathrm{R}^i\Gamma(\mathbf{\Sigma}^l(M))] \cong \mathrm{coker}[\mathrm{R}^{i-1}\Gamma(\mathbf{\Sigma}^l(M)) \to \mathrm{R}^{i-1}\Gamma(\mathbf{\Delta}_l(M))].$$

Therefore, we have isomorphisms

$$\begin{aligned}
\mathbf{\Sigma}\kappa_l(M) &= \mathbf{\Sigma}\ker[\mathrm{R}^i\Gamma(M) \to \mathrm{R}^i\Gamma(\mathbf{\Sigma}^l(M))] \\
&\cong \mathbf{\Sigma}\,\mathrm{coker}[\mathrm{R}^{i-1}\Gamma(\mathbf{\Sigma}^l(M)) \to \mathrm{R}^{i-1}\Gamma(\mathbf{\Delta}_l(M))] \\
&\cong \mathrm{coker}[\mathbf{\Sigma}\mathrm{R}^{i-1}\Gamma(\mathbf{\Sigma}^l(M)) \to \mathbf{\Sigma}\mathrm{R}^{i-1}\Gamma(\mathbf{\Delta}_l(M))] && (\mathbf{\Sigma} \text{ commutes with taking cokernels}) \\
&\cong \mathrm{coker}[\mathrm{R}^{i-1}\Gamma(\mathbf{\Sigma}(\mathbf{\Sigma}^l(M))) \to \mathrm{R}^{i-1}\Gamma(\mathbf{\Sigma}(\mathbf{\Delta}_l(M)))] && (\text{induction hypothesis}) \\
&\cong \mathrm{coker}[\mathrm{R}^{i-1}\Gamma(\mathbf{\Sigma}^l(\mathbf{\Sigma}(M))) \to \mathrm{R}^{i-1}\Gamma(\mathbf{\Delta}_l(\mathbf{\Sigma}(M)))] && (\mathbf{\Sigma} \text{ commutes with } \mathbf{\Sigma}^l \text{ and } \mathbf{\Delta}_l) \\
&\cong \ker[\mathrm{R}^i\Gamma(\mathbf{\Sigma}(M)) \to \mathrm{R}^i\Gamma(\mathbf{\Sigma}^l(\mathbf{\Sigma}(M)))] \\
&= \kappa_l(\mathbf{\Sigma}(M))
\end{aligned}$$

proving our claim that $\mathbf{\Sigma}\kappa_l(M) \cong \kappa_l(\mathbf{\Sigma}(M))$ for $l \gg 0$.

Now, since $\mathrm{R}^i\Gamma(M)$ is a torsion module, we have $\mathrm{R}^i\Gamma(M) = \mathrm{colim}_{l\in\mathbf{N}}\kappa_l(M)$, using which we get

$$\mathbf{\Sigma}\mathrm{R}^i\Gamma(M) = \mathbf{\Sigma}(\mathrm{colim}\,\kappa_l(M)) \cong \mathrm{colim}\,\mathbf{\Sigma}(\kappa_l(M)) \cong \mathrm{colim}\,\kappa_l(\mathbf{\Sigma}(M)) \cong \mathrm{R}^i\Gamma(\mathbf{\Sigma}(M)).$$

For the second isomorphism, we have used the fact that $\mathbf{\Sigma}$ commutes with colimits and for the third isomorphism, that the functor $\mathbf{\Sigma}$ commutes with κ_l for sufficiently large l, which we proved in the previous paragraph. Therefore, the natural map $F_{i,M}$ is an isomorphism, as required. □

Remark 5.9 To prove Proposition 5.8, we have only used the fact that $\mathrm{Mod}\,R$ is locally noetherian, that the Serre subcategory $\mathrm{Mod}^{\mathrm{tors}}_R$ satisfies Property (Inj), that a torsion R-module M can be realized as the colimit of the kernel of the maps $M \to \mathbf{\Sigma}^l(M)$, the functors Γ and $\mathbf{\Sigma}$ commute with taking filtered colimits, and that the functor $\mathbf{\Sigma}$ commutes with Γ and $\mathbf{\Delta}_l$.

Corollary 5.10 *Let M be a torsion-free R-module. If the natural map $M \to \mathbf{\Sigma}^l(M)$ splits for all l, then $\mathrm{R}^i\Gamma(M) = 0$ for all $i \geq 0$.*

Proof The proof of [33, Proposition 4.21] applies. □

Proposition 5.11 *A finitely generated R-module M is semi-induced if and only if $\mathrm{R}\Gamma(M) = 0$.*

Proof Assume that M is an induced module. Then, the natural map $M \to \mathbf{\Sigma}^l(M)$ splits (Lemma 4.3). Therefore, by Corollary 5.10, all the local cohomology of M vanishes. By dévissage, the functor $\mathrm{R}\Gamma$ vanishes on semi-induced modules as well.

Now, assume $\mathrm{R}\Gamma(M) = 0$. In particular, the R-module M is torsion-free as $\Gamma(M) = 0$. We will show by induction on $t_0(M)$ that M is semi-induced. When $t_0(M) = 0$, we have $\mathbf{\Delta}(M) = 0$ and therefore M is semi-induced by Proposition 4.12. Now, assume $t_0(M) > 0$. By the shift theorem, we have a short exact sequence

$$0 \to M \to \mathbf{\Sigma}^l(M) \to \mathbf{\Delta}_l(M) \to 0$$

with $\mathbf{\Sigma}^l(M)$ being semi-induced. Therefore $\mathrm{R}\Gamma(\mathbf{\Sigma}^l(M)) = 0$ by the previous paragraph. So from the long exact sequence of local cohomology, we also have

$\mathrm{R}\Gamma(\mathbf{\Delta}_I(M)) = 0$. By Proposition 4.1(e), we see that $t_0(\mathbf{\Delta}_I(M)) < t_0(M)$ and so $\mathbf{\Delta}_I(N)$ is semi-induced by the induction hypothesis. Now, by Corollary 2.8, we have that M is semi-induced, as required. □

Definition 5.12 We let $\mathbf{S}\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ be the *saturation functor* (where $\mathbf{S} = S \circ T$). The saturation functor $\mathbf{S}$ is a left exact functor. We let $\eta\colon \mathrm{id} \to \mathbf{S}$ denote the unit of adjunction. An R-module M is *saturated* if the natural map $M \to \mathbf{S}(M)$ is an isomorphism. An R-module M is *derived saturated* if it is saturated and $\mathrm{R}^i\mathbf{S}(M) = 0$ for all i.

The next result holds whenever the Serre subcategory satisfies Property (Inj); see [44, Proposition 4.6]) for details.

Lemma 5.13 *An R-module M is derived saturated if and only if $\mathrm{R}\Gamma(M) = 0$.*

Corollary 5.14 *An R-module is derived saturated if and only if it is semi-induced.*

Proof This follows by combining Lemma 5.13 and Proposition 5.11. □

In characteristic zero, Sam–Snowden proved that $\mathrm{Mod}_R^{\mathrm{tors}}$ is equivalent to $\mathrm{Mod}_R^{\mathrm{gen}}$. Recently, Snowden [47] extended this result to *integral* **GL**-algebras in characteristic zero, i.e., for a **GL**-algebra A with $|A|$ an integral domain, Snowden proved that the generic category $\mathrm{Mod}_A^{\mathrm{gen}} = \mathrm{Mod}_A / \mathrm{Mod}_A^{\mathrm{tors}}$ is equivalent to $\mathrm{Mod}_A^{\mathrm{lf}}$, the category of locally finite length A-modules. We show that this result fails for R in positive characteristic.

Proposition 5.15 *The categories $\mathrm{Mod}_R^{\mathrm{gen}}$ and Mod_R are not equivalent abelian categories in positive characteristic.*

Proof In $\mathrm{Mod}_R^{\mathrm{tors}}$, it is easy to see that every finitely generated object has finite injective dimension. We will show that $T(R)$ has infinite injective dimension in $\mathrm{Mod}_R^{\mathrm{gen}}$. Firstly, the injective dimension of $\bigwedge^i(\mathbf{V})$ in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ is unbounded as i varies (see [49, Page 6]). So R has infinite injective dimension in Mod_R as injective objects in Mod_R are injective in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$. Now, since R is derived saturated, we may take an injective resolution of $T(R)$, and apply S to obtain an injective resolution of R. Therefore the injective dimension of $T(R)$ is at least the injective dimension of R, and so infinite as well. This implies that the two categories are not equivalent. □

Theorem 5.16 *Let M, N be two finitely generated R-modules. The module $\mathrm{Ext}_R^i(M, N)$ is finite length for all i, and vanishes for $i \gg 0$.*

Proof By dévissage, it suffices to show that the conclusions of the theorem hold when M and N are the generators of the derived category given in Theorem 5.4. First, assume N is the simple R-module L_λ. The R-module L_λ has a finite injective resolution by finite length injectives. So $\mathrm{Ext}_R^i(M, L_\lambda)$ vanishes for sufficiently large i and is finite length for all i (since M is finitely generated).

Next, assume N is the induced R-module $R \otimes L_\lambda$ and M is the simple R-module L_μ. The module N is derived saturated (Proposition 5.11), so $\mathrm{Ext}_R^i(L_\mu, R \otimes L_\lambda)$ vanishes for all i ([44, Proposition 4.7]). Finally, assume $M = R \otimes L_\mu$ and and $N = R \otimes L_\lambda$

are both induced modules. The $\mathbf{GL}$-representation L_μ has a finite projective resolution $P_\bullet$ in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$, see e.g., [49, Theorem 1]. The complex $R \otimes P_\bullet$ is a finite projective resolution of $R \otimes L_\lambda$. Therefore, the conclusion of the theorem holds in this case as well. □

5.3 Semi-orthogonal decomposition

Using Property (Inj), we obtain a semi-orthogonal decomposition

$$\mathrm{D}^+(\mathrm{Mod}_R) = \langle \mathrm{D}^+(\mathrm{Mod}_R^{\mathrm{tors}}), \mathrm{D}^+(\mathrm{Mod}_R^{\mathrm{gen}}) \rangle$$

of the derived category $\mathrm{D}^+(\mathrm{Mod}_R)$ (see below for definitions and [44, Proposition 4.9] for details). In this section, we prove the analogous result for $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$ (Theorem 5.19). We emphasize that for $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$, the semi-orthogonal decomposition is not a formal consequence of Property (Inj). It will follow from our next theorem on the finiteness of local cohomology, which we prove using the results of the previous two sections.

Theorem 5.17 *(Finiteness of local cohomology) For a finitely generated R-module M, the R-module $\mathrm{R}^i\Gamma(M)$ is finite length for all i, and is zero for sufficiently large i.*

Proof For torsion modules and semi-induced modules, the conclusions of the theorem hold by Corollary 3.5 and Proposition 5.11 respectively. Since the torsion and semi-induced modules generate $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$, the result holds for all finitely generated R-modules. □

We now recall the definition of a semi-orthogonal decomposition of a triangulated category. We refer the reader to [44, Section 4] for a succinct reference.

Definition 5.18 A *semi-orthogonal decomposition* of a a triangulated category $\mathcal{T}$ is a sequence $\langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \rangle$ of full triangulated subcategories such that

(1) $\mathrm{Hom}_{\mathcal{T}}(M_i, M_j) = 0$ for objects $M_i \in \mathcal{A}_i$, $M_j \in \mathcal{A}_j$ with $i < j$, and
(2) the smallest triangulated subcategory containing $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ is $\mathcal{T}$.

Theorem 5.19 *We have a semi-orthogonal decomposition*

$$\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R) = \langle \mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R^{\mathrm{tors}}), \mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R^{\mathrm{gen}}) \rangle.$$

where $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R^{\mathrm{gen}})$ is identified as a subcategory of $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$ using the functor $\mathrm{R}S$.

Proof This is a consequence Theorem 5.17 (see [44, Proposition 4.15] for more details). □

We can now prove a generalization of Proposition 5.3.

Proposition 5.20 *Given an object M in $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$, there exists a distinguished triangle $\mathrm{R}\Gamma(M) \to M \to \mathrm{R}\mathbf{S}(M) \to$ in $\mathrm{D}^b_{\mathrm{fg}}(\mathrm{Mod}_R)$.*

Proof This follows by combining [44, Proposition 4.6] with Theorem 5.17. □

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References

1. Bibby, C., Gadish, N.: Combinatorics of orbit configuration spaces. Sém. Lothar. Combin. 80B:Art. 72, 11 (2018)
2. Bik, A., Danelon, A., Draisma, J., Eggermont, R.H.: Universality of high-strength tensors. Vietnam J. Math. **50**(2), 557–580 (2022)
3. Bik, A., Draisma, J., Eggermont, R.H., Snowden, A.: The geometry of polynomial representations. Int. Math. Res. Not. IMRN, 14131–14195 (2023)
4. Brouwer, A.E., Draisma, J.: Equivariant Gröbner bases and the Gaussian two-factor model. Math. Comput. **80**(274), 1123–1133 (2011). arxiv:0908.1530
5. Church, T., Ellenberg, J.: Homology of FI-modules. Geom. Topol. **21**(4), 2373–2418 (2017)
6. Church, T., Ellenberg, J.S., Farb, B.: FI-modules and stability for representations of symmetric groups. Duke Math. J. **164**(9), 1833–1910 (2015)
7. Cohen, D.E.: On the laws of a metabelian variety. J. Algebra **5**(3), 267–273 (1967)
8. Djament, A.: Des propriétés de finitude des foncteurs polynomiaux. Fund. Math. **233**(3), 197–256 (2016). arxiv:1308.4698
9. Draisma, J.: Finiteness for the k-factor model and chirality varieties. Adv. Math. **223**(1), 243–256 (2010). arxiv:0811.3503
10. Draisma, J., Eggermont, R.H.: Finiteness results for Abelian tree models. J. Eur. Math. Soc. (JEMS) **17**(4), 711–738 (2015). arxiv:1207.1282
11. Erman, D., Sam, S.V., Snowden, A.: Generalizations of Stillman's conjecture via twisted commutative algebra. Int. Math. Res. Not. IMRN **2021**(16), 12281–12304 (2019)
12. Faith, C.: Algebra: Rings, Modules and Categories I, volume 190. Springer Science & Business Media (2012)
13. Friedlander, E.M., Suslin, A.: Cohomology of finite group schemes over a field. Invent. Math. **127**(2), 209–270 (1997)
14. Gabriel, P.: Des catégories abéliennes. Bull. Soc. Math. France **90**, 323–448 (1962)
15. Gan, W.L., Li, L.: An inductive machinery for representations of categories with shift functors. Trans. Am. Math. Soc. **371**(12), 8513–8534 (2019)
16. Gan, W.L., Li, L.: Bounds on homological invariants of VI-modules. Mich. Math. J. **69**(2), 273–284 (2020)
17. Ganapathy, K.: GL-algebras in positive characteristic II: The polynomial ring (in preparation)
18. Gandini, F.: Degree bounds for invariant skew polynomials. Preprint (2021). arxiv:2108.01767
19. Gandini, F.: Resolutions of ideals of subspace arrangements. J. Commut. Algebra **14**(3), 319–338 (2022)
20. Güntürkün, S., Snowden, A.: The representation theory of the increasing monoid. Mem. Am. Math. Soc. **286**(1420), 134 (2023)
21. Harman, N.: Stability and periodicity in the modular representation theory of symmetric groups. Preprint (2015). arxiv:1509.06414
22. Kujawa, J.: The Steinberg tensor product theorem for GL(m|n). Contemp. Math. **413**, 123 (2006)
23. Laudone, R.P.: Syzygies of secant ideals of Plücker-embedded Grassmannians are generated in bounded degree. Preprint (2018). arxiv:1803.04259
24. Le, D.V., Nagel, U., Nguyen, H.D., Römer, T.: Codimension and projective dimension up to symmetry. Math. Nachr. **293**(2), 346–362 (2020)
25. Le, D.V., Nagel, U., Nguyen, H.D., Römer, T.: Castelnuovo–Mumford regularity up to symmetry. Int. Math. Res. Not. **2021**(14), 11010–11049 (2021)
26. Li, L., Yu, N.: Filtrations and homological degrees of FI-modules. J. Algebra **472**, 369–398 (2017)
27. Maraj, A., Nagel, U.: Equivariant Hilbert series for hierarchical models. Algebr. Stat. **12**(1), 21–42 (2021)

28. Miller, J., Nagpal, R., Patzt, P.: Stability in the high-dimensional cohomology of congruence subgroups. Compos. Math. **156**(4), 822–861 (2020)
29. Miller, J., Wilson, J.: Higher-order representation stability and ordered configuration spaces of manifolds. Geom. Topol. **23**(5), 2519–2591 (2019)
30. Murai, S.: Betti tables of monomial ideals fixed by permutations of the variables. Trans. Am. Math. Soc. **373**(10), 7087–7107 (2020)
31. Nagel, U., Römer, T.: FI- and OI-modules with varying coefficients. J. Algebra **535**, 286–322 (2019)
32. Nagpal, R.: FI-modules and the cohomology of modular representations of symmetric groups. Ph.D. thesis, The University of Wisconsin-Madison (2015). arxiv:1505.04294
33. Nagpal, R.: VI-modules in nondescribing characteristic, part I. Algebra Number Theory **13**(9), 2151–2189 (2019)
34. Nagpal, R., Sam, S.V., Snowden, A.: Noetherianity of some degree two twisted commutative algebras. Selecta Math. (N.S.) **22**(2), 913–937 (2016)
35. Nagpal, R., Sam, S.V., Snowden, A.: Regularity of FI-modules and local cohomology. Proc. Am. Math. Soc. **146**(10), 4117–4126 (2018)
36. Nagpal, R., Sam, S.V., Snowden, A.: Noetherianity of some degree two twisted skew-commutative algebras. Selecta Math. (N.S.) **25**(1), 1–26 (2019)
37. Nagpal, R., Snowden, A.: Symmetric subvarieties of infinite affine space. Preprint (2020). arxiv:2011.09009
38. Nagpal, R., Snowden, A.: Symmetric ideals of the infinite polynomial ring. Preprint (2021). arxiv:2107.13027
39. Putman, A., Sam, S.V.: Representation stability and finite linear groups. Duke Math. J. **166**(13), 2521–2598 (2017)
40. Ramos, E.: Generalized representation stability and FI_d-modules. Proc. Am. Math. Soc. **145**(11), 4647–4660 (2017)
41. Sam, S.V., Snowden, A.: Introduction to twisted commutative algebras. Preprint (2012). arxiv:1209.5122
42. Sam, S.V., Snowden, A.: GL-equivariant modules over polynomial rings in infinitely many variables. Trans. Am. Math. Soc. **368**(2), 1097–1158 (2016)
43. Sam, S.V., Snowden, A.: Gröbner methods for representations of combinatorial categories. J. Am. Math. Soc. **30**(1), 159–203 (2017)
44. Sam, S.V., Snowden, A.: GL-equivariant modules over polynomial rings in infinitely many variables II. Forum Math. Sigma **7**, 71 (2019)
45. Sam, S.V., Snowden, A.: Sp-equivariant modules over polynomial rings in infinitely many variables. Trans. Am. Math. Soc. **375**(3), 1671–1701 (2022)
46. Snowden, A.: Syzygies of Segre embeddings and Δ-modules. Duke Math. J. **162**(2), 225–277 (2013). arxiv:1006.5248
47. Snowden, A.: Stable representation theory: beyond the classical groups. Preprint (2021). arxiv:2109.11702
48. Snowden, A.: The spectrum of a twisted commutative algebra. Proc. Lond. Math. Soc. (3), **128**(1):Paper No. e12576, 22 (2024)
49. Totaro, B.: Projective resolutions of representations of GL(n). J. Reine Angew. Math. **482**, 1–13 (1997)