Selecta Mathematica (2023) 29:63
https://doi.org/10.1007/s00029-023-00867-6

Selecta Mathematica
New Series

The ν^+-equivalence classes of genus one knots

Kouki Sato[1]

Accepted: 29 June 2023 / Published online: 17 August 2023

Abstract

The ν^+-equivalence is an equivalence relation on the knot concordance group. This relation can be seen as a certain stable equivalence on knot Floer complexes CFK^∞, and many concordance invariants derived from Heegaard Floer theory are invariant under the relation. In this paper, we show that any genus one knot is ν^+-equivalent to one of the trefoil, its mirror and the unknot.

Keywords Heegaard Floer homology · Knot Floer homology · Knot concordance

Mathematics Subject Classification 57M27 · 57M25

1 Introduction

Throughout this paper, all manifolds are assumed to be smooth, compact, connected, orientable and oriented unless otherwise stated.

1.1 Background and the main theorem

Heegaard Floer homology [16] is a powerful set of invariants for 3- and 4-manifolds and knots in 3-manifolds. In particular, the $\mathbb{Z}^2$-filtered chain complex $CFK^\infty(K)$ [15] associated to any knot K in S^3 is a very effective tool in studying knots and Dehn surgeries along knots. Indeed, from $CFK^\infty(K)$, we can compute

- The knot Floer homology $\widehat{HFK}(K)$ [15], and so we can detect the genus and fibredness of K [6, 11, 14],
- The Floer homology groups $\widehat{HF}$, HF^∞ and $HF^\pm$ and correction terms $d(-, \mathfrak{s})$ of all Dehn surgeries along K [18, 19], and

✉ Kouki Sato
satokou@meijo-u.ac.jp

1 Meijo University, Nagoya, Japan

- Many knot concordance invariants including ν^+, τ, Υ, Υ^2, and so on. (See [4, 8] for details.)

In this paper, to improve the understanding of CFK^∞, we study ν^+-*equivalence* (denoted $\overset{\nu^+}{\sim}$) introduced by Hom [4] and Kim-Park [7]. Here, two knots K_1 and K_2 are ν^+-*equivalent* if $\nu^+(K_1\#(-K_2^*)) = \nu^+(K_2\#(-K_1^*)) = 0$, where $-K$ and K^* denote the inverse and the mirror of K respectively, and ν^+ is a $\mathbb{Z}_{\geq 0}$-valued concordance invariant defined by Hom-Wu [5]. This relation is an equivalence relation on knots, and if two knots are concordant then they are ν^+-equivalent. (We call the equivalence classes ν^+-*classes*.) By the following Hom's theorem, ν^+-equivalence can be seen as a 'stable' filtered chain homotopy equivalence on CFK^∞.

Theorem 1.1 (Hom [4]) *Two knots K_1 and K_2 are ν^+-equivalent if and only if we have the following $\mathbb{Z}^2$-filtered chain homotopy equivalence:*

$$CFK^\infty(K_1) \oplus A_1 \simeq CFK^\infty(K_2) \oplus A_2,$$

where A_1, A_2 are acyclic, i.e., $H_*(A_1) = H_*(A_2) = 0$.

This theorem shows that determining the ν^+-class of knots is meaningful in terms of CFK^∞. Moreover, the ν^+-class of a knot K determines all correction terms of all Dehn surgeries along K and many concordance invariants including ν^+, τ, Υ and Υ^2 of K, and hence classifying the ν^+-classes is useful for computing these invariants. (These arguments are explained carefully in Sect. 2.10.)

The aim of this paper is to classify the ν^+-classes of genus one knots by using the τ-invariant [13]; in fact, we will see that only three ν^+-classes are realized by genus one knots. To state our theorem, we set some notations. For any knot K, let $[K]_{\nu^+}$ denote the ν^+-class of K and $g(K)$ the genus of K. For coprime integers $p, q > 0$, let $T_{p,q}$ denote the (p, q)-torus knot.

Theorem 1.2 *For any knot K with $g(K) = 1$, we have*

$$[K]_{\nu^+} = \begin{cases} [T_{2,3}]_{\nu^+} & \text{if} \quad \tau(K) = 1 \\ [\text{unknot}]_{\nu^+} & \text{if} \quad \tau(K) = 0 \\ [(T_{2,3})^*]_{\nu^+} & \text{if} \quad \tau(K) = -1 \end{cases}.$$

In other words, any genus one knot is ν^+-equivalent to one of the trefoil, its mirror and the unknot.

Since the τ-invariant is relatively understood, Theorem 1.2 enables us to determine the ν^+-class of many concrete examples. For instance, Hedden [2] gives a formula for the τ-invariant of the positive t-twisted Whitehead double of a knot K (denoted by $D_+(K, t)$). By Theorem 1.2, we can generalize his formula to a formula for the ν^+-class of $D_+(K, t)$.

Corollary 1.3 *For any knot K and $t \in \mathbb{Z}$, we have*

$$[D_+(K,t)]_{\nu^+} = \begin{cases} [\textit{unknot}]_{\nu^+} & \textit{for} \quad t \geq 2\tau(K) \\ [T_{2,3}]_{\nu^+} & \textit{for} \quad t < 2\tau(K) \end{cases}.$$

Next, let us consider the quotient set $\mathcal{C}_{\nu^+} := \{\text{knots in } S^3\}/\overset{\nu^+}{\sim}$. Note that since $\overset{\nu^+}{\sim}$ is weaker than knot concordance and the ν^+-invariant has the sub-additivity, we can identify $\mathcal{C}_{\nu^+}$ with a quotient group of the knot concordance group $\mathcal{C}$. So it is natural to ask how different these groups are. To give an observation of the question, we set $\mathcal{F}_g$ to be the subgroup of $\mathcal{C}$ generated by the knots with genus at most g. Let $\pi_{\nu^+} : \mathcal{C} \to \mathcal{C}_{\nu^+}$ be the projection, and then the sequence $\{\mathcal{F}_g\}_{g \in \mathbb{Z}_{\geq 0}}$ gives filtrations

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{C}$$

and

$$0 = \pi_{\nu^+}(\mathcal{F}_0) \subset \pi_{\nu^+}(\mathcal{F}_1) \subset \pi_{\nu^+}(\mathcal{F}_2) \subset \cdots \subset \mathcal{C}_{\nu^+}.$$

It is easy to show that $\mathcal{F}_1$ contains $\mathbb{Z}^\infty$ as a summand. (For instance, compute the ω-signature for the "twisted doubles" of the unknot. We refer to [9].) Therefore, combining it with Theorem 1.2, we have the following proposition, which shows a big gap between $\mathcal{C}$ and $\mathcal{C}_{\nu^+}$.

Proposition 1.4 *$\mathcal{F}_1$ contains $\mathbb{Z}^\infty$ as a summand, while $\pi_{\nu^+}(\mathcal{F}_1)$ is isomorphic to $\mathbb{Z}$.*

In knot concordance theory, there are few kinds of filtrations with each level finitely generated. Hence we suggest the following question.

Question *For each $g \in \mathbb{Z}_{\geq 0}$, is $\pi_{\nu^+}(\mathcal{F}_g)$ finitely generated?*

1.2 The idea of proof: estimating ν^+-classes

In order to prove Theorem 1.2, we use a partial order on $\mathcal{C}_{\nu^+}$ (denoted $\leq$) introduced in the author's paper [22]. We first study this partial order geometrically to give the following estimate for the ν^+-class of any knot K. Here $g_4(K)$ denotes the 4-genus of K, and we note that this estimate depends on $g_4(K)$ rather than $g(K)$.

Theorem 1.5 *For any knot K, we have*

$$-g_4(K)[T_{2,3}]_{\nu^+} \leq [K]_{\nu^+} \leq g_4(K)[T_{2,3}]_{\nu^+}.$$

Next, we study the $\mathbb{Z}^2$-filtered structure of CFK^∞ with $g(K) = 1$ algebraically to obtain another estimate, and combine it with Theorem 1.5 to prove Theorem 1.2. As another consequence of such estimates, we have the following discriminant using the Υ-invariant [17].

Theorem 1.6 *The equality $[K]_{\nu^+} = -g(K)[T_{2,3}]_{\nu^+}$ holds if and only if $\Upsilon_K(1) = g(K)$.*

1.3 Formal knot complexes and new concordance invariants

To study the algebraic aspects of ν^+-classes deeply, we consider an algebraic generalization of CFK^∞ called *formal knot complexes*. (The notion is originally considered

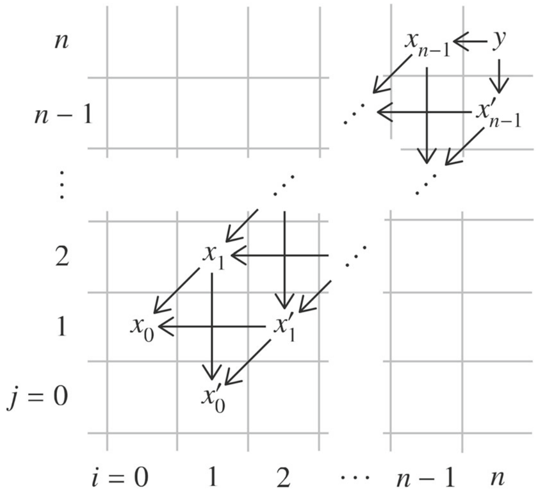


Fig. 1 A formal knot complex C^n with genus one

in [8].) In particular, we establish the category of such complexes, and obtain *the formal knot monoid* $\mathcal{K}^f$ and *the formal knot concordance group* $\mathcal{C}^f$, which are analogies of the knot monoid $\mathcal{K}$ and the knot concordance group $\mathcal{C}$, respectively. Concretely, these monoids are related as follows.

Theorem 1.7 *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{[K]\mapsto[CFK^\infty(K)]} & \mathcal{K}^f \\ {\scriptstyle [K]\mapsto[K]_c}\downarrow & & \downarrow{\scriptstyle [C]\mapsto[C]_{\nu^+}} \\ \mathcal{C} & \xrightarrow[{[K]_c\mapsto[CFK^\infty(K)]_{\nu^+}}]{} & \mathcal{C}^f \end{array}$$

Here, the bottom map coincides with π_{ν^+}. In particular, the image of the bottom map is $\mathcal{C}_{\nu^+}$.

Moreover, we also introduce the genus of formal knot complexes, and define the genus filtration

$$0 = \mathcal{F}_0^f \subset \mathcal{F}_1^f \subset \mathcal{F}_2^f \subset \cdots \subset \mathcal{C}^f,$$

where $\pi_{\nu^+}(\mathcal{F}_g) \subset \mathcal{F}_g^f$. For example, Fig. 1 depicts an infinite family of genus one formal knot complexes, and hence $[C^n]_{\nu^+} \in \mathcal{F}_1^f$ for each $n \in \mathbb{Z}_{>0}$. Here we note that C^1 is $CFK^\infty(T_{2,3})$.

We prove that the $[C^n]_{\nu^+}$ are mutually distinct, which implies that Theorem 1.2 cannot be proved purely algebraically.

Theorem 1.8 *The ν^+-classes $\{[C^n]_{\nu^+}\}_{n=1}^\infty$ are mutually distinct in $\mathcal{C}^f$, while $\tau(C^n) = 1$ for any n. In particular, the complement $\mathcal{F}_1^f \setminus \pi_{\nu^+}(\mathcal{F}_1)$ is infinite.*

In addition, we will show that if a formal knot complex C is realized as CFK^∞ for some knot K, then the genus of C is at least $g(K)$. Since C^n has genus one and $\tau(C^n) = 1$ but cannot be realized by any genus one knot, we have the following result, which is related to the geography problem discussed in [3].

Corollary 1.9 *The formal knot complexes $\{C^n\}_{n=2}^\infty$ cannot be realized by any knot in S^3.*

In order to distinguish the complexes $\{C^n\}$, we introduce an infinite family $\{\mathcal{G}_k\}_{k=0}^\infty$ of invariants of ν^+-classes, where $\mathcal{G}_k(C)$ consists of finitely many subsets of $\mathbb{Z}^2$. Since the ν^+-class of knots is a knot concordance invariant, the family $\{\mathcal{G}_k\}_{k=0}^\infty$ also gives a new family of knot concordance invariants. In particular, the primary invariant $\mathcal{G}_0$ has the following property.

Theorem 1.10 *For any knot K, the following assertions hold:*

1. *$\mathcal{G}_0(K)$ determines all correction terms of all Dehn surgeries along K.*
2. *$\mathcal{G}_0(K)$ determines all of ν^+, τ and Υ.*
3. *$[K]_{\nu^+} = 0$ if and only if $\mathcal{G}_0(K)$ has $\{(i, j) \in \mathbb{Z}^2 \mid i \le 0,\ j \le 0\}$ as the unique element.*

The definition of $\mathcal{G}_k$ and explicit formulas for computing the above invariants from $\mathcal{G}_0(K)$ are given in Sect. 5. In the section, we also discuss the relationship between our secondary invariant $\mathcal{G}_1$ and the Υ^2-invariant [8].

Organization

In Sect. 2, we establish the category of formal knot complexes, and construct the monoid $\mathcal{K}^f$ and the abelian group $\mathcal{C}^f$. Theorem 1.7 is also proved in this section. In Sect. 3, we prove Theorem 1.5. In Sect. 4, we discuss algebraic estimates for ν^+-classes, and prove Theorems 1.2 and 1.6. In Sect. 5, we introduce the invariants $\{\mathcal{G}_k\}$, and prove Theorem 1.8, Corollary 1.9 and Theorem 1.10.

2 Category of formal knot complexes

In this section, we establish the category of formal knot complexes.

2.1 Poset filtered chain complexes

Let P be a *poset*, i.e. a set P with partial order $\le$. For example, we often consider the partial order $\le$ on $\mathbb{Z}^2$ given by $(i, j) \le (k, l)$ if $i \le k$, $j \le l$. For a given poset P, a *closed region* $R \subset P$ is a subset such that for any $x \in P$, if there exists an element $y \in R$ satisfying $x \le y$, then $x \in R$. We denote the set of closed regions of P by $\mathcal{CR}(P)$.

Let $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ and $\mathscr{R}$ be an $\mathbb{F}$-algebra. In this paper, we say that (C, ∂) is a *chain complex C over $\mathscr{R}$* if (C, ∂) satisfies the following:

- C is an $\mathscr{R}$-module and $\partial\colon C \to C$ is an $\mathscr{R}$-linear map with $\partial \circ \partial = 0$.

- As an $\mathbb{F}$-vector space, C is decomposed into $\bigoplus_{n\in\mathbb{Z}} C_n$ and satisfies $\partial(C_n) \subset C_{n-1}$.

(Remark that the $\mathscr{R}$-action does not preserve the grading in general. We often abbreviate (C, ∂) to C.) Then, we say that C is *P-filtered* if a subcomplex C_R of C over $\mathbb{F}$ is associated to each closed region $R \subset P$ so that if $R \subset R'$ then $C_R \subset C_{R'}$. (Here we remark that C_R is not an $\mathscr{R}$-submodule of C in general.) We call the set $\{C_R\}_{R\in\mathcal{CR}(P)}$ a *P-filtration* on C. For instance, a $\mathbb{Z}$-filtration $\{C_{\{i\leq m\}}\}_{\{i\leq m\}\in\mathcal{CR}(\mathbb{Z})}$ is identified with an increasing sequence

$$0 \subset \cdots \subset \mathcal{F}_m \subset \mathcal{F}_{m+1} \subset \cdots \subset C$$

of subcomplexes by $\mathcal{F}_m = C_{\{i\leq m\}}$. Moreover, For two $\mathbb{Z}$-filtrations $\{\mathcal{F}^1_i\}_{i\in\mathbb{Z}}$ and $\{\mathcal{F}^2_j\}_{j\in\mathbb{Z}}$ on C, the set

$$\{C_R\}_{R\in\mathcal{CR}(\mathbb{Z}^2)} := \left\{ \sum_{(i,j)\in R} \mathcal{F}^1_i \cap \mathcal{F}^2_j \right\}_{R\in\mathcal{CR}(\mathbb{Z}^2)}$$

defines a $\mathbb{Z}^2$-filtration on C. We call it *the $\mathbb{Z}^2$-filtration induced by the ordered pair* $(\{\mathcal{F}^1_i\}_{i\in\mathbb{Z}}, \{\mathcal{F}^2_j\}_{j\in\mathbb{Z}})$. For a complex C with an induced $\mathbb{Z}^2$-filtration $(\{\mathcal{F}^1_i\}, \{\mathcal{F}^2_j\})$, C^r denotes C with the induced $\mathbb{Z}^2$-filtration $(\{\mathcal{F}^2_i\}, \{\mathcal{F}^1_j\})$.

For any two P-filtered chain complexes C and C', a map $f : C \to C'$ is *P-filtered* if $f(C_R) \subset C'_R$ for any closed region R. Two P-filtered chain complexes C and C' are *P-filtered homotopy equivalent* (and denoted $C \simeq C'$) if there exists a chain homotopy equivalence map $f : C \to C'$ over $\mathscr{R}$ such that the map, its inverse and all chain homotopies are P-filtered and graded. (Then f is called a *P-filtered homotopy equivalence map*. Particularly, we call the above f a *P-filtered isomorphism* if f is a chain isomorphism.) The following lemma immediately follows from the definition of P-filtered homotopy equivalence.

Proposition 2.1 *Let C and C' be P-filtered chain complexes. If $C \simeq C'$, then for any closed regions $R \subset R'$, we have an isomorphism between the long exact sequences of $\mathscr{R}$-modules:*

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial_*} & H_*(C_R) & \xrightarrow{i_*} & H_*(C_{R'}) & \xrightarrow{p_*} & H_*(C_{R'}/C_R) & \xrightarrow{\partial_*} \cdots \\
& \cong\downarrow & & \cong\downarrow & & \cong\downarrow & \\
\cdots \xrightarrow{\partial_*} & H_*(C'_R) & \xrightarrow{i_*} & H_*(C'_{R'}) & \xrightarrow{p_*} & H_*(C'_{R'}/C'_R) & \xrightarrow{\partial_*} \cdots
\end{array}$$

Here, $i\colon C_R \to C_{R'}$ (resp. $p\colon C_{R'} \to C_{R'}/C_R$) denote the inclusion (resp. the projection). Moreover, the above isomorphism induces an isomorphism between the long exact sequences of graded $\mathbb{F}$-vector spaces:

$$
\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial_{*,n+1}} & H_n(C_R) & \xrightarrow{i_{*,n}} & H_n(C_{R'}) & \xrightarrow{p_{*,n}} & H_n(C_{R'}/C_R) & \xrightarrow{\partial_{*,n}} \cdots \\
& \cong\downarrow & & \cong\downarrow & & \cong\downarrow & \\
\cdots \xrightarrow{\partial_{*,n+1}} & H_n(C'_R) & \xrightarrow{i_{*,n}} & H_n(C'_{R'}) & \xrightarrow{p_{*,n}} & H_n(C'_{R'}/C'_R) & \xrightarrow{\partial_{*,n}} \cdots
\end{array}
$$

2.2 Formal knot complexes

Now we state the precise definition of formal knot complex, and discuss several basic properties of it.

2.2.1 Definition

Let $\Lambda := \mathbb{F}[U, U^{-1}]$. We call a tuple

$$(C, \partial, \{C_n\}_{n\in\mathbb{Z}}, \{\mathcal{F}_j^{\mathrm{Alex}}\}_{j\in\mathbb{Z}}, \{\mathcal{F}_i^{\mathrm{Alg}}\}_{i\in\mathbb{Z}})$$

a *formal knot complex* if it satisfies the following seven conditions;

1. (C, ∂) is a chain complex over Λ with decomposition $C = \bigoplus_{n\in\mathbb{Z}} C_n$. The grading of a homogeneous element x is denoted $\mathrm{gr}(x)$ and called the *Maslov grading* of x.
2. $\{\mathcal{F}_j^{\mathrm{Alex}}\}_{j\in\mathbb{Z}}$ is a $\mathbb{Z}$-filtration on C. This filtration is called *Alexander filtration*, and the filtration level of an element $x \in C$ is denoted $\mathrm{Alex}(x)$ (i.e. $\mathrm{Alex}(x) := \min\{j \mid x \in \mathcal{F}_j^{\mathrm{Alex}}\}$).
3. Similarly, $\{\mathcal{F}_i^{\mathrm{Alg}}\}_{i\in\mathbb{Z}}$ is a $\mathbb{Z}$-filtration on C, called the *algebraic filtration*, and filtration levels of elements are denoted $\mathrm{Alg}(x)$. When we regard C as a $\mathbb{Z}^2$-filtered complex, we use the $\mathbb{Z}^2$-filtration induced by the ordered pair $(\{\mathcal{F}_i^{\mathrm{Alg}}\}_{i\in\mathbb{Z}}, \{\mathcal{F}_j^{\mathrm{Alex}}\}_{j\in\mathbb{Z}})$.
4. The action of U lowers Maslov grading by 2 and Alexander and algabraic filtration levels by 1.
5. As a Λ-module, C is freely and finitely generated by elements $\{x_k\}_{1\leq k\leq r}$ such that
   - each x_k is homogeneous with respect to the Maslov grading,
   - $\{U^{\mathrm{Alex}(x_k)} x_k\}_{1\leq k\leq r}$ is a free basis for $\mathcal{F}_0^{\mathrm{Alex}}$ as an $\mathbb{F}[U]$-module, and
   - $\{U^{\mathrm{Alg}(x_k)} x_k\}_{1\leq k\leq r}$ is a free basis for $\mathcal{F}_0^{\mathrm{Alg}}$ as an $\mathbb{F}[U]$-module.

   We call such $\{x_k\}_{1\leq k\leq r}$ a *filtered basis*.
6. There exists a $\mathbb{Z}^2$-filtered homotopy equivalence map $\iota : C \to C^r$.
7. Regard Λ as a chain complex with trivial boundary map, and define the Maslov grading by

$$\Lambda_n = \begin{cases} \{0, U^{-n/2}\} & (n: \text{ even}) \\ 0 & (n: \text{ odd}) \end{cases}$$

   and the Alexander and algebraic filtrations by

$$\mathcal{F}_i^{\mathrm{Alex}}(\Lambda) = \mathcal{F}_i^{\mathrm{Alg}}(\Lambda) = U^{-i} \cdot \mathbb{F}[U].$$

Then there exists a $\mathbb{Z}$-filtered homotopy equivalence map f_{Alex} (resp. f_{Alg}) $: C \to \Lambda$ over Λ with respect to the Alexander (resp. algebraic) filtration.

We often abbreviate the tuple

$$(C, \partial, \{C_n\}, \{\mathcal{F}_j^{Alex}\}, \{\mathcal{F}_i^{\text{Alg}}\})$$

to C or (C, ∂).

Remark Note that $\left\{U^{\text{Alex}(x_k)-j} x_k\right\}_{1\le k\le r}$ (resp. $\left\{U^{\text{Alg}(x_k)-i} x_k\right\}_{1\le k\le r}$) is a free basis for $\mathcal{F}_j^{\text{Alex}}$ (resp. $\mathcal{F}_i^{\text{Alg}}$) as an $\mathbb{F}[U]$-module. In particular, the equalities

$$U^k(\mathcal{F}_j^{\text{Alex}}) = \mathcal{F}_{j-k}^{\text{Alex}} \text{ and } U^k(\mathcal{F}_i^{\text{Alg}}) = \mathcal{F}_{i-k}^{\text{Alg}}$$

hold for any $i, j, k \in \mathbb{Z}$. (These facts also imply that for any element $x \in C$, both $\text{Alex}(x)$ and $\text{Alg}(x)$ are finite.) Similarly, $\left\{U^{\frac{\text{gr}(x_k)-n}{2}} x_k\right\}_{k\in[n]}$ is a basis for C_n as an $\mathbb{F}$-vector space, where $[n]$ is a subset of $\{1, \dots r\}$ consisting of elements with $\text{gr}(x_k) \equiv n$ (mod 2), and the equality $U^k(C_n) = C_{n-2k}$ holds.

As the simplest example, the tuple

$$(\Lambda, \text{zero map}, \{\Lambda_n\}_{n\in\mathbb{Z}}, \{\mathcal{F}_j^{\text{Alex}}(\Lambda)\}_{j\in\mathbb{Z}}, \{\mathcal{F}_i^{\text{Alg}}(\Lambda)\}_{i\in\mathbb{Z}})$$

is a formal knot complex. In addition, it is easy to see that the following lemmas hold.

Lemma 2.2 *For any formal knot complex C, the complex C^r is also a formal knot complex.*

Lemma 2.3 *Let $(\bar{C}, \bar{\partial})$ be a chain complex over $\mathbb{F}$ generated by a finite basis $\{x_k\}_{1\le k\le r}$ with functions*

$$\text{Alex}\colon \{x_k\}_{1\le k\le r} \to \mathbb{Z} \textit{ and } \text{Alg}\colon \{x_k\}_{1\le k\le r} \to \mathbb{Z}$$

satisfying the following:

- *The sequences*

$$\bar{\mathcal{F}}_j^{\text{Alex}} := \text{span}_{\mathbb{F}}\{x_k \mid \text{Alex}(x_k) \le j\} \textit{ and } \bar{\mathcal{F}}_i^{\text{Alg}} := \text{span}_{\mathbb{F}}\{x_k \mid \text{Alg}(x_k) \le i\}$$

  define $\mathbb{Z}$-filtrations on $\bar{C}$, respectively.
- *For the induced $\mathbb{Z}^2$-filtration $(\{\bar{\mathcal{F}}_i^{\text{Alg}}\}, \{\bar{\mathcal{F}}_j^{\text{Alex}}\})$ on $\bar{C}$, we have a $\mathbb{Z}^2$-filtered homotopy equivalence $\bar{C} \simeq \bar{C}^r$.*
- *Regard $\mathbb{F}$ as a chain complex over $\mathbb{F}$ with trivial boundary map and grading $\mathbb{F} = \mathbb{F}_0$, and define a $\mathbb{Z}$-filtration by $\bar{\mathcal{F}}_i(\mathbb{F}) = \mathbb{F}$ if and only if $i \ge 0$. Then we have $\mathbb{Z}$-filtered homotopy equivalences $\bar{C} \simeq \mathbb{F}$ with respect to both $\{\bar{\mathcal{F}}_j^{\text{Alex}}\}$ and $\{\bar{\mathcal{F}}_i^{\text{Alg}}\}$.*

If we set

- $C := \bar{C} \otimes_{\mathbb{F}} \Lambda$ *and* $\partial := \bar{\partial} \otimes 1$,
- $C_n := \bigoplus_{m\in\mathbb{Z}}(\bar{C}_{n+2m} \otimes_{\mathbb{F}} U^m)$, *and*
- $\mathcal{F}_j^{\text{Alex}} := \sum_{m\in\mathbb{Z}}(\bar{\mathcal{F}}_{j+m}^{\text{Alex}} \otimes_{\mathbb{F}} U^m\,\mathbb{F}[U])$ *and* $\mathcal{F}_i^{\text{Alg}} := \sum_{m\in\mathbb{Z}}(\bar{\mathcal{F}}_{i+m}^{\text{Alg}} \otimes_{\mathbb{F}} U^m\,\mathbb{F}[U])$,

then the tuple

$$(C, \partial, \{C_n\}_{n\in\mathbb{Z}}, \{\mathcal{F}_j^{\text{Alex}}\}_{j\in\mathbb{Z}}, \{\mathcal{F}_i^{\text{Alg}}\}_{i\in\mathbb{Z}})$$

is a formal knot complex.

In [15], Ozsváth and Szabó associate the $\mathbb{Z}^2$-filtered homotopy type of a formal knot complex $CFK^\infty(K)$ to any knot K, and prove that it is an isotopy invariant. To simplify notation, we write C^K for $CFK^\infty(K)$.

Theorem 2.4 ([15]) *If two knots K and J are isotopic, then $C^K \simeq C^J$.*

Moreover, it is proved that the inverse has the same homotopy type as the original one.

Theorem 2.5 ([15]) *For a knot K, we have $C^{-K} \simeq (C^K)^r \simeq C^K$.*

2.2.2 Relationship to abstract infinity complex

Here, we compare formal knot complex with Hedden-Watson's *abstract infinity complex*. First, a *graded, bifiltered complex* is a chain complex over $\mathbb{F}$ which admits a basis $\mathcal{B}$ with functions:

$$m\colon \mathcal{B} \to \mathbb{Z} \text{ and } \mathcal{F}\colon \mathcal{B} \to \mathbb{Z}^2$$

such that for any $a, b \in \mathcal{B}$, if the coefficient of a in ∂b is non-zero, then

$$m(a) = m(b) - 1 \text{ and } \mathcal{F}(a) \leq \mathcal{F}(b).$$

In other words, $C_n := \text{span}_{\mathbb{F}}\{a \in \mathcal{B} \mid m(a) = n\}$ ($n \in \mathbb{Z}$) defines a grading and $C_R := \text{span}_{\mathbb{F}}\{a \in \mathcal{B} \mid \mathcal{F}(a) \in R\}$ ($R \in \mathcal{CR}(\mathbb{Z}^2)$) defines a $\mathbb{Z}^2$-filtration.

Definition ([3, Definition 6.1]) An *abstract infinity complex* is a graded, bifiltered complex $(C, \partial, \mathcal{F})$ satisfying

1. (C, ∂) is freely generated as a chain complex over Λ by a finite set of graded, bifiltered homogeneous generators.
2. Acting by U shifts the grading by -2 and the bifiltration by $(-1, -1)$.
3. $H_*(C, \partial) \cong \Lambda$, where $1 \in \Lambda$ has grading 0.
4. The complex $(C, \partial, \mathcal{F}^r)$, where $\mathcal{F}^r$ is the bifiltration function $\mathcal{F}^r(i, j) := \mathcal{F}(j, i)$, is $\mathbb{Z}^2$-filtered homotopy equivalent to $(C, \partial, \mathcal{F})$.

Proposition 2.6 *Any formal knot complex is an abstract infinity complex.*

Proof For a given formal knot complex C, we take a filtered basis $\{x_k\}_{1\le k\le r}$ and set

- $\mathcal{B} := \{U^l x_k \mid_{l\in\mathbb{Z}}^{1\le k\le r}\}$,
- $m : \mathcal{B} \to \mathbb{Z} : U^l x_k \mapsto \mathrm{gr}(U^l x_k)$, and
- $\mathcal{F} : \mathcal{B} \to \mathbb{Z}\times\mathbb{Z} : U^l x_k \mapsto (\mathrm{Alg}(U^l x_k), \mathrm{Alex}(U^l x_k))$.

Then $(C, \partial, \mathcal{F})$ satisfies the all conditions for being an abstract infinity complex. □

On the other hand, in general, an abstract infinity complex does not satisfy the condition (7) in the definition of formal knot complex. For instance, Λ with grading shifted by $2n$ is an abstract infinity complex, but it is not $\mathbb{Z}$-filtered homotopy equivalent to the original Λ with respect to either Alexander or algebraic filtration.

2.2.3 Basic properties

Here, we discuss several basic properties of formal knot complexes. We first consider a change of filtered basis.

Lemma 2.7 *Let C be a formal knot complex and $\{x_k\}_{1\le k\le r}$ a filtered basis for C.*

1. *For any $l \in \mathbb{Z}$ and $a \in \{1,\dots,r\}$, the set $\{x_k\}_{k\ne a}^{1\le k\le r} \cup \{U^l x_a\}$ is also a filtered basis for C.*
2. *For $a, b \in \{1,\dots,r\}$ with $a \ne b$, if $\mathrm{gr}(x_a) = \mathrm{gr}(x_b)$, $\mathrm{Alex}(x_a) \ge \mathrm{Alex}(x_b)$ and $\mathrm{Alg}(x_a) \ge \mathrm{Alg}(x_b)$, then the set $\{x_k\}_{k\ne a}^{1\le k\le r} \cup \{x_a + x_b\}$ is also a filtered basis for C. Moreover, $\mathrm{Alex}(x_a + x_b) = \mathrm{Alex}(x_a)$ and $\mathrm{Alg}(x_a + x_b) = \mathrm{Alg}(x_a)$.*

Proof It is obvious that both $\{x_k\}_{k\ne a}^{1\le k\le r} \cup \{U^l x_a\}$ and $\{x_k\}_{k\ne a}^{1\le k\le r} \cup \{x_a + x_b\}$ are free bases for C as a Λ-module. Therefore, the first assertion follows from $U^l x_a \in C_{\mathrm{gr}(x_a)-2l}$, $\mathrm{Alex}(U^l x_a) = \mathrm{Alex}(x_a) - l$ and $\mathrm{Alg}(U^l x_a) = \mathrm{Alg}(x_a) - l$.

We consider the second assertion. Since $x_a + x_b \in C_{\mathrm{gr}(x_a)} = C_{\mathrm{gr}(x_b)}$, the element $x_a + x_b$ is homogeneous. Next, let $j_a := \mathrm{Alex}(x_a)$, and then $x_a + x_b$ lies in $\mathcal{F}^{\mathrm{Alex}}_{j_a}$. Here we claim that $x_a + x_b \notin \mathcal{F}^{\mathrm{Alex}}_{j_a-1}$. Assume that $x_a + x_b \in \mathcal{F}^{\mathrm{Alex}}_{j_a-1}$. Then $U^{j_a-1}(x_a + x_b) = U^{j_a-1}x_a + U^{j_a-1}x_b \in \mathcal{F}^{\mathrm{Alex}}_0$, and we have a linear combination

$$U^{j_a-1}x_a + U^{j_a-1}x_b = \sum_{1\le k\le r} p_k(U) U^{\mathrm{Alex}(x_k)} x_k$$

where $p_k(U) \in \mathbb{F}[U]$. However, the minimal degree of $p_a(U)U^{\mathrm{Alex}(x_a)} = p_a(U)U^{j_a}$ is at least j_a, and hence we have $U^{j_a-1} \ne p_a(U)U^{j_a}$. This contradicts the fact that $\{x_k\}_{1\le k\le r}$ is a free basis for C as a Λ-module. Therefore, we have $x_a + x_b \notin \mathcal{F}^{\mathrm{Alex}}_{j_a-1}$ and $\mathrm{Alex}(x_a + x_b) = j_a$. Now, it is easy to check that $\{U^{\mathrm{Alex}(x_k)}x_k\}_{k\ne a}^{1\le k\le r} \cup \{U^{\mathrm{Alex}(x_a+x_b)}(x_a + x_b)\}$ is a free basis for $\mathcal{F}^{\mathrm{Alex}}_0$ as an $\mathbb{F}[U]$-module. Similarly, we can check that $\{U^{\mathrm{Alg}(x_k)}x_k\}_{k\ne a}^{1\le k\le r} \cup \{U^{\mathrm{Alg}(x_a+x_b)}(x_a + x_b)\}$ is a free basis for $\mathcal{F}^{\mathrm{Alg}}_0$ as an $\mathbb{F}[U]$-module. □

Next we consider the rank of formal knot complexes.

Lemma 2.8 *For any formal knot complex C, the rank of C as a Λ-module is odd.*

Proof Since there exists a chain homotopy equivalence map from C to Λ such that the map, its inverse and all chain homotopies are graded and filtered with respect to the Maslov grading and the algebraic filtration, we have $H_*(\mathcal{F}_0^{\text{Alg}}/\mathcal{F}_{-1}^{\text{Alg}}) = H_0(\mathcal{F}_0^{\text{Alg}}/\mathcal{F}_{-1}^{\text{Alg}}) \cong \mathbb{F}$. In particular, the Euler characteristic of $\mathcal{F}_0^{\text{Alg}}/\mathcal{F}_{-1}^{\text{Alg}}$ is 1. Here, as an $\mathbb{F}$-vector space, $\{U^{\text{Alg}(x_k)}x_k\}_{1\leq k\leq r}$ is a basis for $\mathcal{F}_0^{\text{Alg}}/\mathcal{F}_{-1}^{\text{Alg}}$, and hence k is odd. This completes the proof. □

Finally, by using a fixed filtered basis $\{x_k\}_{1\leq k\leq r}$, we consider a decomposition $C = \bigoplus_{(i,j)\in\mathbb{Z}^2} C_{(i,j)}$ as an $\mathbb{F}$-vector space, where $C_{(i,j)}$ is defined by

$$C_{(i,j)} := \text{span}_{\mathbb{F}} \left\{ U^l x_k \;\middle|\; (\text{Alg}(U^l x_k), \text{Alex}(U^l x_k)) = (i, j) \right\}.$$

We call it *the decomposition of C induced by* $\{x_k\}_{1\leq k\leq r}$.

Lemma 2.9 *For any $R \in \mathcal{CR}(\mathbb{Z}^2)$, the equality*

$$C_R = \bigoplus_{(i,j)\in R} C_{(i,j)}$$

holds.

Proof By the definitions of C_R and filtered basis, we see that

$$C_R = \sum_{(i,j)\in R} (\mathcal{F}_i^{\text{Alg}} \cap \mathcal{F}_j^{\text{Alex}})$$

and

$$\mathcal{F}_i^{\text{Alg}} \cap \mathcal{F}_j^{\text{Alex}} = \text{span}_{\mathbb{F}[U]} \left\{ U^{\max\{\text{Alg}(x_k)-i, \text{Alex}(x_k)-j\}} x_k \right\}_{1\leq k\leq r}.$$

Therefore, if $(i, j) \in R$ and $U^l x_k \in C_{(i,j)}$, then

$$l = \text{Alg}(x_k) - i = \text{Alex}(x_k) - j = \max\{\text{Alg}(x_k) - i, \text{Alex}(x_k) - j\},$$

and hence $U^l x_k \in \mathcal{F}_i^{\text{Alg}} \cap \mathcal{F}_j^{\text{Alex}} \subset C_R$. This implies $C_R \supset \bigoplus_{(i,j)\in R} C_{(i,j)}$.

Conversely, if $(i, j) \in R$ and $l \geq \max\{\text{Alg}(x_k) - i, \text{Alex}(x_k) - j\}$, then

$$U^l x_k \in C_{(\text{Alg}(x_k)-l, \text{Alex}(x_k)-l)}$$

and

$$(\text{Alg}(x_k) - l, \text{Alex}(x_k) - l) \leq (i, j).$$

This implies $U^l x_k \in \bigoplus_{(i,j)\in R} C_{(i,j)}$, and hence $C_R \subset \bigoplus_{(i,j)\in R} C_{(i,j)}$. □

As a corollary, we have the following useful lemma.

Lemma 2.10 *For any $R, R' \in \mathcal{CR}(\mathbb{Z}^2)$, we have $C_{R\cup R'} = C_R + C_{R'}$.*

Proof By Lemma 2.9, we see that

$$C_{R\cup R'} = \bigoplus_{(i,j)\in R\cup R'} C_{(i,j)} = (\bigoplus_{(i,j)\in R} C_{(i,j)}) + (\bigoplus_{(i,j)\in R'} C_{(i,j)}) = C_R + C_{R'}.$$

□

2.3 Commutative monoid structure

In this subsection, we check that the tensor product of formal knot complexes is also a formal knot complex.

Let $\mathcal{K}^f$ be the set of the $\mathbb{Z}^2$-filtered homotopy equivalence classes of formal knot complexes.

Proposition 2.11 *For any two formal knot complexes C, C', the tuple*

$$\Big(C \otimes_\Lambda C', \partial \otimes 1 + 1 \otimes \partial, \Big\{ \mathrm{span}_{\mathbb{F}}\, p\Big(\bigcup_{m\in\mathbb{Z}} C_m \times C'_{n-m}\Big)\Big\},$$
$$\Big\{\mathrm{span}_{\mathbb{F}}\, p\left(\mathcal{F}_0^{\mathrm{Alex}} \times \mathcal{F}_j^{\mathrm{Alex}}\right)\Big\}, \Big\{\mathrm{span}_{\mathbb{F}}\, p\left(\mathcal{F}_0^{\mathrm{Alg}} \times \mathcal{F}_i^{\mathrm{Alg}}\right)\Big\}\Big)$$

is a formal knot complex, where $p : \Lambda^{C\times C'} \twoheadrightarrow C \otimes_\Lambda C'$ is the projection. Moreover, the set $\mathcal{K}^f$ with product

$$\mathcal{K}^f \times \mathcal{K}^f \to \mathcal{K}^f : ([C], [C']) \mapsto [C \otimes_\Lambda C']$$

is a commutative monoid.

Remark Note that $p\left(\mathcal{F}_{j_1}^{\mathrm{Alex}} \times \mathcal{F}_{j_2}^{\mathrm{Alex}}\right) = p\left(\mathcal{F}_{j'_1}^{\mathrm{Alex}} \times \mathcal{F}_{j'_2}^{\mathrm{Alex}}\right)$ if $j_1 + j_2 = j'_1 + j'_2$, and hence the definition of the Alexander (resp. algebraic) filtration is symmetric.

Proof The fact that $(C \otimes_\Lambda C', \partial \otimes 1 + 1 \otimes \partial)$ is a chain complex follows from ordinary arguments in homological algebra. Let $\{x_k\}_{1\le k\le r}$ (resp. $\{x'_l\}_{1\le l\le s}$) be a filtered basis for C (resp. C'). Then $\left\{x_k \otimes x'_l \,|_{1\le l\le s}^{1\le k\le r}\right\}$ is a free basis for $C \otimes_\Lambda C'$, and

$$\left\{U^n(x_k \otimes x'_l) \;\middle|\; 1 \le k \le r,\ 1 \le l \le s,\ n \in \mathbb{Z}\right\}$$

is a basis for $C \otimes_\Lambda C'$ as an $\mathbb{F}$-vector space. In particular, the subspace

$$(C \otimes_\Lambda C')_n := \mathrm{span}_{\mathbb{F}}\, p\Big(\bigcup_{m\in\mathbb{Z}} C_m \times C'_{n-m}\Big)$$

is generated by

$$\left\{ U^{\frac{\mathrm{gr}(x_k)+\mathrm{gr}(x'_l)-n}{2}}(x_k \otimes x'_l) \right\}_{(k,l)\in[n]},$$

where $[n]$ is a subset of $\{1,\dots,r\} \times \{1,\dots,s\}$ such that $(k,l) \in [n]$ if and only if $\mathrm{gr}(x_k) + \mathrm{gr}(x'_l) \equiv n \pmod 2$. This implies that $C \otimes_\Lambda C' = \bigoplus_{n\in\mathbb{Z}} (C \otimes_\Lambda C')_n$ as an $\mathbb{F}$-vector space, $\partial((C \otimes_\Lambda C')_n) \subset (C \otimes_\Lambda C')_{n-1}$ and $U((C \otimes_\Lambda C')_n) \subset (C \otimes_\Lambda C')_{n-2}$. Therefore, the first condition and a part of the fourth and fifth conditions hold.

Next, it is obvious that $\left\{ \mathrm{span}_\mathbb{F}\, p\left(\mathcal{F}_0^{\mathrm{Alex}} \times \mathcal{F}_j^{\mathrm{Alex}}\right)\right\}_{j\in\mathbb{Z}}$ gives an increasing sequence of subcomplexes, and we see that $\left\{ U^{\mathrm{Alex}(x_k)+\mathrm{Alex}(x'_l)-j}(x_k \otimes x'_l)\big|_{1\le l\le s}^{1\le k\le r} \right\}$ is a free basis for $\mathrm{span}_\mathbb{F}\, p\left(\mathcal{F}_0^{\mathrm{Alex}} \times \mathcal{F}_j^{\mathrm{Alex}}\right)$ as a $\mathbb{F}[U]$-module. Hence the second condition and a part of the fourth and fifth conditions hold. Similarly, we can verify that the third condition and the remaining part of the fourth and fifth conditions hold.

Next we consider the seventh condition. Here we note that it is easy to check that for the trivial case (i.e. the case of $C = C' = \Lambda$), the seventh condition holds. Indeed, the canonical identification $\Lambda \otimes_\Lambda \Lambda \cong \Lambda$ and its inverse are graded and filtered chain isomorphisms (with respect to both filtrations).

Let f_{Alex} (resp. f'_{Alex}) be a chain homotopy equivalence map from C (resp. C') to Λ satisfying the seventh condition with respect to the Alexander filtration. Then the composition of $f_{\mathrm{Alex}} \otimes f'_{\mathrm{Alex}} : C \otimes_\Lambda C' \to \Lambda \otimes_\Lambda \Lambda$ with the canonical identification $\Lambda \otimes_\Lambda \Lambda \cong \Lambda$ is a chain homotopy equivalence map such that the map, its inverse and all chain homotopies are graded and filtered with respect to the grading $\{(C \otimes_\Lambda C')_n\}_{n\in\mathbb{Z}}$ and the filtration $\left\{ \mathrm{span}_\mathbb{F}\, p\left(\mathcal{F}_0^{\mathrm{Alex}} \times \mathcal{F}_j^{\mathrm{Alex}}\right)\right\}_{j\in\mathbb{Z}}$. Therefore, the seventh condition holds with respect to the Alexander filtration. In the same way, we can also prove the seventh condition with respect to the algebraic filtration, and verify that $C \otimes_\Lambda \Lambda \simeq C$, $C \otimes_\Lambda C' \simeq C' \otimes_\Lambda C$, and if $C \simeq C''$ then $C \otimes_\Lambda C' \simeq C'' \otimes_\Lambda C'$.

Now, to prove the proposition, it suffices to prove the sixth condition, and this follows from taking $\iota \otimes \iota'$, where $\iota : C \to C^r$ (resp. $\iota' : C' \to (C')^r$) is a map satisfying the sixth condition for C (resp. C'). This completes the proof. □

Now, let $\mathcal{K}$ be the monoid of the isotopy class of knots. Then we see that the connected sum formula of CFK^∞ gives a monoid homomorphism $\mathcal{K} \to \mathcal{K}^f$.

Theorem 2.12 ([15, Theorem 7.1]) *The map $\mathcal{K} \to \mathcal{K}^f : [K] \mapsto [C^K]$ is a monoid homomorphism. Equivalently, the equality $[C^{K\#J}] = [C^K \otimes_\Lambda C^J]$ holds.*

2.4 The dual of a formal knot complex

In this subsection, we check that the dual of a formal knot complex is also a formal knot complex.

Let C be a formal knot complex. Since C is freely generated by a filtered basis $\{x_k\}_{1\le k\le r}$ as a Λ-module, the dual $C^* := \mathrm{Hom}_\Lambda(C, \Lambda)$ is freely generated by the dual basis $\{x_k^*\}_{1\le k\le r}$. We use the dual basis to define the Maslov grading and two filtrations on C^*.

Here we note that C^* is an $\mathbb{F}$-vector space and $\{U^l x_k^* \mid l \in \mathbb{Z}, 1 \leq k \leq r\}$ is a basis for C^* as an $\mathbb{F}$-vector space. Hence we can define an $\mathbb{F}$-linear isomorphism $\Phi : C \to C^*$ by $\Phi(U^l x_k) = U^{-l} x_k^*$. (Remark that since C is infinite-dimensional $\mathbb{F}$-vector space, C^* is not isomorphic to $\mathrm{Hom}_{\mathbb{F}}(C, \mathbb{F})$.) We call Φ *the dual isomorphism induced by* $\{x_k\}_{1\leq k\leq r}$.

Next, let $C/\mathcal{F}_j^{\mathrm{Alex}}$ (resp. $C/\mathcal{F}_{-i}^{\mathrm{Alg}}$) denote the subspace of C (as an $\mathbb{F}$-vector space) generated by $\{U^l x_k\}_{1\leq k\leq r}^{l\leq \mathrm{Alex}(x_k)-j-1}$ (resp. $\{U^l x_k\}_{1\leq k\leq r}^{l\leq \mathrm{Alg}(x_k)-i-1}$). Then we have

$$\begin{array}{ll} C = \mathcal{F}_j^{\mathrm{Alex}} \oplus \big(C/\mathcal{F}_j^{\mathrm{Alex}}\big) & \big(\text{resp. } C = \mathcal{F}_i^{\mathrm{Alg}} \oplus \big(C/\mathcal{F}_i^{\mathrm{Alg}}\big)\big), \\ \big(C/\mathcal{F}_{j+1}^{\mathrm{Alex}}\big) \subset \big(C/\mathcal{F}_j^{\mathrm{Alex}}\big) & \big(\text{resp. } \big(C/\mathcal{F}_{i+1}^{\mathrm{Alg}}\big) \subset \big(C/\mathcal{F}_i^{\mathrm{Alg}}\big)\big), \text{ and} \\ U\big(C/\mathcal{F}_j^{\mathrm{Alex}}\big) = \big(C/\mathcal{F}_{j-1}^{\mathrm{Alex}}\big) & \big(\text{resp. } U\big(C/\mathcal{F}_i^{\mathrm{Alg}}\big) = \big(C/\mathcal{F}_{i-1}^{\mathrm{Alg}}\big)\big). \end{array}$$

In particular, we see that $\Phi\big(C/\mathcal{F}_j^{\mathrm{Alex}}\big)$ $\big($resp. $\Phi\big(C/\mathcal{F}_i^{\mathrm{Alg}}\big)\big)$ is a free $\mathbb{F}[U]$-module generated by $\big\{U^{-\mathrm{Alex}(x_k)+j+1} x_k^*\big\}_{1\leq k\leq r}$ (resp. $\big\{U^{-\mathrm{Alg}(x_k)+i+1} x_k^*\big\}_{1\leq k\leq r}$). Now, the formal knot complex structure of C^* is described as follows.

Proposition 2.13 *Let $\partial^* : C^* \to C^*$ denote the dual of the differential ∂ on C. Then, the tuple*

$$\Big(C^*, \partial^*, \{\Phi(C_{-n})\}, \big\{\Phi\big(C/\mathcal{F}_{-j-1}^{\mathrm{Alex}}\big)\big\}, \big\{\Phi\big(C/\mathcal{F}_{-i-1}^{\mathrm{Alg}}\big)\big\}\Big)$$

is a formal knot complex. Moreover, for any formal knot complexes C_1, C_2, if $C_1 \simeq C_2$ then $C_1^* \simeq C_2^*$.

We call the formal knot complex C^* *the dual of* C. Before proving Proposition 2.13, we prove the following lemmas. Here, $\varepsilon : \Lambda \to \mathbb{F}$ is an $\mathbb{F}$-linear map defined by $\varepsilon(p(U)) = p(0)$ for each $p(U) \in \Lambda$ (i.e. ε maps a Laurent polynomial to its constant term).

Lemma 2.14 *We have the equalities*

$$\begin{array}{l} \Phi(C_n) = \Big\{\varphi \in C^* \;\Big|\; \varepsilon \circ \varphi\big(\bigoplus_{m\neq n} C_m\big) = \{0\}\Big\}, \\ \Phi\big(C/\mathcal{F}_j^{\mathrm{Alex}}\big) = \Big\{\varphi \in C^* \;\Big|\; \varepsilon \circ \varphi\big(\mathcal{F}_j^{\mathrm{Alex}}\big) = \{0\}\Big\}, \ \textit{and} \\ \Phi\big(C/\mathcal{F}_i^{\mathrm{Alg}}\big) = \Big\{\varphi \in C^* \;\Big|\; \varepsilon \circ \varphi\big(\mathcal{F}_i^{\mathrm{Alg}}\big) = \{0\}\Big\}. \end{array}$$

In particular, the subspaces $\Phi(C_{-n})$, $\Phi\big(C/\mathcal{F}_{-j-1}^{\mathrm{Alex}}\big)$ and $\Phi\big(C/\mathcal{F}_{-i-1}^{\mathrm{Alg}}\big)$ are independent of Φ. (We often denote them by C_n^*, $\mathcal{F}_j^{\mathrm{Alex}}(C^*)$ and $\mathcal{F}_i^{\mathrm{Alex}}(C^*)$ respectively.)

Proof We first note that $\Phi(C_n)$ is generated by $\Big\{U^{-\frac{\mathrm{gr}(x_k)-n}{2}} x_k^*\Big\}_{k\in[n]}$. Now, Suppose that φ is in $\Phi(C_n)$, and then we have an $\mathbb{F}$-linear combination

$$\varphi = \sum_{k\in[n]} a_k U^{-\frac{\mathrm{gr}(x_k)-n}{2}} x_k^*.$$

Thus, for any element $x = \sum_{1 \le k \le r} p_k(U)x_k \in \bigoplus_{m \ne n} C_m$, we have

$$\varphi(x) = \sum_{k \in [n]} a_k U^{-\frac{\mathrm{gr}(x_k)-n}{2}} p_k(U).$$

Here, since x is in $\bigoplus_{m \ne n} C_m$, the coefficient of $U^{\frac{\mathrm{gr}(x_k)-n}{2}}$ in $p_k(U)$ is zero. This implies that

$$\varepsilon \circ \varphi(x) = \sum_{k \in [n]} a_k \varepsilon \left(U^{-\frac{\mathrm{gr}(x_k)-n}{2}} p_k(U) \right) = 0.$$

Conversely, suppose that $\varphi = \sum_{1 \le k \le r} q_k(U) x_k^* \in C^*$ satisfies $\varepsilon \circ \varphi(\bigoplus_{m \ne -n} C_m) = \{0\}$. Here we note that the coefficient of U^l in $q_k(U)$ is zero if and only if $\varepsilon \circ \varphi(U^{-l} x_k) = 0$. In addition, for any $k \in [n]$, $U^{-l} x_k$ is in $\bigoplus_{k \ne -n} C_k$ if and only if $l \ne -\frac{\mathrm{gr}(x_k)-n}{2}$, and hence we have $q_k(U) = a_k U^{-\frac{\mathrm{gr}(x_k)-n}{2}}$ for some $a_k \in \mathbb{F}$. Otherwise, $U^l x_k \in \bigoplus_{k \ne -n} C_k$ for any l, and hence $q_k(U) = 0$. As a consequence, we have $\varphi = \sum_{k \in [n]} a_k U^{-\frac{\mathrm{gr}(x_k)-n}{2}} x_k^*$. In a similar way, we can also prove the assertions for $\Phi\big(C/\mathcal{F}_j^{\mathrm{Alex}}\big)$ and $\Phi\big(C/\mathcal{F}_i^{\mathrm{Alg}}\big)$. □

Lemma 2.15 *Let C, C' be formal knot complexes and $f : C \to C'$ be a Λ-linear map. Define a map $f^* : C'^* \to C^*$ by $\varphi \mapsto \varphi \circ f$.*

1. *Fix $k \in \mathbb{Z}$. If $f(C_n) \subset C'_{n+k}$ for any n, then $f^*(\Phi(C'_{n+k})) \subset \Phi(C_n)$.*
2. *If $f\big(\mathcal{F}_j^{\mathrm{Alex}}(C)\big) \subset \mathcal{F}_j^{\mathrm{Alex}}(C')$, then $f^*\big(\Phi\big(C'/\mathcal{F}_j^{\mathrm{Alex}}\big)\big) \subset \Phi\big(C/\mathcal{F}_j^{\mathrm{Alex}}\big)$.*
3. *If $f\big(\mathcal{F}_i^{\mathrm{Alg}}(C)\big) \subset \mathcal{F}_i^{\mathrm{Alg}}(C')$, then $f^*\big(\Phi\big(C'/\mathcal{F}_i^{\mathrm{Alg}}\big)\big) \subset \Phi\big(C/\mathcal{F}_i^{\mathrm{Alg}}\big)$.*

Proof Lemma 2.14 implies that for any $\varphi \in \Phi(C'_{n+k})$, the equalities

$$\varepsilon \circ (f^*\varphi) \left(\bigoplus_{m \ne n} C_m \right) = \varepsilon \circ \varphi \left(f \left(\bigoplus_{m \ne n} C_m \right) \right) \subset \varepsilon \circ \varphi \left(\bigoplus_{m \ne n+k} C'_m \right) = \{0\}$$

hold, and hence $f^*\varphi \in \Phi(C_n)$. Similarly, we can prove the second and third assertions in Lemma 2.15. □

Proof of Proposition 2.13 The first, second, third and forth conditions immediately follow from the arguments above Proposition 2.13, the above two lemmas and the equality $U\Phi = \Phi U^{-1}$, and so we next consider the fifth condition. We prove that $\{x_k^*\}_{1 \le k \le r}$ is a filtered basis. First, x_k^* is in $\Phi(C_{-\mathrm{gr}(x_k)})$ and hence it is homogeneous. Next, since

$$U\Phi\big(C/\mathcal{F}_{-j-1}^{\mathrm{Alex}}\big) = \Phi U^{-1}\big(C/\mathcal{F}_{-j-1}^{\mathrm{Alex}}\big) = \Phi\big(C/\mathcal{F}_{-j}^{\mathrm{Alex}}\big) \subset \Phi\big(C/\mathcal{F}_{-j-1}^{\mathrm{Alex}}\big),$$

$\Phi\big(C/\mathcal{F}_{-j-1}^{\mathrm{Alex}}\big)$ is a $\mathbb{F}[U]$-module. In addition, $\{U^{-\mathrm{Alex}(x_k)} x_k^*\}_{1 \le k \le r}$ is a free basis for $\Phi\big(C/\mathcal{F}_{-1}^{\mathrm{Alex}}\big)$ as a $\mathbb{F}[U]$-module, and $x_k^* \in \Phi\big(C/\mathcal{F}_{-j-1}^{\mathrm{Alex}}\big)$ if and only if $j =$

$-\mathrm{Alex}(x_k)$. This implies that $\{x_k^*\}_{1\le k\le r}$ satisfies the fifth condition with respect to the Alexander filtration. In a similar way, we can also prove that $\{x_k^*\}_{1\le k\le r}$ satisfies the condition with respect to the algebraic filtration. Thus, the fifth condition holds.

Next, we consider the seventh condition. Let $f_{\mathrm{Alex}} : C \to \Lambda$ be a chain homotopy equivalence map satisfying the seventh condition with respect to the Alexander filtration, and g_{Alex} the inverse of f_{Alex}. Then the dual $g_{\mathrm{Alex}}^* : C^* \to \Lambda^*$ is a chain homotopy equivalence map over Λ, and Lemma 2.15 implies that the duals of f_{Alex}, g_{Alex} and all chain homotopies are graded with respect to the pair

$$\big(\{\Phi(C_{-n})\}_{n\in\mathbb{Z}}, \{\Phi(\Lambda_{-n})\}_{n\in\mathbb{Z}}\big),$$

and filtered with respect to the pair

$$\Big(\big\{\Phi\big(C/\mathcal{F}^{\mathrm{Alex}}_{-i-1}\big)\big\}_{i\in\mathbb{Z}}, \big\{\Phi\big(\Lambda/\mathcal{F}^{\mathrm{Alex}}_{-i-1}\big)\big\}_{i\in\mathbb{Z}}\Big).$$

Moreover, if we define a Λ-linear map $\Psi : \Lambda \to \Lambda^*$ by $\Psi(1) = 1^*$, then Ψ is a chain isomorphism satisfying

$$\begin{aligned}\Psi(\Lambda_n) &= \begin{cases}\{0, U^{-n/2}\cdot 1^*\} & (n : \text{ even})\\ 0 & (n : \text{ odd})\end{cases}\\ &= \begin{cases}\{0, \Phi(U^{n/2})\} & (n : \text{ even})\\ 0 & (n : \text{ odd})\end{cases}\\ &= \Phi(\Lambda_{-n})\end{aligned}$$

and

$$\begin{aligned}\Psi(\mathcal{F}^{\mathrm{Alex}}_i(\Lambda)) &= \mathrm{span}_{\mathbb{F}}\left\{U^l\cdot 1^* \;\middle|\; l \ge -i\right\}\\ &= \mathrm{span}_{\mathbb{F}}\left\{\Phi(U^l) \;\middle|\; l \le i\right\}\\ &= \Phi\big(C/\mathcal{F}^{\mathrm{Alex}}_{-i-1}\big).\end{aligned}$$

These imply that Ψ and the inverse Ψ^{-1} are graded with respect to the pair

$$\big(\{\Lambda_n\}_{n\in\mathbb{Z}}, \{\Phi(\Lambda_{-n})\}_{n\in\mathbb{Z}}\big),$$

and filtered with respect to the pair

$$\Big(\{\mathcal{F}^{\mathrm{Alex}}_i\}_{i\in\mathbb{Z}}, \big\{\Phi\big(\Lambda/\mathcal{F}^{\mathrm{Alex}}_{-i-1}\big)\big\}_{i\in\mathbb{Z}}\Big).$$

As a consequence, the composition $\Psi^{-1}\circ g^*_{\mathrm{Alex}} : C^* \to \Lambda$ satisfies the seventh condition with respect to the Alexander filtration. In the same way, we can prove the seventh condition with respect to the algebraic filtration. In addition, the sixth condition also follows from similar arguments.

Finally, we consider the last assertion in Proposition 2.13. Suppose that C_1, C_2 are formal knot complexes and $f : C_1 \to C_2$ is a $\mathbb{Z}^2$-filtered homotopy equivalence map. Then Lemma 2.15 implies that the dual $f^* : C_2^* \to C_1^*$ is a $\mathbb{Z}^2$-filtered homotopy equivalence map. This completes the proof. □

For knot complexes, the dual complex corresponds to the mirror. (Note that the knot Floer homology $\widehat{HFK}$ is treated in [15, Proposition 3.7], while the same proof can be applied to CFK^∞.)

Theorem 2.16 ([15, Proposition 3.7]) *For any knot K, the equality $[C^{K^*}] = [(C^K)^*]$ holds.*

In particular, by combining the above theorem with Theorem 2.5, we have

$$[C^{-K^*}] = [(C^K)^*].$$

This fact is important in terms of knot concordance. About dual complexes, we give three more lemmas.

Lemma 2.17 *Let C be a formal knot complex. Then the $\mathbb{F}$-linear map $\varepsilon_n : C^*_{-n} \to \mathrm{Hom}_{\mathbb{F}}(C_n, \mathbb{F})$ defined by $\varphi \mapsto \varepsilon \circ \varphi$ is a cochain isomorphism (where we see $\{C^*_{-n}\}_{n\in\mathbb{Z}}$ as a graded cochain complex over $\mathbb{F}$). In particular, we have $\mathbb{F}$-linear isomorphisms*

$$H_{-n}(C^*) \cong H^n(C_*; \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(H_n(C_*), \mathbb{F}),$$

where the first isomorphism is the isomorphism induced from ε_n.

Proof The equalities $\partial^*(\varepsilon_n\varphi) = \varepsilon \circ \varphi \circ \partial = \varepsilon_{n+1}(\partial^*\varphi)$ show that $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ is a cochain map. We prove that ε_n is an $\mathbb{F}$-linear isomorphism. Let $\{x_k\}_{1\le k\le r}$ be a filtered basis for C and Φ the dual isomorphism induced by $\{x_k\}_{1\le k\le r}$. Then we see that $\left\{\varepsilon \circ (U^{-\frac{\mathrm{gr}(x_k)-n}{2}} x_k^*)\right\}_{k\in[n]}$ coinsides with the dual basis for $\left\{U^{\frac{\mathrm{gr}(x_k)-n}{2}} x_k\right\}_{k\in[n]}$. Here we note that $\left\{U^{-\frac{\mathrm{gr}(x_k)-n}{2}} x_k^*\right\}_{k\in[n]}$ is a basis for C^*_{-n}, and hence ε_n is an isomorphism. □

Lemma 2.18 *For any formal knot complex C, the Λ-linear map $\Xi : C \to C^{**}$ defined by $\Xi(x)(\varphi) = \varphi(x)$ ($x \in C$, $\varphi \in C^*$) is a $\mathbb{Z}^2$-filtered isomorphism. In particular, $C^{**} \simeq C$.*

Proof It is easy to check that Ξ is a chain isomorphism over Λ. Moreover, for a fixed filtered basis $\{x_k\}_{1\le k\le r}$ for C, let $\Phi : C \to C^*$ (resp. $\Phi^* : C^* \to C^{**}$) be the dual isomorphism induced by $\{x_k\}_{1\le k\le r}$ (resp. $\{x_k^*\}_{1\le k\le r}$), and then $\Phi^* \circ \Phi = \Xi$. Hence we have

$$\begin{aligned}
&\Xi(C_n) = \Phi^*(\Phi(C_n)) = \Phi^*(C^*_{-n}) = C^{**}_n,\\
&\Xi(\mathcal{F}^{\mathrm{Alex}}_j(C)) = \Phi^*(\Phi(\mathcal{F}^{\mathrm{Alex}}_j(C))) = \Phi^*(C^*/\mathcal{F}^{\mathrm{Alex}}_{-j-1}) = \mathcal{F}^{\mathrm{Alex}}_j(C^{**}), \text{ and}\\
&\Xi(\mathcal{F}^{\mathrm{Alg}}_i(C)) = \Phi^*(\Phi(\mathcal{F}^{\mathrm{Alg}}_i(C))) = \Phi^*(C^*/\mathcal{F}^{\mathrm{Alg}}_{-i-1}) = \mathcal{F}^{\mathrm{Alg}}_i(C^{**}).
\end{aligned}$$

This completes the proof. □

Lemma 2.19 *Any two formal knot complexes C and C', the Λ-linear map $\Gamma : C^* \otimes C'^* \to (C \otimes C')^*$ defined by $\Gamma(\varphi \otimes \psi)(x \otimes y) = \varphi(x)\psi(y)$ ($\varphi \in C^*, \psi \in C'^*, x \in C, y \in C'$) is a $\mathbb{Z}^2$-filtered isomorphism. In particular, $(C \otimes C')^* \simeq C^* \otimes C'^*$.*

Proof The proof is similar to Lemma 2.18. □

2.5 Stabilizers

Let (A, ∂) be a chain complex over Λ. We call a tuple

$$\left(A, \partial, \{A_n\}_{n\in\mathbb{Z}}, \{\mathcal{F}_j^{\mathrm{Alex}}\}_{j\in\mathbb{Z}}, \{\mathcal{F}_i^{\mathrm{Alg}}\}_{i\in\mathbb{Z}}\right)$$

a *stabilizer* if it satisfies the conditions (1) to (6) in the definition of formal knot complex and the following:

Condition There exists a chain homotopy Φ_{Alex} (resp. Φ_{Alg}) on C connecting the identity and the zero-map which is $\mathbb{Z}$-filtered with respect to the Alexander filtration (resp. the algebraic filtration).

Remark The above condition does not imply $A \simeq 0$. The relation $A \simeq 0$ is corresponding to the existence of chain homotopies Φ_{Alex} and Φ_{Alg} satisfying the above condition and $\Phi_{\mathrm{Alex}} = \Phi_{\mathrm{Alg}}$.

Let C (resp. C') be a chain complex over Λ satisfying the conditions (1) to (6) for being a formal knot complex and $\{x_k\}_{1\le k\le r}$ (resp. $\{x'_l\}_{1\le l\le s}$) a filtered basis for C (resp. C'). Then the tuple

$$\Big(C \oplus C', \partial \oplus \partial', \{C_n \oplus C'_n\}_{n\in\mathbb{Z}},$$
$$\{\mathcal{F}_j^{\mathrm{Alex}}(C) \oplus \mathcal{F}_j^{\mathrm{Alex}}(C')\}_{j\in\mathbb{Z}}, \{\mathcal{F}_i^{\mathrm{Alg}}(C) \oplus \mathcal{F}_i^{\mathrm{Alg}}(C')\}_{i\in\mathbb{Z}}\Big)$$

also satisfies the conditions (1) to (6) for being a formal knot complex, where $\{(x_k, 0)\}_{1\le k\le r} \cup \{(0, x'_l)\}_{1\le l\le s}$ is a filtered basis for the tuple. We abbreviate the tuple to $C \oplus C'$.

Lemma 2.20 *Let A be a chain complex over Λ satisfying the conditions (1) to (6) for being a formal knot complex. Then A is a stabilizer if and only if $H_*(\mathcal{F}_0^{\mathrm{Alex}}) = H_*(\mathcal{F}_0^{\mathrm{Alg}}) = 0$.*

Proof It is obvious that if A is a stabilizer, then $H_*(\mathcal{F}_0^{\mathrm{Alex}}) = H_*(\mathcal{F}_0^{\mathrm{Alg}}) = 0$. We prove the converse. Suppose that the equalities $H_*(\mathcal{F}_0^{\mathrm{Alex}}) = H_*(\mathcal{F}_0^{\mathrm{Alg}}) = 0$ hold. Then, since $U : \mathcal{F}_0^{\mathrm{Alex}} \to \mathcal{F}_{-1}^{\mathrm{Alex}}$ is a chain isomorphism, we have $H_*(\mathcal{F}_{-1}^{\mathrm{Alex}}) = 0$ and $H_*\big(\mathcal{F}_0^{\mathrm{Alex}} / \mathcal{F}_{-1}^{\mathrm{Alex}}\big) = 0$. Let $\{x_k\}_{1\le k\le r}$ be a filtered basis for A. By Lemma 2.7, we may assume that $\mathrm{Alex}(x_k) = 0$ for any k. Then we see $\mathcal{F}_0^{\mathrm{Alex}} / \mathcal{F}_{-1}^{\mathrm{Alex}} = \mathrm{span}_{\mathbb{F}}\{px_k\}_{1\le k\le r}$, where $p : \mathcal{F}_0^{\mathrm{Alex}} \to \mathcal{F}_0^{\mathrm{Alex}} / \mathcal{F}_{-1}^{\mathrm{Alex}}$ is the projection. Moreover, it follows from $H_*\big(\mathcal{F}_0^{\text{Alex}} / \mathcal{F}_{-1}^{\text{Alex}}\big) = 0$ that r is even and there exists a subset $\{k_1, k_2, \dots, k_{r/2}\}$ of $\{1, \dots, r\}$ such that

$$\text{span}_{\mathbb{F}} \left\{px_{k_1}, \dots, px_{k_{r/2}}, \partial(px_{k_1}), \dots, \partial(px_{k_{r/2}})\right\} = \mathcal{F}_0^{\text{Alg}} / \mathcal{F}_{-1}^{\text{Alg}}.$$

This implies that $\text{Alex}(\partial x_{k_i}) = 0$ for any $1 \le i \le r/2$ and

$$\text{span}_{\Lambda} \left\{x_{k_1}, \dots, x_{k_{r/2}}, \partial x_{k_1}, \dots, \partial x_{k_{r/2}}\right\} = A.$$

Now, define a Λ-linear map $\Phi_{\text{Alex}} : A \to A$ by $x_{k_i} \mapsto 0$ and $\partial x_{k_i} \mapsto x_{k_i}$. Then, it is not hard to check that $\Phi(C_n) \subset C_{n+1}$, $\Phi(\mathcal{F}_i^{\text{Alex}}) \subset \mathcal{F}_i^{\text{Alex}}$, and $\Phi \circ \partial + \partial \circ \Phi$ is equal to the identity on A. This proves the condition for being a stabilizer with respect to the Alexander filtration. In the same way, we can prove the condition for being a stabilizer with respect to the algebraic filtration. □

In addition, we can easily check that the following lemmas hold.

Lemma 2.21 *For two stabilizers A and A', the direct sum $A \oplus A'$ is also a stabilizer. Moreover, for a formal knot complex C, the direct sum $C \oplus A$ is also a formal knot complex.*

Lemma 2.22 *For two stabilizers A and A', and a formal knot complex C, the tensor products $A \otimes_\Lambda A'$ and $C \otimes_\Lambda A$ are also stabilizers.*

Lemma 2.23 *For a stabilizer A, the dual A^* is also a stabilizer.*

2.6 ν^+-invariant

For any formal knot complex C, we have

$$H_n(C) \cong H_n(\Lambda) \cong \begin{cases} \mathbb{F} \ (n : \text{even}) \\ 0 \ \ (\text{otherwise}) \end{cases}.$$

In particular, $H_0(C) \cong \mathbb{F}$. A cycle $x \in C$ is called a *homological generator* if x is homogeneous with $\text{gr}(x) = 0$ and the homology class $[x] \in H_0(C)$ is non-zero. We define the ν^+-*invariant* of C by

$$\nu^+(C) := \min\left\{m \in \mathbb{Z}_{\ge 0} \;\middle|\; C_{\{i \le 0,\, j \le m\}} \text{ contains a homological generator}\right\}.$$

Remark The above definition of ν^+ is originally that of ν^-. However, these invariants are the same, and hence we may define ν^+ as above.

Note that the equality

$$\nu^+(C) = \min\left\{m \in \mathbb{Z}_{\ge 0} \;\middle|\; i_{*,0} : H_0(C_{\{i \le 0,\, j \le m\}}) \to H_0(C) \text{ is surjective}\right\}$$

holds, and hence the value $\nu^+(C)$ is invariant under $\mathbb{Z}^2$-filtered homotopy equivalence.

Proposition 2.24 $\nu^+(C \otimes_\Lambda C') \leq \nu^+(C) + \nu^+(C')$.

Proof Note that $C_{\{i\leq 0,\, j\leq m\}} = \mathcal{F}_0^{\mathrm{Alg}} \cap \mathcal{F}_m^{\mathrm{Alex}}$, and hence there exists a homological generator $x \in C$ (resp. $x' \in C'$) lying in $\mathcal{F}_0^{\mathrm{Alg}} \cap \mathcal{F}_{\nu^+(C)}^{\mathrm{Alex}}$ (resp. $\mathcal{F}_0^{\mathrm{Alg}} \cap \mathcal{F}_{\nu^+(C')}^{\mathrm{Alex}}$). This implies that $x \otimes x' \in C \otimes_\Lambda C'$ is lying in

$$p\big(\mathcal{F}_0^{\mathrm{Alg}} \times \mathcal{F}_0^{\mathrm{Alg}}\big) \cap p\big(\mathcal{F}_{\nu^+(C)}^{\mathrm{Alex}} \times \mathcal{F}_{\nu^+(C')}^{\mathrm{Alex}}\big) \subset \mathcal{F}_0^{\mathrm{Alg}} \cap \mathcal{F}_{\nu^+(C)+\nu^+(C')}^{\mathrm{Alex}} \\ = (C \otimes_\Lambda C')_{\{i\leq 0,\, j\leq \nu^+(C)+\nu^+(C')\}}.$$

Moreover, it is easily seen that $x \otimes x'$ is a homogeneous cycle with $\mathrm{gr}(x \otimes x') = 0$ (and so $[x \otimes x'] \in H_0(C \otimes_\Lambda C')$), and the Künneth formula $H_*(C) \otimes_\Lambda H_*(C') \hookrightarrow H_*(C \otimes_\Lambda C')$ implies that $[x \otimes x']$ is non-zero. Therefore, $x \otimes x'$ is a homological generator, and this completes the proof. □

It is easy to see that the value of ν^+ is unchanged under stabilization.

Lemma 2.25 *For any formal knot complex C and stabilizer A, we have $\nu^+(C \oplus A) = \nu^+(C)$.*

Moreover, ν^+ also has the following property.

Lemma 2.26 *For any formal knot complex C, we have*

$$\nu^+(C \otimes_\Lambda C^*) = 0.$$

Proof Let $\{x_k\}_{1\leq k\leq r}$ be a filtered basis for C. Then, the element $x = \sum_{1\leq k\leq r} x_k \otimes x_k^*$ is lying in $(C \otimes_\Lambda C^*)_{\{i\leq 0,\, j\leq 0\}}$ and homogeneous with $\mathrm{gr}(x) = 0$. We prove that this x is a homological generator.

Let $(a_{lk})_{1\leq l,k\leq r}$ be the matrix of $\partial : C \to C$ with respect to $\{x_k\}_{1\leq k\leq r}$, i.e. $\partial x_k = \sum_{1\leq l\leq r} a_{lk} x_l$. Then its transpose $(a_{kl})_{1\leq l,k\leq r}$ is the matrix of $\partial^* : C^* \to C^*$ with respect to $\{x_k^*\}_{1\leq k\leq r}$, and we have

$$(\partial \otimes 1 + 1 \otimes \partial^*)(x_k \otimes x_k^*) = \sum_{1\leq l\leq r} a_{lk} x_l \otimes x_k^* + \sum_{1\leq l\leq r} a_{kl} x_k \otimes x_l^*.$$

This implies that

$$\big(\text{the coefficient of } x_l \otimes x_k^* \text{ in } (\partial \otimes 1 + 1 \otimes \partial^*)(x)\big) = 2a_{lk} = 0$$

for any $1 \leq l, k \leq r$. Hence x is a cycle.

Next, we prove that the homology class of x is non-zero. It is obvious that $\sum_{1\leq k\leq r} x_k^* \otimes x_k \in C^* \otimes_\Lambda C$ is also a cycle. Here, by using the chain isomorphisms Ξ and Γ in Lemmas 2.18 and 2.19, we can identify $C^* \otimes C$ with $(C \otimes C^*)^*$ by $(\varphi \otimes y)(z \otimes \psi) = \varphi(z)\psi(y)$ $(y, z \in C, \varphi, \psi \in C^*)$. (In other words, $\sum_{1\leq k\leq r} x_k^* \otimes x_k$ can be seen as a cocycle.) Now, it follows from Lemma 2.8 that r is odd, and hence we have

$$\left(\sum_{1\le k\le r} x_k^* \otimes x_k\right)(x) = \sum_{1\le k\le r} \left(x_k^*(x_k)\right)^2 = r = 1 \in \Lambda.$$

This implies that the homology class of x is non-zero. □

The following proposition is originally proved by Hom [4] in the case of knot complexes.

Proposition 2.27 ([4, Proposition 3.11]) *For a formal knot complex C, the equalities $\nu^+(C) = \nu^+(C^*) = 0$ holds if and only if we have the $\mathbb{Z}^2$-filtered homotopy equivalence*

$$C \simeq \Lambda \oplus A,$$

where A is a stabilizer.

The proof in [4] is naturally generalized to the case of formal knot complexes. To prove Proposition 2.27, we use the following lemma.

Lemma 2.28 *The inequality $\nu^+(C^*) \le m$ holds if and only if the projection $p_{*,0}$: $H_0(C) \to H_0(C/C_{\{i\le -1 \text{ or } j\le -m-1\}})$ is injective.*

Proof Let $\{x_k\}_{1\le k\le r}$ be a filtered basis for C and Φ denote the dual isomorphism induced by $\{x_k\}_{1\le k\le r}$. We first assume that $\nu^+(C^*) \le m$. Then there exists a homological generator $\varphi \in C_0^*$ lying in

$$\begin{aligned}\mathcal{F}_0^{\text{Alg}}(C^*) \cap \mathcal{F}_m^{\text{Alex}}(C^*) &= \Phi\big(C/\mathcal{F}_{-1}^{\text{Alg}}\big) \cap \Phi\big(C/\mathcal{F}_{-m-1}^{\text{Alex}}\big) \\ &= \operatorname{span}_{\mathbb{F}[\mathrm{U}]} \left\{U^{\max\{-\operatorname{Alg}(x_k), -\operatorname{Alex}(x_k)-m\}} x_k^* \;\middle|\; 1 \le k \le r\right\}.\end{aligned}$$

In particular, we have $\varepsilon \circ \varphi(C_{\{i\le -1 \text{ or } j\le -m-1\}}) = \varepsilon \circ \varphi(\mathcal{F}_{-1}^{\text{Alg}} + \mathcal{F}_{-m-1}^{\text{Alex}}) = 0$, and $\varepsilon \circ \varphi$ is decomposed as $\varepsilon \circ \varphi = \widetilde{\varphi} \circ p$ where $\widetilde{\varphi} \in \operatorname{Hom}_{\mathbb{F}}(C/C_{\{i\le -1 \text{ or } j\le -m-1\}}, \mathbb{F})$ is a cocycle and $p : C \to C/C_{\{i\le -1 \text{ or } j\le -m-1\}}$ is the projection. Now, let $x \in C_0$ be a homological generator. Then we have $\widetilde{\varphi}(p(x)) = (\varepsilon \circ \varphi)(x) = 1$. This implies that the homology class $[p(x)] \in H_0(C/C_{\{i\le -1 \text{ or } j\le -m-1\}})$ is non-zero, and hence $p_{*,0}$ is injective.

Conversely, suppose that $p_{*,0}$ is injective. Let $x \in C_0$ be a homological generator, and then we have $p_{*,0}([x]) \ne 0$. In addition, $\dim_{\mathbb{F}}(C/C_{\{i\le -1 \text{ or } j\le -m-1\}})_0$ is finite, and hence we can take a finite $\mathbb{F}$-basis for $H_0(C/C_{\{i\le -1 \text{ or } j\le -m-1\}})$ containing $p_{*,0}([x])$. Thus, by using the identification

$$\operatorname{Hom}_{\mathbb{F}}\left(H_0(C/C_{\{i\le -1 \text{ or } j\le -m-1\}}), \mathbb{F}\right) \cong H^0(C/C_{\{i\le -1 \text{ or } j\le -m-1\}}; \mathbb{F}),$$

we can take a cocycle $\psi \in \operatorname{Hom}_{\mathbb{F}}((C/C_{\{i\le -1 \text{ or } j\le -m-1\}})_0, \mathbb{F})$ whose cohomology class is the dual $(p_{*,0}([x]))^*$. Moreover, the map ε_0 in Lemma 2.17 is bijective, and hence we can take the inverse $\varphi := \varepsilon_0^{-1}(\psi \circ p) \in C_0^*$. Note that since $\varepsilon \circ \varphi(x) = \psi(p(x)) = 1$, the element $\varphi \in C_0^*$ is a homological generator. Moreover, the equalities

$$\varepsilon \circ \varphi\big(\mathcal{F}_{-1}^{\mathrm{Alg}} + \mathcal{F}_{-m-1}^{\mathrm{Alex}}\big) = \psi \circ p\big(\mathcal{F}_{-1}^{\mathrm{Alg}} + \mathcal{F}_{-m-1}^{\mathrm{Alex}}\big) = \{0\}$$

hold, and hence φ lies in $\Phi\big(C/\mathcal{F}_{-1}^{\mathrm{Alg}}\big) \cap \Phi\big(C/\mathcal{F}_{-m-1}^{\mathrm{Alex}}\big) = C^*_{\{i\le 0,\, j\le m\}}$. This proves that $\nu^+(C^*) \le m$. □

Proof of Proposition 2.27 It immediately follows from Lemma 2.25 that if $C \simeq \Lambda \oplus A$ where A is a stabilizer, then $\nu^+(C) = \nu^+(C^*) = 0$. To prove its converse, we will prove that if $\nu^+(C) = \nu^+(C^*) = 0$, then there exists a filtered basis $\{x_k\}_{1\le k\le r}$ such that C is decomposed into $\mathrm{span}_\Lambda\{x_1\} \oplus \mathrm{span}_\Lambda\{x_k\}_{2\le k\le r}$ as a chain complex. In the situation, the restriction of ∂ on $\mathrm{span}_\Lambda\{x_1\}$ is the zero map, and hence it follows from Lemma 2.20 that $\mathrm{span}_\Lambda\{x_1\}$ is a formal knot complex with $\mathrm{span}_\Lambda\{x_1\} \simeq \Lambda$ and $\mathrm{span}_\Lambda\{x_k\}_{2\le k\le r}$ is a stabilizer.

Suppose that $\nu^+(C) = \nu^+(C^*) = 0$, and let $\{x_k\}_{1\le k\le r}$ be a filtered basis for C. By Lemma 2.7, we may assume that $\mathrm{gr}(x_k) = 0$ for $k \in \{1, \dots, r_0\}$ and $\mathrm{gr}(x_k) = 1$ for $k \in \{r_0+1, \dots r\}$. Set $r_1 := r - r_0$ and $y_l := x_{r_0+l}$ $(1 \le l \le r_1)$. Then, by the definition of ν^+ and Lemma 2.28, there exists a homological generator $x = \sum_{1\le k\le r_0} a_k x_k \in C_0$ such that $x \in C_{\{i\le 0,\, j\le 0\}}$, and the homology class of $p(x)$ is non-zero, where $p : C \to C/C_{\{i\le -1 \text{ or } j\le -1\}}$ is the projection. This implies that

- If $a_k \ne 0$, then $x_k \in C_{\{i\le 0,\, j\le 0\}}$, and
- There exists a number $k \in \{1, \dots, r_0\}$ with $a_k \ne 0$ and $x_k \notin C_{\{i\le -1 \text{ or } j\le -1\}}$.

As a consequence, we have $k' \in \{1, \dots, r_0\}$ such that $a_{k'} \ne 0$ and $\mathrm{Alg}(x_{k'}) = \mathrm{Alex}(x_{k'}) = 0$. Moreover, since the inequalities

$$\mathrm{Alg}(x_k) \le 0 = \mathrm{Alg}(x_{k'})$$

and

$$\mathrm{Alex}(x_k) \le 0 = \mathrm{Alex}(x_{k'})$$

hold for any $k \in \{1, \dots, r_0\}$ with $a_k \ne 0$, it follows from Lemma 2.7 that $\{x\} \cup \{x_k\}_{k\ne k'}^{1\le k\le r_0} \cup \{y_l\}_{1\le l\le r_1}$ is a filtered basis. We reorder $\{x_k\}_{k\ne k'}^{1\le k\le r_0}$ as $\{x_k\}_{2\le k\le r_0}$.

Next, we will change $\{x_k\}_{2\le k\le r_0}$ into $\{x'_k\}_{2\le k\le r_0}$ so that $\{x\} \cup \{x'_k\}_{2\le k\le r_0} \cup \{y_l\}_{1\le l\le r_1}$ is still a filtered basis and $\partial\big(\{y_l\}_{1\le l\le r_1}\big) \subset \mathrm{span}_{\mathbb{F}}\{x'_k\}_{2\le k\le r_0}$. Then, we can conclude that both $\mathrm{span}_\Lambda\{x\}$ and $\mathrm{span}_\Lambda\big(\{x'_k\}_{2\le k\le r_0} \cup \{y_l\}_{1\le l\le r_1}\big)$ are subcomplexes, and this will complete the proof. To obtain such $\{x'_k\}$, we first note that

$$\{px\} \cup \big\{px_k \,\big|\, 2 \le k \le r_0 \text{ and } \mathrm{Alg}(x_k), \mathrm{Alex}(x_k) \ge 0\big\}$$

is a basis for $p(C_0)$. We reorder $\{x_k\}_{2\le k\le r_0}$ so that $\{px\} \cup \{px_k\}_{2\le k\le r'_0}$ is a basis for $p(C_0)$. (Here $r'_0 := \dim_{\mathbb{F}} p(C_0)$.) Let $(a_{kl})_{1\le l\le r_1}^{1\le k\le r'_0}$ be the matrix of $p \circ (\partial|_{C_1}) : C_1 \to p(C_0)$ with respect to the pair $\big(\{y_l\}_{1\le l\le r_1}, \{px_k\}_{2\le k\le r'_0} \cup \{px\}\big)$, i.e. $p \circ \partial(y_l) =$

$\sum_{1\le k\le r'-1} a_{kl}\, px_{k+1} + a_{r'l}\, px$. Then we can replace $\{y_l\}_{1\le l\le r_1}$ with a basis $\{y'_l\}_{1\le l\le r_1}$ so that the corresponding matrix $(a'_{kl})^{1\le k\le r'_0}_{1\le l\le r_1}$ is in reduced column echelon form. Here, since $[px] \neq 0$ in $H_0(C/C_{\{i\le -1 \text{ or } j\le -1\}})$, px is not contained in $p\circ\partial(C_1)$ and the last row of $(a'_{kl})^{1\le k\le r'_0}_{1\le l\le r_1}$ does not contain any leading coefficient. In particular, if $a_{r'_0 l} \neq 0$, then there exists a number k_l in $\{1,\dots,r'_0-1\}$ such the k_l-th row contains the l-th leading coefficient. (Namely, $a_{k_l l'} = \delta_{ll'}$, where $\delta_{ll'}$ is the Kronecker delta.) Now, we define a set $\{x'_k\}_{2\le k\le r_0}$ by

$$x'_k = \begin{cases} x_k + x & (\text{if } k = k_l - 1 \text{ for some } l \in \{1,\dots,r_1\} with\, a_{r'_0 l} \neq 0) \\ x_k & (\text{otherwise}) \end{cases}.$$

Then, it follows from Lemma 2.7 that $\{x\}\cup\{x'_k\}_{2\le k\le r_0}\cup\{y_l\}_{1\le l\le r_1}$ is a filtered basis. Moreover, the replacement of $\{px_k\}_{2\le k\le r'_0}$ with $\{px'_k\}_{2\le k\le r'_0}$ changes $(a'_{kl})^{1\le k\le r'_0}_{1\le l\le r_1}$ so that the last row is a zero vector. This implies that

$$\begin{aligned} p\circ\partial\big(\{y_l\}_{1\le l\le r_1}\big) &\subset p\circ\partial\big(\operatorname{span}_{\mathbb{F}}\{y_l\}_{1\le l\le r_1}\big) \\ &= p\circ\partial\big(\operatorname{span}_{\mathbb{F}}\{y'_l\}_{1\le l\le r_1}\big) \subset \operatorname{span}_{\mathbb{F}}\{px'_k\}_{2\le k\le r'_0}, \end{aligned}$$

and hence we have

$$\partial\big(\{y_l\}_{1\le l\le r_1}\big) \subset p^{-1}\big(\operatorname{span}_{\mathbb{F}}\{px'_k\}_{2\le k\le r'_0}\big) = \operatorname{span}_{\mathbb{F}}\{x'_k\}_{2\le k\le r_0}.$$

This completes the proof. □

Corollary 2.29 *Let C and C' be formal knot complexes. If $\nu^+(C) = \nu^+(C^*) = 0$, then $\nu^+(C'\otimes_\Lambda C) = \nu^+(C')$.*

Proof By Proposition 2.27, we have $C \simeq \Lambda\oplus A$. Here, Lemma 2.22 says that $C'\otimes_\Lambda A$ is a stabilizer, and it is easy to show that $C'\otimes_\Lambda(\Lambda\oplus A) \simeq C'\oplus(C'\otimes_\Lambda A)$. Therefore, by Lemma 2.25, we have

$$\nu^+(C'\otimes_\Lambda C) = \nu^+\Big(C'\oplus(C'\otimes_\Lambda A)\Big) = \nu^+(C').$$

□

Here we refer to the following theorem of Hom and Wu, which is one of the most important facts for obtaining concordance invariants from CFK^∞.

Theorem 2.30 ([5]) *For a knot K, the inequality $\nu^+(C^K) \le g_4(K)$ holds. In particular, if K is a slice knot, then $\nu^+(C^K) = \nu^+((C^K)^*) = 0$.*

2.7 ν^+-equivalence

Two elements $[C], [C'] \in \mathcal{K}^f$ are ν^+-*equivalent* (and denoted $[C] \overset{\nu^+}{\sim} [C']$ or $C \overset{\nu^+}{\sim} C'$) if $\nu^+(C \otimes_\Lambda C'^*) = \nu^+(C^* \otimes_\Lambda C') = 0$. Note that by Propositions 2.11 and 2.13, the values $\nu^+(C \otimes_\Lambda C'^*)$ and $\nu^+(C^* \otimes_\Lambda C')$ are independent of the choice of representatives.

Proposition 2.31 *The relation* $\overset{\nu^+}{\sim}$ *is an equivalence relation on* $\mathcal{K}^f$.

Proof The reflexivity (i.e. $[C] \overset{\nu^+}{\sim} [C]$) follows from Lemma 2.26. The symmetry ($[C] \overset{\nu^+}{\sim} [C']$ if and only if $[C'] \overset{\nu^+}{\sim} [C]$) directly follows from the definition. We prove the transitivity. Suppose that $[C_1] \overset{\nu^+}{\sim} [C_2]$ and $[C_2] \overset{\nu^+}{\sim} [C_3]$. Then, Proposition 2.24, Lemma 2.26 and Corollary 2.29 imply

$$\begin{aligned}
\nu^+(C_1 \otimes_\Lambda C_3^*) &= \nu^+\Big((C_1 \otimes_\Lambda C_3^*) \otimes_\Lambda (C_2 \otimes_\Lambda C_2^*)\Big) \\
&= \nu^+\Big((C_1 \otimes_\Lambda C_2^*) \otimes_\Lambda (C_2 \otimes_\Lambda C_3^*)\Big) \\
&\leq \nu^+(C_1 \otimes_\Lambda C_2^*) + \nu^+(C_2 \otimes_\Lambda C_3^*) = 0.
\end{aligned}$$

Similarly, we can prove that $\nu^+(C_1^* \otimes C_3) = 0$ holds. □

We call the equivalence class of a formal knot complex C under $\overset{\nu^+}{\sim}$ *the* ν^+- *equivalence class* or ν^+-*class of* C, and denote it by $[C]_{\nu^+}$. Then, we can see that Hom's stable homotopy theorem in [4] is naturally generalized to formal knot complexes.

Theorem 2.32 ([4]) *Two formal knot complexes* C *and* C' *are* ν^+-*equivalent if and only if we have the* $\mathbb{Z}^2$-*filtered homotopy equivalence*

$$C \oplus A \simeq C' \oplus A',$$

where A, A' *are stabilizers.*

Proof It follows from Lemma 2.25 and Proposition 2.27 that $C \overset{\nu^+}{\sim} C'$ if and only if $C \otimes_\Lambda C'^* \simeq \Lambda \oplus A$ where A is a stabilizer. Thus, if $C \overset{\nu^+}{\sim} C'$, then there exist stabilizers A_1, A_2 so that $C^* \otimes_\Lambda C' \simeq \Lambda \oplus A_1$ and $C \otimes_\Lambda C^* \simeq \Lambda \oplus A_2$, and we have

$$\begin{aligned}
C \oplus (C \otimes_\Lambda A_1) &\simeq C \otimes_\Lambda (\Lambda \oplus A_1) \simeq C \otimes_\Lambda (C^* \otimes_\Lambda C') \\
&\simeq (C \otimes_\Lambda C^*) \otimes_\Lambda C' \simeq (\Lambda \oplus A_2) \otimes_\Lambda C' \simeq C' \oplus (C' \otimes_\Lambda A_2).
\end{aligned}$$

Conversely, if $C \oplus A \simeq C' \oplus A'$, then

$$\begin{aligned}
\nu^+(C \otimes_\Lambda C'^*) &= \nu^+\Big((C \otimes_\Lambda C'^*) \oplus (A \otimes_\Lambda C'^*)\Big) = \nu^+\Big((C \oplus A) \otimes_\Lambda C'^*\Big) \\
&= \nu^+\Big((C' \oplus A') \otimes_\Lambda C'^*\Big) = \nu^+\Big((C' \otimes_\Lambda C'^*) \oplus (A' \otimes_\Lambda C'^*)\Big)
\end{aligned}$$

$$= \nu^+(C' \otimes_\Lambda C'^*) = 0.$$

Similarly, we can prove $\nu^+(C^* \otimes_\Lambda C') = 0$. □

Here, due to Theorem 2.30, the ν^+-class of C^K can be seen as a knot concordance invariant of K.

Corollary 2.33 ([4]) *For a knot K, $[K]_{\nu^+} := [C^K]_{\nu^+}$ is a knot concordance invariant of K.*

Proof If two knots K and J are concordant, then both $K\#(-J^*)$ and $(-K^*)\#J$ are slice knots. Thus, by Theorem 2.30, we have

$$\nu^+\left(C^K \otimes_\Lambda (C^J)^*\right) = \nu^+\left(C^{K\#(-J^*)}\right) = 0$$

and

$$\nu^+\left((C^K)^* \otimes_\Lambda C^J\right) = \nu^+\left(C^{(-K^*)\#J}\right) = 0.$$

□

2.8 Formal knot concordance group

Now, *the formal knot concordance group* $\mathcal{C}^f$ is obtained as follows.

Proposition 2.34 *The quotient set $\mathcal{C}^f := \mathcal{K}^f / \overset{\nu^+}{\sim}$ with product*

$$\mathcal{C}^f \times \mathcal{C}^f \to \mathcal{C}^f : \left([C]_{\nu^+}, [C']_{\nu^+}\right) \to [C \otimes_\Lambda C']_{\nu^+}$$

is an abelian group. In particular, the projection $\mathcal{K}^f \twoheadrightarrow \mathcal{C}^f$ is a monoid homomorphism.

Proof We first verify that the product is well-defined. Suppose that $[C]_{\nu^+} = [C'']_{\nu^+}$, and then $\nu^+(C \otimes_\Lambda C''^*) = \nu^+(C^* \otimes_\Lambda C'') = 0$. Thus, it follows from Proposition 2.24 and Lemma 2.26 that

$$\nu^+\left((C \otimes_\Lambda C') \otimes_\Lambda (C'' \otimes_\Lambda C')^*\right) = \nu^+\left((C \otimes_\Lambda C''^*) \otimes_\Lambda (C' \otimes_\Lambda C'^*)\right) = 0$$

and

$$\nu^+\left((C \otimes_\Lambda C')^* \otimes_\Lambda (C'' \otimes_\Lambda C')\right) = \nu^+\left((C^* \otimes_\Lambda C'') \otimes_\Lambda (C' \otimes_\Lambda C'^*)\right) = 0.$$

Similarly, we can prove that if $[C']_{\nu^+} = [C'']_{\nu^+}$ then $[C \otimes_\Lambda C']_{\nu^+} = [C \otimes_\Lambda C'']_{\nu^+}$. Now, the commutativity immediately follows from $C \otimes_\Lambda C' \simeq C' \otimes_\Lambda C$, and obviously the projection $\mathcal{K}^f \twoheadrightarrow \mathcal{C}^f$ is a monoid homomorphism. □

As a consequence, we have the following theorem, which is stated in Sect. 1 as Theorem 1.7.

Theorem 2.35 *The map $\mathcal{C} \to \mathcal{C}^f : [K]_c \mapsto [C^K]_{\nu^+}$ is a well-defined group homomorphism. As a consequence, we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{[K]\mapsto[C^K]} & \mathcal{K}^f \\ {\scriptstyle [K]\mapsto[K]_c}\downarrow & & \downarrow{\scriptstyle [C]\mapsto[C]_{\nu^+}} \\ \mathcal{C} & \xrightarrow[{[K]_c\mapsto[C^K]_{\nu^+}}]{} & \mathcal{C}^f \end{array}$$

2.9 Partial order on $\mathcal{C}^f$

In this subsection, we introduce a partial order on $\mathcal{C}^f$, which is a generalization of the partial order on $\mathcal{C}_{\nu^+}$ defined in [22]. Here, as a new observation, we give an interpretation of the ν^+-equivalence and the partial order on $\mathcal{C}^f$ using quasi-isomorphisms.

For two ν^+-classes $[C]_{\nu^+}, [C']_{\nu^+} \in \mathcal{C}^f$, we denote $[C]_{\nu^+} \leq [C']_{\nu^+}$ if the equality $\nu^+(C \otimes_\Lambda C'^*) = 0$ holds.

Proposition 2.36 *The relation $\leq$ is a partial order on $\mathcal{C}^f$.*

Proof This immediately follows from Proposition 2.24, Lemma 2.26 and the definition of $\overset{\nu^+}{\sim}$. □

For two formal knot complexes, a chain map $f : C \to C'$ over Λ is a $\mathbb{Z}^2$-*filtered quasi-isomorphism* if f is $\mathbb{Z}^2$-filtered, graded, and induces an isomorphism $f_* : H_*(C) \to H_*(C')$. Then, the ν^+-equivalence and the partial order on $\mathcal{C}^f$ can be translated into the words of the existence of $\mathbb{Z}^2$-filtered quasi-isomorphisms.

Theorem 2.37 *Two formal knot complexes C and C' are ν^+-equivalent if and only if there exist $\mathbb{Z}^2$-filtered quasi-isomorphisms*

$$f : C \to C' \text{ and } g : C' \to C.$$

Theorem 2.38 *Two ν^+-classes $[C]_{\nu^+}$ and $[C']_{\nu^+}$ satisfy $[C]_{\nu^+} \geq [C']_{\nu^+}$ if and only if there exists a $\mathbb{Z}^2$-filtered quasi-isomorphism $C \to C'$.*

To prove these theorems, we first prove the following lemma.

Lemma 2.39 *Let C and C' be formal knot complexes. If there exists a $\mathbb{Z}^2$-filtered quasi-isomorphism $f : C \to C'$, then $[C]_{\nu^+} \geq [C']_{\nu^+}$.*

Proof Note that under the hypothesis of the lemma, $f \otimes id_{C^*} : C \otimes_\Lambda C^* \to C' \otimes_\Lambda C^*$ is also a $\mathbb{Z}^2$-filtered quasi-isomorphism. Moreover, by Lemma 2.26, we can take a homological generator x of $C \otimes_\Lambda C^*$ lying in $(C \otimes_\Lambda C^*)_{\{i \leq 0,\, j \leq 0\}}$. Now, we see that $f \otimes id_{C^*}(x)$ is a homological generator of $C' \otimes_\Lambda C^*$ lying in $(C' \otimes_\Lambda C^*)_{\{i \leq 0,\, j \leq 0\}}$, and hence $\nu^+(C' \otimes C^*) = 0$. □

Proof of Theorem 2.37 It directly follows from Lemma 2.39 that the existence of f and g implies $C \overset{\nu^+}{\sim} C'$. We prove the converse. Suppose that $C \overset{\nu^+}{\sim} C'$. Then, by Theorem 2.32, we have a $\mathbb{Z}^2$-filtered homotopy equivalence map

$$f' : C \oplus A \overset{\simeq}{\longrightarrow} C' \oplus A',$$

where A, A' are stabilizers. Let $i : C \hookrightarrow C \oplus A$ be the inclusion and $p : C' \oplus A' \twoheadrightarrow C'$ the projection. Then, all of i, f' and p are $\mathbb{Z}^2$-filtered quasi-isomorphisms, and hence we have the $\mathbb{Z}^2$-filtered quasi-isomorphism

$$f := p \circ f' \circ i : C \to C'.$$

Similarly, we can construct a $\mathbb{Z}^2$-filtered quasi-isomorphism $g : C' \to C$. □

Proof of Theorem 2.38 By Lemma 2.39, we only need to prove that $[C]_{\nu^+} \geq [C']_{\nu^+}$ implies the existence of a $\mathbb{Z}^2$-filtered quasi-isomorphism $C \to C'$. Suppose that $[C]_{\nu^+} \geq [C']_{\nu^+}$. Then the equality $\nu^+(C' \otimes_\Lambda C^*) = 0$ holds, and hence $(C' \otimes_\Lambda C^*)_{\{i \leq 0,\, j \leq 0\}}$ contains a homological generator x. Hence, if we define a Λ-linear map

$$f\colon \Lambda \to C' \otimes_\Lambda C^*$$

so that $f(1) = x$, then f is a $\mathbb{Z}^2$-filtered quasi-isomorphism. In addition, the map

$$f \otimes 1\colon C \to C' \otimes_\Lambda C^* \otimes_\Lambda C$$

is also a $\mathbb{Z}^2$-filtered quasi-isomorphism. Moreover, since $(C' \otimes_\Lambda C^* \otimes_\Lambda C) \overset{\nu^+}{\sim} C'$, Theorem 2.37 gives a $\mathbb{Z}^2$-filtered quasi-isomorphism

$$C' \otimes_\Lambda C^* \otimes_\Lambda C \to C'.$$

By combining these maps, we obtain the desired quasi-isomorphism. □

When one wants to construct a $\mathbb{Z}^2$-filtered quasi-isomorphism concretely, the following lemma is useful.

Lemma 2.40 *Let C and C' be formal knot complexes and $f : C \to C'$ be a chain map over Λ such that*

- *f maps a homological generator C to that of C', and*
- *for a filtered basis $\{x_k\}_{1 \leq k \leq r}$ of C and any k, we have*

$$\big(\mathrm{Alg}(f x_k), \mathrm{Alex}(f x_k)\big) \leq \big(\mathrm{Alg}(x_k), \mathrm{Alex}(x_k)\big).$$

Then, f is a $\mathbb{Z}^2$-filtered quasi-isomorphism.

Proof Since $\mathcal{F}^{\text{Alex}}_j(C) = \text{span}_{\mathbb{F}[U]}\left\{U^{\text{Alex}(x_k)-j}x_k\right\}_{1\le k\le r}$, we have

$$f\left(\mathcal{F}^{\text{Alex}}_j(C)\right) = \text{span}_{\mathbb{F}[U]}\left\{U^{\text{Alex}(x_k)-j}fx_k\right\}_{1\le k\le r} \subset \text{span}_{\mathbb{F}[U]}\left\{U^{\text{Alex}(fx_k)-j}fx_k\right\}_{1\le k\le r} \subset \mathcal{F}^{\text{Alex}}_j(C').$$

Similarly, we have $f\left(\mathcal{F}^{\text{Alg}}_i(C)\right) \subset \mathcal{F}^{\text{Alg}}_i(C')$. Now, for any $R \in \mathcal{CR}(\mathbb{Z}^2)$, we see that

$$\begin{aligned} f(C_R) &= f\left(\sum_{(i,j)\in R}\mathcal{F}^{\text{Alg}}_i(C)\cap\mathcal{F}^{\text{Alex}}_j(C)\right)\\ &\subset \sum_{(i,j)\in R} f\left(\mathcal{F}^{\text{Alg}}_i(C)\right)\cap f\left(\mathcal{F}^{\text{Alex}}_j(C)\right)\\ &\subset \sum_{(i,j)\in R}\mathcal{F}^{\text{Alg}}_i(C')\cap\mathcal{F}^{\text{Alex}}_j(C') = C'_R. \end{aligned}$$

It is easy to see that f is a quasi-isomorphism. □

Set $\mathcal{C}_{\nu^+} := \text{Im}(\mathcal{C} \to \mathcal{C}^f\colon [K]_c \mapsto [C^K]_{\nu^+})$. Then $\mathcal{C}_{\nu^+}$ is naturally identified with a quotient group of $\mathcal{C}$, and the partial order on $\mathcal{C}^f$ induces a partial order on $\mathcal{C}_{\nu^+}$. We note that the induced partial order coincides with the order defined in author's paper [22]. In particular, Proposition 1.5 in [22] is naturally generalized to $\mathcal{C}^f$.

Proposition 2.41 ([22, Proposition 1.5]) *The partial order on $\mathcal{C}^f$ has the following properties:*

1. *For elements $x, y, z \in \mathcal{C}^f$, if $x \le y$, then $x + z \le y + z$.*
2. *For elements $x, y \in \mathcal{C}^f$, if $x \le y$, then $-y \le -x$.*

On the other hand, for the case of $\mathcal{C}_{\nu^+}$, we also have the following geometric estimates. (Here, *full-twist operations* are defined as follows. Let K be a knot and D a disk in S^3 which intersects K in its interior. By performing (-1)-surgery along ∂D, we obtain a new knot J in S^3 from K. Let $n = \text{lk}(K, \partial D)$. Since reversing the orientation of D does not affect the result, we may assume that $n \ge 0$. Then we say that K is deformed into J by *a positive full-twist with n-linking*, and call such an operation a *full-twist operation*.)

Theorem 2.42 ([22, Theorem 1.6]) *Suppose that a knot K is deformed into a knot J by a positive full-twist with n-linking.*

1. *If $n = 0$ or 1, then $[J]_{\nu^+} \le [K]_{\nu^+}$.*
2. *If $n \ge 3$, then $[J]_{\nu^+} \not\le [K]_{\nu^+}$. In particular, if the geometric intersection number between K and D is equal to n, then $[J]_{\nu^+} > [K]_{\nu^+}$.*

2.10 Invariants of ν^+-classes

In this subsection, we review the V_k-sequence [12], the τ-invariant [13], the Υ-invariant [17] and the Υ^2-invariant [8] as invariants of formal knot complexes under

ν^+-equivalence. Here we use $\mathbb{Z}^2$-filtered quasi-isomorphisms to prove the invariance of them.

2.10.1 V_k-sequence

The V_k-sequence defined by Ni and Wu [12] is a family of $\mathbb{Z}_{\geq 0}$-valued invariants which is parametrized by $\mathbb{Z}_{\geq 0}$. Concretely, for a formal knot complex C and $k \in \mathbb{Z}_{\geq 0}$, the value $V_k(C)$ is defined by

$$V_k(C) = \dim_{\mathbb{F}} \Big(\operatorname{coker} \big(i_* \colon H_*(C_{\{i \leq 0,\, j \leq k\}}) \to H_*(C_{\{i \leq 0\}}) \big) \Big).$$

In particular, we have the equality

$$\nu^+(C) = \min \left\{ k \in \mathbb{Z}_{\geq 0} \;\middle|\; V_k(C) = 0 \right\}.$$

Moreover, we can use homological generators to determine $V_k(C)$.

Lemma 2.43 *For any $k \in \mathbb{Z}_{\geq 0}$, the equality*

$$V_k(C) = \min \left\{ m \in \mathbb{Z}_{\geq 0} \;\middle|\; C_{\{i \leq m,\, j \leq k+m\}} \text{ contains a homological generator} \right\}$$

holds.

Proof Denote the value of the right-hand side of the equality in Lemma 2.43 by $V'_k(C)$. We first prove that $V_k(C) \geq V'_k(C)$. Since $H_*(C_{\{i \leq 0\}}) \cong \mathbb{F}[U]$ and the map $i_* \colon H_*(C_{\{i \leq 0,\, j \leq k\}}) \to H_*(C_{\{i \leq 0\}})$ is a $\mathbb{F}[U]$-linear map, if $\operatorname{Im} i_{*,2m} = H_{2m}(C_{\{i \leq 0\}})$ then $\operatorname{Im} i_{*,2n} = H_{2n}(C_{\{i \leq 0\}})$ for any $n \leq m$. This implies that

$$i_{*,-2V_k(C)} \colon H_{-2V_k(C)}(C_{\{i \leq 0,\, j \leq k\}}) \to H_{-2V_k(C)}(C_{\{i \leq 0\}})$$

is surjective. Moreover, the map $i_{*,n} \colon H_n(C_{\{i \leq 0\}}) \to H_n(C)$ is an isomorphism for any $n \leq 0$. Consequently, we see that there exists a cycle $x \in C_{-2V_k(C)}$ lying in $C_{\{i \leq 0,\, j \leq k\}}$ such that the homology class $[x] \in H_{-2V_k(C)}(C)$ is non-zero. This implies that $U^{-V_k(C)}x \in C_0$ is a homological generator lying in $C_{\{i \leq V_k(C),\, j \leq k+V_k(C)\}}$. Therefore, we have $V_k(C) \geq V'_k(C)$.

Conversely, since $C_{\{i \leq V'_k(C),\, j \leq k+V'_k(C)\}}$ contains a homological generator x, the cycle $U^{V'_k(C)}x \in C_{-2V'_k(C)}$ is lying in $C_{\{i \leq 0,\, j \leq k\}}$. This implies that the map

$$i_{*,-2V'_k(C)} \colon H_{-2V'_k(C)}(C_{\{i \leq 0,\, j \leq k\}}) \to H_{-2V'_k(C)}(C_{\{i \leq 0\}})$$

is surjective, and hence $V_k(C) \leq V'_k(C)$. □

Now, we can easily see that V_k is a well-defined map on $\mathcal{C}^f$ and preserve the partial order.

Corollary 2.44 *If $[C]_{\nu^+} \leq [C']_{\nu^+}$, then $V_k(C) \leq V_k(C')$ for any $k \geq 0$. In particular, V_k is a well-defined map $\mathcal{C}^f \to \mathbb{Z}_{\geq 0}$.*

Proof Suppose that $[C]_{\nu^+} \leq [C']_{\nu^+}$. Then we have a $\mathbb{Z}^2$-filtered quasi-isomorphism $f\colon C' \to C$. Here, by using Lemma 2.43, we can take a homological generator $x \in C'$ lying in $C'_{\{i \leq V_k(C'),\, j \leq k+V_k(C')\}}$. Then, $f(x)$ is a homological generator of C lying in $C_{\{i \leq V_k(C'),\, j \leq k+V_k(C')\}}$. This completes the proof. □

In addition, we also have the following properties of V_k.

Corollary 2.45 *For any $k \in \mathbb{Z}_{\geq 0}$, we have*

$$V_k(C) - 1 \leq V_{k+1}(C) \leq V_k(C).$$

In particular, for any $0 \leq k \leq \nu^+(C)$, the inequality $V_k(C) + k \leq \nu^+(C)$ holds.

Proof The first assertion immediately follows from the fact that

$$\begin{aligned} C_{\{i \leq m-1,\, j \leq (k+1)+(m-1)\}} &\subset C_{\{i \leq m,\, j \leq k+m\}} \\ &\subset C_{\{i \leq m,\, j \leq (k+1)+m\}}. \end{aligned}$$

Next, for any $0 \leq k \leq \nu^+(C)$, we see that

$$V_k \leq V_{k+1}(C) + 1 \leq \cdots \leq V_{\nu^+(C)} + (\nu^+(C) - k) = \nu^+(C) - k.$$

This completes the proof. □

Moreover, we have a connected sum inequality for V_k. (For knot complexes, it is given in [1].)

Corollary 2.46 *For any formal knot complexes C, C' and $k, k' \in \mathbb{Z}_{\geq 0}$, we have*

$$V_{k+k'}(C \otimes_\Lambda C') \leq V_k(C) + V_{k'}(C').$$

Proof By Lemma 2.43, we have a homological generator $x \in C$ (resp. $x' \in C'$) which is lying in $C_{\{i \leq V_k(C),\, j \leq k+V_k(C)\}}$ (resp. $C_{\{i \leq V_{k'}(C'),\, j \leq k+V_{k'}(C')\}}$). This implies that $x \otimes x'$ is a homological generator of $C \otimes_\Lambda C'$ lying in

$$(C \otimes_\Lambda C')_{\{i \leq V_k(C)+V_{k'}(C'),\, j \leq (k+k')+V_k(C)+V_{k'}(C')\}}.$$

This completes the proof. □

For the case of knot complexes, $V_k(K) := V_k(C^K)$ is an important invariant because it completely determines all correction terms of all positive Dehn surgeries along K. To state the fact precisely, we fix several notations. For coprime integers $p, q > 0$, let $S^3_{p/q}(K)$ denote the p/q-surgery along K. Note that there is a canonical identification between the set of Spin^c structures over $S^3_{p/q}(K)$ and $\{i \mid 0 \leq i \leq p-1\}$. This identification can be made explicit by the procedure in [19, Sect. 4, Section 7]. Let $d(S^3_{p/q}(K), i)$ denote the correction term of $S^3_{p/q}(K)$ with the i-th Spin^c structure $(0 \leq i \leq p-1)$.

Proposition 2.47 ([12, Proposition 1.6]) *The equality*

$$d(S^3_{p/q}(K), i) = d(S^3_{p/q}(O), i) - 2\max\left\{V_{\lfloor \frac{i}{q} \rfloor}(K), V_{\lfloor \frac{p+q-1-i}{q} \rfloor}(K)\right\}$$

holds, where O denotes the unknot and $\lfloor \cdot \rfloor$ is the floor function.

2.10.2 τ-invariant

Let C be a formal knot complex. Define

$$\widehat{\mathrm{C}} := C_{\{i\le 0\}}/C_{\{i\le -1\}}$$

and

$$\widehat{\mathcal{F}}_m := C_{\{i\le 0,\, j\le m\}}/C_{\{i\le -1,\, j\le m\}}$$

for any $m \in \mathbb{Z}$. Then we see $H_*(\widehat{\mathrm{C}}) = H_0(\widehat{\mathrm{C}}) \cong \mathbb{F}$, and $\{\widehat{\mathcal{F}}_m\}_{m\in\mathbb{Z}}$ is an increasing sequence of subcomplexes on $\widehat{\mathrm{C}}$, i.e. a $\mathbb{Z}$-filtration on $\widehat{\mathrm{C}}$. We call a cycle $x \in \widehat{\mathrm{C}}$ a *hat-generator* if x is homogeneous with $\mathrm{gr}(x) = 0$ and the homology class $[x] \in H_0(\widehat{\mathrm{C}})$ is non-zero. We define the *τ-invariant* of C by

$$\tau(C) := \min\left\{m \in \mathbb{Z} \;\middle|\; \widehat{\mathcal{F}}_m \text{ contains a hat-generator}\right\}.$$

We can use homological generators to determine $\tau(C)$ like $V_k(C)$.

Lemma 2.48 *The equality*

$$\tau(C) = \min\left\{m \in \mathbb{Z}_{\ge 0} \;\middle|\; C_{\{i\le -1\}\cup\{i\le 0,\, j\le m\}} \textit{contains a homological generator}\right\}$$

holds.

Proof Denote the value of the right-hand side of the equality in Lemma 2.48 by $\tau'(C)$. We first prove that $\tau(C) \ge \tau'(C)$. By the definition of $\tau(C)$, there exists a chain $x \in C_{\{i\le -1\}\cup\{i\le 0,\, j\le \tau(C)\}}$ such that $p(x) \in \widehat{\mathcal{F}}_{\tau(C)}$ is a hat-generator, where $p\colon C_{\{i\le 0\}} \to \widehat{\mathrm{C}}$ is the projection. Moreover, since the induced map

$$p_{*,0}\colon H_0(C_{\{i\le 0\}}) \to H_0(\widehat{\mathrm{C}})$$

is an isomorphism, there exists a 0-chain $y \in C_{\{i\le -1\}}$ such that $\partial y = \partial x$. In particular, $x - y$ is a homological generator of C lying in $C_{\{i\le -1\}\cup\{i\le 0,\, j\le \tau(C)\}}$. (Note that $p_{*,0}([x-y]) = [p(x-y)] = [p(x)] \ne 0$.) Therefore, we have $\tau(C) \ge \tau'(C)$.

Conversely, since $C_{\{i\le -1\}\cup\{i\le 0,\, j\le \tau'(C)\}}$ contains a homological generator x' and the above map $p_{*,0}$ is an isomorphism, $p(x')$ is a hat-generator lying in $\widehat{\mathcal{F}}_{\tau'(C)}$. This gives $\tau(C) \le \tau'(C)$. □

Now, by the same arguments as the proof of Corollary 2.44, we have the following.

Corollary 2.49 *If* $[C]_{\nu^+} \leq [C']_{\nu^+}$, *then* $\tau(C) \leq \tau(C')$. *In particular,* τ *is a well-defined map* $\mathcal{C}^f \to \mathbb{Z}$.

In addition, τ is related to ν^+ as follows.

Corollary 2.50 *The inequality* $\tau(C) \leq \nu^+(C)$ *holds.*

Proof This follows from $C_{\{i \leq 0,\, j \leq \nu^+(C)\}} \subset C_{\{i \leq -1\} \cup \{i \leq 0,\, j \leq \nu^+(C)\}}$. □

One of the most important properties of τ-invariant is the following additivity.

Proposition 2.51 τ *is a group homomorphism as a map* $\mathcal{C}^f \to \mathbb{Z}$.

Proof Let C and C' be formal knot complexes. Then we can see from Proposition 2.11 that the $\mathbb{Z}$-filtered homotopy equivalence

$$\left(\widehat{C \otimes_\Lambda C'}, \{\widehat{\mathcal{F}}_m\}\right) \simeq \left(\widehat{\mathrm{C}} \otimes_{\mathbb{F}} \widehat{\mathrm{C}}', \left\{ \mathrm{span}_{\mathbb{F}}\, p\Big(\bigcup_{\mu+\mu'=m} \widehat{\mathcal{F}}_\mu \times \widehat{\mathcal{F}}'_{\mu'} \Big)\right\}\right)$$

holds, where $p \colon \mathbb{F}^{\widehat{\mathrm{C}} \times \widehat{\mathrm{C}}'} \twoheadrightarrow \widehat{\mathrm{C}} \otimes_{\mathbb{F}} \widehat{\mathrm{C}}'$ is the projection. Next, let $x \in \widehat{\mathcal{F}}_{\tau(C)}$ (resp. $x' \in \widehat{\mathcal{F}}'_{\tau(C')}$) be a hat-generator. Then, in a similar way to the proof of Proposition 2.27, we have the $\mathbb{Z}$-filtered homotopy equivalence

$$\widehat{\mathrm{C}} \simeq \mathrm{span}_{\mathbb{F}}\{x\} \oplus A (\text{ resp. } \widehat{\mathrm{C}}' \simeq \mathrm{span}_{\mathbb{F}}\{x'\} \oplus A'),$$

where A and A' are acyclic $\mathbb{Z}$-filtered chain complexes. Consequently, the $\mathbb{Z}$-filtered homotopy equivalence

$$\widehat{C \otimes_\Lambda C'} \simeq \mathrm{span}_{\mathbb{F}}\{x \otimes x'\} \oplus A''$$

holds for some acyclic $\mathbb{Z}$-filtered chain complex A'', and this implies that $\tau(C \otimes_\Lambda C') = \tau(C) + \tau(C')$. □

As a consequence, we have the original τ-invariant for knots.

Corollary 2.52 ([13]) *The map* $[K]_c \mapsto \tau(C^K)$ *is a group homomorphism as a map* $\mathcal{C} \to \mathbb{Z}$.

2.10.3 Υ-invariant

For any $t \in [0, 2]$ and $s \in \mathbb{R}$, the set

$$R^t(s) := \left\{ (i, j) \in \mathbb{Z}^2 \;\middle|\; \left(1 - \frac{t}{2}\right) i + \frac{t}{2} j \leq s \right\}$$

is a closed region. Hence, if we denote $C_{R^t(s)}$ by $\mathcal{F}_s^t$, then we have an $\mathbb{R}$-filtration $\{\mathcal{F}_s^t\}_{s\in\mathbb{R}}$ of C. We define

$$\upsilon_C(t) := \min\left\{s \in \mathbb{R} \;\middle|\; \mathcal{F}_s^t \text{ contains a homological generator}\right\}$$

and

$$\Upsilon_C(t) := -2\upsilon_C(t).$$

Remark This definition of Υ is due to Livingston [10] rather than the original one [17].

Since there exist finitely many homological generators of C and their Alexander and algebraic filtrations are finite, $\upsilon_C(t)$ and $\Upsilon_C(t)$ are finite values. In the same way as V_k and τ, we can prove the following proposition.

Proposition 2.53 *If $[C]_{\nu^+} \le [C']_{\nu^+}$, then $\Upsilon_C(t) \ge \Upsilon_{C'}(t)$ for any $t \in [0, 2]$. In particular, $\Upsilon(t) \colon [C]_{\nu^+} \mapsto \Upsilon_C(t)$ is a well-defined map $\mathcal{C}^f \to \mathbb{R}$ for any $t \in [0, 2]$.*

In addition, we can see Υ as a linear approximation of V_k in the following sense.

Proposition 2.54 *For any $t \in [0, 2]$ and $k \in \mathbb{Z}_{\ge 0}$, the inequality*

$$\Upsilon_C(t) \ge -kt - 2V_k(C)$$

holds. In particular, $\Upsilon_C(t) \ge -\nu^+(C)t$ holds.

Proof This follows from $C_{\{i \le V_k(C),\, j \le k+V_k(C)\}} \subset C_{\{(1-\frac{t}{2})i+\frac{t}{2}j \le V_k(C)+\frac{t}{2}k\}}$. □

Moreover, The additivity of $\Upsilon(t)$ is also obtained in the same way as τ.

Proposition 2.55 *For any $t \in [0, 2]$, $\Upsilon(t)$ is a group homomorphism as a map $\mathcal{C}^f \to \mathbb{R}$.*

We can generalize the following properties of the original Υ-invariant to formal knot complexes. The proof is similar to [10, Theorem 8.1].

Proposition 2.56 *For any formal knot complexes C, the following properties hold.*

1. *The map $\Upsilon_C \colon [0, 2] \to \mathbb{R}$, $t \mapsto \Upsilon_C(t)$ is a continuous linear function.*
2. *For any regular point t of Υ_C and filtered basis $\{x_k\}_{1\le k\le r}$, there exists an element $x_l \in \{x_k\}_{1\le k\le r}$ with $\mathrm{gr}(x_l) = 0$ such that*

$$\Upsilon_C(t') = -2\,\mathrm{Alg}(x_l) + (\mathrm{Alg}(x_l) - \mathrm{Alex}(x_l))t'$$

at any point t' nearby t.

3. *Let t be a singular point of Υ_C and $\{x_k\}_{1\le k\le r}$ a filtered basis. Then there exists two elements $x_l, x_{l'} \in \{x_k\}_{1\le k\le r}$ with $\mathrm{gr}(x_l) = \mathrm{gr}(x_{l'}) = 0$ such that*
   - $\mathrm{Alex}(x_l) - \mathrm{Alex}(x_{l'}) = (1 - \frac{2}{t})(\mathrm{Alg}(x_l) - \mathrm{Alg}(x_{l'}))$,

- *the equality*

$$\Upsilon_C(t') = -2\,\mathrm{Alg}(x_l) + (\mathrm{Alg}(x_l) - \mathrm{Alex}(x_l))t'$$

holds at any point t' *nearby* t *satisfying* $t' < t$, *and*

- *the equality*

$$\Upsilon_C(t') = -2\,\mathrm{Alg}(x_{l'}) + (\mathrm{Alg}(x_{l'}) - \mathrm{Alex}(x_{l'}))t'$$

holds at any point t' *nearby* t *satisfying* $t' > t$.

As a consequence of the above arguments, we have the following corollaries. Here, $\mathrm{PL}([0,2],\mathbb{R})$ denotes the set of continuous piecewise linear functions on $[0, 2]$.

Corollary 2.57 *The map* $\Upsilon\colon [C]_{\nu^+} \mapsto \Upsilon_C$ *is a group homomorphism as a map* $\mathcal{C}^f \to \mathrm{PL}([0,2],\mathbb{R})$.

Corollary 2.58 *The map* $[K]_c \mapsto \Upsilon_{C^K}$ *is a group homomorphism as a map* $\mathcal{C} \to \mathrm{PL}([0,2],\mathbb{R})$.

Here we mention that the gradient of Υ_C nearby 0 is equal to $-\tau(C)$. The proof is the same as [10, Theorem 14.1].

Proposition 2.59 *For any sufficiently small* $t > 0$, *we have* $\Upsilon_C(t) = -\tau(C)t$.

2.10.4 Υ^2-invariant

Let C be a formal knot complex, $\{x_k\}_{1\le k\le r}$ a filtered basis and $\{C_{(i,j)}\}$ the induced decomposition of C. Define *the support of* $\{C_{(i,j)}\}$ by

$$\mathcal{P} := \left\{(i,j) \in \mathbb{Z}^2 \;\middle|\; C_{(i,j)} \neq 0\right\}.$$

In addition, consider *the support line for* $\mathcal{F}^t_s$ by

$$\mathcal{L}^t_s := \left\{(i,j) \in \mathbb{Z}^2 \;\middle|\; \left(1 - \frac{t}{2}\right) i + \frac{t}{2} j = s\right\}.$$

Now, for any $t \in [0, 2]$, set

$$\mathcal{P}_t := \mathcal{P} \cap \mathcal{L}^t_{\upsilon_C(t)}.$$

Then, we see that $\mathcal{P}_t \neq \emptyset$ for any t. Moreover, from Proposition 2.56, we have the following proposition.

Proposition 2.60 *The following assertions hold:*

1. *For any* $t \in [0, 2]$ *and small* $\delta > 0$, *the intersection* $\mathcal{P}_t \cap \mathcal{P}_{t-\delta}$ *(resp.* $\mathcal{P}_t \cap \mathcal{P}_{t+\delta}$*) has exactly one point. (We denote these points by* p^-_t *and* p^+_t, *respectively.)*
2. *The function* Υ_K *has a singularity at* t *if and only if* $p^-_t \neq p^+_t$.

In light of this proposition, for small $\delta > 0$, we set

$$\mathcal{Z}_t^{\pm}(C) := \left\{\text{homological generator in } \mathcal{F}^{t\pm\delta}_{\upsilon_C(t\pm\delta)}\right\}.$$

If $\mathcal{Z}_t^-(C) \cap \mathcal{Z}_t^+(C) = \emptyset$, then for any $s \in [0, 2]$, we define

$$\upsilon^2_{C,t}(s) := \min\left\{r \in \mathbb{R} \;\middle|\; \exists z^{\pm} \in \mathcal{Z}_t^{\pm}(C), [z^-] = [z^+] \text{ in } H_0(\mathcal{F}^t_{\upsilon_C(t)} + \mathcal{F}^s_r)\right\}.$$

Now, we can define the Υ^2-*invariant* of C as

$$\Upsilon^2_{C,t}(s) := \begin{cases} -2(\upsilon^2_{C,t}(s) - \upsilon_C(t)) & \text{if } \mathcal{Z}^-(C) \cap \mathcal{Z}^+(C) = \emptyset \\ \infty & \text{if } \mathcal{Z}^-(C) \cap \mathcal{Z}^+(C) \neq \emptyset \end{cases}.$$

From the view point of $\mathbb{Z}^2$-filtered quasi-isomorphism, we have the following inequality.

Proposition 2.61 *If* $[C]_{\nu^+} \leq [C']_{\nu^+}$ *and* $\Upsilon_C|_{[t-\varepsilon,t+\varepsilon]} = \Upsilon_{C'}|_{[t-\varepsilon,t+\varepsilon]}$ *for given* $t \in (0, 2)$ *and some* $\varepsilon > 0$, *then* $\Upsilon^2_{C,t}(s) \geq \Upsilon^2_{C',t}(s)$ *for any* $s \in [0, 2]$.

Proof Take $0 < \delta < \varepsilon$ sufficiently small so that Propsition 2.60 holds at given t for both C and C'. Let $z'^{\pm} \in \mathcal{Z}_t^{\pm}(C')$ such that

$$[z'^-] = [z'^+] \text{ in } H_0(\mathcal{F}^t_{\upsilon_{C'}(t)}(C') + \mathcal{F}^s_{\upsilon^2_{C',t}(s)}(C')),$$

and $f : C' \to C$ a $\mathbb{Z}^2$-filtered quasi-isomorphism. Since $\upsilon_C(t \pm \delta) = \upsilon_{C'}(t \pm \delta)$, we see $f(z'^{\pm}) \in \mathcal{Z}_t^{\pm}(C)$. Now, we have the equalities

$$[f(z'^-)] = f_*([z'^-]) = f_*([z'^+]) = [f(z'^+)]$$

as elements of $H_0(\mathcal{F}^t_{\upsilon_C(t)}(C) + \mathcal{F}^s_{\upsilon^2_{C',t}(s)}(C))$. Hence, if $\mathcal{Z}_t^-(C) \cap \mathcal{Z}_t^+(C) = \emptyset$, then C' also satisfies $\mathcal{Z}_t^-(C') \cap \mathcal{Z}_t^+(C') = \emptyset$ and we have the inequality

$$\upsilon^2_{C,t}(s) \leq \upsilon^2_{C',t}(s),$$

which gives the desired inequality. Otherwise, $\Upsilon^2_{C,t}(s) = \infty$, and hence the desired inequality obviously holds. □

As a corollary, we have the invariance of Υ^2. (Note that Υ^2 is originally given as an invariant of formal knot complexes in [8].)

Theorem 2.62 ([8, Theorem 4.8]) *For any* $t \in (0, 2)$ *and* $s \in [0, 2]$, *the map*

$$\Upsilon^2_t(s) \colon [C]_{\nu^+} \mapsto \Upsilon^2_{C,t}(s)$$

is well-defined as a map $\mathcal{C}^f \to \mathbb{R} \cup \{\infty\}$. *In particular,* $\Upsilon^2_{K,t}(s) := \Upsilon^2_{C^K,t}(s)$ *is a knot concordance invariant.*

We also mention the following sub-additivity of $\Upsilon^2_{C,t}(t)$.

Theorem 2.63 ([8, Theorem 5.1]) *For any formal knot complexes C, C' and $t \in (0,2)$, we have*

$$\Upsilon^2_{C\otimes_\Lambda C',t}(t) \geq \min\{\Upsilon^2_{C,t}(t), \Upsilon^2_{C',t}(t)\}.$$

3 Geometric estimates

In this section, we prove the following theorem.

Theorem 1.5 *For any knot K, we have*

$$-g_4(K)[T_{2,3}]_{\nu^+} \leq [K]_{\nu^+} \leq g_4(K)[T_{2,3}]_{\nu^+}.$$

To prove the theorem, we consider replacing a given knot K several times. We start with the following lemma.

Lemma 3.1 *For any knot K, there exists a knot K' concordant to K which bounds a ribbon surface with genus $g_4(K)$.*

Proof Let F be a surface in $B^4 \cong (S^3 \times [0,1])/(S^3 \times \{1\})$ with genus $g_4(K)$ and $\partial F = K \subset S^3 \times \{0\}$. Then, a similar argument to [21, Lemma2.1] shows that F can be isotoped to a surface F' in B^4 such that the composition $f : F' \hookrightarrow (S^3 \times [0,1])/(S^3 \times \{1\}) \overset{p_2}{\twoheadrightarrow} [0,1]$ is a Morse function, and f satisfies

1. All births happen at time $\frac{1}{6}$ (we denote the number of births by b),
2. b saddles happen at time $\frac{2}{6}$,
3. The time $\frac{3}{6}$ is a regular value and $f^{-1}(\frac{3}{6})$ is connected,
4. The remaining saddles happen at time $\frac{4}{6}$, and
5. All deaths happen at $\frac{5}{6}$.

In particular, we see that $f^{-1}([0,\frac{3}{6}])$ is a (ribbon) concordance from K to $K' := f^{-1}(\frac{3}{6})$, and $f^{-1}([\frac{3}{6},1]$ is a ribbon surface in $(S^3 \times [\frac{3}{6},1])/(S^3 \times \{1\}) \cong B^4$ whose boundary is K' and genus is $g_4(K)$. This completes the proof. □

Next, by using full-twists, we construct a surface embedded in S^3.

Lemma 3.2 *If a knot K bounds a genus g ribbon surface, then there exists a knot K' with genus g which is obtained from K only by adding positive full-twists with 1-linking.*

Proof Suppose that K bounds a genus g ribbon surface F with n ribbon singularities. Then, for proving the lemma, it suffices to find a positive full-twist with 1-linking deforming K into a knot K' which bounds a genus g ribbon surface with $n-1$ ribbon singularities.

Let Σ_g be an abstract genus g surface with $\partial\Sigma_g \cong S^1$, and $f : \Sigma_g \to S^3$ an immersion with $f(\Sigma_g) = F$. Choose a ribbon singularity b on F. Then $f^{-1}(b)$

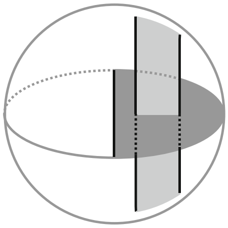


Fig. 2 A local picture near $f(a) \cup b$

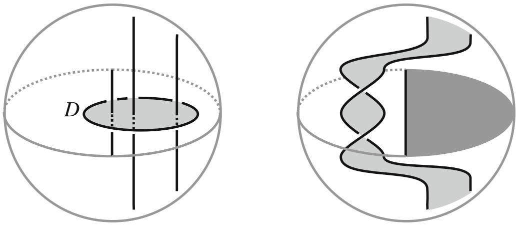


Fig. 3 A disk D and a surface F'

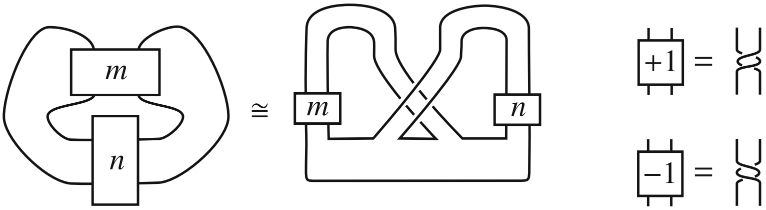


Fig. 4 The (m, n)-twist knot $K_{m,n}$

consists of two arcs in Σ_g, one of which is properly embedded and the other is lying in $\mathrm{Int}\, \Sigma_g$. Denote the arc in $\mathrm{Int}\, \Sigma_g$ by $\widetilde{b}$, and take an arc a in Σ_g such that $\mathrm{Int}\, a$ avoids the preimage of all ribbon singularities on F, and one end of a is in $\partial \widetilde{b}$ and the other is in $\partial \Sigma_g$. Then $f(a)$ is an arc in F which connects b to ∂F, and $\mathrm{Int}\, f(a)$ avoids all singularities on F. Thus, we can take a (small) tubular neighborhood N of $f(a) \cup b$ such that $(N, F \cap N)$ is diffeomorphic to the pair of the 3-ball and the immersed surface shown in Fig. 2.

Now, we take a twisting disk D as shown in the left-hand side of Fig. 3. After adding a positive full-twist along D, we have a new ribbon surface F' which coincides with F in $S^3 \backslash N$, and $(N, F' \cap N)$ is diffeomorphic to the pair of the 3-ball and the embedded surface shown in the right-hand side of Fig. 3. By the construction, it is obvious that $K' := \partial F'$ is obtained from K by a positive full-twist with 1-linking, and F' is a genus g ribbon surface with $n - 1$ ribbon singularities. This completes the proof. □

For $m, n \in \mathbb{Z}$, let $K_{m,n}$ denote the (m, n)-twist knot, whose diagram is shown in Fig. 4. Then, the final replacement is stated as follows.

Lemma 3.3 *Any genus g knot is deformed into the knot $K_{m_1,n_1} \# \cdots \# K_{m_g,n_g}$ only by adding positive full-twists with* 0*-linking, where $m_i, n_i \in \mathbb{Z}_{>0}$ (for all $i \in \{1, \ldots, g\}$).*

Proof Let K be a genus g knot and F a genus g surface with boundary K. By an isotopy, we can assume that F is of the form of Fig. 5, where L is obtained from a

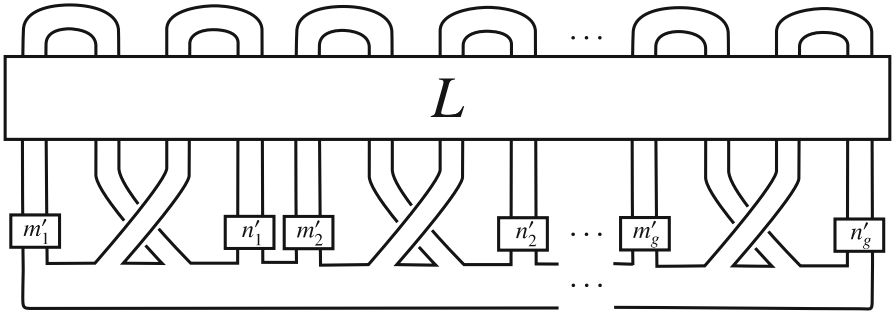


Fig. 5 A description of F by a string link

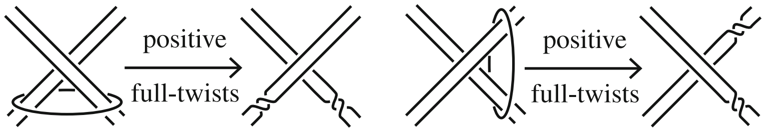


Fig. 6 Pass moves with framings changing

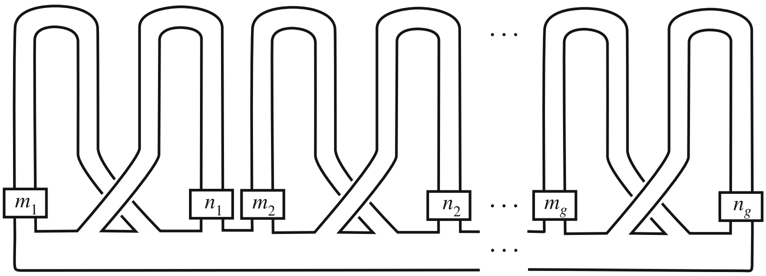
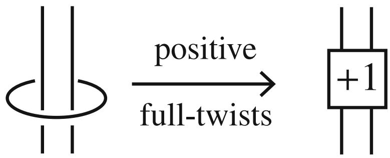


Fig. 7 A surface F'

Fig. 8 A positive full-twist increasing a framing



string link (with $4g$ strings) by parallelizing the string link with arbitrary framings. (The framings are characterized by a choice of $\{m'_1, n'_1, \ldots, m'_g, n'_g\}$.) Then, as shown in Fig. 6, positive full-twists with 0-linking can realize both directions of pass moves with framings changing, and hence such full-twists can deform F into a surface F' with new framings $\{m_1, n_1, \ldots, m_g, n_g\}$, which is shown in Fig. 7. Moreover, by adding positive full-twists with 0-linking as shown in Fig. 8, we may assume that all m_i, n_i are positive. Here it is obvious that the boundary of F' is $K_{m_1,n_1} \# \cdots \# K_{m_g,n_g}$, and this fact completes the proof. □

Here we note that all $K_{m,n}$ are 2-bridge knots and hence alternating knots. For alternating knots, the following strong classification theorem of ν^+-classes follows from [20, Section 3.1].

Theorem 3.4 ([20, Section 3.1]) *For any alternating knot K, we have $[K]_{\nu^+} = -\frac{\sigma(K)}{2}[T_{2,3}]_{\nu^+}$, where $\sigma(K)$ is the knot signature of K.*

Now we can determine the ν^+-classes of the $K_{m,n}$.

Lemma 3.5 *For any $m, n > 0$, $[K_{m,n}]_{\nu^+} = -[T_{2,3}]_{\nu^+}$.*

Proof It is easy to verify that for any $m, n > 0$, we have $\sigma(K_{m,n}) = 2$. Therefore, Theorem 3.4 completes the proof. □

Now we prove Theorem 1.5.

Proof of Theorem 1.5 Fix a knot K. Then, Lemma 3.1 provides a knot K' such that $[K']_{\nu^+} = [K]_{\nu^+}$ and K' bounds a ribbon surface with genus $g_4(K)$. Moreover, it follows from Lemmas 3.2 and 3.3 that there exists a sequence of finitely many positive full-twists with 0 or 1-linking which deforms K into $K_{m_1,n_1}\#\cdots\#K_{m_{g_4(K)},n_{g_4(K)}}$ for some $m_i, n_i \in \mathbb{Z}_{>0}$ ($i \in \{1,\ldots,g_4(K)\}$). Therefore, by Theorem 2.42 and Lemma 3.5, we have

$$[K]_{\nu^+} = [K']_{\nu^+} \geq \sum_{1\leq i\leq g_4(K)} [K_{m_i,n_i}]_{\nu^+} = -g_4(K)[T_{2,3}]_{\nu^+}.$$

Since $g_4(-K^*) = g_4(K)$, we also have

$$-[K]_{\nu^+} = [-K^*]_{\nu^+} \geq -g_4(K)[T_{2,3}]_{\nu^+}.$$

This completes the proof. □

4 Algebraic estimates

In this section, we establish several algebraic estimate for the ν^+-classes, and prove Theorems 1.2 and 1.6.

4.1 Genus of a formal knot complex

We first define the genus of formal knot complexes.

4.1.1 Maximal and minimal degrees

For a formal knot complex C, set

$$\mathrm{Mdeg}(C) := \min\left\{m \in \mathbb{Z} \;\middle|\; \widehat{\mathcal{F}}_m = \widehat{\mathrm{C}}\right\}$$

and

$$\mathrm{mdeg}(C) := \min\left\{m \in \mathbb{Z} \;\middle|\; \widehat{\mathcal{F}}_m \neq 0\right\}.$$

(For the definition of $\{\widehat{\mathcal{F}}_m\}_{m\in\mathbb{Z}}$, see Sect. 2.10.2.) Let $\{x_k\}_{1\le k\le r}$ be a filtered basis for C. The finiteness of the above values follows from the following lemma.

Lemma 4.1 *The equalities*

$$\mathrm{Mdeg}(C) = \max_{1\le k\le r}\{\mathrm{Alex}(x_k) - \mathrm{Alg}(x_k)\}$$

and

$$\mathrm{mdeg}(C) = \min_{1\le k\le r}\{\mathrm{Alex}(x_k) - \mathrm{Alg}(x_k)\}$$

hold.

Proof From the definition of $\{\widehat{\mathcal{F}}_m\}_{m\in\mathbb{Z}}$, we can see that

$$\widehat{\mathcal{F}}_m = \mathrm{span}_{\mathbb{F}}\left\{U^{\mathrm{Alg}(x_k)}x_k \;\middle|\; \mathrm{Alex}(x_k) - \mathrm{Alg}(x_k) \le m\right\}.$$

This completes the proof. □

Corollary 4.2 *The equalities*

$$\mathrm{Mdeg}(C^*) = -\,\mathrm{mdeg}(C) \textit{ and } \mathrm{mdeg}(C^*) = -\,\mathrm{Mdeg}(C)$$

hold.

Proof As shown in the proof of Proposition 2.13, we can take a filtered basis $\{x_k^*\}_{1\le k\le r}$ such that

$$\mathrm{Alex}(x_k^*) = -\,\mathrm{Alex}(x_k) \text{ and } \mathrm{Alg}(x_k^*) = -\,\mathrm{Alg}(x_k).$$

This completes the proof. □

Moreover, about the decomposition $\{C_{(i,j)}\}_{(i,j)\in\mathbb{Z}^2}$ induced by a filtered basis $\{x_k\}$, we have the following lemma.

Lemma 4.3 *The support* $\{(i, j) \mid C_{(i,j)} \neq 0\}$ *is contained in the set*

$$\{\mathrm{mdeg}(C) \le j - i \le \mathrm{Mdeg}(C)\}.$$

Proof If $U^l x_k$ is lying in $C_{(i,j)}$, then

$$\mathrm{Alex}(x_k) - \mathrm{Alg}(x_k) = \mathrm{Alex}(U^l x_k) - \mathrm{Alg}(U^l x_k) = j - i.$$

Therefore, by Lemma 4.1, we have $\mathrm{mdeg}(C) \le j - i \le \mathrm{Mdeg}(C)$. □

For a coordinate $(k, l) \in \mathbb{Z}^2$, set

$$R_{(k,l)} := \{(i, j) \in \mathbb{Z}^2 \mid i \leq k \text{ and } j \leq l\},$$

and then $R_{(k,l)} \in \mathcal{CR}$. For any subset $S \subset \mathbb{Z}^2$, define the *closure* of S by

$$cl(S) := \bigcup_{(i,j)\in S} R_{(i,j)}.$$

Then we also have $cl(S) \in \mathcal{CR}(\mathbb{Z}^2)$. In addition, the equality

$$cl(S) = \bigcap_{R\in\mathcal{CR}(\mathbb{Z}^2),\, S\subset R} R$$

holds. For any $R \in \mathcal{CR}(\mathbb{Z}^2)$ and $m, M \in \mathbb{Z}$ with $m \leq M$, define

$$S^R_{m,M} := \{(i, j) \in R \mid m \leq j - i \leq M\}.$$

Then, as a corollary of Lemma 4.3, we have the following.

Corollary 4.4 *For any formal knot complex C and $R \in \mathcal{CR}(\mathbb{Z}^2)$, the equality*

$$C_R = C_{cl(S^R_{\mathrm{mdeg}(C),\mathrm{Mdeg}(C)})}$$

holds.

Proof Since $R \supset cl(S^R_{\mathrm{mdeg}(C),\mathrm{Mdeg}(C)})$, obviously we have

$$C_R \supset C_{cl(S^R_{\mathrm{mdeg}(C),\mathrm{Mdeg}(C)})}.$$

Next we prove the converse. Fix a filtered basis $\{x_k\}_{1\leq k\leq r}$ and denote the induced decomposition by $\{C_{(i,j)}\}$. By Lemma 2.9, it suffices to show that for any $(i, j) \in R\backslash cl(S^R_{\mathrm{mdeg}(C),\mathrm{Mdeg}(C)})$, the equality $C_{(i,j)} = 0$ holds. Indeed, for any such coordinate (i, j), at least one of the inequalities

$$j - i < \mathrm{mdeg}(C) \text{ and } j - i > \mathrm{Mdeg}(C)$$

holds. Therefore, it follows from Lemma 4.3 that $C_{(i,j)} = 0$. □

4.1.2 Genus of a formal knot complex

Now we define the *genus of a formal knot complex* C by

$$g(C) := \max\{\mathrm{Mdeg}(C), -\mathrm{mdeg}(C)\}.$$

Then it is obvious that $g(C) \geq 0$, and Corollary 4.2 gives $g(C^*) = g(C)$. Moreover, for knot complexes, we have the following.

Theorem 4.5 ([14], [15, Section 5]) *For any knot K, the equality*

$$g(K) = \min\{g(C) \mid C \in [C^K]\}$$

holds.

Moreover, by definition, we have $-g(C) \leq \mathrm{mdeg}(C) \leq \mathrm{Mdeg}(C) \leq g(C)$. Hence Corollary 4.4 gives the following.

Corollary 4.6 *For any formal knot complex C and $R \in \mathcal{CR}(\mathbb{Z}^2)$, the equality*

$$C_R = C_{cl(S^R_{-g(C),g(C)})}$$

holds.

The following lemma is useful for reducing C_R in concrete situations.

Lemma 4.7 *The following assertions hold:*

1. *For any $k \in \mathbb{Z}$, we have $C_{\{i \leq k\}} = C_{R_{(k,g(C)+k)}}$.*
2. *For any $l \in \mathbb{Z}$, we have $C_{\{j \leq l\}} = C_{R_{(g(C)+l,l)}}$.*

Proof Here we verify the assertion (1). For any $k \in \mathbb{Z}$, we see

$$\begin{aligned} S^{\{i \leq k\}}_{-g(C),g(C)} &= \{i \leq k\} \cap \{-g(C) \leq j - i \leq g(C)\} \\ &\subset \{i \leq k\} \cap \{j \leq g(C) + i\} \\ &\subset \{i \leq k\} \cap \{j \leq g(C) + k\} = R_{(k,g(C)+k)}. \end{aligned}$$

Therefore, we have $\{i \leq k\} \supset R_{(k,g(C)+k)} \supset cl(S^{\{i \leq k\}}_{-g(C),g(C)})$, and hence Corollary 4.6 gives $C_{\{i \leq k\}} = C_{R_{(k,g(C)+k)}}$. Similarly, we can verify the assertion (2). □

4.2 Comparison with $[(T_{2,2g+1})^*]_{\nu^+}$

For $g \in \mathbb{Z}_{\geq 0}$, let $T_{2,2g+1}$ be the $(2, 2g+1)$-torus knot. These knots are alternating knots such that $\sigma(T_{2,2g+1}) = -2g$, and hence it follows from Theorem 3.4 that $[T_{2,2g+1}]_{\nu^+} = g[T_{2,3}]_{\nu^+}$. In this subsection, we consider comparing ν^+-classes with $[(T_{2g+1})^*]_{\nu^+}$. First, we recall that the knot complex $C^{(T_{2,2g+1})^*}$ has a filtered basis

$$\{a_k, b_l \mid 0 \leq k \leq g,\ 0 \leq l \leq g-1\}$$

satisfying:

- $\mathrm{gr}(a_k) = 0$ and $\mathrm{gr}(b_l) = -1$.
- $(\mathrm{Alg}(a_k), \mathrm{Alex}(a_k)) = (-g+k, -k)$ and $(\mathrm{Alg}(b_l), \mathrm{Alex}(b_l)) = (-g+l, -l-1)$.
- $\partial a_k = b_{k-1} + b_k$ and $\partial b_l = 0$, where $b_{-1} = b_g = 0$.

Here we note that $a := a_0 + \cdots + a_g$ is a unique homological generator of $C^{(T_{2,2g+1})^*}$. For any $g \in \mathbb{Z}_{\geq 0}$, define

$$R^g := \bigcup_{0 \leq n \leq g} R_{(-g+n,-n)}.$$

Then we have the following sufficient condition for satisfying the inequality $[C]_{\nu^+} \leq [(T_{2,2g+1})^*]_{\nu^+}$.

Proposition 4.8 *For any formal knot complex* C*, if* C_{R^g} *contains a homological generator, then the inequality*

$$[C]_{\nu^+} \leq [(T_{2,2g+1})^*]_{\nu^+}$$

holds.

Proof Fix a filtered basis and denote the induced decomposition by $\{C_{(i,j)}\}$. Define the subsets $S_k \subset \mathbb{Z}^2$ $(k = 0, 1, \ldots, g)$ by

$$S_0 := R_{(-g,0)}$$

and

$$S_k := \{i = -g + k,\ j \leq -k\}$$

for $1 \leq k \leq g$. Then $R^g = \amalg_{0 \leq k \leq g} S_k$, and hence we can uniquely decompose a homological generator $z \in C_{R^g}$ into a linear combination $z = \sum_{k=0}^{g} z_k$, where $z_k \in \bigoplus_{(i,j) \in S_k} C_{(i,j)}$. We denote $y_l := \partial(z_0 + \ldots + z_l)$ for any $0 \leq l \leq g-1$.

Claim 1 *y_l is lying in $C_{R_{(-g+l,-l-1)}}$.*

Proof Since z is a cycle, we see $y_l = \partial(z_0 + \cdots + z_l) = \partial(z_{l+1} + \cdots + z_g)$. Moreover, since the relations

$$\bigcup_{0 \leq k \leq l} S_k = \bigcup_{0 \leq k \leq l} R_{(-g+k,-k)}$$

and

$$\bigcup_{l+1 \leq k \leq g} S_k \subset \bigcup_{l+1 \leq k \leq g} R_{(-g+k,-k)}$$

hold, we have $y_l \in \left(C_{\bigcup_{0\le k\le l} R_{(-g+k,-k)}}\right) \cap \left(C_{\bigcup_{l+1\le k\le g} R_{(-g+k,-k)}}\right)$. Here, Lemma 2.9 gives

$$\begin{aligned}
&\left(C_{\bigcup_{0\le k\le l} R_{(-g+k,-k)}}\right) \cap \left(C_{\bigcup_{l+1\le k\le g} R_{(-g+k,-k)}}\right)\\
&= \left(\bigoplus_{(i,j)\in\bigcup_{0\le k\le l} R_{(-g+k,-k)}} C_{(i,j)}\right) \cap \left(\bigoplus_{(i,j)\in\bigcup_{l+1\le k\le g} R_{(-g+k,-k)}} C_{(i,j)}\right)\\
&= C_{(\bigcup_{0\le k\le l} R_{(-g+k,-k)})\cap(\bigcup_{l+1\le k\le g} R_{(-g+k,-k)})} = C_{R_{(-g+l,-l-1)}}.
\end{aligned}$$

□

Now, we define a Λ-linear map $f\colon C^{(T_{2,2g+1})^*} \to C$ by

$$fa_k = z_k \text{ and } fb_l = y_l.$$

Then we can check that f is a chain map over Λ. (Notice that $\partial z_k = \partial(z_0 + \cdots + z_{k-1}) + \partial(z_0 + \cdots + z_k) = y_{k-1} + y_k$.) Moreover, by Claim 1, we have

$$(\mathrm{Alg}(fa_k), \mathrm{Alex}(fa_k)) \le (-g+k, -k)$$

and

$$(\mathrm{Alg}(fb_l), \mathrm{Alex}(fb_l)) \le (-g+l, -l-1).$$

In addition, $f(a) = f(a_0 + \cdots + a_g) = z_0 + \cdots + z_g = z$. Now, Lemma 2.40 proves that f is a $\mathbb{Z}^2$-filtered quasi-isomorphism. □

4.3 An estimate of genus one complexes

Here, we consider an estimate for genus one formal knot complexes.

Theorem 4.9 *Let C be a formal knot complex with $g(C) = 1$.*

1. *If $\tau(C) = 1$, then $[C]_{\nu^+} \ge [T_{2,3}]_{\nu^+}$.*
2. *If $\tau(C) = 0$, then $[C]_{\nu^+} = 0$.*
3. *If $\tau(C) = -1$, then $[C]_{\nu^+} \le -[T_{2,3}]_{\nu^+}$.*

Proof By Lemma 2.48, we have a homological generator lying in $C_{\{i\le -1\}\cup R_{(0,\tau(C))}}$. Moreover, Lemmas 2.10 and 4.7 imply that

$$C_{\{i\le-1\}\cup R_{(0,\tau(C))}} = C_{\{i\le -1\}} + C_{R_{(0,\tau(C))}} = C_{R_{(-1,0)}} + C_{R_{(0,\tau(C))}} = C_{R_{(-1,0)}\cup R_{(0,\tau(C))}}.$$

As a result, we have a homological generator in $C_{R_{(-1,0)}\cup R_{(0,\tau(C))}}$.

First, suppose that $\tau(C) = 0$. Then $C_{R_{(-1,0)}\cup R_{(0,\tau(C))}} = C_{R_{(0,0)}}$. This proves $\nu^+(C) = 0$. Moreover, since $\tau(C^*) = -\tau(C) = 0$ and $g(C^*) = g(C) = 1$, we also have $\nu^+(C^*) = 0$. Therefore, the assertion (2) holds.

Next, suppose that $\tau(C) = -1$. Then $C_{R_{(-1,0)} \cup R_{(0,\tau(C))}} = C_{R_{(-1,0)} \cup R_{(0,-1)}} = C_{R^1}$. Therefore, it follows from Proposition 4.8 that

$$[C]_{\nu^+} \leq [(T_{2,3})^*]_{\nu^+} = -[T_{2,3}]_{\nu^+},$$

and the assertion (3) holds.

Finally, the assertion (1) follows from the assertion (3) and the fact that $\tau(C^*) = -\tau(C) = -1$ and $[C^*]_{\nu^+} = -[C]_{\nu^+}$. □

Now we can prove the main theorem.

Theorem 1.2 *For any knot K with $g(K) = 1$, we have*

$$[K]_{\nu^+} = \begin{cases} \left[T_{2,3}\right]_{\nu^+} & \text{if } \tau(K) = 1 \\ [\text{unknot}]_{\nu^+} = 0 & \text{if } \tau(K) = 0 \\ \left[(T_{2,3})^*\right]_{\nu^+} = -[T_{2,3}]_{\nu^+} & \text{if } \tau(K) = -1 \end{cases}.$$

In other words, any genus one knot is ν^+-equivalent to one of the trefoil, its mirror and the unknot.

Proof Let K be a genus one knot. Then, by Theorem 1.5, we have

$$-[T_{2,3}]_{\nu^+} \leq [K]_{\nu^+} \leq [T_{2,3}]_{\nu^+}.$$

Moreover, by Theorem 4.5, we can take a knot complex C^K with $g(C^K) = 1$. Hence, Theorem 4.9 gives

$$[K]_{\nu^+} \begin{cases} \geq \left[T_{2,3}\right]_{\nu^+} & \text{if } \tau(K) = 1 \\ = 0 & \text{if } \tau(K) = 0 \\ \leq -[T_{2,3}]_{\nu^+} & \text{if } \tau(K) = -1 \end{cases}.$$

This completes the proof. □

4.4 An estimate using Υ

Here we show an estimate which is obtained by using Υ.

Theorem 4.10 *If $\Upsilon_C(1) = g(C)$, then $[C]_{\nu^+} \leq -g(C)[T_{2,3}]_{\nu^+}$.*

Proof By the definition of Υ, we have a homological generator which lies in $C_{\{i+j \leq -g(C)\}}$. Here, we note that

$$\{i + j \leq -g(C)\} \subset \{i \leq -g(C)\} \cup R^{g(C)} \cup \{j \leq -g(C)\}.$$

Moreover, Lemmas 2.10 and 4.7 imply that

$$C_{\{i \leq -g(C)\} \cup R^{g(C)} \cup \{j \leq -g(C)\}} = C_{\{i \leq -g(C)\}} + C_{R^{g(C)}} + C_{\{j \leq -g(C)\}}$$

$$= C_{R_{(-g(C),0)}} + C_{R^{g(C)}} + C_{R_{(0,-g(C))}} = C_{R^{g(C)}}.$$

As a result, we have a homological generator in $C_{R^{g(C)}}$. Therefore, Proposition 4.8 proves that $[C]_{\nu^+} \leq [(T_{2,2g(C)+1})^*]_{\nu^+} = -g(C)[T_{2,3}]_{\nu^+}$. □

Now we can prove the following discriminant.

Theorem 4.11 *The equality* $[K]_{\nu^+} = -g(K)[T_{2,3}]_{\nu^+}$ *holds if and only if* $\Upsilon_K(1) = g(K)$.

Proof If $[K]_{\nu^+} = -g(K)[T_{2,3}]_{\nu^+}$, then $\Upsilon_K(1) = -g(K)\Upsilon_{T_{2,3}}(1) = g(K)$. Let us prove the converse. For any knot K, by Theorem 1.5, we have

$$[K]_{\nu^+} \geq -g(K)[T_{2,3}]_{\nu^+}.$$

Moreover, by Theorem 4.5, we can take a knot complex C^K with $g(C^K) = g(K)$. Hence, if $\Upsilon_K(1) = g(K)$, then Theorem 4.10 gives

$$[K]_{\nu^+} \leq -g(K)[T_{2,3}]_{\nu^+}.$$

This completes the proof. □

5 New concordance invariants

In this section, we discuss new invariants $\{\mathcal{G}_n\}$ of ν^+-classes whose values are finite subsets of $\mathcal{CR}(\mathbb{Z}^2)$.

5.1 The invariants $\widetilde{\mathcal{G}}_0$ and $\mathcal{G}_0$

As seen in Sect. 2.10, many invariants introduced in previous work can be translated into the words of closed regions containing a homological generator. From the view point, it is natural to consider the universal set

$$\widetilde{\mathcal{G}}_0(C) := \{R \in \mathcal{CR}(\mathbb{Z}^2) \mid C_R \text{ contains a homological generator}\}.$$

In fact, it behaves very naturally in terms of filtered quasi-isomorphism.

Theorem 5.1 *If* $[C]_{\nu^+} \leq [C']_{\nu^+}$, *then* $\widetilde{\mathcal{G}}_0(C) \supset \widetilde{\mathcal{G}}_0(C')$.

Proof By Theorem 2.38, we have a $\mathbb{Z}^2$-filtered quasi-isomorphism $f : C' \to C$. Therefore, for any element $R \in \widetilde{\mathcal{G}}_0(C')$ and a homological generator $x \in C'_R$, we see that C_R also contains a homological generator $f(x)$, and hence $R \in \widetilde{\mathcal{G}}_0(C)$. □

As a corollary, we have the invariance of $\widetilde{\mathcal{G}}_0$. Here $\mathcal{P}(\mathcal{CR}(\mathbb{Z}^2))$ denotes the power set of $\mathcal{CR}(\mathbb{Z}^2)$.

Corollary 5.2 *$\widetilde{\mathcal{G}}_0(C)$ is invariant under ν^+-equivalence. In particular,*

$$\widetilde{\mathcal{G}}_0\colon [C]_{\nu^+} \mapsto \widetilde{\mathcal{G}}_0(C)$$

is a well-defined map $\mathcal{C}^f \to \mathcal{P}(\mathcal{CR}(\mathbb{Z}^2))$.

By definition, $\widetilde{\mathcal{G}}_0(C)$ obviously has the following property.

Proposition 5.3 *For any $R \in \widetilde{\mathcal{G}}_0(C)$ and $R' \in \mathcal{CR}(\mathbb{Z}^2)$, if $R \subset R'$, then $R' \in \widetilde{\mathcal{G}}_0(C)$.*

In particular, we see that $\widetilde{\mathcal{G}}_0(C)$ is an infinite set. To extract an essential part of $\widetilde{\mathcal{G}}_0$, we consider the minimalization of $\widetilde{\mathcal{G}}_0$.

For a subset $\mathcal{S} \subset \mathcal{CR}(\mathbb{Z}^2)$, an element $R \in \mathcal{S}$ is *minimal in $\mathcal{S}$* if it satisfies

$$\text{if } R' \in \mathcal{S} \text{ and } R' \subset R, \text{ then } R' = R.$$

Define the map

$$\min\colon \mathcal{P}(\mathcal{CR}(\mathbb{Z}^2)) \to \mathcal{P}(\mathcal{CR}(\mathbb{Z}^2))$$

by

$$\mathcal{S} \mapsto \{R \in \mathcal{S} \mid R \text{ is minimal in } \mathcal{S}\}.$$

Now we define $\mathcal{G}_0(C)$ by

$$\mathcal{G}_0(C) := \min \widetilde{\mathcal{G}}_0(C).$$

The invariance of $\mathcal{G}_0(C)$ under $\overset{\nu^+}{\sim}$ immediately follows from Corollary 5.2.

Here, for referring later, we prove the following lemma.

Lemma 5.4 *Let $\mathcal{S} \subset \mathcal{CR}(\mathbb{Z}^2)$ be a non-empty finite subset. Then, for any $R \in \mathcal{S}$, there exists an element $R' \in \min \mathcal{S}$ with $R' \subset R$. In particular, $\min \mathcal{S}$ is non-empty.*

Proof We prove the lemma by the induction of the order of S. If $|S| = 1$, then $\min \mathcal{S} = \mathcal{S}$, and the assertion obviously holds.

Assume that for any subsets of $\mathcal{CR}(\mathbb{Z}^2)$ with order n, the assertion holds. Let $\mathcal{S} \subset \mathcal{CR}(\mathbb{Z}^2)$ be a subset with order $n+1$. if any element of $\mathcal{S}$ is minimal in $\mathcal{S}$, then the assertion holds for $\mathcal{S}$. Suppose that there exist elements $R, R' \subset \mathcal{S}$ such that $R' \subsetneq R$. Then, since $\mathcal{S}\backslash\{R\}$ has order n, the assertion holds for $\mathcal{S} \setminus \{R\}$. In particular, we have an element $R'' \in \min(\mathcal{S}\backslash\{R\})$ with $R'' \subset R'$. Here we note that $R'' \in \min \mathcal{S}$, since $R \not\subset R''$ and R'' is not required to satisfy $R = R''$. Moreover, we have $R'' \subset R' \subset R$. This implies that the assertion holds for $\mathcal{S}$, and completes the proof. ◻

5.2 Finiteness of $\mathcal{G}_0$

In this subsection, we show that $\mathcal{G}_0(C)$ is a finite set for any formal knot complex C.

5.2.1 The region of a chain

For a non-zero element $p = p(U) \in \Lambda$, denote the lowest degree of p by $l(p)$. Let C be a formal knot complex, and $\{x_k\}_{1\leq k\leq r}$ a filtered basis for C. For any non-zero chain $x = \sum_{1\leq k\leq r} p_k(U)x_k$, we define *the region of* x as

$$R_x := cl\left\{(\mathrm{Alg}(U^{l(p_k)}x_k), \mathrm{Alex}(U^{l(p_k)}x_k)) \;\middle|\; \begin{matrix} 1 \leq k \leq r \\ p_k(U) \neq 0 \end{matrix}\right\}.$$

Then we see that $R_x \in \mathcal{CR}(\mathbb{Z}^2)$ and $x \in C_{R_x}$. The following lemma implies that R_x does not depend on the choice of $\{x_k\}$.

Lemma 5.5 *The equality*

$$R_x = \bigcap_{R\in\mathcal{CR},\, x\in C_R} R$$

holds. In particular, $x \in C_R$ *if and only if* $R_x \subset R$.

Proof It is obvious that $R_x \supset \bigcap_{R\in\mathcal{CR},\, x\in C_R} R$. We prove the converse. Let $\{C_{(i,j)}\}$ be the decomposition of C induced by $\{x_k\}$, and take $R \in \mathcal{CR}(\mathbb{Z}^2)$ with $x \in C_R$. Then, since $C_R = \bigoplus_{(i,j)\in R} C_{(i,j)}$ and

$$C_{(i,j)} = \mathrm{span}_{\mathbb{F}}\{U^l x_k \mid (\mathrm{Alg}(U^l x_k), \mathrm{Alex}(U^l x_k)) = (i,j)\},$$

we see that

$$\left(\mathrm{Alg}(U^{l(p_k)}x_k), \mathrm{Alex}(U^{l(p_k)}x_k)\right) \in R$$

for any $k \in \{1, \ldots, r\}$ with $p_k(U) \neq 0$. This completes the proof. □

Lemma 5.6 *For any* $\mathbb{Z}^2$*-filtered chain map* $f : C \to C'$ *and* $x \in C$*, we have* $R_{f(x)} \subset R_x$.

Proof Since $x \in C_{R_x}$, we see

$$f(x) \in f(C_{R_x}) \subset C'_{R_x}.$$

Hence, Lemma 5.5 proves $R_{f(x)} \subset R_x$. □

5.2.2 The regions of homological generators

For a formal knot complex C, define

$$\widetilde{\mathrm{gen}}_0(C) := \{\text{homological generator of } C\},$$
$$\widetilde{\mathcal{G}}'_0(C) := \{R_x \in \mathcal{CR}(\mathbb{Z}^2) \mid x \in \widetilde{\mathrm{gen}}_0(C)\},$$

and

$$\mathcal{G}'_0(C) := \min \widetilde{\mathcal{G}}'_0(C).$$

In addition, for $R \in \mathcal{G}_0(C)$, set

$$\mathrm{gen}_0(C; R) := \{x \in \widetilde{\mathrm{gen}}_0(C) \mid R_x = R\},$$

and call $x \in \mathrm{gen}_0(C; R)$ *a realizer of* R. Notice that since $\dim_{\mathbb{F}} C_0 < \infty$, C has finitely many homological generators, and hence both $\widetilde{\mathcal{G}}'_0(C)$ and $\mathcal{G}'_0(C)$ are finite and non-empty. Therefore, the following theorem implies the finiteness and non-emptiness of $\mathcal{G}_0(C)$.

Theorem 5.7 *The equality* $\mathcal{G}_0(C) = \mathcal{G}'_0(C)$ *holds.*

Proof We first prove $\mathcal{G}_0(C) \supset \mathcal{G}'_0(C)$. Note that since $x \in C_{R_x}$ for any homological generator x, we have $\widetilde{\mathcal{G}}_0(C) \supset \widetilde{\mathcal{G}}'_0(C)$. Take $R_x \in \mathcal{G}'_0(C)$, and suppose that $R \in \widetilde{\mathcal{G}}_0(C)$ and $R \subset R_x$. Then, there exists a homological generator x' in C_R, and hence Lemma 5.5 implies $R_{x'} \subset R \subset R_x$. Here, since $R_{x'} \in \widetilde{\mathcal{G}}'_0(C)$ and R_x is minimal in $\widetilde{\mathcal{G}}'_0(C)$, we have $R_{x'} = R = R_x$. This proves $R_x \in \mathcal{G}_0(C)$, and hence $\mathcal{G}_0(C) \supset \mathcal{G}'_0(C)$.

Next we prove $\mathcal{G}_0(C) \subset \mathcal{G}'_0(C)$. For a given element $R \in \mathcal{G}_0(C)$, we first need to prove that $R \in \widetilde{\mathcal{G}}'_0(C)$. Here, in a similar way to the above arguments, we see that there exists a homological generator x such that $R_x \subset R$. Moreover, since R_x is also in $\widetilde{\mathcal{G}}_0(C)$ and R is minimal in $\widetilde{\mathcal{G}}_0(C)$, we have $R = R_x \in \widetilde{\mathcal{G}}'_0(C)$. Now, the minimality of R in $\widetilde{\mathcal{G}}'_0(C)$ immediately follows from the minimality in $\widetilde{\mathcal{G}}_0(C)$. Therefore, we have $R \in \mathcal{G}'_0(C)$, and hence $\mathcal{G}_0(C) \subset \mathcal{G}'_0(C)$. □

As a corollary, we have the following useful property of $\mathcal{G}_0(C)$.

Corollary 5.8 *For any formal knot complex C and* $R \in \mathcal{CR}(\mathbb{Z}^2)$*,the following holds:*

$$R \in \widetilde{\mathcal{G}}_0(C) \Leftrightarrow \exists R' \in \mathcal{G}_0(C),\, R' \subset R.$$

Proof Suppose that $R \in \widetilde{\mathcal{G}}_0(C)$. Then, by the definition of $\widetilde{\mathcal{G}}_0(C)$ and Lemma 5.5, there exists a homological generator $x \in C$ with $R_x \subset R$. Moreover, since $R_x \in \widetilde{\mathcal{G}}'_0(C)$ and $\widetilde{\mathcal{G}}'_0(C)$ is a non-empty finite set, Lemma 5.4 gives an element $R' \in \mathcal{G}'_0(C) = \mathcal{G}_0(C)$ with $R' \subset R_x \subset R$. The converse follows from Proposition 5.3. □

Here we also mention the relationship of $\mathcal{G}_0(C)$ to the partial order on $\mathcal{C}^f$.

Proposition 5.9 *If* $[C]_{\nu^+} \leq [C']_{\nu^+}$*, then for any* $R' \in \mathcal{G}_0(C')$*, there exists an element* $R \in \mathcal{G}_0(C)$ *with* $R \subset R'$.

Proof For any $R' \in \mathcal{G}_0(C')$, Theorem 5.1 shows $R' \in \widetilde{\mathcal{G}}_0(C)$. Now, by Corollary 5.8, we have an element $R \in \mathcal{G}_0(C)$ with $R \subset R'$. □

5.3 Higher invariants $\mathcal{G}_n$

Here, we discuss higher invariants.

5.3.1 The secondary invariant $\mathcal{G}_1$

For a formal knot complex C, suppose that $\mathcal{G}_0(C)$ has distinct two elements R_1 and R_2. Under the hypothesis, we define the secondary invariant $\mathcal{G}_1(C; R_1, R_2)$ as follows. First, set

$$\widetilde{\mathrm{gen}}_1(C; R_1, R_2) := \{x \in C_1 \mid \exists z_i \in \mathrm{gen}_0(C; R_i), \partial x = z_1 + z_2\},$$

and

$$\widetilde{\mathcal{G}}'_1(C; R_1, R_2) := \{R_x \mid x \in \widetilde{\mathrm{gen}}_1(C; R_1, R_2)\}.$$

Then we define $\mathcal{G}_1(C; R_1, R_2)$ by

$$\mathcal{G}_1(C; R_1, R_2) := \min \widetilde{\mathcal{G}}'_1(C; R_1, R_2).$$

Here, for $R \in \mathcal{G}_1(C; R_1, R_2)$, we also define *the realizers of* R by

$$\mathrm{gen}_1(C; R_1, R_2; R) := \{x \in \widetilde{\mathrm{gen}}_1(C; R_1, R_2) \mid R_x = R\}.$$

Note that the above notions are independent of the order of $\{R_1, R_2\}$.

Lemma 5.10 *$\mathcal{G}_1(C; R_1, R_2)$ is a non-empty finite set.*

Proof Take an arbitrary realizer $z_i \in \mathrm{gen}_0(C; R_i)$ for each $i = 1, 2$. Then we see that $0 \neq [z_1] = [z_2] \in H_0(C) \cong \mathbb{F}$, and hence there exists a 1-chain $x \in C_1$ such that $\partial x = z_1 + z_2$. Moreover, $\dim_{\mathbb{F}}(C_1) < \infty$. These facts shows that $\widetilde{\mathrm{gen}}_1(C; R_1, R_2)$ is non-empty and finite. Combining this fact with Lemma 5.4, we see that $\mathcal{G}_1(C; R_1, R_2)$ is non-empty and finite. □

Theorem 5.11 *Suppose that $[C]_{\nu^+} \leq [C']_{\nu^+}$ and $\mathcal{G}_0(C) \cap \mathcal{G}_0(C')$ has distinct two elements R_1 and R_2. Then, for any $R' \in \mathcal{G}_1(C'; R_1, R_2)$, there exists an element $R \in \mathcal{G}_1(C; R_1, R_2)$ with $R \subset R'$.*

Proof Take $z_i \in \mathrm{gen}_0(C'; R_i)$ $(i = 1, 2)$ and $x \in \mathrm{gen}_1(C'; R_1, R_2; R')$ such that $\partial x = z_1 + z_2$. Let $f : C' \to C$ be a $\mathbb{Z}^2$-filtered quasi-isomorphism. Then we see from the assumption and Lemma 5.6 that

$$R_{f(z_i)}, R_i \in \widetilde{\mathcal{G}}'_0(C)$$

and

$$R_{f(z_i)} \subset R_{z_i} = R_i.$$

Moreover, R_i is minimal in $\widetilde{\mathcal{G}}'_0(C)$, and hence we have $R_{f(z_i)} = R_i$. In particular, $f(z_i) \in \mathrm{gen}_0(C; R_i)$. Here, note that

$$\partial(f(x)) = f(\partial x) = f(z_1) + f(z_2),$$

and hence $f(x) \in \widetilde{\mathrm{gen}}_1(C; R_1, R_2)$ and $R_{f(x)} \in \mathcal{G}_1(C; R_1, R_2)$. Now, Lemma 5.4 and Lemma 5.6 give an element $R \in \mathcal{G}_1(C; R_1, R_2)$ with

$$R \subset R_{f(x)} \subset R_x = R'.$$

□

Corollary 5.12 *For any $[C]_{\nu^+} \in \mathcal{C}^f$ and distinct two elements $R_1, R_2 \in \mathcal{G}_0(C)$, $\mathcal{G}_1(C; R_1, R_2) \in \mathcal{P}(\mathcal{CR}(\mathbb{Z}^2))$ is an invariant of the ν^+-class $[C]_{\nu^+}$.*

Proof Suppose that $[C]_{\nu^+} = [C']_{\nu^+}$. Then, since $\mathcal{G}_0(C) = \mathcal{G}_0(C')$, we have

$$R_1, R_2 \in \mathcal{G}_0(C) \cap \mathcal{G}_0(C').$$

Let $R \in \mathcal{G}_1(C; R_1, R_2)$. Since $[C]_{\nu^+} \geq [C']_{\nu^+}$, Theorem 5.11 gives an element $R' \in \mathcal{G}_1(C'; R_1, R_2)$ with $R' \subset R$. Moreover, since $[C]_{\nu^+} \leq [C']_{\nu^+}$, we also have $R'' \in \mathcal{G}_1(C; R_1, R_2)$ with $R'' \subset R' \subset R$. Here, since R is minimal in $\mathcal{G}_1(C; R_1, R_2)$, we have

$$R'' = R' = R,$$

and hence $R = R' \in \mathcal{G}_1(C'; R_1, R_2)$. This proves $\mathcal{G}_1(C; R_1, R_2) \subset \mathcal{G}_1(C'; R_1, R_2)$.

In the same way, we also have $\mathcal{G}_1(C; R_1, R_2) \supset \mathcal{G}_1(C'; R_1, R_2)$. □

5.3.2 Higher invariants $\mathcal{G}_n$ with $n \geq 2$

Now we construct more higher invariants $\mathcal{G}_n$ by induction. Let n be an integer with $n \geq 2$, and assume that

$$\begin{array}{ll} \exists R_1^0, R_2^0 \in \mathcal{G}_0(C) & \text{with } R_1^0 \neq R_2^0, \\ \exists R_1^1, R_2^1 \in \mathcal{G}_1(C; \{R_1^0, R_2^0\}) & \text{with } R_1^1 \neq R_2^1, \\ \cdots & \\ \exists R_1^{n-1}, R_2^{n-1} \in \mathcal{G}_{n-1}(C; \{R_1^j, R_2^j\}_{j=0}^{n-2}) & \text{with } R_1^{n-1} \neq R_2^{n-1}. \end{array}$$

Then, we define

$$\widetilde{\mathrm{gen}}_n(C; \{R_1^j, R_2^j\}_{i=0}^{n-1}) := \left\{ x \in C_n \;\middle|\; \begin{array}{r} \exists z_i \in \mathrm{gen}_{n-1}(C; \{R_1^j, R_2^j\}_{j=0}^{n-2}; R_i^{n-1}) \\ \text{s.t.} \begin{cases} \partial z_1 = \partial z_2 \\ \partial x = z_1 + z_2 \end{cases} \end{array} \right\},$$

$$\widetilde{\mathcal{G}}'_n(C; \{R_1^j, R_2^j\}_{j=0}^{n-1}) := \{R_x \mid x \in \widetilde{\mathrm{gen}}_n(C; \{R_1^j, R_2^j\}_{j=0}^{n-1})\}, \text{ and}$$

$$\mathcal{G}_n(C;\{R_1^j,R_2^j\}_{j=0}^{n-1}) := \min \widetilde{\mathcal{G}}'_n(C;\{R_1^j,R_2^j\}_{j=0}^{n-1}).$$

In addition, for $R \in \mathcal{G}_n(C;\{R_1^j,R_2^j\}_{j=0}^{n-1})$, we define

$$\mathrm{gen}_n(C;\{R_1^j,R_2^j\}_{j=0}^{n-1};R) := \{x \in \widetilde{\mathrm{gen}}_n(C;\{R_1^j,R_2^j\}_{j=0}^{n-1}) \mid R_x = R\}.$$

Unlike the cases of $\mathcal{G}_0$ and $\mathcal{G}_1$, it is unknown whether $\mathcal{G}_n(C;\{R_1^j,R_2^j\}_{j=0}^{n-1})$ is empty or not, while we see that it is finite. (This is caused by the condition $\partial z_1 = \partial z_2$.) However, if $\mathcal{G}_n(C;\{R_1^j,R_2^j\}_{j=0}^{n-1})$ is non-empty, then we can show that it is invariant under ν^+-equivalence. (As a consequence, the emptiness of $\mathcal{G}_n$ is also an invariant of ν^+-classes.)

Theorem 5.13 *Suppose that $[C]_{\nu^+} \le [C']_{\nu^+}$ and the intersection*

$$\mathcal{G}_k(C;\{R_1^j,R_2^j\}_{j=1}^{k-1}) \cap \mathcal{G}_k(C';\{R_1^j,R_2^j\}_{j=1}^{k-1})$$

has distinct two elements R_1^k and R_2^k (where $k = 0, 1, \ldots, n-1$, and $\{R_1^j,R_2^j\}_{j=0}^{-1} = \emptyset$). Then, for any $R' \in \mathcal{G}_n(C';\{R_1^j,R_2^j\}_{j=1}^{n-1})$, there exists an element

$$R \in \mathcal{G}_n(C;\{R_1^j,R_2^j\}_{j=1}^{n-1})$$

with $R \subset R'$. In particular, the non-emptiness of $\mathcal{G}_n(C';\{R_1^j,R_2^j\}_{j=1}^{n-1})$ implies the non-emptiness of $\mathcal{G}_n(C;\{R_1^j,R_2^j\}_{j=1}^{n-1})$.

Proof The proof follows from arguments exactly the same as the proof of Theorem 5.11. (We only need to care about the fact that

$$f(z_i) \in \mathrm{gen}_{n-1}(C;\{R_1^j,R_2^j\}_{j=0}^{n-2};R_i^{n-1}),$$

but this also can be proved by induction.) □

Corollary 5.14 *For any $[C]_{\nu^+} \in \mathcal{C}^f$ and sequence of distinct two elements $R_1^k, R_2^k \in \mathcal{G}_k(C;\{R_1^j,R_2^j\}_{j=0}^{k-1})$ $(k = 0, 1, \ldots, n-1)$, the element*

$$\mathcal{G}_n(C;\{R_1^j,R_2^j\}_{j=1}^{n-1}) \in \mathcal{P}(\mathcal{CR}(\mathbb{Z}^2))$$

is an invariant of the ν^+-class $[C]_{\nu^+}$.

Proof The proof follows from arguments exactly the same as the proof of Corollary 5.12. (In fact, we only need to replace some symbols suitably.) □

5.4 Relationship to other invariants

In this subsection, we study the relationship of the new invariants $\mathcal{G}_0$ and $\mathcal{G}_1$ to the invariants reviewed in Sect. 2.10.

5.4.1 Relationship of $\mathcal{G}_0$ to ν^+, V_k, τ and Υ

We first discuss the relationship of $\mathcal{G}_0$ to ν^+. Here, recall that $R_{(k,l)}$ is defined by

$$R_{(k,l)} := \{(i, j) \in \mathbb{Z}^2 \mid (i, j) \leq (k, l)\}.$$

Proposition 5.15 *For any formal knot complex C, the invariants $\nu^+(C)$ and $\nu^+(C^*)$ are determined from $\mathcal{G}_0(C)$ by the formulas*

$$\nu^+(C) = \min\{m \in \mathbb{Z}_{\geq 0} \mid \exists R \in \mathcal{G}_0(C), R \subset R_{(0,m)}\}$$

and

$$\nu^+(C^*) = \min\{m \in \mathbb{Z}_{\geq 0} \mid \forall R \in \mathcal{G}_0(C), R \supset R_{(0,-m)}\}.$$

Proof We can see that the equality

$$\nu^+(C) = \min\left\{m \in \mathbb{Z}_{\geq 0} \;\middle|\; R_{(0,m)} \in \widetilde{\mathcal{G}}_0(C)\right\}$$

holds. Therefore, the first assertion immediately follows from Corollary 5.8.

Next, by Lemma 2.28, the inequality $\nu^+(C^*) > m$ holds if and only if there is a homological generator $x \in C$ with $R_x \subset \{i \leq -1 \text{ or } j \leq -m-1\}$. Here, we note that $R_x \subset \{i \leq -1 \text{ or } j \leq -m-1\}$ if and only if $R_x \not\supset R_{(0,-m)}$. Therefore, we have

$$\nu^+(C^*) = \min\left\{m \in \mathbb{Z}_{\geq 0} \;\middle|\; \forall R_x \in \widetilde{\mathcal{G}}'_0(C), R_x \supset R_{(0,-m)}\right\}.$$

Moreover, Lemma 5.4 implies that any $R_x \in \widetilde{\mathcal{G}}'_0(C)$ includes $R_{(0,-m)}$ if and only if any $R_x \in \mathcal{G}_0(C)$ includes $R_{(0,-m)}$. This completes the proof. □

From Proposition 5.15, we see that $\mathcal{G}_0$ detects the zero element as a ν^+-class.

Theorem 5.16 *For any formal knot complex C, the following holds:*

$$[C]_{\nu^+} = 0 \Leftrightarrow \mathcal{G}_0(C) = \{R_{(0,0)}\}$$

Proof By the invariance of $\mathcal{G}_0$ under $\overset{\nu^+}{\sim}$ and easy computation $\mathcal{G}_0(\Lambda) = \{R_{(0,0)}\}$, it is obvious that $[C]_{\nu^+} = 0$ implies $\mathcal{G}_0(C) = \{R_{(0,0)}\}$. Moreover, the converse immediately follows from Proposition 5.15, since the unique element $R := R_{(0,0)} \in \mathcal{G}_0(C)$ satisfies $R \subset R_{(0,0)}$ and $R \supset R_{(0,0)}$. □

On the other hand, we will see in Sect. 5.5 that $\mathcal{G}_0$ is not a perfect invariant of ν^+-classes.

We can also translate the invariants V_k, τ and Υ as follows:

$$\begin{aligned} V_k(C) &= \min\left\{m \in \mathbb{Z}_{\geq 0} \;\middle|\; R_{(m,k+m)} \in \widetilde{\mathcal{G}}_0(C)\right\} \\ \tau(C) &= \min\left\{m \in \mathbb{Z} \;\middle|\; (\{i \leq -1\} \cup R_{(0,m)}) \in \widetilde{\mathcal{G}}_0(C)\right\} \\ \Upsilon_C(t) &= -2\left(\min\left\{s \in \mathbb{R} \;\middle|\; R^t(s) \in \widetilde{\mathcal{G}}_0(C)\right\}\right) \end{aligned}$$

(Here, recall $R^t(s) := \{(i,j) \in \mathbb{Z}^2 \mid (1-t/2)i + (t/2)j \leq s\}$.) Therefore, we have the following formulas.

Proposition 5.17 *For any formal knot complex C, the invariants $V_k(C)$, $\tau(C)$ and $\Upsilon_C(t)$ are determined from $\mathcal{G}_0(C)$ by the formulas:*

$$\begin{aligned} V_k(C) &= \min\left\{m \in \mathbb{Z}_{\geq 0} \;\middle|\; \exists R \in \mathcal{G}_0(C),\, R \subset R_{(m,k+m)}\right\} \\ \tau(C) &= \min\left\{m \in \mathbb{Z} \;\middle|\; \exists R \in \mathcal{G}_0(C),\, R \subset (\{i \leq -1\} \cup R_{(0,m)})\right\} \\ \Upsilon_C(t) &= -2\left(\min\left\{s \in \mathbb{R} \;\middle|\; \exists R \in \mathcal{G}_0(C),\, R \subset R^t(s)\right\}\right) \end{aligned}$$

5.4.2 Relationship of $\mathcal{G}_1$ to Υ^2

Next, we discuss the relationship of $\mathcal{G}_1$ to Υ^2. (Precisely, we compare $\mathcal{G}_1$ with υ^2 rather than Υ^2.) Let

$$\mathcal{G}_0^{t\pm}(C) := \{R \in \mathcal{G}_0(C) \mid R \subset \mathcal{F}^{t\pm\delta}_{\upsilon_C(t\pm\delta)}\},$$

and then we see that the inequality

$$\mathcal{Z}_t^{\pm}(C) \supset \bigcup_{R \in \mathcal{G}_0^{t\pm}(C)} \mathrm{gen}_0(C; R)$$

holds for each sign. (Remark that it does not become the equality in general, since we might have $x \in \widetilde{\mathrm{gen}}_0(C)$ such that $R \subsetneq R_x \subset \mathcal{F}^{t\pm\delta}_{\upsilon_C(t\pm\delta)}$ for some $R \in \mathcal{G}_0^{t\pm}(C)$. Such x is lying in $\mathcal{Z}^{\pm}(C)$ but not in the right-hand side.) In particular, $\mathcal{Z}_t^-(C) \cap \mathcal{Z}_t^+(C) = \emptyset$ only if $\mathcal{G}_0^{t-}(C) \cap \mathcal{G}_0^{t+}(C) = \emptyset$.

For any $t \in (0,2)$, we set

$$\mathcal{G}_1^t(C) := \bigcup_{R^{\pm} \in \mathcal{G}_0^{t\pm}(C),\, R^- \neq R^+} \mathcal{G}_1(C; R^-, R^+).$$

Then, we have the following inequality. (In light of the inequality, we can regard $\upsilon^2_{C,t}$ as a linear approximation of $\mathcal{G}_1^t(C)$.)

Proposition 5.18 *For any formal knot complex C, $t \in (0,2)$ and $s \in [0,2]$, the inequality*

$$\upsilon^2_{C,t}(s) \leq \min\left\{r \in \mathbb{R} \;\middle|\; \exists R \in \mathcal{G}_1^t(C),\, R \subset \left(R^t(\upsilon_C(t)) \cup R^s(r)\right)\right\}.$$

holds.

Proof Denote the right-hand side of the inequality in Proposition 5.18 by the symbol $\upsilon^2_{\mathcal{G}^t_1(C)}(s)$. Then, we can take $R \in \mathcal{G}^t_1(C)$ with $R \subset (R^t(\upsilon_C(t)) \cup R^s(\upsilon^2_{\mathcal{G}^t_1(C)}(s)))$. Moreover, by the definition of $\mathcal{G}^t_1(C)$, there exist elements $R^\pm \in \mathcal{G}^{t\pm}_0(C)$ such that $R^- \neq R^+$ and $R \in \mathcal{G}_1(C; R^-, R^+)$. This implies that we have a homological generator

$$z^\pm \in \mathrm{gen}_0(C; R^\pm) \subset \mathcal{Z}^\pm_t(C)$$

for each sign and 1-chain $x \in C_{R_x} \subset C_{R^t(\upsilon_C(t)) \cup R^s(\upsilon^2_{\mathcal{G}^t_1(C)}(s))}$ such that $\partial x = z^- - z^+$. Here, by Lemma 2.10, we see

$$C_{R^t(\upsilon_C(t)) \cup R^s(\upsilon^2_{\mathcal{G}^t_1(C)}(s))} = C_{R^t(\upsilon_C(t))} + C_{R^s(\upsilon^2_{\mathcal{G}^t_1(C)}(s))} = \mathcal{F}^t_{\upsilon_C(t)} + \mathcal{F}^s_{\upsilon^2_{\mathcal{G}^t_1(C)}(s)},$$

and hence $[z^-] - [z^+] = [\partial x] = 0 \in H_0(\mathcal{F}^t_{\upsilon_C(t)} + \mathcal{F}^s_{\upsilon^2_{\mathcal{G}^t_1(C)}(s)})$. This shows the desired inequality $\upsilon^2_{C,t}(s) \leq \upsilon^2_{\mathcal{G}^t_1(C)}(s)$. □

5.5 Genus one complexes with no realizing knot

In this subsection, we define the complexes C^n precisely, and prove Theorem 1.8 and Corollary 1.9.

For any $n \in \mathbb{Z}_{>0}$, we define an $\mathbb{F}$-vector space $\bar{C}^n$ with a basis $\{x_k, x'_k, y\}_{k=0}^{n-1}$ and $\mathbb{F}$-linear map $\bar{\partial} \colon \bar{C}^n \to \bar{C}^n$ as follows:

$$\begin{cases} \mathrm{gr}(x_k) = \mathrm{gr}(x'_k) = k & (0 \leq k \leq n-1) \\ \mathrm{gr}(y) = n \end{cases}$$

$$\begin{cases} \bar{\partial} x_0 = \bar{\partial} x'_0 = 0 \\ \bar{\partial} x_k = \bar{\partial} x'_k = x_{k-1} + x'_{k-1} & (1 \leq k \leq n-1) \\ \bar{\partial} y = x_{n-1} + x'_{n-1} \end{cases}$$

$$\begin{cases} (\mathrm{Alg}(x_k), \mathrm{Alex}(x_k)) = (k, k+1) & (0 \leq k \leq n-1) \\ (\mathrm{Alg}(x'_k), \mathrm{Alex}(x'_k)) = (k+1, k) & (0 \leq k \leq n-1) \\ (\mathrm{Alg}(y), \mathrm{Alex}(y)) = (n, n) \end{cases}$$

Then we can check that $(\bar{C}, \bar{\partial})$ satisfies all conditions of Lemma 2.3. Figure 1 in Sect. 1 depicts the complex $(\bar{C}^n, \bar{\partial}$.) Therefore, we have a formal knot complex (C, ∂) which is related to $(\bar{C}, \bar{\partial})$ as described in Lemma 2.3. Note that C^1 coincides with the knot complex for the right-hand trefoil $T_{2,3}$. Moreover, $g(C^n) = 1$ for any n.

Proposition 5.19 *For any $n \in \mathbb{Z}_{>0}$, C^n satisfies the following:*

1. $\mathcal{G}_0(C^n) = \{R_{(0,1)}, R_{(1,0)}\}$.

2. $\mathcal{G}_k(C^n; \{R_{(j,j+1)}, R_{(j+1,j)}\}_{j=0}^{k-1}) = \{R_{(k,k+1)}, R_{(k+1,k)}\}$ $(1 \le k \le n-1)$.
3. $\mathcal{G}_n(C^n; \{R_{(j,j+1)}, R_{(j+1,j)}\}_{j=0}^{n-1}) = \{R_{(n,n)}\}$.

Proof Obviously, we see that

$$\widetilde{\mathrm{gen}}_0(C^n) = \{x_0, x_0'\}$$

and

$$\widetilde{\mathcal{G}}_0'(C^n) = \{R_{x_0}, R_{x_0'}\} = \{R_{(0,1)}, R_{(1,0)}\}.$$

Moreover, both $R_{(0,1)}$ and $R_{(1,0)}$ are minimal in $\{R_{(0,1)}, R_{(1,0)}\}$, and hence we have $\mathcal{G}_0(C^n) = \{R_{(0,1)}, R_{(1,0)}\}$ and $\mathrm{gen}_0(C^n) = \{x_0, x_0'\}$.

Next, fix $m \in \{0, 1, \dots, n-2\}$, and assume that the assertion (2) holds for any $1 \le k \le m$. Then the equalities

$$\mathrm{gen}_m(C^n; \{R_{(j,j+1)}, R_{(j+1,j)}\}_{j=0}^{m-1}; R_{(m,m+1)}) = \{x_m\}$$

and

$$\mathrm{gen}_m(C^n; \{R_{(j,j+1)}, R_{(j+1,j)}\}_{j=0}^{m-1}; R_{(m+1,m)}) = \{x_m'\}$$

must hold. Now we see

$$\widetilde{\mathrm{gen}}_{m+1}(C^n; \{R_{(j,j+1)}, R_{(j+1,j)}\}_{j=0}^{m}) = \{x_{m+1}, x_{m+1}'\},$$

and hence we can conclude

$$\mathcal{G}_{m+1}(C^n; \{R_{(j,j+1)}, R_{(j+1,j)}\}_{j=0}^{m}) = \{R_{(m+1,m+2)}, R_{(m+2,m+1)}\}.$$

This proves the assertion (2). Similarly, we can prove the assertion (3). □

Now, we can easily prove the following theorems from the above computation.

Theorem 1.8 *The ν^+-classes $\{[C^n]_{\nu^+}\}_{n=1}^{\infty}$ are mutually distinct in $\mathcal{C}^f$, while $\tau(C^n) = 1$ for any n. In particular, the complement $\mathcal{F}_1^f \setminus \pi_{\nu^+}(\mathcal{F}_1)$ is infinite.*

Proof The first half assertion directly follows from Proposition 5.19. Moreover, since $\tau(k[T_{2,3}]_{\nu^+}) = k$, $\tau(C^n) = 1$ and $[C^n]_{\nu^+} \ne [C^1]_{\nu^+} = [T_{2,3}]_{\nu^+}$ for any $k \in \mathbb{Z}$ and $n \ge 2$, we have $[C^n]_{\nu^+} \ne k[T_{2,3}]_{\nu^+}$. This proves the second half assertion. □

Corollary 1.9 *The formal knot complexes $\{C^n\}_{n=2}^{\infty}$ cannot be realized by any knot in S^3.*

Proof If there exists a knot K with $[C^K] = [C^n]$ for some $n \ge 2$, then it follows from Proposition 5.19 and Theorem 4.5 that $\tau(K) = \tau(C^n) = 1$ and $g(K) = 1$. (Note that

$1 = g(C^n) \geq \min\{g(C) \mid C \in [C^n] = [C^K]\} = g(K) \geq \tau(K) = 1$.) Therefore, by Theorem 1.2, we have

$$[C^n]_{\nu^+} = [K]_{\nu^+} = [T_{2,3}]_{\nu^+} = [C^1]_{\nu^+},$$

which contradicts to Theorem 1.8. □

Acknowledgements The authors would like to thank Jennifer Hom, Min Hoon Kim and JungHwan Park for many interesting conversations about the present work. The author was supported by JSPS KAKENHI Grant Number 18J00808.

References

1. Bodnár, J., Celoria, D., Golla, M.: A note on cobordisms of algebraic knots. Algebr. Geom. Topol. **17**(4), 2543–2564 (2017)
2. Hedden, M.: Knot Floer homology of Whitehead doubles. Geom. Topol. **11**, 2277–2338 (2007)
3. Hedden, M., Watson, L.: On the geography and botany of knot floer homology. Selecta Math. **24**(2), 997–1037 (2018)
4. Hom, J.: A survey on Heegaard Floer homology and concordance. J. Knot Theory Ramif. **26**(2), 1740015 (2017)
5. Hom, J., Zhongtao, W.: Four-ball genus bounds and a refinement of the Ozváth-Szabó tau invariant. J. Symplectic Geom. **14**(1), 305–323 (2016)
6. Juhász, A.: Floer homology and surface decompositions. Geom. Topol. **12**(1), 299–350 (2008)
7. Kim, M.H., Park, K.: An infinite-rank summand of knots with trivial Alexander polynomial. J. Symplectic Geom. **16**(6), 1749–1771 (2018)
8. Kim, S.-G., Livingston, C.: Secondary upsilon invariants of knots. Q. J. Math. **69**(3), 799–813 (2018)
9. Lickorish, W.B.R.: *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, (1997)
10. Livingston, C.: Notes on the knot concordance invariant upsilon. Algebr. Geom. Topol. **17**(1), 111–130 (2017)
11. Ni, Y.: Knot Floer homology detects fibred knots. Invent. Math. **170**(3), 577–608 (2007)
12. Ni, Y., Zhongtao, W.: Cosmetic surgeries on knots in S^3. J. Reine Angew. Math. **706**, 1–17 (2015)
13. Ozsváth, P., Szabó, Z.: Knot Floer homology and the four-ball genus. Geom. Topol. **7**, 615–639 (2003)
14. Ozsváth, P., Szabó, Z.: Holomorphic disks and genus bounds. Geom. Topol. **8**, 311–334 (2004)
15. Ozsváth, P., Szabó, Z.: Holomorphic disks and knot invariants. Adv. Math. **186**(1), 58–116 (2004)
16. Ozsváth, P., Szabó, Z.: Holomorphic disks and topological invariants for closed three-manifolds. Ann. Math. **159**(3), 1027–1158 (2004)
17. Ozsváth, P.S., Stipsicz, A.I., Szabó, Z.: Concordance homomorphisms from knot Floer homology. Adv. Math. **315**, 366–426 (2017)
18. Ozsváth, P.S., Szabó, Z.: Knot Floer homology and integer surgeries. Algebr. Geom. Topol. **8**(1), 101–153 (2008)
19. Ozsváth, P.S., Szabó, Z.: Knot Floer homology and rational surgeries. Algebr. Geom. Topol. **11**(1), 1–68 (2011)
20. Petkova, I.: Cables of thin knots and bordered Heegaard Floer homology. Quantum Topol. **4**(4), 377–409 (2013)
21. Sarkar, S.: Grid diagrams and the Ozsváth-Szabó tau-invariant. Math. Res. Lett. **18**(6), 1239–1257 (2011)
22. Sato, K.: A full-twist inequality for the ν^+-invariant. Topol. Appl. **245**, 113–130 (2018)