



On tropical intersection theory

Andreas Mihatsch¹

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Abstract

We develop a tropical intersection formalism of forms and currents that extends classical tropical intersection theory in two ways. First, it allows to work with arbitrary polyhedra, also non-rational ones. Second, it allows for smooth differential forms as coefficients. The intersection product in our formalism can be defined through the diagonal intersection method of Allermann–Rau or the fan displacement rule. We prove with a limiting argument that both definitions agree.

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1 Introduction

For their “Tropical approach to non-archimedean Arakelov theory” [9], Gubler–Künnemann combine tropical intersection theory and smooth differential forms into

✉ Andreas Mihatsch
mihatsch@math.uni-bonn.de

¹ Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

their formalism of so-called δ -forms. They use these to develop a calculus of Green currents on non-archimedean spaces that is related to intersection theory on formal models. The strength of their approach is that δ -forms are simpler to work with than formal models, leading to a computationally accessible handle for certain arithmetic intersection problems.

The present paper contributes to these ideas through the development of a more general and concise theory of δ -forms. This is a purely tropical endeavor: δ -Forms are a natural generalization of tropical cycles and have the same formal properties. For example, they admit pull-backs, push-forwards and a tropical intersection product called the \wedge -product. δ -Forms also encompass Lagerberg's smooth forms [11] and obey the same kind of differential calculus. They furthermore come with a *boundary operator* that generalizes the frequently used corner locus constructions of Esterov [5] and Francois [6], also cf. Gubler–Künnemann [9]. Moreover, our formalism allows *non-rational* polyhedra throughout. For tropical cycles, this generalization had already been obtained by Esterov [5].

We now provide a more detailed description of δ -forms and our results. *Smooth forms* are always meant in the sense of Lagerberg in the following, cf. [11] or §2.1. Recall that a *current* is a continuous linear form on the space of smooth forms with compact support. A smooth form α on \mathbb{R}^n and a polyhedron $\sigma \subseteq \mathbb{R}^n$ define a *current of integration* $(\alpha \wedge \sigma)(\eta) := \int_{\sigma} \alpha \wedge \eta$. (The polyhedron really needs to be weighted for this to work which will be explained below.) A current is called *polyhedral* if it is a locally finite sum $\sum_{i \in I} \alpha_i \wedge \sigma_i$ of such integration currents. In particular, polyhedral currents are entirely combinatorial objects. The following is our main definition.

Definition 1.1 A δ -form on \mathbb{R}^n is a polyhedral current T on \mathbb{R}^n such that both derivatives $d'T$ and $d''T$ are again polyhedral.

The differentials d' and d'' here are taken in the sense of currents, i.e. as the duals of d' and d'' for smooth forms. δ -Forms turn out to be stable under d' and d'' . Additional structure is then provided by defining a δ -form $T = \sum_{i \in I} \alpha_i \wedge \sigma_i$ to be of *tridegree* (p, q, r) if the α_i may be chosen of bidegree (p, q) and the σ_i of codimension r . Then d' naturally decomposes as $d' = d'_p - \partial'$, where d'_p is trihomogeneous of tridegree $(1, 0, 0)$ and ∂' trihomogeneous of tridegree $(0, -1, 1)$. The first summand d'_p is the so-called *polyhedral derivative* $d'_p(\alpha \wedge \sigma) = (d'\alpha) \wedge \sigma$ of Gubler–Künnemann, while ∂' is the above-mentioned boundary operator. The latter is closely related to boundary integration of differential forms and to the corner locus construction, cf. (4.13). A similar decomposition $d'' = d''_p - \partial''$ exists for d'' .

Next, we come to the combinatorial description of δ -forms.

Theorem 1.2 A polyhedral current $T = \sum_{i \in I} \alpha_i \wedge \sigma_i$ is a δ -form if and only if the datum $(\alpha_i, \sigma_i)_{i \in I}$ is balanced in the sense of tropical geometry.

We formulate the relevant balancing condition in (1.1) below. Note that Theorem 1.2 has precursors in the literature: Lagerberg [11, Proposition 4.7], Gubler [8, Proposition 3.8] and Gubler–Künnemann [9, Proposition 2.16] (in successive level of generality) essentially prove it whenever the α_i are smooth functions. Cast in our terminology, they show that the tropical cycles of codimension r with smooth coefficients are exactly the δ -forms of tridegree $(0, 0, r)$.

Theorem 1.2 makes δ -forms behave like tropical cycles and we show that Allermann–Rau’s construction of an intersection product [1] goes through without substantial change. This leads to our main result which is clearly inspired by Gubler–Künnemann’s [9, Proposition 4.15].

Theorem 1.3 *There is a graded-commutative \wedge -product of δ -forms that extends the \wedge -product of smooth forms and the intersection product of tropical cycles. The derivatives d', d'' , the polyhedral derivatives d'_p, d''_p and the boundary derivatives ∂', ∂'' all satisfy the Leibniz rule for \wedge .*

A more precise characterization of the \wedge -product may be found in the main text, cf. Theorem 4.1. We also show that the \wedge -product can be computed by the *fan displacement rule*, cf. Proposition 4.21. Recall that for intersections of tropical cycles, this rule goes back to Fulton–Sturmfels [7] and Mikhalkin [12]. Its equality with Allermann–Rau’s intersection product was shown independently by Rau [14] and Katz [10]. Our proof is similar to the combinatorial one of Rau and based on the observation that the \wedge -product suitably commutes with limits, cf. §4.3.

We next explain the tropical formalism for possibly non-rational polyhedra. For a polyhedron $\sigma \subseteq \mathbb{R}^n$, denote by $N_\sigma \subseteq \mathbb{R}^n$ the linear space spanned by all the elements $x - y$, $x, y \in \sigma$. Given a facet $\tau \subset \sigma$, the subspace $N_\tau \subset N_\sigma$ is of codimension 1. If σ is *rational*, then $(N_\tau \cap \mathbb{Z}^n) \subset (N_\sigma \cap \mathbb{Z}^n)$ is a sublattice of corank 1 and a *normal vector* for $\tau \subset \sigma$ is any vector $n_{\sigma,\tau} \in N_\sigma \cap \mathbb{Z}^n$ that generates $(N_\sigma \cap \mathbb{Z}^n)/(N_\tau \cap \mathbb{Z}^n)$ and points in direction of σ . For the general situation, we consider *weighted* polyhedra instead. A weight for σ is simply a generator $\mu_\sigma \in \det N_\sigma$ up to sign. Equivalently, it is a choice of Haar measure on M_σ . Given a facet inclusion $\tau \subset \sigma$ and respective weights μ_τ and μ_σ , a normal vector is any $n_{\sigma,\tau} \in N_\sigma$ that satisfies $\mu_\sigma = \mu_\tau \wedge n_{\sigma,\tau}$ and points in direction of σ . The two definitions are linked by the observation that every rational polyhedron σ has a natural weight, namely the unique-up-to-sign generator of $\det_{\mathbb{Z}}(N_\sigma \cap \mathbb{Z}^n)$. The balancing condition (1.1) in Theorem 1.2 is now a literal adaption of the classical balancing condition.

Definition 1.4 Consider a polyhedral complex \mathcal{T} , weights $(\mu_\sigma)_{\sigma \in \mathcal{T}}$ for its polyhedra and smooth forms $(\alpha_\sigma)_{\sigma \in \mathcal{T}}$, $\alpha_\sigma \in A(\sigma)$. Here, $A(\sigma)$ denotes the smooth forms on σ . This datum is called *balanced* if for all $\tau \in \mathcal{T}$,

$$\sum_{\sigma \in \mathcal{T}, \tau \subset \sigma \text{ a facet}} \alpha_\sigma|_\tau \otimes n_{\sigma,\tau} \text{ lies in } A(\tau) \otimes_{\mathbb{R}} N_\tau. \tag{1.1}$$

We next elucidate on the intersection theory of weighted polyhedra. Recall that given two properly intersecting *rationally defined* subspaces $N_1, N_2 \subseteq \mathbb{R}^n$, one defines their intersection multiplicity as the lattice index $[\mathbb{Z}^n : (N_1 \cap \mathbb{Z}^n) + (N_2 \cap \mathbb{Z}^n)]$. In the not necessarily rational case, still assuming proper intersection, one instead considers weights μ_1, μ_2 for N_1, N_2 and endows the intersection $N_1 \cap N_2$ with the unique weight ν such that $\mu_1 \otimes \mu_2 = \nu \otimes \mu_{\text{std}}$ under the canonical-up-to-sign identification $\det(V_1 \oplus V_2) = \det((V_1 \cap V_2) \oplus \mathbb{R}^n)$. Here μ_{std} is the standard weight on \mathbb{R}^n . This rule extends to a full description of the \wedge -product of transversally intersecting δ -forms and underlies the fan displacement rule.

Finally, a weight μ for σ is also the precise datum needed to define the integral $\int_{[\sigma, \mu]} \eta$ of a (compactly supported) form η over σ . So in the definition of polyhedral current above, all polyhedra were silently weighted. For this natural reason, weights implicitly occur in Lagerberg [11] and Chambert-Loir–Ducros [3]. In fact, the *calibrages* from [3] are the same as our weights with an additional sign.

Tropical intersection theory has also been extended from \mathbb{R}^n to more general combinatorial spaces. We will not address such questions here but take them up in our related work [13]. More precisely, we develop there a theory of δ -forms on so-called tropical spaces with applications to non-archimedean Arakelov theory.

Layout

Section 2 contains a summary of Lagerberg’s theory of differential forms and introduces the formalism of weights, normal vectors and fiber integration. Section 3 is dedicated to the definition of δ -forms and to the proof of Theorem 1.2. Section 4 contains the main result Theorem 1.3 and some additional properties of δ -forms. The fan displacement rule is Proposition 4.21 and will be proved in Section 4.3.

2 Forms and currents

2.1 Smooth forms

Let $C^\infty(\mathbb{R}^n)$ and $\Omega^p(\mathbb{R}^n)$ denote the smooth functions and “usual” real smooth p -forms on \mathbb{R}^n . We fix the hosting space \mathbb{R}^n for now and simply write C^∞ and Ω^p . The *smooth forms* in this paper, whose definition is due to Lagerberg [11], are the elements of the exterior algebra

$$A := A(\mathbb{R}^n) := \bigwedge_{C^\infty}^* (\Omega^1 \oplus \Omega^1). \tag{2.1}$$

There is a bigrading $A = \bigoplus_{p,q} A^{p,q}$, where $A^{p,q}$ is the piece $\Omega^p \otimes_{C^\infty} \Omega^q$. Elements $\alpha \in A^{p,q}$ are called bihomogeneous of bidegree (p, q) and homogeneous of degree $\deg \alpha = p + q$.

Being an exterior algebra, A is endowed with a natural \wedge -product. It is bihomogeneous in the sense that $A^{p,q} \wedge A^{s,t} \subseteq A^{p+s, q+t}$. It is also graded-commutative, meaning that

$$\alpha \wedge \beta = (-1)^{\deg \alpha \deg \beta} \beta \wedge \alpha \tag{2.2}$$

whenever α and β are homogeneous.

We use the terminology of [3] for differential operators. Write $d_{\text{std}} : C^\infty \rightarrow \Omega^1$ for the usual differential. Given $f \in C^\infty$, we put

$$d' f = (d_{\text{std}} f, 0), \quad d'' f = (0, d_{\text{std}} f) \in \Omega^1 \oplus \Omega^1. \tag{2.3}$$

Denoting by x_1, \dots, x_n the standard coordinates on \mathbb{R}^n , any $\alpha \in A^{p,q}$ is now in a unique way of the form

$$\alpha = \sum_{I, J \subseteq \{1, \dots, n\}, |I|=p, |J|=q} \varphi_{I, J}(x_1, \dots, x_n) d'x_I \wedge d''x_J$$

with $\varphi_{I, J} \in C^\infty$. The above $d' : C^\infty \rightarrow A^{1,0}$ and $d'' : C^\infty \rightarrow A^{0,1}$ extend to A in a unique way that satisfies the *Leibniz rule*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta, \quad d \in \{d', d''\}. \tag{2.4}$$

Concretely, this extension is given as

$$d(\varphi d'x_I \wedge d''x_J) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i} dx_i \wedge d'x_I \wedge d''x_J, \quad d \in \{d', d''\}.$$

The so-defined $d', d'' : A \rightarrow A$ are bihomogeneous of bidegree $(1, 0)$ resp. $(0, 1)$.

Given an affine-linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, there is a natural pull-back map $f^* : A^{p,q}(\mathbb{R}^m) \rightarrow A^{p,q}(\mathbb{R}^n)$ which stems from usual pull-back of differential forms. It commutes with \wedge, d' and d'' .

The integral of an (n, n) -form η with compact support is defined as follows. Write $\eta = \varphi d'x_1 \wedge d''x_1 \wedge \dots \wedge d'x_n \wedge d''x_n$ and put

$$\int_{\mathbb{R}^n} \eta := \int_{\mathbb{R}^n} \varphi \tag{2.5}$$

where the right hand side is defined in terms of the Lebesgue integral for the standard volume on \mathbb{R}^n . It is immediate that, for an affine linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} f^* \eta = |\det f| \int_{\mathbb{R}^n} \eta. \tag{2.6}$$

There is, in particular, no implicit choice of orientation involved. This also reflects in the fact that the forms $d'x_i \wedge d''x_i$ have degree 2, hence pairwise commute, so the function φ in (2.5) is independent of coordinate ordering.

Let $D = D(\mathbb{R}^n)$ denote the space of *currents*, i.e. the topological dual of compactly supported forms A_c , cf. [11, Sect. 1.1]. The topological aspect of the definition will never play a role in this paper. Write $D = \bigoplus_{p,q} D^{p,q}$ where $D^{p,q}$ is dual to $A_c^{n-p, n-q}$. Currents $T \in D^{p,q}$ are said to be of bidegree (p, q) and of degree $\deg T = p + q$. There is an injective map $A^{p,q} \rightarrow D^{p,q}, \alpha \mapsto [\alpha]$, defined by

$$[\alpha](\eta) := \int_{\mathbb{R}^n} \alpha \wedge \eta.$$

With the following sign conventions one defines derivatives $d' : D^{p,q} \rightarrow D^{p+1,q}, d'' : D^{p,q} \rightarrow D^{p,q+1}$ as well as a product $\wedge : A^{p,q} \times D^{s,t} \rightarrow D^{p+s, q+t}$:

$$\begin{aligned} (dT)(\eta) &= (-1)^{\deg T + 1} T(d\eta), \quad d \in \{d', d''\}, \\ (\alpha \wedge T)(\eta) &= (-1)^{\deg \alpha \deg T} T(\alpha \wedge \eta). \end{aligned} \tag{2.7}$$

Then it follows that, for homogeneous α, β, T and $d \in \{d', d''\}$,

$$\begin{aligned} (d[\alpha])(\eta) &= (-1)^{\deg \alpha + 1} \int \alpha \wedge d\eta = \int d\alpha \wedge \eta = [d\alpha](\eta), \\ (\alpha \wedge [\beta])(\eta) &= (-1)^{\deg \alpha \deg \beta} \int \beta \wedge \alpha \wedge \eta = \int \alpha \wedge \beta \wedge \eta = [\alpha \wedge \beta](\eta), \end{aligned} \tag{2.8}$$

for every compactly supported test form η , so the inclusion $A \rightarrow D$ commutes with \wedge, d' and d'' . Furthermore, the Leibniz rule extends:

$$d(\alpha \wedge T) = d\alpha \wedge T + (-1)^{\deg \alpha} \alpha \wedge dT, \quad d \in \{d', d''\}. \tag{2.9}$$

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be an affine linear map. Then there is a push-forward map $D_c^{p,q}(\mathbb{R}^n) \rightarrow D^{p+m-n, q+m-n}(\mathbb{R}^m)$ from currents with compact support. It is defined by

$$(f_*T)(\eta) = T(f^*\eta). \tag{2.10}$$

Since $f^*(d\eta) = d(f^*\eta)$ for $\eta \in A_c(\mathbb{R}^m)$ and $d \in \{d', d''\}$, one obtains the identity $f_*(dT) = d(f_*T)$ by duality, cf. (2.7).

2.2 Polyhedral currents

By *polyhedron* in \mathbb{R}^n we mean a subset σ that may be written as the intersection of finitely many (not necessarily rational) half-spaces. Denote by N_σ the linear space spanned by all $x - y, x, y \in \sigma$ and by $M_\sigma := N_\sigma^\vee$ its \mathbb{R} -dual. The *dimension* of σ is the dimension of N_σ .

A *polyhedral complex* is a locally finite set of polyhedra \mathcal{T} which is stable under taking faces and is such that $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2 for every $\sigma_1, \sigma_2 \in \mathcal{T}$. The d -dimensional (resp. r -codimensional in \mathbb{R}^n) polyhedra of a polyhedral complex are denoted by \mathcal{T}_d (resp. \mathcal{T}^r).

Let $C^\infty(\sigma)$ denote the smooth functions on σ , i.e. all $\varphi: \sigma \rightarrow \mathbb{R}$ such that there is some smooth function $\tilde{\varphi} \in C^\infty(\mathbb{R}^n)$ with $\tilde{\varphi}|_\sigma = \varphi$. Smooth (p, q) -forms on σ are defined by an analogous restriction process. Let $L_\sigma = x + N_\sigma, x \in \sigma$, be the smallest affine linear space containing σ . There is a well-defined space of (p, q) -forms $A^{p,q}(L_\sigma)$ because (p, q) -forms transform naturally under affine linear maps. Then

$$A^{p,q}(\sigma) := C^\infty(\sigma) \otimes_{C^\infty(L_\sigma)} A^{p,q}(L_\sigma).$$

Equivalently, it is the space of smooth (p, q) -forms on the interior σ° of σ in L_σ that come by restriction from $A^{p,q}(L_\sigma)$. Note that $A^{p,q}(\sigma) = 0$ if $\dim \sigma < \max\{p, q\}$ and that there is a restriction map $A^{p,q}(\sigma) \rightarrow A^{p,q}(\tau), \alpha \mapsto \alpha|_\tau$ for every inclusion of polyhedra $\tau \subseteq \sigma$ which commutes with \wedge, d' and d'' . The following definition of weight is inspired by Chambert-Loir–Ducros’ definition of a calibration, cf. [3, Sect. 1.5].

Definition 2.1 A *weight* for a polyhedron σ is a generator $\mu \in \det N_\sigma$ up to sign. The convention for 0-dimensional polyhedra here is that the determinant of the 0-space is \mathbb{R} itself and that a weight is a *positive* scalar. The pair $[\sigma, \mu]$ is called a *weighted polyhedron*. A *weighted polyhedral complex* is the datum of a polyhedral complex \mathcal{T} together with weights $(\mu_\sigma)_{\sigma \in \mathcal{T}}$ for all its polyhedra.

Equivalently, a weight for σ is the choice of a Haar measure for M_σ . The bijection is defined as follows: Given $0 \neq \mu \in \det N_\sigma$, choose a family of $x_i \in N_\sigma$ such that $\mu = x_1 \wedge \dots \wedge x_{\dim \sigma}$ up to sign. Then endow M_σ with the Haar measure $\text{Vol}(U) = \text{Vol}((x_1, \dots, x_{\dim \sigma})(U))$, where the volume on the right hand side is taken for the standard Lebesgue (Haar) measure on $\mathbb{R}^{\dim \sigma}$.

We denote by $\mu^\vee \in \det M_\sigma$ the dual of μ .

Example 2.2 Every *rational* polyhedron $\sigma \subseteq \mathbb{R}^n$ has a natural weight with respect to the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$. Namely $N_\sigma \cap \mathbb{Z}^n$ is a lattice in N_σ and the choice of a generator $\mu_0 \in \det_{\mathbb{Z}}(N_\sigma \cap \mathbb{Z}^n)$ is unique up to sign. Every other weight is in a unique way of the form $\lambda \mu_0, \lambda > 0$.

Let $[\sigma, \mu]$ be a weighted polyhedron of dimension d and let $\eta \in A_c^{d,d}(\sigma)$. Pick any coordinates $x_1, \dots, x_d \in M_\sigma$ such that $\mu^\vee = x_1 \wedge \dots \wedge x_d$ and write $\eta = \varphi d'x_1 \wedge d''x_1 \wedge \dots \wedge d'x_d \wedge d''x_d$. (The x_i are defined up to translation on L_σ , so their differentials $d'x_i$ and $d''x_i$ are canonical.) Then set

$$\int_{[\sigma, \mu]} \eta := \int_\sigma \varphi \tag{2.11}$$

where the right hand side is the Lebesgue integral with respect to the volume defined by the choice of isomorphism $(x_1, \dots, x_d): N_\sigma \cong \mathbb{R}^d$. The transformation rule (2.6) ensures that this is well-defined. In this way, $[\sigma, \mu]$ is viewed as element of $D^{r,r}$, where $r = n - d$ is the codimension of σ .

The following definitions are due to Gubler–Künnemann, cf. [9, Definition 2.3]. A *polyhedral current* is a current that is a locally finite sum of currents of the form $\alpha \wedge [\sigma, \mu]$. Deviating from their notation, we write $P \subseteq D$ for the space of all polyhedral currents and $P^{p,q,r} \subseteq D^{p+r,q+r}$ for those which are locally finite sums of $\alpha \wedge [\sigma, \mu]$ with σ of codimension r and $\alpha \in A^{p,q}(\sigma)$. One easily checks the direct sum decomposition $P = \bigoplus_{p,q,r} P^{p,q,r}$. We also say that elements of $P^{p,q,r}$ are trihomogeneous of tridegree (p, q, r) .

Remark 2.3 When presenting a polyhedral current T as a locally finite sum $T = \sum_{i \in I} \alpha_i \wedge [\sigma_i, \mu_i]$, the datum of all $(\alpha_i, \sigma_i, \mu_i)_{i \in I}$ is unique up to locally finitely many operations of the following kinds: Subdividing the σ_i , adding/removing terms with $\alpha = 0$, replacing (α, σ, μ) by $(\lambda \alpha, \sigma, \lambda^{-1} \mu)$ for some $\lambda > 0$, and exchanging $(\alpha_1, \sigma, \mu) + (\alpha_2, \sigma, \mu)$ and $(\alpha_1 + \alpha_2, \sigma, \mu)$.

Definition 2.4 Let T be a polyhedral current, say $T = \sum_{i \in I} \alpha_i \wedge [\sigma_i, \mu_i]$. Its *polyhedral derivatives* are the polyhedral currents

$$d'_p T := \sum_{i \in I} (d' \alpha_i) \wedge [\sigma_i, \mu_i], \quad d''_p T := \sum_{i \in I} (d'' \alpha_i) \wedge [\sigma_i, \mu_i].$$

It has been remarked before, cf. [9, Remark 2.4 (iii)], that $d'_p T$ and $d' T$ resp. $d''_p T$ and $d'' T$ need not coincide. The derivatives $d' T$ and $d'' T$ may even be non-polyhedral, cf. Example 2.10 below.

A polyhedral complex \mathcal{T} is *subordinate* to T if there is a presentation of the form $T = \sum_{\sigma \in \mathcal{T}} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$. With such \mathcal{T} fixed, the α_σ and μ_σ are uniquely determined up to the replacement of $(\alpha_\sigma, \mu_\sigma)$ by $(\lambda \alpha_\sigma, \lambda^{-1} \mu_\sigma)$, where $\lambda > 0$.

2.3 Functoriality

For an exact sequence of finite dimensional \mathbb{R} -vector spaces

$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0,$$

there is a canonical isomorphism $\det N_2 = \det N_1 \otimes \det N_3$. So given weights μ_i for N_i for two out of $\{N_1, N_2, N_3\}$, they uniquely determine a weight for the third space through the relation

$$\mu_2 = \mu_1 \wedge \mu_3 := \mu_1 \wedge \widetilde{\mu}_3 \tag{2.12}$$

where $\widetilde{\mu}_3 \in \bigwedge^{\dim N_3} N_2$ is any lift of μ_3 .

There is a space $PS(\sigma)$ of *piecewise smooth forms* on a polyhedron σ . By definition, a piecewise smooth form is the datum of a polyhedral complex \mathcal{T} with $\sigma = \cup_{\rho \in \mathcal{T}} \rho$ and smooth forms $\alpha_\rho \in A(\rho)$, $\rho \in \mathcal{T}$, such that $\alpha_\rho|_\tau = \alpha_\tau$ for all $\tau \subseteq \rho$; up to subdivision. We write $PS^{p,q}(\sigma)$ for those with all α_ρ of bidegree (p, q) . If μ is a weight for σ and $\alpha = (\alpha_\rho)_{\rho \in \mathcal{T}} \in PS(\sigma)$ as before, we define the polyhedral current

$$\alpha \wedge [\sigma, \mu] = \sum_{\rho \in \mathcal{T}, \dim \rho = \dim \sigma} \alpha_\rho \wedge [\rho, \mu]. \tag{2.13}$$

Here μ defines a weight for ρ because $N_\rho = N_\sigma$ for dimension reasons. For fixed μ , this defines an embedding $PS(\sigma) \subseteq D(\mathbb{R}^n)$.

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be an affine linear map and $\sigma \subseteq \mathbb{R}^n$ a polyhedron. Then $f(\sigma)$ is again a polyhedron. Let μ be a weight on σ and ν a weight on $f(\sigma)$. Then $K := \ker((f - f(0))|_{N_\sigma}: N_\sigma \rightarrow N_{f(\sigma)})$ acquires a canonical weight δ through (2.12) and there is a natural fiber integration map for forms with compact support, $f_{\delta,*}: A_c^{p,q}(\sigma) \rightarrow PS^{p-k,q-k}(f(\sigma))$, where $k = \dim K$. It satisfies the projection formula

$$\int_{[\sigma, \mu]} \alpha \wedge f^* \eta = \int_{[f(\sigma), \nu]} (f_{\delta,*} \alpha) \wedge \eta$$

which determines it uniquely. In other words, fiber integration provides a representative for the push-forward from (2.10),

$$f_*(\alpha \wedge [\sigma, \mu]) = (f_{\delta,*} \alpha) \wedge [f(\sigma), \nu]. \tag{2.14}$$

In particular, the push-forward of a polyhedral current (with relatively compact support) is polyhedral again. Note that if $\alpha \wedge [\sigma, \mu] \in P_c^{p,q,r}$ and $\dim K = k$ as before, then $f_*(\alpha \wedge [\sigma, \mu]) \in P^{p-k,q-k,r+m-n+k}$. So f_* is not trihomogeneous, but only bihomogeneous.

Example 2.5 If $T = \varphi \wedge [\sigma, \mu] \in P^{0,0,r}$ is a weighted polyhedron with smooth coefficient (cf. [9, Sect.1]), then $f_*T \neq 0$ only if $f|_\sigma$ is injective. In this case we find $f_*(\varphi \wedge [\sigma, \mu]) = (\varphi \circ f^{-1}) \wedge [f(\sigma), f(\mu)]$, which is precisely the classical push-forward of weighted polyhedra in tropical geometry that underlies e.g. the Sturmfels–Tevelev multiplicity formula [15].

Given a *surjective* affine linear map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and a current T on \mathbb{R}^m , we may now also define a *pull-back current* $f^*T \in D(\mathbb{R}^n)$. Namely the fiber integral $f_*\eta$ of a smooth form $\eta \in A_c^{p,q}(\mathbb{R}^n)$ (with respect to the standard weights on \mathbb{R}^n and \mathbb{R}^m) is again smooth and we put

$$(f^*T)(\eta) := T(f_*\eta), \quad \eta \in A_c(\mathbb{R}^n).$$

If $T = \alpha \wedge [\sigma, \mu]$, then one easily finds $f^*T = f^*\alpha \wedge [f^{-1}\sigma, \nu]$, where $\nu = \delta \wedge \mu$ for the natural weight δ on $\ker(f - f(0))$. Since $f_*(d\eta) = d(f_*\eta)$ for $\eta \in A_c(\mathbb{R}^n)$ and $d \in \{d', d''\}$, it follows by duality that $f^*(dT) = d(f^*T)$ for any current T .

Example 2.6 Assume $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is bijective and affine linear. Let μ be the standard weight on \mathbb{R}^n . Then $f(\mu) = |\det f| \mu$. The fiber weight δ on $\ker(f - f(0)) = \{0\}$ is thus $|\det f|$ and fiber integration is given by

$$f_{\delta,*}(\alpha) = |\det f| f^{-1,*}(\alpha). \tag{2.15}$$

The transformation rule (2.6) implies that this satisfies the projection formula:

$$\begin{aligned} \int_{[\mathbb{R}^n, \mu]} \alpha \wedge f^*\eta &= \int_{[\mathbb{R}^n, \mu]} f^*(f^{-1,*}(\alpha) \wedge \eta) \stackrel{(2.6)}{=} |\det f| \int_{[\mathbb{R}^n, \mu]} f^{-1,*}(\alpha) \wedge \eta \\ &= \int_{[\mathbb{R}^n, \mu]} f_{\delta,*}(\alpha) \wedge \eta. \end{aligned}$$

Interchanging the roles of α and η , the equality of leftmost and rightmost term shows

$$f^*(\alpha \wedge [\mathbb{R}^n, \mu]) = f^*(\alpha) \wedge [\mathbb{R}^n, \mu]. \tag{2.16}$$

2.4 Stokes' theorem

By its very definition, a (p, q) -form α on \mathbb{R}^n may be viewed as an alternating form in $p + q$ variables on $\mathbb{R}^n \times \mathbb{R}^n$ with values in C^∞ . Given $w \in \mathbb{R}^n \times \mathbb{R}^n$, the *contraction* (α, w) of α with w (interior derivative) is defined as the multilinear form resulting from inserting and fixing w as the first entry of α . This operation is characterized by

the Leibniz rule

$$(\alpha \wedge \beta, w) = (\alpha, w) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (\beta, w) \tag{2.17}$$

and the identities

$$(d'\varphi, w) = \frac{\partial \varphi}{\partial w_1}, \quad (d''\varphi, w) = \frac{\partial \varphi}{\partial w_2}, \quad \varphi \in C^\infty, \quad w = (w_1, w_2). \tag{2.18}$$

Recall that a *facet* of a polyhedron is a face of codimension 1.

Definition 2.7 Let $[\sigma, \mu]$ be a weighted polyhedron and $\tau \subset \sigma$ a facet that is endowed with a weight ν . Then there is a unique vector $\bar{n}_{\sigma, \tau} \in N_\sigma / N_\tau$ that points in direction of σ and is such that $\mu = \nu \wedge \bar{n}_{\sigma, \tau}$ in the sense of (2.12). A *normal vector* for $\tau \subset \sigma$ is any choice of lift $n_{\sigma, \tau} \in N_\sigma$.

Assume $m = \dim \sigma$. The (first) *boundary integral* of $\alpha \in A_c^{m-1, m}(\sigma)$ over τ is defined as

$$\int_{\partial'_\tau[\sigma, \mu]} \alpha := - \int_{[\tau, \nu]} (\alpha, n''_{\sigma, \tau})|_\tau. \tag{2.19}$$

The convention in notation here is $v' = (v, 0)$ and $v'' = (0, v)$ for any vector $v \in \mathbb{R}^n$. The restriction $(\alpha, n''_{\sigma, \tau})|_\tau$ is independent of the choice of normal vector and the whole expression is independent of the choice of ν . Define the (first) *boundary integral* of σ as

$$\int_{\partial'[\sigma, \mu]} \alpha := \sum_{\tau \subset \sigma \text{ a facet}} \int_{\partial'_\tau[\sigma, \mu]} \alpha. \tag{2.20}$$

The definition of the (second) boundary integral differs by a sign which is motivated by Example 2.9 below. For $\beta \in A_c^{m, m-1}(\sigma)$, put

$$\int_{\partial''[\sigma, \mu]} \beta := \int_{[\tau, \nu]} (\beta, n'_{\sigma, \tau})|_\tau, \quad \int_{\partial''[\sigma, \mu]} \beta = \sum_{\tau \subset \sigma \text{ a facet}} \int_{\partial''_\tau[\sigma, \mu]} \beta. \tag{2.21}$$

Proposition 2.8 ([3, Lemma 1.5.7], Stokes' Theorem) *Let $[\sigma, \mu]$ be an m -dimensional weighted polyhedron and let $\alpha \in A_c^{m-1, m}(\sigma)$ and $\beta \in A_c^{m, m-1}(\sigma)$. Then*

$$\int_{[\sigma, \mu]} d'\alpha = \int_{\partial'[\sigma, \mu]} \alpha, \quad \int_{[\sigma, \mu]} d''\beta = \int_{\partial''[\sigma, \mu]} \beta.$$

Example 2.9 Proposition 2.8 is essentially just the following statement. For every smooth function $\rho: [0, 1] \rightarrow \mathbb{R}$,

$$\int_0^1 \rho'(x) d'x \wedge d''x = \rho(1) - \rho(0) = - \int_0^1 \rho'(x) d''x \wedge d'x.$$

The differing signs explain the sign change from (2.19) to (2.21).

Example 2.10 Proposition 2.8 says $d'[\sigma, \mu] = -\partial'[\sigma, \mu]$ and $d''[\sigma, \mu] = -\partial''[\sigma, \mu]$ as currents, but these derivatives are never polyhedral if $\dim \sigma > 0$. (The difference in sign with Stokes' Theorem comes from (2.7).) Namely they have support on the union of facets $\partial\sigma$ of σ . If $\dim \sigma = m$, then $\partial\sigma$ is an $(m - 1)$ -dimensional polyhedral set, so $\eta|_{\partial\sigma} = 0$ for every $\eta \in A^{m-1,m}(\sigma)$ resp. $\eta \in A^{m,m-1}(\sigma)$, but not necessarily

$$\int_{\partial'[\sigma,\mu]} \eta = 0 \quad \text{resp.} \quad \int_{\partial''[\sigma,\mu]} \eta = 0.$$

3 δ -Forms

We consider forms, currents and polyhedra on \mathbb{R}^n in the following.

Definition 3.1 A δ -form is a polyhedral current T such that both $d'T$ and $d''T$ are again polyhedral.

This definition turns out to be equivalent to the familiar concept of balancing for T .

Definition 3.2 Let \mathcal{T} be a polyhedral complex, $\mu_\sigma, \sigma \in \mathcal{T}$, a family of weights for its polyhedra and $\alpha_\sigma \in A(\sigma), \sigma \in \mathcal{T}$ a family of smooth forms. This datum is called *balanced*, if the following two equivalent conditions are met.

- (1) For all polyhedra $\tau \in \mathcal{T}$, the sum

$$\sum_{\sigma \in \mathcal{T}, \tau \subset \sigma \text{ a facet}} \alpha_\sigma|_\tau \otimes n_{\sigma,\tau} \in A(\tau) \otimes_{\mathbb{R}} \mathbb{R}^n. \tag{3.1}$$

lies in the subspace $A(\tau) \otimes_{\mathbb{R}} N_\tau$. The normal vectors $n_{\sigma,\tau}$ here are taken for the weights μ_σ and μ_τ .

- (2) For every polyhedron $\tau \in \mathcal{T}$, every affine linear function z with *constant* restriction $z|_\tau$ and normal vectors $n_{\sigma,\tau}$ as before,

$$\sum_{\sigma \in \mathcal{T}, \tau \subset \sigma \text{ a facet}} \frac{\partial z}{\partial n_{\sigma,\tau}} \alpha_\sigma|_\tau = 0. \tag{3.2}$$

Since $z|_\tau$ is constant, this expression does not depend on the choices of the $n_{\sigma,\tau}$.

Proof (Proof of the equivalence of (1) and (2).) Assume $z|_\tau$ to be constant and consider the pairing

$$A(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{R}^n \longrightarrow A(\tau), \alpha \otimes v \longmapsto (d'z \wedge \alpha, v')|_\tau.$$

It follows from the Leibniz rule that

$$(d'z \wedge \alpha, v')|_\tau = (\partial z / \partial v) \cdot \alpha|_\tau,$$

so the pairing factors through $A(\tau) \otimes_{\mathbb{R}} \mathbb{R}^n$ and is simply the $A(\tau)$ -linear extension of $v \mapsto \partial z / \partial v$. The proof is now the statement that a vector v lies in N_τ if and only if $\partial z / \partial v = 0$ for every affine linear function z with constant restriction $z|_\tau$. \square

Formulation (1) is closer to the usual condition of balancing in tropical geometry but makes implicit use of the existence of the ambient space \mathbb{R}^n . Formulation (2) in turn is more suitable for generalizations to abstract polyhedral complexes, cf. [13].

Being balanced is stable under the four operations in Remark 2.3, so only depends on the current $T = \sum_{\sigma \in \mathcal{T}} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$. Also note that (3.1) and (3.2) are trihomogeneous in α and that only polyhedra of a fixed dimension occur. One obtains that $T = \sum_{p,q,r} T^{p,q,r}$, with $T^{p,q,r}$ of tridegree (p, q, r) , is balanced if and only if each $T^{p,q,r}$ is.

Theorem 3.3 *A polyhedral current T is a δ -form if and only if it is balanced. In particular, it is a δ -form if and only if $T^{p,q,r}$ is a δ -form for all (p, q, r) .*

Furthermore, T is already a δ -form if one out of $d'T, d''T$ is again polyhedral.

Proof

(1) We first assume that T is of tridegree (p, q, r) . Let \mathcal{T} be a weighted polyhedral complex subordinate to T , say

$$T = \sum_{\sigma \in \mathcal{T}^r} \alpha_\sigma \wedge [\sigma, \mu_\sigma], \quad \alpha_\sigma \in A^{p,q}(\sigma),$$

and let $\eta \in A_c^{n-p-r, n-q-r}$ be a test form. One obtains from the Leibniz rule and Stokes' Theorem that

$$\begin{aligned} (d'T - d'_p T)(\eta) &= \sum_{\tau \in \mathcal{T}^{r+1}} \int_{[\tau, \mu_\tau]} \sum_{\sigma \in \mathcal{T}^r, \tau \subset \sigma} (\alpha_\sigma \wedge \eta, n''_{\sigma, \tau})|_\tau \\ &= \sum_{\tau} \left[\sum_{\tau \subset \sigma} ((\alpha_\sigma, n''_{\sigma, \tau}) \wedge [\tau, \mu_\tau])(\eta) + (-1)^{\deg \alpha} \int_{[\tau, \mu_\tau]} \sum_{\tau \subset \sigma} \alpha_\sigma \wedge (\eta, n''_{\sigma, \tau})|_\tau \right]. \end{aligned} \tag{3.3}$$

The individual contractions $(\alpha_\sigma, n''_{\sigma, \tau})$ and $(\eta, n''_{\sigma, \tau})$ depend on the choices of normal vectors, but the total expression does not. We henceforth fix the choices $n_{\sigma, \tau}$. The terms $((\alpha_\sigma, n''_{\sigma, \tau}) \wedge [\tau, \mu_\tau])(\eta)$ always define polyhedral currents. So the statement to prove is that T is balanced if and only if the following is a polyhedral current,

$$\eta \longmapsto (-1)^{\deg \alpha} \sum_{\tau} \int_{[\tau, \mu_\tau]} \sum_{\tau \subset \sigma} \alpha_\sigma \wedge (\eta, n''_{\sigma, \tau})|_\tau.$$

(2) Assume first that T is balanced, fix some τ and write

$$\sum_{\tau \subset \sigma} \alpha_\sigma|_\tau \otimes n_{\sigma, \tau} = \sum_{i \in I} \beta_i \otimes v_i, \quad \beta_i \in A^{p,q}(\tau), \quad v_i \in N_\tau, \tag{3.4}$$

according to (3.1). Then

$$\sum_{\tau \subset \sigma} \alpha_\sigma \wedge (\eta, n''_{\sigma, \tau})|_\tau = \sum_{i \in I} \beta_i \wedge (\eta, v''_i)|_\tau. \tag{3.5}$$

By the Leibniz rule,

$$\beta_i \wedge (\eta, v''_i) = (-1)^{\deg \beta} (\beta_i \wedge \eta, v''_i) + (-1)^{\deg \beta + 1} (\beta_i, v''_i) \wedge \eta. \tag{3.6}$$

Since $\beta_i \wedge \eta$ is of bidegree $(\dim \tau, \dim \tau + 1)$, the first summand vanishes. The (integral over $[\tau, \mu_\tau]$ of the) second summand defines a polyhedral current in η . Taking the sum over i and τ shows that $d'T$ is a polyhedral current. The same argument works for $d''T$, proving that a trihomogeneous balanced polyhedral current is a δ -form.

(3) Conversely assume that T is *not* balanced, our claim being that $d'T$ is *not* polyhedral. (We still assume that T has tridegree (p, q, r) currently.) Generally, if S is a polyhedral current, C some polyhedral sets with $\text{Supp } S \subseteq C$ and $\eta \in A_c$ a test form, then $\eta|_C = 0$ implies $S(\eta) = 0$. In the situation at hand, we have already seen that $\text{Supp}(d'T - d'_p T)$ is contained in the codimension $r + 1$ skeleton $\bigcup_{\tau \in \mathcal{T}^{r+1}} \tau$ and our approach is to construct a test form η with $\eta|_\tau = 0$ for all τ but $(d'T - d'_p T)(\eta) \neq 0$. Pick τ and z such that (3.2) is not satisfied, i.e. z is an affine linear function with constant restriction $z|_\tau$ and such that

$$\beta := \sum_{\tau \subset \sigma} \frac{\partial z}{\partial n_{\sigma, \tau}} \alpha_\sigma|_\tau \neq 0. \tag{3.7}$$

There exists a bump test form $\bar{\eta} \in A_c^{\dim \tau - p, \dim \tau - q}$ with the two properties that $\text{Supp } \bar{\eta} \cap \tau' \neq \emptyset, \tau' \in \mathcal{T}^{r+1}$, only for $\tau' = \tau$ and

$$\int_{[\tau, \mu_\tau]} \beta \wedge \bar{\eta} \neq 0.$$

Then the τ -contribution to (3.3) for the test form $\eta = (-1)^{\deg \alpha} d''z \wedge \bar{\eta}$ is simply

$$\sum_{\tau \subset \sigma} (d''z \wedge \alpha_\sigma \wedge \bar{\eta}, n''_{\sigma, \tau})|_\tau = \beta \wedge \bar{\eta}.$$

Here we combined the Leibniz rule for $(\ , n''_{\sigma, \tau})$ with the properties $d''z|_\tau = 0$ and $(d''z, n''_{\sigma, \tau}) = \partial z / \partial n_{\sigma, \tau}$. Thus $(d'T - d'_p T)(d''z \wedge \bar{\eta}) \neq 0$ even though $d''z \wedge \bar{\eta}|_{\tau'} = 0$ for every $\tau' \in \mathcal{T}^{r+1}$. So $d'T$ cannot be polyhedral and hence T is not a δ -form. Note that arguments (2) and (3) show the stronger statement that a trihomogeneous T is balanced if and only if *one out of* $d'T$ and $d''T$ is polyhedral, i.e. they prove the last statement of Theorem 3.3 for trihomogeneous T .

(4) Now consider a general δ -form $T = \sum_{p, q, r} T^{p, q, r}$ with $T^{p, q, r}$ of the indicated tridegree. Our claim is that each $T^{p, q, r}$ is a δ -form. Since $T^{p_1, q_1, r}$ and $T^{p_2, q_2, r}$ lie in

different bidegrees as currents for $(p_1, q_1) \neq (p_2, q_2)$ and since d' and d'' are bihomogeneous, it is enough to prove that all $T^r := \sum_{p,q} T^{p,q,r}$ are δ -forms. Polyhedral currents may be added ad libitum, so it is sufficient to show that all $(d'T^r - d'_p T^r)$ and $(d''T^r - d''_p T^r)$ are polyhedral. Assume for the sake of contradiction that there is some r_0 with $d'T^{r_0}$ not polyhedral and assume that r_0 is chosen minimal. Then the previous arguments imply that there is some point

$$x \in \text{Supp}(d'T^{r_0} - d'_p T^{r_0}) \setminus \text{Supp}\left(\sum_{r>r_0} d'T^r - d'_p T^r\right)$$

that has no open neighborhood $x \in U$ such that $d'T^{r_0}|_U$ is polyhedral. (Take some $x \in \text{Supp } \beta$ where β is as in (3.7).) Using minimality of r_0 , we conclude that $d'T$ cannot be polyhedral. The same argument applies with d'' instead of d' . Thus we obtain that T is a δ -form, if and only if each $T^{p,q,r}$ is a δ -form, if and only if each $T^{p,q,r}$ is balanced, if and only if T is balanced. \square

Definition 3.4

(1) We denote by $B^{p,q,r} = B^{p,q,r}(\mathbb{R}^n)$ the space of δ -forms of the indicated tridegree and by $B = \bigoplus_{p,q,r} B^{p,q,r}$ the space of all δ -forms. Bidegrees and degrees of δ -forms are meant in the sense of currents: A δ -form T of tridegree (p, q, r) is of bidegree $(p + r, q + r)$ and degree $\text{deg}(T) = p + q + 2r$. We sometimes write $B^{p,q} := \bigoplus_r B^{p-r,q-r,r}$ for the bigrading by bidegree.

(2) Since $d'd'T = 0$, Theorem 3.3 implies that $d'T$ is a δ -form. Similarly for d'' , so one obtains derivatives

$$d' : B^{p,q} \longrightarrow B^{p+1,q}, \quad d'' : B^{p,q} \longrightarrow B^{p,q+1}.$$

The balancing condition (3.1) is stable under d'_p and d''_p , so the polyhedral derivatives restrict to operators

$$d'_p : B^{p,q,r} \longrightarrow B^{p+1,q,r}, \quad d''_p : B^{p,q,r} \longrightarrow B^{p,q+1,r}.$$

Define the boundary operators $\partial' := d'_p - d'$ and $\partial'' := d''_p - d''$. It will be explained below, cf. (3.9), that these are trihomogeneous in the sense

$$\partial' : B^{p,q,r} \longrightarrow B^{p,q-1,r+1}, \quad \partial'' : B^{p,q,r} \longrightarrow B^{p-1,q,r+1}.$$

Lemma 3.5 (1) *The boundary derivatives satisfy*

$$\begin{aligned} 0 &= \partial' \partial' = \partial'' \partial'', \\ 0 &= \partial' \partial'' + \partial'' \partial', \\ 0 &= \partial' d'_p + d'_p \partial' = \partial'' d''_p + d''_p \partial'', \\ 0 &= \partial' d''_p + d''_p \partial'' + \partial'' d'_p + d'_p \partial'. \end{aligned}$$

- (2) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be an affine linear map. Assume that $T \in B(\mathbb{R}^n)$ has relatively compact support with respect to f . Then $f_*T \in B(\mathbb{R}^m)$ is also a δ -form.
- (3) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a surjective affine linear map and $S \in B(\mathbb{R}^m)$. Then $f^*S \in B(\mathbb{R}^n)$ is also a δ -form.

Proof

(1) The necessary observation is that d'_p, ∂', d''_p and ∂'' are all trihomogeneous of different tridegrees. The stated relations then follow from the standard identities $(d')^2 = (d'')^2 = 0$ and $d'd'' = -d''d'$.

(2) and (3) follow from the fact that f_* and f^* both commute with d' and d'' and preserve the property of being polyhedral, cf. Sect. 2.3. □

Example 3.6 Every polyhedral current T of tridegree $(n - r, q, r)$ or $(p, n - r, r)$ is a δ -form. This follows from the observation that then all terms $\alpha_\sigma|_\tau$ in (3.1) vanish. Alternatively, one argues that $d'T = 0$ resp. $d''T = 0$ because T is of bidegree $(n, q + r)$ resp. $(p + r, n)$ as current and applies Theorem 3.3.

Lemma 3.7 *The δ -forms $B^{p,q,0}(\mathbb{R}^n)$ are precisely the currents that are of the form $\alpha \wedge [\mathbb{R}^n, \mu_{\text{std}}]$ for a piecewise smooth (p, q) -form $\alpha \in PS^{p,q}(\mathbb{R}^n)$.*

Proof Assume that $T \in P^{p,q,0}$ and write $T = \sum_{\sigma \in \mathcal{T}^0} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$ for a weighted subordinate polyhedral complex \mathcal{T} . We may assume all occurring $\mu_\sigma = \mu_{\text{std}}$ because $N_\sigma = \mathbb{R}^n$ for every n -dimensional σ . Any $\tau \in \mathcal{T}^1$ is then a facet of precisely two $\sigma_1, \sigma_2 \in \mathcal{T}^0$. One may choose $n_{\sigma_2, \tau} = -n_{\sigma_1, \tau}$. The balancing condition (3.1) is then equivalent to $\alpha_{\sigma_1}|_\tau = \alpha_{\sigma_2}|_\tau$ for all such $\tau \subset \sigma_1, \sigma_2$, which is equivalent to the $(\alpha_\sigma)_{\sigma \in \mathcal{T}^0}$ defining a piecewise smooth form. □

Example 3.8 Let $\alpha \in PS(\mathbb{R}^n)$ be piecewise smooth and $T \in P(\mathbb{R}^n)$ a polyhedral current. Let \mathcal{T} be a polyhedral complex that is subordinate to both T and α , say

$$\alpha = (\alpha_\sigma)_{\sigma \in \mathcal{T}^0}, \quad T = \sum_{\rho \in \mathcal{T}} \beta_\rho \wedge [\rho, \mu_\rho].$$

Define their product as

$$\alpha T := \sum_{\rho \in \mathcal{T}} \alpha|_\rho \wedge \beta_\rho \wedge [\rho, \mu_\rho]. \tag{3.8}$$

The restriction $\alpha|_\rho$ here is well-defined by the piecewise smooth property. If T is a δ -form, then αT is also a δ -form since the balancing condition (3.1) is PS -linear. For example, the δ -preforms from [9, Sect. 2] are precisely the sums of products αT where $\alpha \in A(\mathbb{R}^n)$ is smooth and $T \in B^{0,0,r}$ a tropical cycle.

We end this section by providing three ways to compute $\partial'T$. The case of ∂'' is the same by symmetry; it merely requires paying attention to the difference in signs of (2.19) and (2.21). Throughout, we assume that $T \in B^{p,q,r}$, say with presentation

$T = \sum_{\sigma \in \mathcal{T}^r} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$ for a subordinate weighted polyhedral complex \mathcal{T} . The proof of Theorem 3.3 shows that $\partial' T = \sum_{\tau \in \mathcal{T}^{r+1}} \beta_\tau \wedge [\tau, \mu_\tau]$ for certain β_τ which we would like to determine.

(1) Implicit in the proof of Theorem 3.3 is the following formula. Fix τ and write

$$\sum_{\tau \subset \sigma \text{ a facet}} \alpha_\sigma|_\tau \otimes n_{\sigma, \tau} = \sum_{i \in I} \beta_i \otimes v_i, \quad \beta_i \in A(\tau), \quad v_i \in N_\tau$$

as in (3.4). Then (3.3), together with (3.5) and (3.6), implies

$$\beta_\tau = \sum_{i \in I} (\beta_i, v_i'') - \sum_{\tau \subset \sigma} (\alpha_\sigma, n''_{\sigma, \tau})|_\tau. \tag{3.9}$$

(2) The next formula for β_τ is more in line with formulation (3.2) of the balancing condition. We use the definition $\partial' := d'_p - d'$ for all polyhedral currents in the following. Pick any affine linear map $f: \mathbb{R}^n \rightarrow \mathbb{R}^{\dim \tau + 1}$ such that $f|_\sigma$ is injective for every $\tau \subset \sigma \in \mathcal{T}^r$. Let $C = \bigcup_{\tau \subset \sigma} \sigma$ be the polyhedral set formed by all $\sigma \in \mathcal{T}^r$ containing τ . Denote by Z its boundary in the topological space $\bigcup_{\sigma \in \mathcal{T}^r} \sigma$. The current $S = \sum_{\tau \subset \sigma} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$ has support contained in C and is a δ -form away from Z . Since $f|_C$ has finite fibers, $\text{Supp } S$ is relatively compact over \mathbb{R}^n , so the push-forward $f_* S$ is defined. It is a δ -form away from $f(Z)$. Moreover $f_*(d'_p S) = d'_p(f_* S)$ because $f|_S$ has finite fibers. It follows that $f_*(\partial' S) = \partial' f_*(S)$. Writing $\partial' f_*(S) = \gamma \wedge [f(\tau), f(\mu_\tau)]$ away from $f(Z)$ shows

$$\beta_\tau = f^* \gamma.$$

Now note that $(f_* S)|_{\mathbb{R}^{\dim \tau + 1} \setminus f(Z)}$ lies in $B^{p, q, 0}(\mathbb{R}^{\dim \tau + 1} \setminus f(Z))$, i.e. $f_* S$ is given by a piecewise smooth form away from $f(Z)$ by Lemma 3.7. This makes the determination of γ very simple: $f_* S$ is described near $f(\tau) \setminus f(Z)$ by smooth forms $\omega_i \in A^{p, q}(\rho_i)$ on two $(\dim \tau + 1)$ -dimensional polyhedra ρ_1, ρ_2 with $\rho_1 \cap \rho_2 = f(\tau)$. These satisfy $\omega_1|_{f(\tau)} = \omega_2|_{f(\tau)}$. Picking the normal vectors in the above (3.9) as $n := n_{\rho_1, f(\tau)} = -n_{\rho_2, f(\tau)}$ eliminates the first sum in (3.9) and shows

$$\gamma = (\omega_2, n'') - (\omega_1, n''). \tag{3.10}$$

(3) For the third and final formula, choose a tuple of coordinate functions $x_1, \dots, x_{n-r-1}: \mathbb{R}^n \rightarrow \mathbb{R}$ that restrict to a basis of M_τ . For each σ containing τ , choose a non-constant affine linear function $z_\sigma: \sigma \rightarrow \mathbb{R}$ such that $z_\sigma|_\tau$ is constant. (For example, one may choose an affine linear $z: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $z|_\tau$ is constant but $z_\sigma = z|_\sigma$ non-constant for every $\sigma \supset \tau$.) Then every α_σ can be uniquely expressed as

$$\alpha_\sigma = \alpha_\sigma^{(1)} + d' z_\sigma \wedge \alpha_\sigma^{(2)} + d'' z_\sigma \wedge \alpha_\sigma^{(3)} + d' z_\sigma \wedge d'' z_\sigma \wedge \alpha_\sigma^{(4)}$$

with the $\alpha_\sigma^{(j)}$ all $C^\infty(\sigma)$ -linear combination of $d'x_I \wedge d''x_J$. Our claim is that

$$\beta_\tau = - \sum_{\tau \subset \sigma \text{ a facet}} \frac{\partial z_\sigma}{\partial n_{\sigma,\tau}} \alpha_\sigma^{(3)}|_\tau. \tag{3.11}$$

Note that already the individual summands are independent of the chosen z_σ .

Proof (Proof of the claim.) In light of (3.3), we need to show that the following identity holds for all smooth forms $\eta \in A^{n-p-r-1, n-q-r}(\mathbb{R}^n)$ of complementary degree:

$$\sum_{\tau \subset \sigma \text{ a facet}} (\alpha_\sigma \wedge \eta, n''_{\sigma,\tau})|_\tau = \sum_{\tau \subset \sigma \text{ a facet}} \frac{\partial z_\sigma}{\partial n_{\sigma,\tau}} \alpha_\sigma^{(3)} \wedge \eta|_\tau. \tag{3.12}$$

Since $d'z_\sigma|_\tau = d''z_\sigma|_\tau = 0$, it is immediately clear that

$$(\alpha_\sigma \wedge \eta, n''_{\sigma,\tau})|_\tau = (\alpha_\sigma^{(1)} \wedge \eta, n''_{\sigma,\tau})|_\tau + \frac{\partial z_\sigma}{\partial n_{\sigma,\tau}} \alpha_\sigma^{(3)} \wedge \eta|_\tau.$$

Our task is thus to show

$$\sum_{\tau \subset \sigma \text{ a facet}} (\alpha_\sigma^{(1)} \wedge \eta, n''_{\sigma,\tau})|_\tau = 0. \tag{3.13}$$

Pick coordinate functions $y_1, \dots, y_{r+1}: \mathbb{R}^n \rightarrow \mathbb{R}$ that extend x_1, \dots, x_{n-r-1} to a basis and that are constant along τ . Then $d'y_i|_\tau = d''y_i|_\tau = 0$. So if η is of the form $d'y_i \wedge \tilde{\eta}$, then already $(\alpha_\sigma^{(1)} \wedge \eta, n''_{\sigma,\tau})|_\tau = 0$. Similarly, if η is a C^∞ -linear combination of monomials $d'x_I \wedge d''x_J$, then already $\alpha_\sigma^{(1)} \wedge \eta = 0$ because this form is of degree $(n-r-1, n-r)$.

It is thus left to show (3.13) for forms $\eta = d''y_i \wedge \tilde{\eta}$. We obtain that

$$\begin{aligned} \sum_{\tau \subset \sigma \text{ a facet}} (\alpha_\sigma^{(1)} \wedge d''y_i \wedge \tilde{\eta}, n''_{\sigma,\tau})|_\tau &= (-1)^{p+q} \sum_{\tau \subset \sigma \text{ a facet}} \frac{\partial y_i}{\partial n_{\sigma,\tau}} \alpha_\sigma^{(1)} \wedge \tilde{\eta}|_\tau \\ &= (-1)^{p+q} \sum_{\tau \subset \sigma \text{ a facet}} \frac{\partial y_i}{\partial n_{\sigma,\tau}} \alpha_\sigma \wedge \tilde{\eta}|_\tau. \end{aligned} \tag{3.14}$$

The last expression vanishes by the balancing condition (3.2). □

4 Intersection theory

4.1 Main result

The definition of the \wedge -product of δ -forms is based on two specific constructions. The first is the product of piecewise smooth and δ -forms from Example 3.8: By Lemma

3.7, every δ -form of tridegree $(p, q, 0)$ is of the form $\alpha \wedge [\mathbb{R}^n, \mu_{\text{std}}]$ for a (unique) piecewise smooth (p, q) -form α . We write α by abuse of notation and define

$$\alpha \wedge T := \alpha T, \quad \alpha \in B^{\bullet, \bullet, 0}, \quad T \in B. \tag{4.1}$$

The second construction is the exterior product of currents, cf. [4, Sect. I.2], defined as follows. Given homogeneous currents $T_1 \in D(\mathbb{R}^n)$ and $T_2 \in D(\mathbb{R}^m)$, it is the unique current $T_1 \boxtimes T_2 \in D(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$(T_1 \boxtimes T_2)(p_1^* \eta_1 \wedge p_2^* \eta_2) = (-1)^{\text{deg } T_1 \text{ deg } T_2} T_1(\eta_1) \cdot T_2(\eta_2).$$

In particular,

$$d(T_1 \boxtimes T_2) = dT_1 \boxtimes T_2 + (-1)^{\text{deg } T_1} T_1 \boxtimes dT_2, \quad d \in \{d', d''\}. \tag{4.2}$$

The exterior product preserves polyhedral currents which follows from the identity

$$(\alpha_1 \wedge [\sigma_1, \mu_1]) \boxtimes (\alpha_2 \wedge [\sigma_2, \mu_2]) = \alpha_1 \wedge \alpha_2 \wedge [\sigma_1 \times \sigma_2, \mu_1 \wedge \mu_2]. \tag{4.3}$$

Relation (4.2) then implies that the exterior product of δ -forms is a δ -form again. Moreover, one sees that if T_i is of polyhedral tridegree (p_i, q_i, r_i) , then $T_1 \boxtimes T_2$ has tridegree $(p_1 + p_2, q_1 + q_2, r_1 + r_2)$. Separating (4.2) by tridegree provides

$$\begin{aligned} d_P(T_1 \boxtimes T_2) &= d_P T_1 \boxtimes T_2 + (-1)^{\text{deg } T_1} T_1 \boxtimes d_P T_2, \quad d_P \in \{d'_P, d''_P\}, \\ \partial(T_1 \boxtimes T_2) &= \partial T_1 \boxtimes T_2 + (-1)^{\text{deg } T_1} T_1 \boxtimes \partial T_2, \quad \partial \in \{\partial', \partial''\}. \end{aligned} \tag{4.4}$$

We simply write $T_1 \times T_2$ instead of $T_1 \boxtimes T_2$ for δ -forms T_1 and T_2 .

In the following, $\Delta = (\text{id}, \text{id})_*[\mathbb{R}^n, \mu_{\text{std}}] \in B^{0,0,n}(\mathbb{R}^n \times \mathbb{R}^n)$ denotes the diagonal viewed as δ -form.

Theorem 4.1 *There is a unique way to define an associative product $\wedge : B \times B \rightarrow B$ that satisfies the Leibniz rules with respect to d' and d'' , extends definition (4.1), and can be computed by restriction to the diagonal, meaning*

$$S \wedge T = p_{1,*}(\Delta \wedge (S \times T)). \tag{4.5}$$

This product has the following additional properties.

- (1) *It is graded commutative and trihomogeneous in the sense $B^{p,q,r} \wedge B^{s,t,u} \subseteq B^{p+s,q+t,r+u}$. In particular, it satisfies the Leibniz rule with respect to the operators $\partial', d'_P, \partial''$ and d''_P .*
- (2) *It commutes with pull-back: If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a surjective affine linear map and if $S, T \in B(\mathbb{R}^m)$ are δ -forms, then $f^*(T \wedge S) = f^*T \wedge f^*S$.*
- (3) *It satisfies the projection formula: If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a surjective affine linear map, $S \in B(\mathbb{R}^m)$ a δ -form, and $T \in B(\mathbb{R}^n)$ a δ -form with compact support over \mathbb{R}^m , then*

$$f_*(T \wedge f^*S) = f_*T \wedge S. \tag{4.6}$$

(4) It coincides with the tropical intersection products from [1, 5, 9] on $\bigoplus_r B^{0,0,r}$ whenever they are defined.

The idea of characterizing and constructing the tropical intersection product through divisor intersections and restriction to the diagonal is due to Allermann–Rau [1].

Proof (Proof of the uniqueness assertion.) If a \wedge -product exists as claimed, the Leibniz rule implies for piecewise smooth α that

$$d' d''(\alpha T) = d' d'' \alpha \wedge T + \alpha \wedge d' d'' T + (-1)^{\deg \alpha} (d' \alpha \wedge d'' T - d'' \alpha \wedge d' T). \tag{4.7}$$

In case of a piecewise linear function φ , the δ -forms $d' \varphi$ resp. $d'' \varphi$ agree with $d'_p \varphi$ resp. $d''_p \varphi$ and are again piecewise smooth, because the contractions in (3.9) vanish for degree reasons. (This applies more generally to piecewise smooth functions.) It follows that

$$d' d'' \varphi \wedge T = d' d''(\varphi T) - \varphi \wedge d' d'' T + d'' \varphi \wedge d' T - d' \varphi \wedge d'' T \tag{4.8}$$

is uniquely determined by the Leibniz rule and the piecewise smooth case. Denote by x_1, \dots, x_n and y_1, \dots, y_n the coordinate functions on $\mathbb{R}^n \times \mathbb{R}^n$ and define $\varphi_i := \max\{x_i, y_i\}$. Then, by [1, Remark 9.2], the diagonal Δ is the product

$$\Delta = d' d'' \varphi_1 \wedge \dots \wedge d' d'' \varphi_n,$$

where the right hand side is a successive application of (4.8). Again by (4.8) as well as the associativity of the \wedge -product, $\Delta \wedge (S \times T)$ is now uniquely determined. Hence $S \wedge T = p_{1,*}(\Delta \wedge (S \times T))$ is uniquely characterized by the stated conditions. \square

The existence statement will be shown in the next section. Here, we give an application of Theorem 4.1 to the definition of a pull-back for all affine linear maps, not just surjective ones. It is specific to δ -forms, meaning it does not extend to polyhedral currents. Its construction is well-known for tropical cycles, cf. [9, Remark 1.4 (v)] for example.

Proposition/Definition 4.2 *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be an affine linear map and $S \in B(\mathbb{R}^m)$ a δ -form. There is a unique δ -form $f^*(S) \in B(\mathbb{R}^n)$, called the pull-back of S along f , that satisfies the projection formula*

$$f_*(T \wedge f^* S) = f_* T \wedge S \tag{4.9}$$

for all $T \in B(\mathbb{R}^n)$ of relatively compact support with respect to f . This pull-back is functorial in f and commutes with \wedge -products as well as with the six operators d' , d'' , d'_p , d''_p , ∂' and ∂'' .

Proof Identity (4.9) determines f^*S uniquely because it determines all its values on test forms η by

$$\begin{aligned} (f^*S)(\eta) &= \int_{\mathbb{R}^n} f^*S \wedge \eta \\ &= \int_{\mathbb{R}^m} f_*(f^*S \wedge \eta) = \int_{\mathbb{R}^m} S \wedge f_*(\eta). \end{aligned}$$

Just from this uniqueness, one may deduce all further properties. For example, for $d \in \{d', d''\}$ and for all homogeneous δ -forms T of relatively compact support with respect to f ,

$$\begin{aligned} f_*(T \wedge d(f^*S)) &= (-1)^{\deg T} f_*(d(T \wedge f^*S) - dT \wedge f^*S) \\ &= (-1)^{\deg T} (d(f_*(T \wedge f^*S)) - f_*(dT \wedge f^*S)) \\ &= (-1)^{\deg T} (d(f_*T \wedge S) - f_*(dT) \wedge S) = f_*T \wedge dS. \end{aligned}$$

So necessarily $d(f^*S) = f^*(dS)$. For commutativity with \wedge -products, we compute

$$\begin{aligned} f_*(T \wedge f^*S_1 \wedge f^*S_2) &= f_*(T \wedge f^*S_1) \wedge S_2 \\ &= f_*T \wedge S_1 \wedge S_2 \end{aligned}$$

and then deduce $f^*S_1 \wedge f^*S_2 = f^*(S_1 \wedge S_2)$. We omit the verification of the remaining properties which are shown similarly.

To show existence of f^* , we consider the graph $\Gamma_f = (\text{id}, f)_*[\mathbb{R}^n, \mu_{\text{std}}]$ as a δ -form on $\mathbb{R}^n \times \mathbb{R}^m$. We claim that the following definition satisfies (4.9):

$$f^*S := p_{1,*}(\Gamma_f \wedge p_2^*S).$$

The next succession of identities verifies that claim. The first four equalities come either by definition or from the projection formula (4.6). The last equality will be explained below.

$$\begin{aligned} f_*(T \wedge f^*S) &= f_*(T \wedge p_{1,*}(\Gamma_f \wedge p_2^*S)) \\ &= f_*(p_{1,*}(p_1^*T \wedge \Gamma_f \wedge p_2^*S)) \\ &= p_{2,*}(p_1^*T \wedge \Gamma_f \wedge p_2^*S) \\ &= p_{2,*}(p_1^*T \wedge \Gamma_f) \wedge S \\ &= f_*T \wedge S. \end{aligned} \tag{4.10}$$

The last equality comes from the identity $p_{2,*}(p_1^*T \wedge \Gamma_f) = f_*T$ which may be seen as follows. The form $p_1^*T \wedge \Gamma_f$ has support contained in $\text{Supp } \Gamma_f$ and has the property $p_{1,*}(p_1^*T \wedge \Gamma_f) = T \wedge p_{1,*}\Gamma_f = T$ by the projection formula (4.6). As $\text{Supp } \Gamma_f \rightarrow \mathbb{R}^n$ is bijective, this means $p_1^*T \wedge \Gamma_f = (\text{id}, f)_*(T)$. It is then merely left to note that $p_2 \circ (\text{id}, f) = f$ and the proof is complete. \square

Example 4.3 Every affine linear map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be factored as $f = h \circ g$, where $g: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is surjective and $h: \mathbb{R}^k \rightarrow \mathbb{R}^m$ injective. Functoriality gives $f^* = g^* \circ h^*$ where g^* is the pull-back of currents from Sect. 2.1.

Put $L = h_*[\mathbb{R}^k, \mu_{\text{std}}] \in B^{0,0,m-k}(\mathbb{R}^m)$. Then (4.9) with $T = 1$ (constant function) comes out as

$$h_*h^*S = h_*(1 \wedge h^*S) = L \wedge S.$$

Especially interesting here is the property of h^* to commute with \wedge -products, cf. Proposition 4.2. It specializes to

$$(L \wedge S_1) \wedge_L (L \wedge S_2) = L \wedge S_1 \wedge S_2,$$

where \wedge_L denotes the wedge product on L .

4.2 Existence of the \wedge -product

This section proves the existence of the \wedge -product. We begin with some Leibniz rule properties of the product with piecewise smooth forms in (4.1).

Lemma 4.4 *Let T be a δ -form.*

(1) *For every homogeneous piecewise smooth form α and each polyhedral derivative $d_P \in \{d'_P, d''_P\}$,*

$$d_P(\alpha \wedge T) = d_P\alpha \wedge T + (-1)^{\text{deg } \alpha} \alpha \wedge d_P T.$$

(2) *For every homogeneous piecewise smooth form $\alpha \in B^{p,0,0}$,*

$$\partial'(\alpha \wedge T) = (-1)^{\text{deg } \alpha} \alpha \wedge \partial' T$$

and hence

$$d'(\alpha \wedge T) = d'\alpha \wedge T + (-1)^{\text{deg } \alpha} \alpha \wedge d' T.$$

(3) *Analogously, for every homogeneous piecewise smooth form $\alpha \in B^{0,q,0}$,*

$$\partial''(\alpha \wedge T) = (-1)^{\text{deg } \alpha} \alpha \wedge \partial'' T$$

and hence

$$d''(\alpha \wedge T) = d''\alpha \wedge T + (-1)^{\text{deg } \alpha} \alpha \wedge d'' T.$$

Proof Identity (1) may be checked polyhedron by polyhedron and, in this way, reduces to the Leibniz rule for smooth forms. Identities (2) and (3) follow from the observation that the contractions in (3.9) are linear (up to the sign $(-1)^{\text{deg } \alpha}$) with respect to multiplication by piecewise smooth functions in the stated degrees. \square

By a *divisor* we mean a d' -closed and d'' -closed δ -form of tridegree $(0, 0, 1)$. These are the tropical cycles with constant coefficients of codimension 1 in classical terminology.

Lemma 4.5 *Let D be a divisor on \mathbb{R}^n . Then there exists a piecewise linear function φ such that $D = d'd''\varphi$. It is unique up to addition of affine linear functions.*

Proof This is entirely due to Lagerberg, cf. [11, Proposition 5.3], we merely give the straightforward reduction to his results. Let $U_1 \subset U_2 \subset \dots$ be a covering of \mathbb{R}^n by convex relatively compact opens. By definition, D is a *locally finite* sum of currents $m \cdot [\sigma, \mu]$ with $m \in \mathbb{R}$. So for each $i \geq 1$, there is a finite linear combination H_i of weighted hyperplanes such that $(D + H_i)|_{U_i}$ is *positive* in the sense that all its coefficients are ≥ 0 . By [11, Proposition 2.4 and Proposition 2.6], there is then a convex function φ'_i on U_i such that $(D + H_i)|_{U_i} = d'd''\varphi'_i$. The lemma is easily seen to hold for hyperplanes and hence the H_i , so we obtain for each i the existence of a convex function φ_i with $D|_{U_i} = d'd''\varphi_i$. Then φ_i is necessarily piecewise linear, cf. [11, Proof of Proposition 5.3]. A piecewise linear function φ is affine linear if and only if $d'd''\varphi = 0$, so the φ_i are determined up to addition of affine linear functions. They may then be chosen compatibly, i.e. such that they satisfy $\varphi_{i+1}|_{U_i} = \varphi_i$, proving the lemma. \square

For affine linear φ and every current T we have by (2.9) the relation

$$d'd''(\varphi T) = \varphi \wedge d'd''T - d''\varphi \wedge d'T + d'\varphi \wedge d''T. \tag{4.11}$$

Definition 4.6 Let D be a divisor and T a δ -form. Choose a piecewise linear function φ with $D = d'd''\varphi$ as in Lemma 4.5 and define

$$D \cdot T := d'd''(\varphi \wedge T) - \varphi \wedge d'd''T + d''\varphi \wedge d'T - d'\varphi \wedge d''T. \tag{4.12}$$

The definition does not depend on the choice of φ by (4.11). The resulting $D \cdot T$ is again a δ -form.

Remark 4.7 The definition collapses to $D \cdot T = d'd''(\varphi T)$ whenever $d'T = d''T = 0$. This identity is well-known in Bedford–Taylor theory, cf. [2] and [3, Sect. 5].

Lemma 4.8 *Let D, T and φ be as above. The following two identities hold:*

$$\begin{aligned} D \cdot T &= d'(d''\varphi \wedge T) + d''\varphi \wedge d'T \\ &= -\partial'(d''\varphi \wedge T) - d''\varphi \wedge \partial'T. \end{aligned} \tag{4.13}$$

Proof Part (3) of Lemma 4.4 shows that

$$d''(\varphi \wedge T) = d''\varphi \wedge T + \varphi \wedge d''T.$$

Substituting this in (4.12) and using part (2) of Lemma 4.4 leads to the first equality of (4.13).

Part (1) of Lemma 4.4 together with the observation $d'_p d''\varphi = 0$ implies that

$$d'_p(d''\varphi \wedge T) = -d''\varphi \wedge d'_p T.$$

Substituting this in the first line of (4.13) gives the second equality. □

We remark that identity (4.13) collapses to the definition of the corner locus [9, Definition 1.10] if T is a tropical cycle. Also, if φ is affine linear, then $d''\varphi$ is a smooth form and sign-commutes with ∂' by Lemma 4.4. Then (4.13) gives $d'd''\varphi \cdot T = 0$ as expected. The identity also shows that if T is of tridegree (p, q, r) , then $D \cdot T$ is of tridegree $(p, q, r + 1)$. Its most important consequence for us, however, is the following simple description of $D \cdot T$.

Lemma 4.9 *Let φ be a piecewise linear function and T a δ -form of tridegree (p, q, r) . Let further \mathcal{T} be a weighted polyhedral complex subordinate to both φ and T , say $T = \sum_{\sigma \in \mathcal{T}^r} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$. Then*

$$d'd''\varphi \cdot T = \sum_{\tau \in \mathcal{T}^{r+1}} \beta_\tau \wedge [\tau, \mu_\tau]$$

with

$$\beta_\tau = \sum_{\tau \subset \sigma \text{ a facet}} \frac{\partial(\varphi - \varphi_\tau)}{\partial n_{\sigma, \tau}} \alpha_\sigma|_\tau, \tag{4.14}$$

where φ_τ is any choice of affine linear function with $(\varphi - \varphi_\tau)|_\tau$ constant.

Proof It is clear that \mathcal{T} is also subordinate to $d'd''\varphi \cdot T$, our task is merely to determine the β_τ . They may be computed locally near every inner point of any given τ . Having some τ fixed, we may replace φ by $\varphi - \varphi_\tau$ because $d'd''\varphi_\tau = 0$. Then $d''(\varphi - \varphi_\tau)|_\tau = 0$, so the term $d''(\varphi - \varphi_\tau) \wedge \partial'T$ in (4.13) vanishes and we are left to find the coefficient of τ in $-\partial'(d''(\varphi - \varphi_\tau) \wedge T)$.

Assume first that $(\varphi - \varphi_\tau)|_\sigma$ is non-constant for every $\sigma \in \mathcal{T}^r$ containing τ . Then we can put $z_\sigma = (\varphi - \varphi_\tau)|_\sigma$ to obtain (4.14) from a literal application of (3.11).

The general case follows since the right hand side of (4.14) is a priori independent of the choice φ_τ by the balancing condition (3.2). □

From here on, many ideas belong to Allermann–Rau [1] and we merely extend them to δ -forms. We will provide references to their paper for comparison.

Lemma 4.10 (Compare [1, Proposition 6.7]) *Given divisors D_1, D_2 and a δ -form T ,*

$$D_1 \cdot (D_2 \cdot T) = D_2 \cdot (D_1 \cdot T).$$

Proof Let φ_i be a piecewise linear function with $D_i = d'd''\varphi_i$. Assume T of tridegree (p, q, r) and let \mathcal{T} be a weighted polyhedral complex that is subordinate to φ_1, φ_2 and T ; write $T = \sum_{\sigma \in \mathcal{T}^r} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$. Fix some $\rho \in \mathcal{T}^{r+2}$ and assume both

$\varphi_1|_\rho$ and $\varphi_2|_\rho$ to vanish. Each $\sigma \in \mathcal{T}^r$ with $\rho \subset \sigma$ has precisely two facets τ, τ' containing ρ and we write $\sigma = \tau + \tau'$ if this relation holds. Define a constant $\chi(\sigma)$ through $\mu_\sigma = \chi(\sigma)n_{\tau,\rho} \wedge n_{\tau',\rho} \wedge \mu_\rho$ in this case. In other words, one may pick $n_{\sigma,\tau} = \chi(\sigma)n_{\tau',\rho}$ whenever $\sigma = \tau + \tau'$. Pick an auxiliary affine linear function z with $z|_\rho = 0$ and $z|_\tau \neq 0$ for all $\rho \subset \tau \in \mathcal{T}^{r+1}$. Define the constants

$$x_\tau := \frac{\partial\varphi_1|_\tau}{\partial n_{\tau,\rho}}, \quad y_\tau := \frac{\partial\varphi_2|_\tau}{\partial n_{\tau,\rho}}, \quad \lambda_\tau := \frac{\partial z}{\partial n_{\tau,\rho}}, \quad \tau \in \mathcal{T}^{r+1}, \quad \rho \subset \tau.$$

Then $\varphi_2 - (y_\tau/\lambda_\tau)z$ vanishes on τ and may be used in formula (4.14) to compute the τ -contribution $\beta_\tau \wedge [\tau, \mu_\tau]$ to $D_2 \cdot T$,

$$\beta_\tau = \sum_{\tau \subset \sigma \text{ a facet}} \frac{\partial(\varphi_2 - (y_\tau/\lambda_\tau)z)}{\partial n_{\sigma,\tau}} \alpha_\sigma|_\tau \tag{4.15}$$

$$= \sum_{\tau \subset \sigma \text{ a facet}} \chi(\sigma)(y_{\tau'} - y_\tau \lambda_{\tau'}/\lambda_\tau) \alpha_\sigma|_\tau. \tag{4.16}$$

The ρ -contribution $\gamma_\rho \wedge [\rho, \mu_\rho]$ to $D_1 \cdot (D_2 \cdot T)$ is then, again using (4.14),

$$\gamma_\rho = \sum_{(\tau,\tau'), \sigma=\tau+\tau' \in \mathcal{T}^r} \chi(\sigma)(x_\tau(y_{\tau'} - y_\tau \lambda_{\tau'}/\lambda_\tau)) \alpha_\sigma|_\rho \tag{4.17}$$

$$= \sum_{\{\tau,\tau'\}, \sigma=\tau+\tau' \in \mathcal{T}^r} \chi(\sigma)(x_\tau y_{\tau'} + x_{\tau'} y_\tau - x_\tau y_\tau \lambda_{\tau'}/\lambda_\tau - x_{\tau'} y_{\tau'} \lambda_\tau/\lambda_{\tau'}) \alpha_\sigma|_\rho. \tag{4.18}$$

The last expression is symmetric with respect to exchange of x and y , proving the lemma. □

Lemma 4.11 *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a surjective affine linear map, T a δ -form on \mathbb{R}^n with compact support over \mathbb{R}^m and D a divisor on \mathbb{R}^m . Then the projection formula holds,*

$$D \cdot f_* T = f_*(f^* D \cdot T).$$

Proof Write $D = d' d'' \varphi$ for a piecewise linear function φ as in Lemma 4.5. Push-forward commutes with both d' and d'' while multiplication with the piecewise smooth forms $\varphi, d' \varphi$ and $d'' \varphi$ on \mathbb{R}^m in the sense of Example 3.8 obviously satisfies the projection formula. The claim now follows directly from Definition 4.6. □

Lemma 4.12 (Compare [1, Lemma 9.4]) *Let T be a δ -form on $\mathbb{R}^c \times \mathbb{R}^m$ and denote by $D_i = p_{12}^*(d' d'' \max\{x_i, y_i\})$ the divisor on $\mathbb{R}^c \times \mathbb{R}^c \times \mathbb{R}^m$ where the i -th coordinates of the first two factors agree, $i = 1, \dots, c$. Let $g(x, z) := (x, x, z)$ be the partial diagonal $\mathbb{R}^c \times \mathbb{R}^m \rightarrow \mathbb{R}^c \times \mathbb{R}^c \times \mathbb{R}^m$. Then*

$$D_1 \cdots D_c \cdot (\mathbb{R}^c \times T) = g_* T.$$

Proof By a recursive argument, it is enough to treat the case $c = 1$. Let \mathcal{T} be a polyhedral complex on $\mathbb{R} \times \mathbb{R}^m$ that is subordinate to T , say $T = \sum_{\sigma \in \mathcal{T}} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$. Assume without loss of generality that $\bigcup_{\sigma \in \mathcal{T}} \sigma = \mathbb{R} \times \mathbb{R}^m$ and define, for each σ ,

$$\tilde{\sigma}_? = \{(x, y, z) \in \mathbb{R} \times \sigma \mid x \text{ ? } y\}, \quad ? \in \{\geq, \leq\}.$$

Let x and y denote the coordinates on the first two factors of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m$. A polyhedral complex structure on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m$ that is subordinate to both $\mathbb{R} \times T$ and the function $\varphi = \max\{0, x - y\}$ is then, for example,

$$S = \bigcup_{\sigma \in \mathcal{T}} \{\tilde{\sigma}_\geq, g(\sigma), \tilde{\sigma}_\leq\}.$$

It becomes a weighted complex by endowing $\tilde{\sigma}_\geq$ and $\tilde{\sigma}_\leq$ with weight $\mu_{\text{std}} \wedge \mu_\sigma$ and $g(\sigma)$ with $g(\mu_\sigma)$. The support of $d'd''\varphi \cdot (\mathbb{R} \times T)$ is contained in φ 's locus of non-linearity $\{x = y\} = \bigcup_{\sigma \in \mathcal{T}} g(\sigma)$. Given a polyhedron $g(\sigma) \in S$, it is the facet of precisely the polyhedra $\tilde{\sigma}_\geq, \tilde{\sigma}_\leq$ and all $g(\rho)$ such that $\sigma \subset \rho$ is a facet. Normal vectors in these cases are $(1, 0, 0)$, $(-1, 0, 0)$ and $g(n_{\rho, \sigma})$, respectively. Using that $\varphi|_{\{x=y\}} = 0$, the contribution $\beta_\sigma \wedge [g(\sigma), g(\mu_\sigma)]$ of $g(\sigma)$ to $d'd''\varphi \cdot (\mathbb{R} \times T)$ is by Lemma 4.9

$$\beta_\sigma = \left(\frac{\partial\varphi|_{x \geq y}}{\partial(1, 0, 0)} + \frac{\partial\varphi|_{x \leq y}}{\partial(-1, 0, 0)} \right) g_*\alpha_\sigma + \sum_{\sigma \subset \rho \text{ a facet}} \frac{\partial\varphi|_{g(\rho)}}{\partial g(n_{\rho, \sigma})} g_*(\alpha_\rho|_\sigma).$$

Since $\varphi|_{\{x \leq y\}} = 0$, only the first term is non-zero and contributes $g_*\alpha_\sigma$ as claimed. \square

Definition 4.13 (Compare [1, Definition 9.3]) Let $D_i = d'd''(\max\{x_i, y_i\})$ denote the divisor on $\mathbb{R}^n \times \mathbb{R}^n$ where the i -th coordinates agree. The \wedge -product of δ -forms $S, T \in B(\mathbb{R}^n)$ is defined as the δ -form

$$S \wedge T := p_{1,*}(D_1 \cdots D_n \cdot (S \times T)).$$

The notational convention (and only possibility) here is that the successive product is evaluated from right to left. We also write $\Delta \cdot (S \times T)$ instead of $D_1 \cdots D_n \cdot (S \times T)$. Note that we have already seen that the order of the D_i does not matter, but only Corollary 4.16 below will prove the independence of the choice of $\{D_1, \dots, D_n\}$ to describe the diagonal.

Lemma 4.14 *The following identity holds for all δ -forms S and T ,*

$$S \times T = p_1^*S \wedge p_2^*T.$$

Proof Let $g: \mathbb{R}^n \times \mathbb{R}^n \rightarrow (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)$ be the diagonal. The first equality in the following is by definition, the second is Lemma 4.12 and the third is the identity $p_{12} \circ g = \text{id}$.

$$\begin{aligned} p_1^*S \wedge p_2^*T &= p_{12,*}(\Delta \cdot (S \times \mathbb{R}^n \times \mathbb{R}^n \times T)) \\ &= p_{12,*}g_*(S \times T) \\ &= S \times T. \end{aligned}$$

□

Lemma 4.15 (Compare [1, Lemma 9.7]) *Let S, T be δ -forms on \mathbb{R}^n and C a divisor. Then*

$$C \cdot (S \wedge T) = (C \cdot S) \wedge T.$$

Proof It follows from Lemma 4.9 that $p_1^*C \cdot (S \times T) = (C \cdot S) \times T$. Then the claim follows from the commutativity in Lemma 4.10 and the projection formula in Lemma 4.11:

$$\begin{aligned} C \cdot p_{1,*}(D_1 \cdots D_n \cdot (S \times T)) &= p_{1,*}(p_1^*C \cdot D_1 \cdots D_n \cdot (S \times T)) \\ &= p_{1,*}(D_1 \cdots D_n \cdot ((C \cdot S) \times T)). \end{aligned}$$

□

Successive application of Lemma 4.15 shows the following corollary.

Corollary 4.16 (Compare [1, Corollary 9.8]) *Let T be a δ -form on \mathbb{R}^n and C, C_1, \dots, C_l divisors. Then*

$$C_1 \cdots C_l \cdot T = (C_1 \cdots C_l) \wedge T.$$

In particular,

$$C \wedge T = C \cdot T, \quad \Delta \wedge (S \times T) = \Delta \cdot (S \times T).$$

Proof of Theorem 4.1. So far, Definition 4.13 provides a well-defined bilinear map $\wedge: B \times B \rightarrow B$. It is left to verify all the properties stated in Theorem 4.1.

(a) The \wedge -product is clearly trihomogeneous in the sense that it restricts to maps $B^{p,q,r} \times B^{s,t,u} \rightarrow B^{p+s,q+t,r+u}$. It is graded-commutative in the sense that $S \wedge T = (-1)^{\deg S \deg T} T \wedge S$ for homogeneous S and T because

$$s^*(S \times T) = (-1)^{\deg S \deg T} T \times S,$$

where $s: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ is the map that switches the two factors.

(b) We claim that the \wedge -product satisfies the projection formula

$$S \wedge f_*T = f_*(f^*S \wedge T)$$

for every *surjective* linear map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$. To check this, we may assume $n = m + c$ and, after a change of coordinates, $f: \mathbb{R}^m \times \mathbb{R}^c \rightarrow \mathbb{R}^m$ being just the projection. Then simply $f^*S = S \times \mathbb{R}^c$. Recall that $D_i = d'd'' \max\{x_i, y_i\}$ on $\mathbb{R}^n \times \mathbb{R}^n$ and $\Delta = D_1 \cdots D_n$. Write $\Delta^m = D_1 \cdots D_m$ and $\Delta^c = D_{m+1} \cdots D_n$. Then the following

equalities hold, as will be explained below.

$$\begin{aligned}
 f_*(f^*S \wedge T) &= p_{1,*}(f, f)_*(\Delta^m \cdot \Delta^c \cdot (f^*S \times T)) \\
 &= p_{1,*}(\delta^m \cdot (f, f)_*(\Delta^c \cdot (S \times \mathbb{R}^c \times T))) \\
 &= p_{1,*}(\delta^m \cdot (f, f)_*(S \times (\Delta^c \cdot (\mathbb{R}^c \times T)))) \tag{4.19} \\
 &= p_{1,*}(\delta^m \cdot (f, f)_*(S \times g_*T)) \\
 &= S \wedge f_*T.
 \end{aligned}$$

The first equality is the definition of the left hand side combined with the identity $f \circ p_1 = p_1 \circ (f, f)$. The second follows from the projection formula for divisor intersection, Lemma 4.11, applied to $\Delta^m = (f, f)^*\delta^m$, where $\delta^m \subset \mathbb{R}^m \times \mathbb{R}^m$ denotes the diagonal. The third equality is the observation $p_2^*C \cdot (X \times Y) = X \times (p_2^*C \cdot Y)$ for any divisor C and δ -forms X, Y , applied successively to the divisor intersection Δ^c . The map g in the next line is the partial diagonal

$$g: \mathbb{R}^m \times \mathbb{R}^c \longrightarrow \mathbb{R}^c \times \mathbb{R}^m \times \mathbb{R}^c, (x, y) \longmapsto (y, x, y)$$

and the identification $\Delta^c \cdot (\mathbb{R}^c \times T) = g_*T$ is Lemma 4.12. The final equality then is the observation $(f, f)_*(S \times g_*T) = S \times f_*T$ which follows from the identity $(f, f) \circ (\text{id}_{\mathbb{R}^m}, g) = (\text{id}_{\mathbb{R}^m}, f)$.

(c) The next claim is that the \wedge -product is associative, $S \wedge (T \wedge U) = (S \wedge T) \wedge U$. Indeed, applying the projection formula (b) repeatedly, one obtains

$$S \wedge (T \wedge U) = p_{1,*}(\Delta \cdot (S \times \mathbb{R}^n \times T \times U))$$

where the intersection takes place on $(\mathbb{R}^n)^4$ and where $\Delta = \prod_{i=1}^n p_{12}^*D_i \cdot p_{23}^*D_i \cdot p_{34}^*D_i$ is the diagonal $\mathbb{R}^n \subset (\mathbb{R}^n)^4$. In exactly the same way,

$$(S \wedge T) \wedge U = p_{1,*}(\Delta \cdot (S \times T \times \mathbb{R}^n \times U)).$$

The two expressions are seen to be equal by switching the middle factors as in Step (a).

(d) Next, we claim that $\alpha \wedge T = \alpha T$ for every piecewise smooth α . This follows from Lemma 4.12 and the fact that multiplication by piecewise smooth forms commutes with divisor intersection. The latter is immediate from Lemma 4.9.

(e) Lemma 4.14 furthermore showed that $S \times T = p_1^*S \wedge p_2^*T$, so the constructed \wedge -product is computed by intersection with the diagonal, cf. (4.5).

(f) We turn to the Leibniz rule. Let C be a divisor and T a δ -form. Our first step is to prove the identity

$$d(C \wedge T) = C \wedge dT, \quad d \in \{d', d''\}. \tag{4.20}$$

We write $d = d_P - \partial$, with $d_P \in \{d'_P, d''_P\}$ and $\partial \in \{\partial', \partial''\}$ suitable, and verify (4.20) for d_P and ∂ separately.

Applying (4.14), the identity $d_P(C \wedge T) = C \wedge d_P T$ is immediate. Writing D as $D = d' d'' \varphi$ for a piecewise linear function φ as in Lemma 4.5 and using (4.13) twice, we have

$$\partial'(C \wedge T) = -\partial' \partial'(d'' \varphi \wedge T) - \partial'(d'' \varphi \wedge \partial' T) = C \wedge \partial' T$$

because $(\partial')^2 = 0$ by Lemma 3.5. Finally, applying Lemma 4.4 (3) and the rule $\partial' \partial'' = -\partial'' \partial'$, we also obtain

$$\begin{aligned} \partial''(C \wedge T) &= -\partial'' \partial'(d'' \varphi \wedge T) - \partial''(d'' \varphi \wedge \partial' T) \\ &= -\partial'(d'' \varphi \wedge \partial'' T) - (d'' \varphi \wedge \partial' \partial'' T) = C \wedge \partial'' T. \end{aligned}$$

This finishes the proof of (4.20). Successive application of the divisor case now yields

$$d(\Delta \wedge (S \times T)) = \Delta \wedge d(S \times T).$$

The Leibniz rule (4.2) for exterior products, coupled with Definition 4.13, completes the proof of the Leibniz rule for d in general. Separating by tridegree provides the Leibniz rules for the other differential operators.

(g) The identity $f^*(S \wedge T) = f^* S \wedge f^* T$ only uses the fact $p_1^* D \cdot (S \times T) = (D \cdot S) \times T$ for divisor intersection. Namely assume $f: \mathbb{R}^m \times \mathbb{R}^c \rightarrow \mathbb{R}^m$ to be the projection and write $p_{12}: \mathbb{R}^m \times \mathbb{R}^c \times \mathbb{R}^m \times \mathbb{R}^c \rightarrow \mathbb{R}^m \times \mathbb{R}^c$ for the projection to the first two factors. Then, in the terminology of Step (b),

$$\begin{aligned} f^* S \wedge f^* T &= p_{12,*}(\Delta^m \cdot \Delta^c \cdot (S \times \mathbb{R}^c \times T \times \mathbb{R}^c)) \\ &= p_{12,*}(\Delta^m \cdot (S \times T)) \times \Delta^c \\ &= (S \wedge T) \times \mathbb{R}^c = f^*(S \wedge T). \end{aligned} \tag{4.21}$$

(h) Finally, the tropical intersection products of Allermann–Rau [1], its extension to smoothly weighted rational polyhedra in [9, Remark 1.4], and the intersection product of Esterov [5] for polynomially weighted (possibly non-rational) polyhedra can all be expressed in terms of divisor intersection and restriction to the diagonal. In these two specific cases, they coincide with our definition. So any two of the mentioned products coincide whenever both are defined. □

4.3 Fan displacement rule

Two linear subspaces $N_1, N_2 \subseteq \mathbb{R}^n$ are said to intersect *transversally* if $N_1 + N_2 = \mathbb{R}^n$. (Equivalently, their intersection is transversal if their codimensions are related by $\text{codim}(N_1 \cap N_2) = \text{codim}(N_1) + \text{codim}(N_2)$.) In the transversal case, there is an exact sequence

$$0 \rightarrow N_1 \cap N_2 \rightarrow N_1 \oplus N_2 \rightarrow \mathbb{R}^n \rightarrow 0.$$

Given weights μ_1 and μ_2 for N_1 and N_2 , respectively, we denote by $\mu_1 \cap \mu_2$ the weight on $N_1 \cap N_2$ that satisfies $(\mu_1 \cap \mu_2) \wedge \mu_{\text{std}} = \mu_1 \wedge \mu_2$ in the sense of (2.12). The next lemma is easily checked.

Lemma 4.17 *Let $[N_1, \mu_1], [N_2, \mu_2] \subseteq \mathbb{R}^n$ be weighted linear subspaces, viewed as δ -forms. Assume that their intersection is transverse. Then*

$$[N_1, \mu_1] \wedge [N_2, \mu_2] = [N_1 \cap N_2, \mu_1 \cap \mu_2].$$

Let \mathcal{T}_1 and \mathcal{T}_2 be polyhedral complexes on \mathbb{R}^n which are pure of codimension r_1 and r_2 , respectively. By this we mean that \mathcal{T}_i agrees with the set of faces of all $\sigma_i \in \mathcal{T}_i^{r_i}$. Then \mathcal{T}_1 and \mathcal{T}_2 are said to intersect transversally if, for all pairs $(\sigma_1, \sigma_2) \in \mathcal{T}^{r_1} \times \mathcal{T}^{r_2}$, the intersection $\sigma_1 \cap \sigma_2$ is either empty or of codimension $r_1 + r_2$ and not contained in the union of boundaries $\partial\sigma_1 \cup \partial\sigma_2$. Note that then N_{σ_1} and N_{σ_2} intersect transversally whenever $\sigma_1 \cap \sigma_2 \neq \emptyset$.

Assume the above \mathcal{T}_i to intersect transversally and let \mathcal{S} be the polyhedral complex of all $\sigma_1 \cap \sigma_2, \sigma_i \in \mathcal{T}_i$. Then \mathcal{S} is pure of codimension $r_1 + r_2$ and every top-dimensional $\tau \in \mathcal{S}^{r_1+r_2}$ determines a unique pair $(\sigma_1, \sigma_2) \in \mathcal{T}_1^{r_1} \times \mathcal{T}_2^{r_2}$ such that $\tau = \sigma_1 \cap \sigma_2$.

Lemma 4.18 *Let \mathcal{T}_1 and \mathcal{T}_2 be transversally intersecting weighted polyhedral complexes of pure codimensions r_1 and r_2 , respectively. Let*

$$T_1 = \sum_{\sigma \in \mathcal{T}_1^{r_1}} \alpha_\sigma \wedge [\sigma, \mu_\sigma] \quad \text{and} \quad T_2 = \sum_{\sigma \in \mathcal{T}_2^{r_2}} \beta_\sigma \wedge [\sigma, \nu_\sigma]$$

be δ -forms. Then

$$T_1 \wedge T_2 = \sum_{(\sigma_1, \sigma_2) \in \mathcal{T}^{r_1} \times \mathcal{T}^{r_2}, \sigma_1 \cap \sigma_2 \neq \emptyset} \alpha_{\sigma_1} \wedge \beta_{\sigma_2} \wedge [\sigma_1 \cap \sigma_2, \mu_{\sigma_1} \cap \nu_{\sigma_2}]. \quad (4.22)$$

Proof Let \mathcal{S} be the polyhedral complex generated by all $\sigma_1 \cap \sigma_2, \sigma_i \in \mathcal{T}_i^{r_i}$. Then

$$\text{Supp}(T_1 \wedge T_2) \subseteq \text{Supp } T_1 \cap \text{Supp } T_2 \subseteq \bigcup_{\tau \in \mathcal{S}^{r_1+r_2}} \tau,$$

so $T_1 \wedge T_2 = \sum_{\tau \in \mathcal{S}^{r_1+r_2}} \gamma_\tau \wedge [\tau, \mu_\tau]$ for certain forms γ_τ and weights μ_τ . Each γ_τ is uniquely determined by its restriction to the relative interior $\tau^\circ \subseteq \tau$. Also, every occurring τ is in a unique way the intersection $\sigma_1 \cap \sigma_2$ of top-dimensional $\sigma_i \in \mathcal{T}_i$. On an open neighborhood of τ° , the situation then agrees with a subspace intersection as in Lemma 4.17, multiplied by $\alpha_{\sigma_1} \wedge \beta_{\sigma_2}$, and the claim follows from Theorem 4.1 and Lemma 4.17. □

Recall that a sequence (resp. net) of currents $(T_i)_{i \in I}$ converges weakly to a current T if for every test form $\eta \in A_c$, the sequence (resp. net) $T_i(\eta)$ converges to $T(\eta)$. Given a current T on \mathbb{R}^n and a vector $v \in \mathbb{R}^n$, we write $\lambda_v(x) = x + v$ and denote by $v + T = \lambda_{v,*}T = \lambda_{-v}^*T$ the v -translated current.

Proposition 4.19 *Let $S, T \in B(\mathbb{R}^n)$ be δ -forms and $v \in \mathbb{R}^n$ a vector. Then there is the weak convergence*

$$S \wedge (\varepsilon v + T) \longrightarrow S \wedge T, \quad \varepsilon \longrightarrow 0.$$

Proof Consider on \mathbb{R} the piecewise smooth form $\rho_\varepsilon = -d''x \wedge [[0, \varepsilon], \mu_{\text{std}}]$. Its boundary $\partial' \rho_\varepsilon = \delta_0 - \delta_\varepsilon$ is the difference of the Dirac measures at 0 and ε . Next, consider the map $f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, (\varepsilon, y) \mapsto y - \varepsilon v$. The Leibniz rule yields

$$\omega_\varepsilon := p_1^*(\delta_0 - \delta_\varepsilon) \wedge f^*T = \partial'(p_1^*\rho_\varepsilon \wedge f^*T) + p_1^*\rho_\varepsilon \wedge f^*\partial'T. \quad (4.23)$$

Note that $p_1^*\rho_\varepsilon$ is piecewise smooth, making the \wedge -products on the right hand side straightforward, cf. Example 3.8. Now Lemma 4.18 implies that

$$p_1^*\delta_\varepsilon \wedge f^*T = \{\varepsilon\} \times (\varepsilon v + T), \quad \varepsilon \in \mathbb{R},$$

and hence

$$S \wedge (T - (\varepsilon v + T)) = S \wedge p_{2,*}\omega_\varepsilon. \quad (4.24)$$

Our task is to show that this expression converges weakly to 0 as $\varepsilon \rightarrow 0$. The projection formula, cf. Theorem 4.1, allows to rewrite (4.24) as

$$p_{2,*}(p_2^*S \wedge \omega_\varepsilon).$$

Now observe that if $(X_i)_{i \in I} \rightarrow X$ is a weakly convergent net of currents on $\mathbb{R} \times \mathbb{R}^n$ with X and all X_i of compact support over \mathbb{R}^n , then $(p_{2,*}X_i)_{i \in I} \rightarrow p_{2,*}X$ by definition (2.10). So it remains to show $p_2^*S \wedge \omega_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Claim. For every polyhedral current $\gamma \in P(\mathbb{R} \times \mathbb{R}^n)$, there is the weak convergence $p_1^*\rho_\varepsilon \wedge \gamma \rightarrow 0$ as $\varepsilon \rightarrow 0$. This is straightforward: It is enough to consider the case $\gamma = \alpha \wedge [\sigma, \mu]$ in which case there are the two possibilities that $p_1(\sigma)$ is of dimension 0 or 1. In the 0-dimensional case, $p_1^*\rho_\varepsilon|_\sigma = 0$ and we are done. In the 1-dimensional case, a simple volume argument shows that for every compactly supported $(\dim \sigma, \dim \sigma)$ -form η ,

$$\int_{[\sigma \cap ([0, \varepsilon] \times \mathbb{R}^n), \mu]} \eta \longrightarrow 0, \quad \varepsilon \longrightarrow 0,$$

which implies the claim.

It follows that $p_2^*S \wedge p_1^*\rho_\varepsilon \wedge f^*\partial'T \rightarrow 0$ as $\varepsilon \rightarrow 0$ and it is only left to show, cf. (4.23), that

$$p_2^*S \wedge \partial'(p_1^*\rho_\varepsilon \wedge f^*T) \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \quad (4.25)$$

Applying the Leibniz rule for ∂' and using again the above claim, one is reduced to proving

$$\partial'(p_2^*S \wedge p_1^*\rho_\varepsilon \wedge f^*T) \longrightarrow 0, \quad \varepsilon \longrightarrow 0.$$

Now for every weakly convergent net of currents $(X_i)_{i \in I} \rightarrow X$, the sequence of derivatives $(d'X_i)_{i \in I} \rightarrow d'X$ converges weakly, which follows immediately from the definition in (2.7). Using the above claim once more, it is hence enough to show

$$d'_p(p_2^*S \wedge p_1^*\rho_\varepsilon \wedge f^*T) \longrightarrow 0, \quad \varepsilon \longrightarrow 0.$$

Applying the Leibniz rule for d'_p and the vanishing $d'_p\rho_\varepsilon = 0$, this last statement follows from yet another application of the above claim. The proof is complete. \square

Let \mathcal{T}_1 and \mathcal{T}_2 be finite polyhedral complexes on \mathbb{R}^n , pure of codimensions r_1 and r_2 respectively. A vector $v \in \mathbb{R}^n$ is called generic for \mathcal{T}_1 and \mathcal{T}_2 if there exists $\varepsilon_0 > 0$ such that \mathcal{T}_1 and $\varepsilon v + \mathcal{T}_2$ intersect transversally for all $0 < \varepsilon < \varepsilon_0$. For not necessarily finite \mathcal{T}_1 and \mathcal{T}_2 , a vector v is called generic if it is generic for all finite subcomplexes of the same pure codimensions. Generic vectors in this sense always exist.

Construction 4.20 Let v be a generic vector for two polyhedral complexes \mathcal{T}_1 and \mathcal{T}_2 that are pure of codimensions r_1 and r_2 , respectively. Let

$$T_1 = \sum_{\sigma \in \mathcal{T}_1^{r_1}} \alpha_\sigma \wedge [\sigma, \mu_\sigma], \quad T_2 = \sum_{\sigma \in \mathcal{T}_2^{r_2}} \beta_\sigma \wedge [\sigma, \nu_\sigma]$$

be δ -forms. Define their v -displacement product with respect to \mathcal{T}_1 and \mathcal{T}_2 as

$$T_1 \cdot_v T_2 := \sum_{(\sigma_1, \sigma_2) \in \mathcal{T}_1^{r_1} \times \mathcal{T}_2^{r_2}, \sigma_1 \cap (\varepsilon v + \sigma_2) \neq \emptyset \text{ for } \varepsilon \searrow 0} \alpha_{\sigma_1} \wedge \beta_{\sigma_2} \wedge [\sigma_1 \cap \sigma_2, \mu_{\sigma_1} \cap \nu_{\sigma_2}]. \tag{4.26}$$

The sum here is over all (σ_1, σ_2) such that $\sigma_1 \cap (\varepsilon v + \sigma_2) \neq \emptyset$ for all sufficiently small ε . This condition implies that also $\sigma_1 \cap \sigma_2 \neq \emptyset$. Since v is generic, it moreover implies that N_{σ_1} and N_{σ_2} intersect transversally. Thus, even though σ_1 and σ_2 may intersect non-transversally (i.e. in a face), $\sigma_1 \cap \sigma_2$ is always of codimension $r_1 + r_2$. Note that v need not be generic for subdivisions of \mathcal{T}_1 and \mathcal{T}_2 anymore, which is why the definition depends on their choice.

Proposition 4.21 Let T_1 and T_2 be δ -forms with subordinate polyhedral complexes \mathcal{T}_1 and \mathcal{T}_2 as above. Assume v is generic for \mathcal{T}_1 and \mathcal{T}_2 . Then the v -displacement product (with respect to the \mathcal{T}_i) computes the \wedge -product,

$$T_1 \cdot_v T_2 = T_1 \wedge T_2. \tag{4.27}$$

In particular, the v -displacement product is independent of the choices $\mathcal{T}_1, \mathcal{T}_2$ and v .

Proof Both sides of (4.27) are computed locally, so we may assume \mathcal{T}_1 and \mathcal{T}_2 to be finite by a partition of unity argument. Lemma 4.18 then shows that for all sufficiently small $\varepsilon > 0$,

$$T_1 \wedge (\varepsilon v + T_2) = \sum_{(\sigma_1, \sigma_2) \in \mathcal{T}_1^{r_1} \times \mathcal{T}_2^{r_2}, \sigma_1 \cap (\varepsilon v + \sigma_2) \neq \emptyset} \alpha_{\sigma_1} \wedge (\varepsilon v + \beta_{\sigma_2}) \wedge [\sigma_1 \cap (\varepsilon v + \sigma_2), \mu_{\sigma_1} \cap \nu_{\sigma_2}].$$

The intersection $\sigma_1 \cap (\varepsilon v + \sigma_2)$ being non-empty and transverse for all sufficiently small ε implies that $\sigma_1 \cap \sigma_2$ is non-empty and of codimension $r_1 + r_2$. Moreover in this case,

$$\alpha_{\sigma_1} \wedge (\varepsilon v + \beta_{\sigma_2}) \wedge [\sigma_1 \cap (\varepsilon v + \sigma_2), \mu_{\sigma_1} \cap \nu_{\sigma_2}] \longrightarrow \alpha_{\sigma_1} \wedge \beta_{\sigma_2} \wedge [\sigma_1 \cap \sigma_2, \mu_{\sigma_1} \cap \nu_{\sigma_2}]$$

in the weak sense. It follows that $T_1 \wedge (\varepsilon v + T_2) \rightarrow T_1 \cdot v T_2$ in the weak sense. Proposition 4.19 on the other hand shows that this limit equals $T_1 \wedge T_2$, proving the proposition. \square

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