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The universality of Hughes-free division rings

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Abstract
Let $E * G$ be a crossed product of a division ring E and a locally indicable group G. Hughes showed that up to $E * G$-isomorphism, there exists at most one Hughes-free division $E * G$-ring. However, the existence of a Hughes-free division $E * G$-ring $\mathcal{D}_{E*G}$ for an arbitrary locally indicable group G is still an open question. Nevertheless, $\mathcal{D}_{E*G}$ exists, for example, if G is amenable or G is bi-orderable. In this paper we study, whether $\mathcal{D}_{E*G}$ is the universal division ring of fractions in some of these cases. In particular, we show that if G is a residually-(locally indicable and amenable) group, then there exists $\mathcal{D}_{E[G]}$ and it is universal. In Appendix we give a description of $\mathcal{D}_{E[G]}$ when G is a RFRS group.

Keywords Locally indicable groups · Universal division ring of fractions · Hughes-free division ring

Mathematics Subject Classification Primary 16S35 · 20F65; Secondary 12E15 · 16S34 · 16K40

1 Introduction

A division R-ring $\phi : R \to \mathcal{D}$ is called **epic** if $\phi(R)$ generates $\mathcal{D}$ as a division ring. Each division R-ring $\mathcal{D}$ induces a Sylvester matrix rank function $\mathrm{rk}_{\mathcal{D}}$ on R. Given a ring R, Cohn introduced the notion of universal division R-ring (see, for example, [4, Section 7.2]). In the language of Sylvester rank functions, an epic division R-ring $\mathcal{D}$ is **universal** if for every division R-ring $\mathcal{E}$, $\mathrm{rk}_{\mathcal{D}} \geq \mathrm{rk}_{\mathcal{E}}$. By a result of Cohn [3, Theorem 4.4.1], the universal epic division R-ring is unique up to R-isomorphism. The universal division R-ring $\mathcal{D}$ is called **universal division ring of fractions of** R if $\mathcal{D}$ is epic and $\mathrm{rk}_{\mathcal{D}}$ is faithful (that is R is embedded in $\mathcal{D}$).

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If R is a commutative domain, then the field of fractions $\mathcal{Q}(R)$ is the universal division R-ring. The situation is much more complicated in the non-commutative setting. For example, Passman [24] gave an example of a Noetherian domain which does not have a universal division ring of fractions. Moreover, we show in Proposition 4.1 that the group algebra $\mathbb{Q}[H]$ does not have a universal division ring of fractions if H is not locally indicable. In this paper we want to study whether a group algebra or, more generally, a crossed product $E * G$, where E is a division ring, has a universal division ring of fractions. Thus, from the previous observation it is natural to consider the case of group algebras and crossed products $E * G$ where G is locally indicable.

Let E be a division ring and G a locally indicable group. Hughes [11] introduced a condition on an epic division $E * G$-rings and showed that up to $E * G$-isomorphism, there exists at most one epic division $E * G$-ring satisfying this condition. We call this division ring, the **Hughes-free division** $E * G$**-ring** and denote it by $\mathcal{D}_{E*G}$. For simplicity, in this paper the Sylvester matrix rank function $\mathrm{rk}_{\mathcal{D}_{E*G}}$ is denoted by rk_{E*G}. We say that a locally indicable group G is **Hughes-free embeddable** if $E * G$ has a Hughes-free division ring for every division ring E and every crossed product $E * G$.

The existence of a Hughes-free division $E * G$-ring is known for several families of locally indicable groups. In the case of amenable locally-indicable groups G, $\mathcal{D}_{E*G} = \mathcal{Q}(E * G)$ is the classical ring of fractions of $E * G$, and in the case of bi-orderable groups G, $\mathcal{D}_{E*G}$ is constructed using the Malcev-Neumann construction [20,23] (see [8]). The existence of $\mathcal{D}_{K[G]}$ is also known for group algebras $K[G]$, where K is of characteristic 0 and G is an arbitrary locally indicable group [15].

In [15, Theorem 8.1] it is shown that if there exists a universal epic division $E * G$-ring and a Hughes-free division $E * G$-ring, they are isomorphic as $E * G$-rings. Following Sánchez (see [25, Definition 6.18]), we say that a locally indicable group G is a **Lewin group** if it is Hughes-free embeddable and for all possible crossed products $E * G$, where E is a division ring, $\mathcal{D}_{E*G}$ is universal (in Sect. 3.3 we will see that this definition is equivalent to the Sánchez one). We conjecture that all locally indicable groups are Lewin.

Conjecture 1 *Let G be a locally indicable group, E a division ring and $R = E * G$ a crossed product of E and G. Then*

(A) the Hughes-free division R-ring $\mathcal{D}_R$ exists and
(B) it is universal division ring of fractions of R.

We want to notice that at this moment it is also an open problem of whether the universal division $E * G$-ring of fractions (if exists) should be Hughes-free.

In this paper we study part (B) of the conjecture in some cases where part (A) is known. Using Theorem 3.7 we can show that Conjecture 1 is valid for the following locally indicable groups.

Theorem 1.1 *Locally indicable amenable groups, residually-(torsion-free nilpotent) groups and free-by-cyclic groups are Lewin groups.*

In the case of group algebras we can prove a stronger result. The metric space $\mathcal{G}_n$ of **marked** n**-generated groups** consists of pairs $(G; S)$, where G is a group and S is an ordered generating set of G of cardinality n. Such pairs are in 1-to-1 correspondence with epimorphisms $F_n \to G$, where F_n is the free group of rank n, and thus the set $\mathcal{G}_n$ can be identified with the set of all normal subgroups of $F = F_n$. The distance between two normal subgroups M_1 and M_2 of F is defined by

$$d(M_1, M_2) = \inf\{e^{-k} : M_1 \cap B_k(1_F) = M_2 \cap B_k(1_F)\},$$

where $B_k(1_F)$ denotes the closed ball of radius k and center 1_F.

We say that a sequence of n-generated groups $\{G_i\}_{i\in\mathbb{N}}$ **converges** to an n-generated group G if $(G_i; S_i) \in \mathcal{G}_n$ converge to $(G; S) \in \mathcal{G}_n$ for some generating sets S_i of G_i ($i \in \mathbb{N}$) and S of G, respectively.

Theorem 1.2 *Let F be a free group freely generated by a finite set S and M and $\{M_i\}_{i\in\mathbb{N}}$ normal subgroups of F. We put $G = F/M$ and $G_i = F/M_i$ and assume that $(G_i, SM_i/M_i)$ converges to $(G, SM/M)$. Assume that for all i, G_i is locally indicable and $\mathcal{D}_{E[G_i]}$ exists. Then G is locally indicable, $\mathcal{D}_{E[G]}$ exists and*

$$\mathrm{rk}_{E[G]} = \lim_{i\to\infty} \mathrm{rk}_{E[G_i]}$$

as Sylvester matrix rank functions on $E[F]$.

As a corollary we obtain the following consequence.

Corollary 1.3 *Let G be a residually-(locally indicable and amenable) group and let E be a division ring. Then $\mathcal{D}_{E[G]}$ exists and it is the universal division ring of fractions of $E[G]$.*

The corollary can be applied to RFRS groups, because they are residually poly-$\mathbb{Z}$. The notion of RFRS groups arose in a work of Agol [1], in connection with the virtual-fibering of 3-manifolds [2], and it abstracts a critical property of the fundamental groups of special cube complexes. Kielak [18] realizes that the main result of [1] can be stated not only for 3-manifold groups but also for virtually RFRS groups. The proof of Kielak uses a new description of $\mathcal{D}_{\mathbb{Q}[G]}$ when G is RFRS. In Sect. 5 we give a description of $\mathcal{D}_{E[G]}$ when G is a RFRS group that generalizes the result of Kielak.

Let us consider now the case of group algebras $K[G]$ where K is a subfield of $\mathbb{C}$ and G is locally indicable. In this case it was shown in [15] that the division closure $\mathcal{D}(K[G], \mathcal{U}(G))$ of $K[G]$ in the algebra of affiliated operators $\mathcal{U}(G)$ is a Hughes-free division $K[G]$-ring. We denote by rk_G the von Neumann rank function (its definition is recalled in Sect. 2.6), and by $\mathrm{rk}_{\{1\}}$ the Sylvester matrix rank function on $\mathbb{Q}[G]$ induced by the homomorphism $\mathbb{Q}[G] \to \mathbb{Q}$ that sends all the elements of G to 1 (in the previous notation $\mathrm{rk}_{\{1\}}$ is $\mathrm{rk}_{\mathbb{Q}}$). In view of Conjecture 1, it is natural to ask for which groups G, $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$. It follows from [26, Proposition 1.9] that if a group G satisfies the condition $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$, then G is locally indicable. Thus, we propose also a weak version of Conjecture 1.

Conjecture 2 *Let G be locally indicable group. Then $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$ as Sylvester matrix rank functions on $\mathbb{Q}[G]$.*

From the discussion in the paragraph before the conjecture, we conclude that Corollary 1.3 has the following consecuence.

Corollary 1.4 *Let G be a residually-(locally indicable and amenable) group. Then* $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$ *as Sylvester matrix rank functions on* $\mathbb{Q}[G]$.

Combining this result with the mentioned above result of Kielak [18], we obtain the following corollary.

Corollary 1.5 *Let G be a finitely generated group which is virtually RFRS. Then the following are equivalent.*

(1) G is virtually fibered, in the sense that it admits a virtual map onto $\mathbb{Z}$ with finitely generated kernel.
(2) G admits a virtual map onto $\mathbb{Z}$ whose kernel has finite first Betti number.

Our next result is another consequence of Corollary 1.4 that generalizes a result of Wise [28, Theorem 1.3],

Corollary 1.6 *Let X be a compact CW-complex with $\pi_1 X$ non-trivial residually-(locally indicable and amenable) group. Then*

$$b_1^{(2)}(\widetilde{X}) \leq b_1(X) - 1 \text{ and } b_p^{(2)}(\widetilde{X}) \leq b_p(X) \text{ if } p \geq 2.$$

The paper is structured as follows. We introduce the basic notions in Sect. 2. In Sect. 3, we prove Theorem 1.1, Theorem 1.2 and Corollary 1.3. In Sect. 4 we study the consequences of the condition $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$ and, in particular, we prove Corollary 1.5 and Corollary 1.6. In Sect. 5 we give an alternative description of the division ring $\mathcal{D}_{E[G]}$ when G is RFRS and E is a division ring.

2 Preliminaries

2.1 Notation and definitions

All rings in this paper are unitary and ring homomorphisms send the identity element to the identity element. By a module we will mean a left module. Let G be a group with trivial element e. We say that a ring R is G-**graded** if R is equal to the direct sum $\oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all g and h in G. If for each $g \in G$, R_g contains an invertible element u_g, then we say that R is a **crossed product** of R_e and G and we will write $R = S * G$ if $R_e = S$. In the following if H is a subgroup of G, $S * H$ will denote the subring of R generated by S and $\{u_h : h \in H\}$.

A ring R may have several different G-gradings. It will be always clear from the context what G-grading we use. However, under some conditions the grading is unique. Assume that $R \cong E * G$, where E is a division ring and G is locally indicable, then by [9], the invertible elements $U(R)$ of R are $\bigcup_{g\in G} R_g \setminus \{0\}$. Hence R_e is the maximal subring in $U(R) \cup \{0\}$ and $G \cong U(R)/(R_e \setminus \{0\})$. Thus, R has a unique grading with R_e is a division ring and G is locally indicable.

An R-**ring** is a pair (S, ϕ) where $\phi : R \to S$ is a homomorphism. We will often omit ϕ if it is clear from the context.

2.2 Ordered groups

A total order $\preceq$ on a group G is **left-invariant** if for any $a, b, g \in G$, if $a \preceq b$ then $ga \preceq gb$. It is **bi-invariant** if, moreover we have $ag \preceq bg$.

Let $\preceq$ be a left-invariant order on a group G. A subgroup H is called **convex** if H contains every element g lying between any two elements of H ($h_1 \preceq g \preceq h_2$ with $h_1, h_2 \in H$). We say that $\preceq$ is **Conradian** if for all elements $f, g \succeq 1$, there exists a natural number n such that $fg^n \succ g$. In fact, one may actually take $n = 2$ ([6, Proposition 3.2.1]). Recall that a group G is **locally indicable** if every finitely generated non-trivial subgroup of G has an infinite cyclic quotient. A useful characterization of locally indicable groups says that they are the groups admitting a Conradian order ([5]). We will need the following important property of a Conradian order.

Proposition 2.1 [6, Corollary 3.2.28] *Let $(G, \preceq)$ be a group with a Conradian order and let N be the proper maximal convex subgroup of G. Then there exists an order preserving homomorphism $\phi : G \to \mathbb{R}$ such that $N = \ker \phi$.*

2.3 Hughes-free division rings

Let E be a division ring and G a locally indicable group. Let $\varphi : E * G \to \mathcal{D}$ be a homomorphism from $E * G$ to a division ring $\mathcal{D}$. We say that a division $E * G$-ring $(\mathcal{D}, \varphi)$ is **Hughes-free** if

(1) $\mathcal{D}$ is the division closure of $\varphi(E * G)$ ($\mathcal{D}$ is epic).
(2) For every non-trivial finitely generated subgroup H of G, a normal subgroup N of H with $H/N \cong \mathbb{Z}$, and $h_1, \dots, h_n \in H$ in distinct cosets of N, the sum $\mathcal{D}_{N,\mathcal{D}}\varphi(u_{h_1}) + \cdots + \mathcal{D}_{N,\mathcal{D}}\varphi(u_{h_n})$ is direct. (Here $\mathcal{D}_{N,\mathcal{D}} = \mathcal{D}(\varphi(E * N), \mathcal{D})$ is the division closure of $\varphi(E * N)$ in $\mathcal{D}$.)

Hughes [11] (see also [7]) showed that up to $E * G$-isomorphism there exists at most one Hughes-free division ring. We denote it by $\mathcal{D}_{E*G}$. The uniqueness of Hughes-free division rings implies that for every subgroup H of G, $\mathcal{D}_{H,\mathcal{D}_{E*G}}$ is Hughes-free as a division $E * H$-ring.

Gräter showed in [8, Corollary 8.3] that $\mathcal{D}_{E*G}$ (if it exists) is **strongly Hughes-free**, that it satisfies the following additional conition:

(2') For every non-trivial subgroup H of G, a normal subgroup N of H and $h_1, \dots, h_n \in H$ in distinct cosets of N, the sum $\mathcal{D}_{N,\mathcal{D}_{E*G}}\varphi(u_{h_1}) + \cdots + \mathcal{D}_{N,\mathcal{D}_{E*G}}\varphi(u_{h_n})$ is direct.

In particular, this implies the following result that we will use often without mentioning it explicitly.

Proposition 2.2 *Let G be a locally indicable group, N a normal subgroup of G and E a division ring. Assume that for a crossed product $E * G$, $\mathcal{D}_{E*G}$ exists. Then the ring R generated by $\mathcal{D}_{N,\mathcal{D}_{E*G}}$ and G has structure of a crossed product $\mathcal{D}_{E*N} * (G/N)$. In particular,*

(1) if N is of finite index in G, then $\mathcal{D}_{E*G} = \mathcal{D}_{E*N} * (G/N)$ and
(2) if G/N is abelian, $\mathcal{D}_{E*G}$ is isomorphic to the classical Ore ring of fractions of $\mathcal{D}_{E*N} * (G/N)$.

2.4 Free division $E * G$-ring of fractions

Let G be group with a Conradian left-invariant order $\preceq$ (so, G is locally indicable). Let E be a division ring. Let $\varphi : E * G \to \mathcal{D}$ be a homomorphism from a crossed product $E * G$ to a division ring $\mathcal{D}$. We say that a division $E * G$-ring $(\mathcal{D}, \varphi)$ is **free with respect to** $\preceq$ if

(1) $\mathcal{D}$ is the division closure of $\varphi(E * G)$.
(2) For every subgroup H of G, and the maximal proper convex subgroup N of H (which is normal by Proposition 2.1), and $h_1, \ldots, h_n \in H$ in distinct cosets of N, the sum $\mathcal{D}_{N,\mathcal{D}}\varphi(u_{h_1}) + \cdots + \mathcal{D}_{N,\mathcal{D}}\varphi(u_{h_n})$ is direct.

This notion was introduced by Gräter in [8].

Remark 2.3 Notice that in part (2) of the definition, we also can assume that H is finitely generated. Indeed, assume (2) holds for finitely generated subgroups, but for some H and $h_1, \ldots, h_n$, there are $d_1, \ldots, d_n \in \mathcal{D}_{N,\mathcal{D}}$, not all equal to zero, such that $d_1\varphi(u_{h_1}) + \cdots + d_n(u_{h_n}) = 0$. Then we can find a finitely generated subgroup of N' of N such that $d_1, \ldots, d_n \in \mathcal{D}_{N',\mathcal{D}}$. Let H' be the subgroup of G generated by $h_1, \ldots, h_n$ and N'. Since $n \geq 2$, $N \cap H'$ is the maximal convex subgroup of H'. This contradicts our assumption that (2) holds for H'.

Gräter proved the following result.

Proposition 2.4 *[8, Corollary 8.3] Let G be a group with a Conradian left-invariant order $\preceq$ and let E be a division ring. A division $E * G$-ring is free with respect to $\preceq$ if and only if it is Hughes-free (and so, it is $E * G$-isomorphic to $\mathcal{D}_{E*G}$).*

2.5 Sylvester matrix rank functions

Let R be a ring. A **Sylvester matrix rank function** rk on R is a function that assigns a non-negative real number to each matrix over R and satisfies the following conditions.

(SMat1) $\mathrm{rk}(M) = 0$ if M is any zero matrix and $\mathrm{rk}(1) = 1$;
(SMat2) $\mathrm{rk}(M_1M_2) \leq \min\{\mathrm{rk}(M_1), \mathrm{rk}(M_2)\}$ for any matrices M_1 and M_2 which can be multiplied;
(SMat3) $\mathrm{rk}\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$ for any matrices M_1 and M_2;
(SMat4) $\mathrm{rk}\begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix} \geq \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$ for any matrices M_1, M_2 and M_3 of appropriate sizes.

We denote by $\mathbb{P}(R)$ the set of Sylvester matrix rank functions on R, which is a compact convex subset of the space of functions on matrices over R. If $\phi : F_1 \to F_2$ is an R-homomorphism between two free finitely generated R-modules F_1 and F_2, then $\mathrm{rk}(\phi)$ is $\mathrm{rk}(A)$ where A is the matrix associated with ϕ with respect to some R-bases of F_1 and F_2. It is clear that $\mathrm{rk}(\phi)$ does not depend on the choice of the bases.

A useful observation is that a ring homomorphism $\varphi : R \to S$ induces a continuous map $\varphi^\sharp : \mathbb{P}(S) \to \mathbb{P}(R)$, i.e., we can pull back any rank function rk on S to a rank function $\varphi^\sharp(\mathrm{rk})$ on R by just defining

$$\varphi^\sharp(\mathrm{rk})(A) = \mathrm{rk}(\varphi(A))$$

for every matrix A over R. We will often abuse the notation and write rk instead of $\varphi^\sharp(\mathrm{rk})$ when it is clear that we speak about the rank function on R.

A division ring $\mathcal{D}$ has a unique Sylvester matrix rank function which we denote by $\mathrm{rk}_\mathcal{D}$. If a Sylvester matrix rank function rk on R takes only integer values, then by a result of P. Malcolmson [21] there are a division ring $\mathcal{D}$ and a homomorphism $\varphi : R \to \mathcal{D}$ such that $\mathrm{rk} = \varphi^\sharp(\mathrm{rk}_\mathcal{D})$. Moreover, if $\mathcal{D}$ is equal to the division closure of $\varphi(R)$ ($\mathcal{D}$ is an epic division R-ring), then $\varphi : R \to \mathcal{D}$ is unique up to isomorphisms of R-rings. We denote the set of integer-valued rank functions on a ring R by $\mathbb{P}_{div}(R)$. In the following, if a rank function on R is induced by a homomorphism to $\mathcal{D}$ we will also use $\mathrm{rk}_\mathcal{D}$ to denote this rank function (in this case the homomorphism will be clear from the context).

Given two Sylvester matrix rank functions on R, rk_1 and rk_2, we will write $\mathrm{rk}_1 \le \mathrm{rk}_2$ if for any matrix A over R, $\mathrm{rk}_1(A) \le \mathrm{rk}_2(A)$. In the case where both functions are integer-valued and come from homomorphisms $\varphi_i : R \to \mathcal{D}_i$ ($i = 1, 2$) from R to epic division rings $\mathcal{D}_1$ and $\mathcal{D}_2$, the condition $\mathrm{rk}_{\mathcal{D}_1} \le \mathrm{rk}_{\mathcal{D}_2}$ is equivalent to the existence of a specialization from $\mathcal{D}_2$ to $\mathcal{D}_1$ in the sense of P. M. Cohn ([3, Subsection 4.1]). We say that an epic division R-ring $\mathcal{D}$ is **universal** if for every epic division R-ring $\mathcal{E}$, $\mathrm{rk}_\mathcal{D} \ge \mathrm{rk}_\mathcal{E}$.

An alternative way to introduce Sylvester rank functions is via Sylvester module rank functions. A **Sylvester module rank function** dim on R is a function that assigns a non-negative real number to each finitely presented R-module and satisfies the following conditions.

(SMod1) $\dim\{0\} = 0$, $\dim R = 1$;
(SMod2) $\dim(M_1 \oplus M_2) = \dim M_1 + \dim M_2$;
(SMod3) if $M_1 \to M_2 \to M_3 \to 0$ is exact then

$$\dim M_1 + \dim M_3 \ge \dim M_2 \ge \dim M_3.$$

There exists a natural bijection between Sylvester matrix and module rank functions over a ring. Given a Sylvester matrix rank function rk on R and a finitely presented R-module $M \cong R^n/R^m A$ (A is a matrix over R), we define the corresponding Sylvester module rank function dim by means of $\dim(M) = n - \mathrm{rk}(A)$. If a Sylvester matrix rank function $\mathrm{rk}_\mathcal{D}$ comes from a division R-ring $\mathcal{D}$, then the corresponding Sylvester module rank function will be denoted by $\dim_\mathcal{D}$. Then $\mathcal{D}$ is the universal epic division R-ring if and only if for every epic division R-ring $\mathcal{E}$ and every finitely presented R-module, $\dim_\mathcal{D}(M) \le \dim_\mathcal{E}(M)$.

By a recent result of Li [19], any Sylvester module rank function on R can be extended to a function (satisfying some natural conditions) on arbitrary modules over R. In the case of an integer-valued Sylvester module rank function $\dim_\mathcal{D}$ and an R-module M we simply have $\dim_\mathcal{D}(M) = \dim_\mathcal{D}(\mathcal{D} \otimes_R M)$.

2.6 Von Neumann rank function

Consider first the case where G is countable. Then G acts by left and right multiplication on the separable Hilbert space $l^2(G)$. A finitely generated **Hilbert** G-**module** is a closed subspace $V \le l^2(G)^n$, invariant under the left action of G. We denote by $\mathrm{proj}_V : l^2(G)^n \to l^2(G)^n$ the orthogonal projection onto V and we define

$$\dim_G V := \mathrm{Tr}_G(\mathrm{proj}_V) := \sum_{i=1}^{n} \langle (\mathbf{1}_i)\,\mathrm{proj}_V, \mathbf{1}_i \rangle_{l^2(G)^n},$$

where $\mathbf{1}_i$ is the element of $l^2(G)^n$ having 1 in the ith entry and 0 in the rest of the entries. The number $\dim_G V$ is the **von Neumann dimension** of V.

Let $A \in \mathrm{Mat}_{n\times m}(\mathbb{C}[G])$ be a matrix over $\mathbb{C}[G]$. The action of A by right multiplication on $l^2(G)^n$ induces a bounded linear operator $\phi_G^A : l^2(G)^n \to l^2(G)^m$. We put

$$\mathrm{rk}_G(A) = \dim_G \overline{\mathrm{Im}\,\phi_G^A}.$$

If G is not countable then rk_G can be defined in the following way. Take a matrix A over $\mathbb{C}[G]$. Then the group elements that appear in A are contained in a finitely generated group H. We will put $\mathrm{rk}_G(A) = \mathrm{rk}_H(A)$. One easily checks that the value $\mathrm{rk}_H(A)$ does not depend on the subgroup H.

Another obvious Sylvester matrix rank function on G arises from the trivial homomorphism $G \to \{1\}$ and it is defined as

$$\mathrm{rk}_{\{1\}}(A) = \mathrm{rk}_{\mathbb{C}}(\overline{A}),$$

where $\overline{A}$ is the matrix over $\mathbb{C}$ obtained from A by sending all the elements of G to 1. More generally, if $\overline{G}$ is a quotient of G, $\mathrm{rk}_{\overline{G}}(A)$ is denoted to be $\mathrm{rk}_{\overline{G}}(\overline{A})$, where $\overline{A}$ is the matrix over $\mathbb{C}[\overline{G}]$ obtained from A by applying the obvious map $\mathbb{C}[G] \to \mathbb{C}[\overline{G}]$.

2.7 The natural extension

Let $R = E * G$ be a crossed product of a division ring E and a group G. Let N be a normal subgroup of G such that G/N is amenable. Consider a transversal $\overline{X}$ of N in G. Since G/N is amenable there are finite subsets $\overline{X}_k$ of $\overline{X}$ such that $\{N\overline{X}_k/N\}$ is a Følner sequence in G/N with respect to the right action. Put $X_k = N\overline{X}_k$.

Let rk be a Sylvester rank function on $E * N$ and assume that rk is invariant under conjugation by the elements $\{u_g\}_{g\in G}$. Observe that if $\mathrm{rk} = \mathrm{rk}_{\mathcal{E}}$ for some epic division $E * N$-ring $\mathcal{E}$, then the conjugation of $E * N$ by any $u_g (g \in G)$ can be extended to a unique automorphism of $\mathcal{E}$. Thus one can consider the crossed product $\mathcal{E} * G/N$ containing $E * G$.

Let $A \in \mathrm{Mat}_{n\times m}(R)$ and let S be the union of supports of the entries of A. For any subset T of G we denote $R_T = \oplus_{t\in T} R_t$. Let $\phi_k : (R_{X_k})^n \to (R_{X_k S})^m$ be the homomorphism of finitely generated free $E * N$-modules induced by the right multiplication by A. Let ω be a non-principal ultrafilter on $\mathbb{N}$. Then we put

$$\widetilde{\mathrm{rk}}_\omega(A) = \lim_\omega \frac{\mathrm{rk}(\phi_i)}{|\overline{X}_i|}. \tag{1}$$

Then $\widetilde{\mathrm{rk}}_\omega$ is a Sylvester rank function on R. The rank function $\widetilde{\mathrm{rk}}_\omega$ has been already studied previously in different situations (see [14,15,17,27]). In [17] it is shown that $\widetilde{\mathrm{rk}}_\omega$ does not depend on ω. Therefore in the following we denote $\widetilde{\mathrm{rk}}_\omega$ by $\widetilde{\mathrm{rk}}$. The Sylvester rank function $\widetilde{\mathrm{rk}}$ is called **the natural extension** of rk. We describe now the cases that appear in this paper.

Proposition 2.5 *Let G be a group with a normal subgroup N such that G/N is amenable. Let E be a division ring, and assume the previous notation. Then the following holds.*

(1) Assume that N and G/N are locally indicable and $\mathrm{rk} = \mathrm{rk}_{\mathcal{E}}$ for some epic division $E * N$-ring $\mathcal{E}$. Then $\widetilde{\mathrm{rk}}$ coincides with $\mathrm{rk}_{\mathcal{Q}(\mathcal{E}*(G/N))}$, where $\mathcal{Q}(\mathcal{E}*(G/N))$ denotes the classical Ore ring of fractions of $\mathcal{E}*(G/N)$.
(2) Assume $E * G = K[G]$, where K is a subfield of $\mathbb{C}$ and $\mathrm{rk} = \mathrm{rk}_N$. Then $\widetilde{\mathrm{rk}}$ is equal to rk_G.
(3) Assume $E * G = K[G]$, where K is a subfield of $\mathbb{C}$ and $\mathrm{rk} = \mathrm{rk}_{\{1\}}$. Then $\widetilde{\mathrm{rk}}$ is equal to $\mathrm{rk}_{G/N}$.

Proof (1) We can extend $\widetilde{\mathrm{rk}}$ to a Sylvester matrix rank function on $\mathcal{E}*(G/N)$ (which we denote also by $\widetilde{\mathrm{rk}}$) using the formula (1). Since G/N is locally indicable, the ring $\mathcal{E}*(G/N)$ is a domain. Thus, by the definition of $\widetilde{\mathrm{rk}}$, $\widetilde{\mathrm{rk}}(a) = 1$ for every $0 \neq a \in \mathcal{E}*(G/N)$. Hence, applying [14, Proposition 5.2], we obtain that $\widetilde{\mathrm{rk}} = \mathrm{rk}_{\mathcal{Q}(\mathcal{E}*(G/N))}$.
The statements (2) and (3) follow from [14, Theorem 12.1]. □

3 On the universality of $\mathcal{D}_{E*G}$

3.1 A general criterion of universality

In this subsection we present a general criterion of universality of a division R-ring. The proof of the following lemma is immediate.

Lemma 3.1 *Let R be a ring and $\mathcal{E}$ a division R-ring. Let M be a finitely generated left R-module. Then the following are equivalent.*

(1) $\dim_{\mathcal{E}}(M) \neq 0$.
(2) $\mathcal{E} \otimes_R M \neq 0$.
(3) $\mathrm{Hom}_R(M, \mathcal{E}) \neq 0$.

The following proposition tells us that in order to check universality of a division R-ring $\mathcal{D}$ it is enough to understand the structure of its finitely generated R-submodules.

Proposition 3.2 *Let R be a ring and $\mathcal{D}$ an epic division R-ring. Then $\mathrm{rk}_{\mathcal{D}}$ is universal in $\mathbb{P}_{div}(R)$ if and only if for every finitely generated left R-submodule L of $\mathcal{D}$ and every division R-ring $\mathcal{E}$, $\dim_{\mathcal{E}}(L) > 0$.*

Proof Assume that $\mathrm{rk}_{\mathcal{D}}$ is universal. Since $\mathrm{Hom}_R(L, \mathcal{D}) \neq 0$, by Lemma 3.1, $\dim_{\mathcal{D}}(L) > 0$ and so

$$\dim_{\mathcal{E}}(L) \geq \dim_{\mathcal{D}}(L) > 0.$$

This proves the "only if" part of the proposition.

Now, consider the "if" part. We want to show that for every finitely generated left R-module M and every division R-ring $\mathcal{E}$, $\dim_{\mathcal{E}}(M) \geq \dim_{\mathcal{D}}(M)$. We will do it by induction on $\dim_{\mathcal{D}}(M)$.

Let $\overline{M}$ be the image of the natural R-homomorphism $\alpha : M \to \mathcal{D} \otimes_R M$ that sends $m \in M$ to $1 \otimes m$. Observe that, since $\mathcal{D} \otimes_R M \cong \mathcal{D} \otimes_R \overline{M}$, $\dim_{\mathcal{D}}(M) = \dim_{\mathcal{D}}(\overline{M})$. We have also that $\dim_{\mathcal{E}}(\overline{M}) \leq \dim_{\mathcal{E}}(M)$. Thus, without loss of generality, we can assume that α is injective.

Now assume that $\dim_{\mathcal{D}}(M) = 1$. Since M is a submodule of $\mathcal{D}$, then $\dim_{\mathcal{E}}(M) > 0$, and so, $\dim_{\mathcal{E}}(M) \geq 1 = \dim_{\mathcal{D}}(M)$. This gives us the base of induction.

Assume that the claim holds if $\dim_{\mathcal{D}}(M) \leq n - 1$. Consider the case $\dim_{\mathcal{D}}(M) = n \geq 2$. Observe that $\dim_{\mathcal{E}}(M) \neq 0$, since M has a nontrivial quotient that lies in $\mathcal{D}$. Hence $\mathcal{E} \otimes_R M \neq \{0\}$. Let $m \in M$ be such that $1 \otimes m$ is not trivial in $\mathcal{E} \otimes_R M$. Then $\dim_{\mathcal{E}}(M/Rm) = \dim_{\mathcal{E}}(M) - 1$. Since we assume that α is injective, $1 \otimes m$ is non-trivial in $\mathcal{D} \otimes_R M$, and so, we also have $\dim_{\mathcal{D}}(M/Rm) = \dim_{\mathcal{D}}(M) - 1$. Applying the inductive assumption we obtain that

$$\dim_{\mathcal{D}}(M) = \dim_{\mathcal{D}}(M/Rm) + 1 \leq \dim_{\mathcal{E}}(M/Rm) + 1 = \dim_{\mathcal{E}}(M).$$

□

3.2 The universality of $\mathcal{D}_{E*G}$ in the amenable case

Let E be a division ring and G a locally indicable group. Proposition 3.2 indicates that in order to prove the universality we have to understand the structure of finitely generated $E*G$-submodules of $\mathcal{D}_{E*G}$. If G is amenable, they are isomorphic to finitely generated left ideals of $E * G$. The following result shows that in the latter case the condition of Proposition 3.2 holds.

Proposition 3.3 *Let $R = E * G$ be a crossed product of a division ring E and a locally indicable group G. Then for every non-trivial finitely generated left ideal L of R and every division R-ring $\mathcal{E}$, $\dim_{\mathcal{E}}(L) > 0$.*

Proof We denote by R_g the gth component of R and let u_g be an invertible element of R_g. For any element $r = \sum_{g \in G} r_g \in R$ ($r_g \in R_g$) denote by $\mathrm{supp}\,(r)$ the elements $g \in G$ for which $r_g \neq 0$ and put $l(r)$ to be equal to the number of non-trivial elements in $\mathrm{supp}\,(r)$. Thus, $l(r) = 0$ means that $r \in R_e$. For a non-trivial finitely generated left ideal L of R we put

$$l(L) = \min\{l(r_1) + \cdots + l(r_s) : \ L = Rr_1 + \cdots + Rr_s\}.$$

Observe that if a set of generators $\{r_1, \ldots, r_s\}$ of L satisfies the equality $l(L) = l(r_1) + \cdots l(r_s)$, then for each i, $l(r_i) = |\mathrm{supp}\,(r_i)| - 1$. (If not, we can change r_i by $u_g^{-1} r_i$ with $g \in \mathrm{supp}(r_i)$ and obtain a contradiction.) Moreover, if all r_i are non-trivial and $L \neq R$, then $s \leq l(L)$. Now, we define

$$s(L) = \max\{s : \ L = Rr_1 + \cdots + Rr_s, l(L) = l(r_1) + \cdots + l(r_s) \text{ and } r_i \text{ are non-trivial}\}.$$

We will prove the proposition by induction on $l(L)$. If $l(L) = 0$, then $L = R$ and we are done. Now assume that the proposition holds if $l(L) \leq n - 1$, and consider the case $l(L) = n \geq 1$.

We will proceed by inverse induction on $s(L)$. Observe that there is no L such that $s(L) \geq l(L) + 1$, so there is nothing to prove in this case. Assume that we can prove the proposition if $l(L) = n$ and $s(L) \geq k + 1$, and consider the case $l(L) = n$ and $s(L) = k$.

Let $r_1, \ldots r_k$ be a set of non-zero generators of L such that $n = l(r_1) + \cdots l(r_k)$. Let H be the group generated by $\cup_{i=1}^k \mathrm{supp}(r_i)$. Since G is locally indicable there exists a surjective $\alpha : H \to \mathbb{Z}$. Let $N = \ker \alpha$ and $t \in H$ such that $\langle t \rangle N = H$. We write

$$r_i = \sum_j u_t^{l_{ij}} r_{ij} \ \text{ with } \ 0 \neq r_{ij} \in E * N.$$

Let L' be a left ideal of R generated by $\{r_{ij}\}$. Observe that

$$\sum_{i,j} l(r_{ij}) \leq \sum_i l(r_i) \ \text{ and } \ |\{r_{ij}\}| > s(L) = k.$$

Thus, we obtain that either $l(L') < l(L)$ or $l(L') = l(L)$ and $s(L') > s(L)$. Hence we can apply the inductive hypothesis and obtain that $\mathrm{rk}_{\mathcal{E}}(L') > 0$. Thus $\mathrm{Hom}_R(L', \mathcal{E}) \neq 0$. Let $0 \neq \phi \in \mathrm{Hom}_R(L', \mathcal{E})$.

Put $S = E * H$. Observe that $S \cong E * N[x^{\pm 1}; \tau]$, where τ is conjugation by u_t. Let $\widetilde{\mathcal{E}}$ be the Ore division ring of fractions of $\mathcal{E}[x^{\pm}; \tau]$, where τ is conjugation by u_t. Then $\widetilde{\mathcal{E}}$ has a natural S-ring structure. We denote by $\dim_{\widetilde{\mathcal{E}}}$ the corresponding Sylvester module rank function on S. By Proposition 2.5(1), $\mathrm{rk}_{\widetilde{\mathcal{E}}}$ is equal to the natural extension of the restriction of $\mathrm{rk}_{\mathcal{E}}$ on $E * N$.

Let L_0 and L_0' be the left ideals of S generated by $\{r_i\}$ and $\{r_{ij}\}$ respectively. We have that $L_0 \leq L_0'$. Every element m of L_0' can be written in a unique way as $m = \sum_j u_t^j m_j$, where $m_j \in E * N \cap L_0'$. We define

$$\widetilde{\phi}(m) = \sum_j x^j \phi(m_j).$$

This defines a homomorphism of left S-modules $\widetilde{\phi} : L_0' \to \widetilde{\mathcal{E}}$. Since ϕ is not trivial, there exists r_{ij} such that $\phi(r_{ij}) \neq 0$. Therefore, $\phi(r_i) \neq 0$. Thus, the restriction of $\widetilde{\phi}$ on L_0 is not trivial. Hence, by Lemma 3.1, $\dim_{\widetilde{\mathcal{E}}}(L_0) > 0$.

Let $\dim'_{\mathcal{E}}$ be the Sylvester module rank function associated to the division S-ring $\mathcal{E}$. Since the restrictions of $\mathrm{rk}_{\mathcal{E}}$ and $\mathrm{rk}_{\widetilde{\mathcal{E}}}$ on $E * N$ coincide, [15, Lemma 8.3] implies that $\mathrm{rk}_{\mathcal{E}} \leq \mathrm{rk}_{\widetilde{\mathcal{E}}}$ as Sylvester matrix rank functions on $E * H$, and so

$$\dim'_{\mathcal{E}}(L_0) \geq \dim_{\widetilde{\mathcal{E}}}(L_0) > 0.$$

Now observe that $L \cong R \otimes_S L_0$. Hence

$$\dim_{\mathcal{E}}(L) = \dim'_{\mathcal{E}}(L_0) > 0$$

and we are done. □

Corollary 3.4 *Let G be an amenable locally indicable group and let E be a division ring. Then $\mathcal{D}_{E*G}$ is the universal division ring of fractions of $E * G$.*

Proof Observe that $E * G$ satisfies the right Ore condition and so $\mathcal{D}_{E*G}$ is isomorphic as $E * G$-ring to the classical ring of fractions $\mathcal{Q}(E * G)$. Since any finitely generated left submodule of $\mathcal{Q}(E * G)$ is isomorphic to a left ideal of $E * G$, Proposition 3.2 and Proposition 3.3 imply the desired result. □

We remark that Corollary 3.4 can be also proved using arguments similar to the ones used in the proof of [10, Lemma 2.1]. Also it is worth to be mentioned here that, by a result of D. Morris [22], a left orderable amenable group is always locally indicable.

3.3 A criterion for a group to be Lewin

In this subsection we will show that in order to prove that a Hughes-free embeddable group G is Lewin, it is enough to consider only group algebras $E[G]$. As before, by rk_E we denote the Sylvester matrix rank function on $E[G]$ induced by the homomorphism $E[G] \to E$ that sends all the group elements from G to 1.

Proposition 3.5 *Let G be a locally indicable group and E a division ring. Assume that for every division ring $\mathcal{E}$,*

(1) $\mathcal{D}_{\mathcal{E}[G]}$ exists and
(2) $\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G]}} \geq \mathrm{rk}_{\mathcal{E}}$ as Sylvester matrix rank functions on $\mathcal{E}[G]$.

If for a crossed product $E * G$, the space $\mathbb{P}_{div}(E * G)$ is not empty, then $E * G$ has the Hughes-free division ring $\mathcal{D}_{E*G}$ and, moreover, $\mathcal{D}_{E*G}$ is universal.

Proof First let us show that $\mathcal{D}_{E*G}$ exists. Let $\phi : E * G \to \mathcal{E}$ be a division $E * G$-ring. Write $R = E * G = \oplus_{g \in G} R_g$. We fix an invertible element $u_g \in R_g$ for each $g \in G$. For every $g_1, g_2 \in G$ we define

$$\alpha(g_1, g_2) = u_{g_1} u_{g_2} u_{g_1 g_2}^{-1} \in E.$$

Observe that $\mathcal{E}$ is a $E * G$-bimodule. This allows us to convert the $\mathcal{E}$-space $\tilde{R} = \oplus_{g\in G}\mathcal{E}v_g$ into a ring by putting

$$v_g a = (\phi(u_g)a\phi(u_g^{-1}))v_g \quad \text{and} \quad v_g v_h = \phi(\alpha(g,h))v_{gh},\ g,h \in G,\ a \in \mathcal{E}.$$

Clearly the ring $\tilde{R}$ has a structure of a crossed product $\tilde{R} = \mathcal{E} * G$. Define the map $\widetilde{\phi} : E * G \to \mathcal{E} * G$ by

$$\widetilde{\phi}(\sum_{g\in G} k_g u_g) = \sum_{g\in G} \phi(k_g)v_g,\ k_g \in E.$$

Then $\widetilde{\phi}$ is a homomorphism.

For each $g \in G$ we put $w_g = \phi(u_g^{-1})v_g \in \mathcal{E} * G$. Then w_g commutes with the elements from $\mathcal{E}$ and for every $g, h \in G$,

$$\begin{aligned} w_g w_h &= \phi(u_g^{-1})v_g\phi(u_h^{-1})v_h = \phi(u_h^{-1})\phi(u_g^{-1})v_g v_h \\ &= \phi(u_h^{-1})\phi(u_g^{-1})\phi(\alpha(g,h))v_{gh} = \phi(u_{gh}^{-1})v_{gh} = w_{gh}. \end{aligned}$$

Thus, we obtain that $\tilde{R} \cong \mathcal{E}[G]$. In particular $\mathcal{D}_{\mathcal{E}*G}$, and so, $\mathcal{D}_{E*G}$ exist and $\widetilde{\phi}^{\#}(\mathrm{rk}_{\mathcal{D}_{\mathcal{E}*G}})$ is equal to $\mathrm{rk}_{\mathcal{D}_{E*G}}$.

Now, we want to show that $\mathcal{D}_{E*G}$ is universal. In other words we want to show that $\mathrm{rk}_{\mathcal{D}_{E*G}} \geq \phi^{\#}(\mathrm{rk}_{\mathcal{E}})$. Let $\psi : \mathcal{E} * G \to \mathcal{E}$ be the map that sends all w_g to 1. Denote by $\mathrm{rk}'_{\mathcal{E}}$ the Sylvester matrix rank function on $\mathcal{E} * G$ induced by ψ. By our assumptions, $\mathrm{rk}'_{\mathcal{E}} \leq \mathrm{rk}_{\mathcal{D}_{\mathcal{E}*G}}$. Now observe that $\phi = \psi \circ \widetilde{\phi}$. Hence

$$\phi^{\#}(\mathrm{rk}_{\mathcal{E}}) = (\psi \circ \widetilde{\phi})^{\#}(\mathrm{rk}_{\mathcal{E}}) = \widetilde{\phi}^{\#}(\psi^{\#}(\mathrm{rk}_{\mathcal{E}})) = \widetilde{\phi}^{\#}(\mathrm{rk}'_{\mathcal{E}}) \leq \widetilde{\phi}^{\#}(\mathrm{rk}_{\mathcal{D}_{\mathcal{E}*G}}) = \mathrm{rk}_{\mathcal{D}_{E*G}}$$

as Sylvester matrix rank functions on $E * G$. □

Corollary 3.6 *Any subgroup of a Lewin group is Lewin.*

The corollary implies that our definition of Lewin group is equivalent to the one of Sánchez ([25, Definition 6.18]).

3.4 Proofs of Theorem 1.2 and Corollary 1.3

Let F be a free group freely generated by a finite set S, and let M and $\{M_i\}_{i\in\mathbb{N}}$ be normal subgroups of F. We put $G = F/M$ and $G_i = F/M_i$ and assume that $(G_i, SM_i/M_i)$ converges to $(G, SM/M)$. Assume that for all i, G_i is locally indicable and $\mathcal{D}_{E[G_i]}$ exists. Since G_i are quotients of F, abusing notation, we will also refer to $\mathrm{rk}_{E[G_i]}$ as a Sylvester matrix rank function on $E[F]$.

Let ω be an arbitrary non-principal ultrafilter on $\mathbb{N}$. We put

$$\mathrm{rk} = \lim_{\omega} \mathrm{rk}_{\mathcal{D}_{E[G_i]}} \in \mathbb{P}_{div}(E[F]).$$

Observe that for every $g \in M$, $\mathrm{rk}(g-1)=0$. Thus, rk is also a Sylvester matrix rank function on $E[G]$. We want to show that rk corresponds to the Sylvester matrix rank function of a Hughes-free division $E * G$-ring. This will prove Theorem 1.2.

For each i we fix a left-invariant Conradian order $\preceq_i$ on G_i. Define an order $\preceq$ on G by

$$fM \preceq hM \quad \text{if } \{i \in \mathbb{N} : \ fM_i \preceq_i hM_i\} \in \omega.$$

The definition does not depend on the choice of representatives, because for every $m \in M$, the set $\{i \in \mathbb{N} : m \in M_i\}$ is in ω. It is also clear that $\preceq$ is left-invariant and Conradian. In particular, this proves that G is locally indicable.

Denote by α_j the canonical homomorphism $F \to G_j$ and extend it to the homomorphism $\alpha_j : E[F] \to \mathcal{D}_{E[G_j]}$. The rank function rk corresponds to the homomorphism

$$\alpha = (\alpha_i) : E[F] \to \prod_{\omega} \mathcal{D}_{E[G_i]} := (\prod_{i\in\mathbb{N}} \mathcal{D}_{E[G_i]})/I_\omega,$$

with $I_\omega = \{(d_i) : \ \lim_\omega \mathrm{rk}_{\mathcal{D}_{E[G_i]}}(d_i) = 0\}$. Observe that $\prod_\omega \mathcal{D}_{E[G_i]}$ is a division ring. We denote by $\mathcal{D}$ the division closure of $\alpha(E[F])$ in $\prod_\omega \mathcal{D}_{E[G_i]}$. As we have observed before, for each $m \in M$, $\alpha(m-1) = 0$. Thus, $\mathcal{D}$ is a epic division $E[G]$-ring. We are going to show that $\mathcal{D}$ is free with respect to $\preceq$. For simplicity, in what follows, for each $j \in \mathbb{N}$, $\mathcal{D}_{E[G_j]}$ is denoted by $\mathcal{D}_j$.

Let H be a finitely generated subgroup of G and let N be the maximal convex subgroup of H. Let $h_1, \dots, h_n \in H$ be in distinct cosets of N. We want to show that $\alpha(h_1), \dots, \alpha(h_n)$ are $\mathcal{D}_{N,\mathcal{D}_\omega}$-linearly independent. Without loss of generality we will assume that $H = G$.

Let L_j/M_j be the maximal convex subgroup of G_j with respect to $\preceq_j$. By Proposition 2.1, since $\preceq_j$ is Conradian, there exists order-preserving homomorphism $\phi_j : G_j \to \mathbb{R}$ such that $\ker\phi_j = L_j/M_j$. Without loss of generality we see ϕ_j as an element of $H^1(F;\mathbb{R})$. We can multiply ϕ_j by a scalar in such way that $\max_{s\in S} |\phi_j(s)| = 1$. Let $\phi = \lim_\omega \phi_j \in H^1(F;\mathbb{R})$ and $L = \ker\phi$. Observe that ϕ is non-trivial, $M \le \ker\phi$ and ϕ is order-preserving with respect to $\preceq$ if we consider it as a map $G \to \mathbb{R}$. In particular, $N = L/M$.

For each i choose $f_i \in F$ such that $h_i = f_i M$. By way of contradiction, assume that there are $d_1, \dots, d_n \in \mathcal{D}_{N,\mathcal{D}}$ such that

$$d_1\alpha(f_1) + \cdots + d_n\alpha(f_n) = 0 \ \text{ in } \mathcal{D} \tag{2}$$

with $d_i \neq 0$ for some $1 \le i \le n$.

Consider the subring R of $\mathcal{D}$ generated by $\mathcal{D}_{[G,G],\mathcal{D}}$ and N. It is a quotient of a crossed product $\mathcal{D}_{[G,G],\mathcal{D}} * (N/[G,G])$. Since $N/[G,G]$ is finitely generated abelian, $\mathcal{D}_{[G,G],\mathcal{D}} * (N/[G,G])$ is left and right Noetherian. Thus, R is also left and right Noetherian. Since R is a domain, $\mathcal{D}_{N,\mathcal{D}}$ is the classical division ring of fractions of R. Hence, without loss of generality we can assume that $d_i \in R$ in (2). Therefore, there are $f_{il} \in L$ and $d_{il} \in \mathcal{D}_{[G,G],\mathcal{D}}$ such that

$$d_i = \sum_l d_{il} \cdot \alpha(f_{il}).$$

Since $h_1, \ldots, h_n \in H$ belong to distinct cosets of N, all values $\phi(f_1), \ldots, \phi(f_n)$ are distinct. Let $\epsilon = \min_{j \neq i} |\phi(f_j) - \phi(f_i)|$. Since for all i, j, $\phi(f_{il}) = 0$, we obtain that

$$\{k \in \mathbb{N} : \ |\phi_k(f_{il})| \leq \frac{\epsilon}{4} \ \text{for all}\, i, l \text{ and } \ |\phi_k(f_j) - \phi_k(f_i)| \geq \frac{3\epsilon}{4} \ \text{ for all } \ i \neq j\} \in \omega.$$

Thus, without loss of generality we assume that for every $k \in \mathbb{N}$, $|\phi_k(f_{il})| \leq \frac{\epsilon}{4}$ for all i, l and $|\phi_k(f_j) - \phi_k(f_i)| \geq \frac{3\epsilon}{4}$ for all $i \neq j$.

Since $d_{il} \in \mathcal{D}_{[G,G],\mathcal{D}}$, d_{il} are in the division closure of $\alpha(E[([F, F])])$. Therefore, we can write

$$d_{il} = (d_{ilk})_k \ \text{ and } \ d_i = \left(\sum_l d_{ilk}\alpha_k(f_{il})\right)_k \in \prod_\omega \mathcal{D}_k, \ \text{ with } \ d_{ilk} \in \mathcal{D}_{[G_j,G_j],\mathcal{D}_j}.$$

Since $d_1\alpha(f_1) + \cdots + d_n\alpha(f_n) = 0$, we obtain that

$$\{k \in \mathbb{N} : \ \sum_{i,l} d_{ilk}\alpha_k(f_{il} \cdot f_i) = 0\} \in \omega.$$

Thus, we can assume that $\sum_{i,l} d_{ilk}\alpha_k(f_{il} \cdot f_i) = 0$ for all $k \in \mathbb{N}$. Observe that since $|\phi_k(f_{il})| \leq \frac{\epsilon}{4}$ and $|\phi_k(f_j) - \phi_k(f_i)| \geq \frac{3\epsilon}{4}$,

$$\phi_k(f_{il_1} \cdot f_i) \neq \phi_k(f_{jl_2} \cdot f_j) \quad \text{if}\, i \neq j.$$

Recall that $\mathcal{D}_k$ is free with respect to $\preceq_k$. In particular, this implies that for all i,

$$\left(\sum_l d_{ilk}\alpha_k(f_{il})\right)\alpha_k(f_i) = \sum_l d_{ilk}\alpha_k(f_{il} \cdot f_i) = 0.$$

Since this holds for all k, $d_i = 0$ for all i. This shows that $\mathcal{D}$ is free with respect to $\preceq$, and so it is Hughes-free by Proposition 2.4. This finishes the proof of Theorem 1.2.

Proof of Corollary 1.3 Without loss of generality we may assume that G is finitely generated. Hence G is a limit of a collection of locally indicable amenable groups $\{G_i\}$. Thus, by Theorem 1.2, for every division ring $\mathcal{E}$, there exists $\mathcal{D}_{\mathcal{E}[G]}$. Moreover, since by Corollary 3.4, $\mathrm{rk}_{\mathcal{E}[G_i]} \geq \mathrm{rk}_{\mathcal{E}}$ as Sylvester matrix rank functions on $\mathcal{E}[G_i]$, Theorem 1.2 also implies that $\mathrm{rk}_{\mathcal{E}[G]} \geq \mathrm{rk}_{\mathcal{E}}$ as Sylvester matrix rank functions on $\mathcal{E}[G]$. Now, by Proposition 3.5, we obtain that $\mathcal{D}_{E[G]}$ is universal. □

3.5 Examples of Lewin groups

The following theorem shows that the groups that appear in Theorem 1.1 are Lewin.

Theorem 3.7 *Let G be a locally indicable group.*

(1) If all finitely generated subgroups of G are Lewin, then G is also Lewin.
(2) Any subgroup of a Lewin group is also Lewin.
(3) G is Lewin if G has a normal Lewin subgroup N such that G/N is amenable and locally indicable.
(4) Any limit in $\mathcal{G}_n$ of Lewin groups which is Hughes-free embeddable is Lewin.
(5) A finite direct product of Lewin groups is Lewin.

Proof The first statement follows directly from the definition of Lewin groups and the second one from Corollary 3.6. Let us prove now part (3).

First observe that G is Hughes-free embeddable by [12] (see also [25, Theorem 6.10]). Let $\mathcal{E}$ be a division ring. Observe that the restriction of $\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G]}}$ on $\mathcal{E}[N]$ is equal to $\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[N]}}$ and $\mathcal{D}_{\mathcal{E}[G]} \cong \mathcal{Q}(\mathcal{D}_{\mathcal{E}[N]} * G/N)$ as $\mathcal{E}[G]$-rings. Thus, by Proposition 2.5(1), $\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G]}} = \widetilde{\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[N]}}}$.

Denote by $\mathrm{rk}'_{\mathcal{E}}$ the Sylvester matrix rank function on $E[N]$ coming from the obvious map $\mathcal{E}[N] \to \mathcal{E}$. Then, again by Proposition 2.5(1), we obtain that $\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G/N]}} = \mathrm{rk}_{\mathcal{Q}(E[G/N])} = \widetilde{\mathrm{rk}'_{\mathcal{E}}}$.

Since N is Lewin, $\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[N]}} \geq \mathrm{rk}'_{\mathcal{E}}$, and so, $\widetilde{\mathrm{rk}_{\mathcal{D}_{E[N]}}} \geq \widetilde{\mathrm{rk}'_{\mathcal{E}}}$. Thus, $\mathrm{rk}_{\mathcal{D}_{E[G]}} \geq \mathrm{rk}_{\mathcal{D}_{E[G/N]}}$ as Sylvester matrix rank functions on $E[G]$. Since G/N is amenable and locally indicable, Corollary 3.4 implies that $\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G/N]}} \geq \mathrm{rk}_{\mathcal{E}}$. Hence $\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G]}} \geq \mathrm{rk}_{\mathcal{E}}$. Using Proposition 3.5, we obtain (3).

The fourth statement follows from Proposition 3.5 and Theorem 1.2.

Consider now the fifth claim. First let us prove that the direct product $G = G_1 \times G_2$ of two Lewin groups G_1 and G_2 is again Lewin. By [12], G is Hughes-free embeddable. Let $\mathcal{E}$ be a division ring. Consider the natural homomorphisms

$$\phi_1 : \mathcal{E}[G] \to \mathcal{E}[G_1],\ \phi_2 : \mathcal{E}[G_1] \to \mathcal{E} \quad \text{and} \quad \phi_3 = \phi_2 \circ \phi_1 : \mathcal{E}[G] \to \mathcal{E}.$$

Since G_2 is Lewin,

$$\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G_1]}[G_2]} \geq \mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G_1]}} \quad \text{in } \mathbb{P}(\mathcal{D}_{\mathcal{E}[G_1]}[G_2]).$$

Therefore, since $\mathcal{D}_{\mathcal{E}[G]} = \mathcal{D}_{\mathcal{D}_{\mathcal{E}[G_1]}[G_2]}$,

$$\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G]}} \geq \phi_1^{\#}(\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G_1]}}) \quad \text{in } \mathbb{P}(\mathcal{E}[G]).$$

Since G_1 is Lewin,

$$\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G_1]}} \geq \phi_2^{\#}(\mathrm{rk}_{\mathcal{E}}) \quad \text{in } \mathbb{P}(\mathcal{E}[G_1]).$$

Hence, we conclude that

$$\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G]}} \geq \phi_1^{\#}(\mathrm{rk}_{\mathcal{D}_{\mathcal{E}[G_1]}}) \geq \phi_1^{\#}(\phi_2^{\#}(\mathrm{rk}_{\mathcal{E}})) = \phi_3^{\#}(\mathrm{rk}_{\mathcal{E}}) \quad \text{in } \mathbb{P}(\mathcal{E}[G]).$$

Since $\mathcal{E}$ is arbitrary, applying Proposition 3.5, we obtain that G is Lewin. The case of two groups implies that (5) holds for an arbitrary finite product of Lewin groups. □

4 Universality of rk_G

As we have already mentioned in Introduction, when G is locally indicable $\mathrm{rk}_G = \mathrm{rk}_{\mathcal{D}_{\mathbb{C}[G]}}$. In this section we compare rk_G with other natural Sylvester matrix rank functions on $\mathbb{C}[G]$.

4.1 The condition $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$

In this subsection we will see several consequences of the condition $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$. Recall that $\mathrm{rk}_{\{1\}}$ is an alternative expression for $\mathrm{rk}_{\mathbb{C}}$ that has appeared in the previous sections. We start with the following useful proposition.

Proposition 4.1 *Let H be a finitely generated group and assume that H is not indicable. Then $\mathrm{rk}_{\{1\}}$ is maximal in $\mathbb{P}(\mathbb{Q}[H])$. In particular, any group G for which $\mathbb{Q}[G]$ has a universal division ring of fractions, is locally indicable.*

Proof Suppose that H has the following presentation.

$$H = \langle x_1, \ldots, x_d \mid r_1, r_2, \ldots \rangle.$$

Reordering the relations $\{r_i\}$ of H, without loss of generality we can assume that the abelianization of the group

$$\widetilde{H} = \langle x_1, \ldots, x_d \mid r_1, r_2, \ldots, r_d \rangle$$

is already finite.

Let F be a free group generated by $x_1, \ldots, x_d$. For each $1 \leq i \leq d$, we write $r_i - 1 = \sum_{j=1}^{d} a_{ij}(x_j - 1)$, where $a_{ij} \in \mathbb{Z}[F]$. Let

$$A = (a_{ij}) \in \mathrm{Mat}_d(\mathbb{Z}[F]) \quad \text{and} \quad B = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_d - 1 \end{pmatrix} \in \mathrm{Mat}_{d\times 1}(\mathbb{Z}[F]).$$

Denote by $\overline{A}$ and $\overline{B}$ the matrices over $\mathbb{Z}[H]$ obtained from A and B, respectively, by applying the obvious homomorphism $\mathbb{Z}[F] \to \mathbb{Z}[H]$. Since $\widetilde{H}$ has finite abelianization, we obtain that

$$\mathrm{rk}_{\{1\}}(A) = d - \dim_{\mathbb{Q}} H_1(\widetilde{H}; \mathbb{Q}) = d.$$

Let $\mathrm{rk} \in \mathbb{P}(\mathbb{Q}[H])$ satisfy $\mathrm{rk} \geq \mathrm{rk}_{\{1\}}$. In particular,

$$\mathrm{rk}(\overline{A}) \geq \mathrm{rk}_{\{1\}}(\overline{A}) = \mathrm{rk}_{\{1\}}(A) = d.$$

Since $AB = \begin{pmatrix} r_1 - 1 \\ \vdots \\ r_d - 1 \end{pmatrix}$, we obtain that $\overline{AB} = 0$. Thus, by [13, Proposition 5.1(3)], $\mathrm{rk}(\overline{B}) = 0$. Therefore, $\mathrm{rk}(a) = 0$ for every $a \in I$, where I is the augmentation ideal of $\mathbb{Q}[H]$. Since $\mathbb{Q}[H]/I$ is a division ring and so it has only one Sylvester matrix rank function, $\mathrm{rk} = \mathrm{rk}_{\{1\}}$. This shows the first part of the proposition.

Assume now that $\mathbb{Q}[G]$ has a universal division ring of fractions $\mathcal{D}$. Let H be a finitely generated subgroup of G. If H is not indicable, then, as we have just proved, the restriction of $\mathrm{rk}_{\mathcal{D}}$ on $\mathbb{Q}[H]$ is equal to $\mathrm{rk}_{\{1\}}$. Since $\mathrm{rk}_{\mathcal{D}}$ is faithful, $H = \{1\}$. □

In the next proposition we will show that the condition $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$ implies that $\mathrm{rk}_G \geq \mathrm{rk}_{\overline{G}}$ for any amenable quotient $\overline{G}$ of G.

Proposition 4.2 *Let G be a group and N a normal subgroup with G/N amenable. Let K be a subfield of $\mathbb{C}$. Assume that $\mathrm{rk}_N \geq \mathrm{rk}_{\{1\}}$ in $\mathbb{P}(K[N])$. Then $\mathrm{rk}_G \geq \mathrm{rk}_{G/N}$ as Sylvester matrix rank functions on $K[G]$.*

Proof By Proposition 2.5, rk_G is the natural extension of rk_N and $\mathrm{rk}_{G/N}$ is the natural extension of $\mathrm{rk}_{\{1\}}$. Since $\mathrm{rk}_N \geq \mathrm{rk}_{\{1\}}$ in $\mathbb{P}(K[N])$, we obtain that $\mathrm{rk}_G \geq \mathrm{rk}_{G/N}$ in $\mathbb{P}(K[G])$ □

Corollary 4.3 *Let G be a group and N a normal subgroup with G/N residually amenable. Let K be a subfield of $\mathbb{C}$. If $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$ in $\mathbb{P}(K[G])$, then $\mathrm{rk}_G \geq \mathrm{rk}_{G/N}$ holds as well.*

Proof Without loss of generality we may assume that G is finitely generated. Then there exists a chain $G = N_0 > N_1 > N_2 > \cdots$ of normal subgroups of G such that G/N_k is amenable and $\cap N_k = N$. By [13, Theorem 1.3],

$$\mathrm{rk}_{G/N} = \lim_{k \to \infty} \mathrm{rk}_{G/N_k} \text{ in } \mathbb{P}(K[G]).$$

By Proposition 4.2, $\mathrm{rk}_G \geq \mathrm{rk}_{G/N_k}$ in $\mathbb{P}(K[G])$ for every k. Hence $\mathrm{rk}_G \geq \mathrm{rk}_{G/N}$ holds as well. □

We conjecture that the corollary holds without the condition that G/N is residually amenable.

Conjecture 3 *Let G be a group and let K be a subfield of $\mathbb{C}$. Assume that $\mathrm{rk}_G \geq \mathrm{rk}_{\{1\}}$ in $\mathbb{P}(K[G])$. Then $\mathrm{rk}_G \geq \mathrm{rk}_{\overline{G}}$ in $\mathbb{P}(K[G])$ for any quotient $\overline{G}$ of G.*

4.2 Proof of Corollary 1.5

It is clear that part (1) of of Corollary 1.5 implies part (2). Kielak proved in [18] that in order to show (1), it is enough to prove that the first L^2-Betti number of G is zero.

Using Theorem 1.1, we will show that the condition (2) of Corollary 1.5 implies that the first L^2-Betti number of G is zero.

First, let us recall the definition of RFRS groups. A group G is called **residually finite rationally solvable** or **RFRS** if there exists a chain $G = H_0 > H_1 > \cdots$ of finite index normal subgroups of G with trivial intersection such that H_{i+1} contains a normal subgroup K_{i+1} of H_i satisfying that H_i/K_{i+1} is torsion-free abelian. The following proposition implies that RFRS groups are residually poly-$\mathbb{Z}$.

Proposition 4.4 *Let G be a finitely generated group, and let*

$$G = H_0 > H_1 > H_2 > \cdots > H_n > \cdots$$

be a chain of finite index normal subgroups of G with $\bigcap_{n=0}^{\infty} H_n = 1$. Suppose that for every $n \geq 0$ there exists a subgroup $K_{n+1} \lhd H_n$ such that $K_{n+1} \leq H_{n+1}$ and H_n/K_{n+1} is poly-$\mathbb{Z}$. Then G is residually poly-$\mathbb{Z}$.

Proof A pro-p version of this result is proved in [16, Proposition 5.1]. The same proof works in our case. We include it for the convenience of the reader.

For $n \geq 1$ let

$$\widetilde{K}_n = \bigcap_{g \in G/H_{n-1}} g K_n g^{-1} \lhd G$$

be the normal core of K_n in G. Since the direct product of poly-$\mathbb{Z}$-groups is poly-$\mathbb{Z}$ and a subgroup of a poly-$\mathbb{Z}$ group is poly-$\mathbb{Z}$, the group $H_{n-1}/\widetilde{K}_n$ is poly-$\mathbb{Z}$ as well.

For every $n \geq 1$ set

$$L_n = \bigcap_{i \leq n} \widetilde{K}_i \lhd G$$

and note that since $\bigcap_{n=0}^{\infty} H_n = 1$, this is a chain of subgroups that satisfies

$$\bigcap_{n=1}^{\infty} L_n \subseteq \bigcap_{n=1}^{\infty} \widetilde{K}_n \subseteq \bigcap_{n=1}^{\infty} K_n \subseteq \bigcap_{n=1}^{\infty} H_{n-1} = 1.$$

We shall argue, by induction on n, that G/L_n is poly-$\mathbb{Z}$. For $n = 1$ we have

$$G/L_1 = G/\widetilde{K}_1 = H_0/\widetilde{K}_1 \text{ is poly-}\mathbb{Z}.$$

Once $n \geq 2$ we have $L_n = L_{n-1} \cap \widetilde{K}_n$, and by induction G/L_{n-1} is poly-$\mathbb{Z}$. Thus, since an extension of two poly-$\mathbb{Z}$ groups is poly-$\mathbb{Z}$, it suffices to show that L_{n-1}/L_n is poly-$\mathbb{Z}$. Indeed, since $K_{n-1} \leq H_{n-1}$, we have that

$$L_{n-1}/L_n = L_{n-1}/L_{n-1} \cap \widetilde{K}_n \cong L_{n-1}\widetilde{K}_n/\widetilde{K}_n \leq H_{n-1}/\widetilde{K}_n \text{ is poly-}\mathbb{Z}.$$

Therefore, we conclude by recalling that a subgroup of a poly-$\mathbb{Z}$ group is poly-$\mathbb{Z}$. □

Now let us prove that the condition (2) of Corollary 1.5 implies that the first L^2-Betti number of G is zero. Let H be a subgroup of finite index such that there exists a normal subgroup N of H with $H/N \cong \mathbb{Z}$ and $H_1(N;\mathbb{Q})$ is finite-dimensional.

Assume that H has the following presentation.

$$H = \langle x_1, \dots, x_d | \, r_1, r_2, \dots \rangle.$$

Observe that $H_1(N;\mathbb{Q}) \cong H_1(H;\mathbb{Q}[H/N])$.

Let F be a free group generated by $x_1, \dots, x_d$ and consider $\mathbb{Q}[H/N]$ as an F-module. Then $H_1(F;\mathbb{Q}[H/N]) \cong \mathbb{Q}[H/N]^{d-1}$ as a $\mathbb{Q}[H/N]$-module. Since $\mathbb{Q}[H/N]$ is a PID, we can reorganize the relations $\{r_i\}$ and without loss of generality we can assume that $H_1(\widetilde{H};\mathbb{Q}[\widetilde{H}/\widetilde{N}])$ is finite-dimensional, where

$$\widetilde{H} = \langle x_1, \dots, x_d | \, r_1, r_2, \dots, r_{d-1} \rangle,$$

$\phi : \widetilde{H} \to H$ is the canonical surjection and $\widetilde{N} = \phi^{-1}(N)$.

For each $1 \le i \le d-1$, we write $r_i - 1 = \sum_{j=1}^{d} a_{ij}(x_j - 1)$, where $a_{ij} \in \mathbb{Z}[F]$. Let

$$A = (a_{ij}) \in \mathrm{Mat}_{(d-1)\times d}(\mathbb{Z}[F]) \text{ and } B = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_d - 1 \end{pmatrix} \in \mathrm{Mat}_{d\times 1}(\mathbb{Z}[F]).$$

Denote by $\overline{A}$ and $\overline{B}$ the matrices over $\mathbb{Z}[H]$ obtained from A and B, respectively, by applying the obvious homomorphism $\mathbb{Z}[F] \to \mathbb{Z}[H]$. Since $H_1(\widetilde{H};\mathbb{Q}[\widetilde{H}/\widetilde{N}])$ is finite-dimensional, we obtain that

$$\mathrm{rk}_{H/N}(\overline{A}) = \mathrm{rk}_{H/N}(A) = \mathrm{rk}_{\widetilde{H}/\widetilde{N}}(A) = d - 1.$$

By Proposition 4.4, H is residually poly-$\mathbb{Z}$. By Corollary 4.3, $\mathrm{rk}_H \ge \mathrm{rk}_{\{1\}}$ in $\mathbb{P}(\mathbb{Q}[H])$. Thus, by Corollary 4.3, $\mathrm{rk}_H(\overline{A}) \ge \mathrm{rk}_{H/N}(A) = d - 1$. Hence, since H is infinite, the sequence

$$l^2(H)^{d-1} \xrightarrow{\phi_H^{\overline{A}}} l^2(H)^d \xrightarrow{\phi_H^{\overline{B}}} l^2(H) \to 0$$

is weakly exact. Therefore, the first L^2-Betti number of H vanishes, and so the first L^2-Betti number of G vanishes as well.

4.3 Proof of Corollary 1.6

Consider the cellular chain complex of $\widetilde{X}$

$$\mathcal{C}(\widetilde{X}) : \quad \dots \mathbb{Z}[\mathcal{C}_{p+1}(\widetilde{X})] \stackrel{\partial_{p+1}}{\to} \mathbb{Z}[\mathcal{C}_p(\widetilde{X})] \stackrel{\partial_p}{\to} \mathbb{Z}[\mathcal{C}_{p-1}(\widetilde{X})] \dots \to \mathbb{Z} \to 0.$$

Since G acts freely on $\widetilde{X}$ and $X = \widetilde{X}/G$ is of finite type, we obtain that $\mathbb{Z}[\mathcal{C}_p(\widetilde{X})] \cong \mathbb{Z}[G]^{n_p}$ is a free $\mathbb{Z}[G]$-module of finite rank and the connected morphisms ∂_p are represented by multiplication by matrices A_p over $\mathbb{Z}[G]$. Hence we obtain the following equivalent representation of $\mathcal{C}(\widetilde{X})$:

$$\mathcal{C}(\widetilde{X}) : \quad \ldots\ldots \mathbb{Z}[G]^{n_{p+1}} \overset{\times A_{p+1}}{\to} \mathbb{Z}[G]^{n_p} \overset{\times A_p}{\to} \mathbb{Z}[G]^{n_{p-1}} \ldots \to \mathbb{Z} \to 0.$$

Therefore, if $p \geq 1$ the pth Betti number of X and the pth L^2-Betti number of $\widetilde{X}$ can be expressed in the following way.

$$b_p(X) = n_p - (\mathrm{rk}_{\{1\}}(A_p) + \mathrm{rk}_{\{1\}}(A_{p+1})) \text{ and } b_p^{(2)}(\widetilde{X}) = n_p - (\mathrm{rk}_G(A_p) + \mathrm{rk}_G(A_{p+1})).$$

Thus, Corollary 1.4 implies that $b_p^{(2)}(\widetilde{X}) \leq b_p(X)$ if $p \geq 2$. If $p = 1$, then $\mathrm{rk}_G(A_1) = 1$ and $\mathrm{rk}_{\{1\}}(A_1) = 0$. Therefore $b_1^{(2)}(\widetilde{X}) \leq b_1(X) - 1$.

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5 Appendix: The universal division ring of fractions of group rings of division rings and RFRS groups

In this section G is assumed to be a finitely generated RFRS group and E is a division ring. By Proposition 4.4, G is residually poly-$\mathbb{Z}$. Therefore, Corollary 1.3 implies that $\mathcal{D}_{E[G]}$ exists and it is universal. In this section we will give an alternative description of $\mathcal{D}_{E[G]}$ (see Theorem 5.10). Our proof follows essentially the argument of Kielak [18], where this description is done when $E = \mathbb{Q}$.

5.1 Characters

A **character** of G is a homomorphism from G to the additive group of real numbers $\mathbb{R}$. The set of characters $\mathrm{Hom}(G, \mathbb{R})$ is denoted also by $H^1(G; \mathbb{R})$. A character ϕ is called **irrational** if $\ker \phi/[G, G]$ is a torsion group.

If H is a subgroup of finite index of G then the restriction map embeds $H^1(G; \mathbb{R})$ into $H^1(H; \mathbb{R})$. In what follows, we will often consider $H^1(G; \mathbb{R})$ as a subset of $H^1(H; \mathbb{R})$.

If H is a normal subgroup of G then G acts on $H^1(H; R)$: for $\phi \in H^1(H; R)$ and $g \in G$, we denote by ϕ^g the character that sends $h \in H$ to $\phi(ghg^{-1})$.

Let $G = H_0 > H_1 > H_2 > \cdots$ be a chain of subgroups of G of finite index and $n \geq 0$. For any $U \subset H^1(H_n; \mathbb{R})$ we denote

$$U_n = U^o \text{ and } U_{k-1} = (\overline{U_k})^o \cap H^1(H_{k-1}; \mathbb{R}) \quad \text{when } 1 \leq k \leq n.$$

We say that U is $(G, \{H_i\}_{i\geq 0})$**-rich** if U_0 contains all the irrational characters of G. When G and $\{H_i\}_{i\geq 0}$ are clear from the context, we will simply say that U is rich.

Lemma 5.1 *Let $G = H_0 > H_1 > H_2 > \cdots$ be a chain of subgroups of G of finite index.*

(1) If U is rich in $H^1(H_n; \mathbb{R})$ and $g \in G$, then U^g is also rich.
(2) The intersection of two rich subsets of $H^1(H_n; \mathbb{R})$ is again rich.

Proof Claim (1) is clear. Let us show the second claim.

First observe that if U and V are two open subsets of $\mathbb{R}^k$, then

$$(\overline{U \cap V})^o = (\overline{U})^o \cap (\overline{V})^o. \tag{3}$$

Indeed, let $x \in (\overline{U})^o \cap (\overline{V})^o$ and let $O(x)$ be a neighborhood of x such that

$$O(x) \subseteq \overline{U} \cap \overline{V}.$$

Consider $y \in O(x)$, and let $O(y)$ be an arbitrary neighborhood of y such that

$$O(y) \subseteq \overline{U} \cap \overline{V}.$$

In particular, there exists $z \in U \cap O(y)$. Recall that U is open. Consider an arbitrary neighborhood $O(z)$ of z such that $O(z) \subseteq U \cap \overline{V}$. Since V is open, $O(z) \cap U \cap V$ is not empty. Hence $z \in \overline{U \cap V}$, and so, $y \in \overline{U \cap V}$ as well. Thus, $O(x) \subseteq \overline{U \cap V}$ and $x \in (\overline{U \cap V})^o$.

Now let U and V be two rich subset of $H^1(H_n; \mathbb{R})$ and let $W = U \cap V$. We put

$$U_n = U^o \text{ and } U_{k-1} = (\overline{U_k})^o \cap H^1(H_{k-1}; \mathbb{R}), \text{ when } 1 \leq k \leq n,$$

and similarly we define V_k and W_k.

Then we have that $W_n = U_n \cap V_n$. Now, assume that we have proved that $W_k = U_k \cap V_k$ for some $k \leq n$. Then we obtain that

$$W_{k-1} = (\overline{W_k})^o \cap H^1(H_{k-1}; \mathbb{R}) = (\overline{U_k \cap V_k})^o \cap H^1(H_{k-1}; \mathbb{R}) \stackrel{(3)}{=} U_{k-1} \cap V_{k-1}.$$

In particular, W_0 contains contains all the irrational characters of G, and so, W is rich. □

We will need the following criterion of richness.

Lemma 5.2 *Let $G = H_0 > H_1 > H_2 > \cdots$ be a chain of subgroups of G of finite index. Take non-negative integers $n \geq k \geq 0$. Let U be an open subset of $H^1(H_k; \mathbb{R})$ and let V be an open subset of $H^1(H_n; \mathbb{R})$. Assume that U is rich and all the irrational characters of U belong to V. Then V is also rich.*

Proof We put $V_n = V^o$ and $V_{i-1} = (\overline{V_i})^o \cap H^1(H_{i-1}; \mathbb{R})$ when $1 \leq i \leq n$. Then by the inverse induction on i, we obtain that all the irrational characters of U belong also to V_i for $n \leq i \leq k$. Hence $\overline{U} \subseteq \overline{V_k}$. This clearly implies that V is rich. □

5.2 Novikov rings

Let $S * G$ be a crossed product and let $\phi \in H^1(G; \mathbb{R})$. Denote by $\| \ \|_\phi$ a norm on $S * G$ defined by

$$\| \sum_i s_i \bar{g}_i \|_\phi = \max\{2^{-\phi(g_i)} : s_i \neq 0\}.$$

Our convention is that $\|0\|_\phi = 0$. Let $\widehat{S * G}^\phi$ be the completion of $S * G$ with respect to the metric induced by the norm $\| \ \|_\phi$. The ring $\widehat{S * G}^\phi$ is called the **Novikov ring of $S * G$ with respect to ϕ**.

Let $N = \ker \phi$. Then ϕ is also a character of G/N and $\widehat{S * G}^\phi$ is canonically isomorphic to $\widehat{(S * N) * G/N}^\phi$. We will not distinguish between these two rings.

Any element of $\widehat{S * G}^\phi$ can be represented in the following form $\sum_{i=1}^\infty a_i g_i$, where $a_i \in S * N$ and $\{\phi(g_i)\}_{i \in \mathbb{N}}$ is an increasing sequence tending to the infinity.

We would like to construct an environment, where we can calculate the intersection $\mathcal{D}_{E[G]} \cap \widehat{E[G]}^\phi$. In order to do this, consider the following commutative diagram of of injective homomorphisms of rings.

$$\begin{array}{ccc} E[G] & \hookrightarrow & \mathcal{D}_{E[G]} \\ \downarrow & & \downarrow^{\alpha_{G,\phi}} \\ \widehat{E[G]}^\phi & \hookrightarrow^{\beta_{G,\phi}} & \widehat{\mathcal{D}_{E[N]} * G/N}^\phi \end{array}, \tag{4}$$

where the maps are defined as follows.

Notice that $\widehat{\mathcal{D}_{E[N]} * G/N}^\phi$ is a division ring and $\mathcal{D}_{E[G]}$ is the classical Ore ring of fractions of $\mathcal{D}_{E[N]} * G/N$. Therefore, the map $\alpha_{G,\phi}$ is the unique extension of the embedding

$$\mathcal{D}_{E[N]} * G/N \hookrightarrow \widehat{\mathcal{D}_{E[N]} * G/N}^\phi.$$

Since Hughes-free division ring is unique, for every subgroup H of G, the division ring $\mathcal{D}_{E[H]}$ can be identified with the division closure of $E[H]$ in $\mathcal{D}_{E[G]}$. Thus, the ring $\widehat{\mathcal{D}_{E[N \cap H]} * (H/(N \cap H))}^\phi$ can be identified with the closure of

$$\mathcal{D}_{E[N \cap H]} * (H/(N \cap H)) \cong \mathcal{D}_{E[N \cap H]} * (HN/N) \subset \mathcal{D}_{E[N]} * G/N$$

in $\widehat{\mathcal{D}_{E[N]} * G/N}^{\phi}$ with respect to the topology induced by $\| \|_{\phi}$. Using this identifications, we obtain that $\alpha_{H,\phi}$ is the restriction of $\alpha_{G,\phi}$. Therefore, in the following we will simply write α_{ϕ} instead of $\alpha_{G,\phi}$.

The map $\beta_{G,\phi}$ can be defined as the the continuous (with respect to $\| \|_{\phi}$) extension of the map

$$E[G] = E[N] * G/N \hookrightarrow \mathcal{D}_{E[N]} * G/N.$$

Let H be a normal subgroup of G of finite index. Then the restriction of ϕ on H is a character of H and $\widehat{E[H]}^{\phi}$ can be identified with the closure of $E[H]$ in $\widehat{E[G]}^{\phi}$ with respect to the topology induced by $\| \|_{\phi}$. It follows from the definitions that $\beta_{H,\phi}$ is the restriction of $\beta_{G,\phi}$ on $\widehat{E[H]}^{\phi}$. Thus, in the following we will simply write β_{ϕ} instead of $\beta_{G,\phi}$.

For any subset S of $H^1(G;\mathbb{R})$ we put

$$\mathcal{D}_{E[G],S} = \{x \in \mathcal{D}_{E[G]} : \alpha_{\phi}(x) \in \operatorname{Im} \beta_{\phi} \text{ for every } \phi \in S\}. \tag{5}$$

If $\phi \in H^1(G;\mathbb{R})$, we will simply write $\mathcal{D}_{E[G],\phi}$ instead of $\mathcal{D}_{E[G],\{\phi\}}$. Therefore, by our definition,

$$\mathcal{D}_{E[G],S} = \bigcap_{\phi \in S} \mathcal{D}_{E[G],\phi}.$$

Proposition 5.3 *Let H be a normal subgroup of G of finite index and let S be a subset of $H^1(G;\mathbb{R})$. Then $\mathcal{D}_{E[H],S}$ is G-invariant and $\mathcal{D}_{E[G],S}$ is equal to the ring generated by $\mathcal{D}_{E[H],S}$ and G. In particular $\mathcal{D}_{E[G],S}$ is a crossed product $\mathcal{D}_{E[H],S} * G/H$.*

Proof It is clear that $\mathcal{D}_{E[H],S}$ and G are contained in $\mathcal{D}_{E[G],S}$.

Now let $x \in \mathcal{D}_{E[G],S}$. Let Q be a transversal of H in G. Since $\mathcal{D}_{E[G]} = \mathcal{D}_{E[H]} * G/H$, we can write

$$x = \sum_{q \in Q} x_q q$$

with $x_q \in \mathcal{D}_{E[H]}$. We want to show that

$$x_q \in \mathcal{D}_{E[H],S} \text{ for all } q \in Q. \tag{6}$$

This will prove the proposition. Observe that this claim does not depend on the choice of Q, because $H \subset \mathcal{D}_{E[H],S}$.

In order to prove (6), it is enough to show that for every $\phi \in S$, $x_q \in \mathcal{D}_{E[H],\phi}$. Put $N = \ker \phi$ and $T = HN$. Let Q_1 be a transversal of H in T and Q_2 a transversal of T in G. We assume that $Q = Q_1 Q_2$. Thus, we obtain that

$$x = \sum_{q_2 \in Q_2} y_{q_2} q_2, \quad \text{where } y_{q_2} = \sum_{q_1 \in Q_1} x_{q_1 q_2} q_1.$$

Since $\ker\phi \le T$ and T has finite index in G,

$$\widehat{E[G]}^{\phi} = \bigoplus_{q_2 \in Q} \widehat{E[T]}^{\phi} q_2.$$

Thus, for all $q_2 \in Q_2$, $y_{q_2} \in \mathcal{D}_{E[T],\phi}$.

Without loss of generality we can also assume that $Q_1 \subset N$. Thus Q_1 is also a transversal of $N \cap H$ in N.

For each $r \in \phi(T) = \phi(H)$, choose, $h_r \in H$ such that $\phi(h_r) = r$. Then there are $r_1 > r_2 > r_3 > \cdots$ such that we can write

$$\alpha_\phi(x_q) = \sum_{i=1}^{\infty} h_{r_i} a_{i,q} \quad \text{with } a_{i,q} \in \mathcal{D}_{E[N\cap H]}.$$

For each $q_2 \in Q_2$, we obtain that

$$\alpha_\phi(y_{q_2}) = \sum_{i=1}^{\infty} h_{r_i} \Big(\sum_{q_1 \in Q_1} a_{i,q_1 q_2} q_1 \Big).$$

Since $\alpha_\phi(y_{q_2}) \in \operatorname{Im}\beta_\phi$, we obtain that for each $i \ge 1$,

$$\sum_{q \in Q} a_{i,q_1 q_2} q_1 \in E[N].$$

Therefore, for each $i \ge 1$ and $q \in Q$, $a_{i,q} \in E[N \cap H]$. This implies, that $\alpha_\phi(x_q) \in \operatorname{Im}\beta_\phi$, and so, $x_q \in \mathcal{D}_{E[H],\phi}$ for every q. □

Let H be a normal subgroup of finite index of G and let S be a subset of $H^1(H;\mathbb{R})$. Then we put

$$\mathcal{D}_{E[G],S} = \sum_{g \in G} \mathcal{D}_{E[H],S}\, g.$$

In view of Proposition 5.3, this definition is coherent with the previous definition of $\mathcal{D}_{E[G],S}$ in (5).

Observe that if S is G-invariant, then $g^{-1}\mathcal{D}_{E[H],S}\, g \subseteq \mathcal{D}_{E[H],S}$ for all g, and so, $\mathcal{D}_{E[G],S}$ is equal to the subring of $\mathcal{D}_{E[G]}$ generated by G and $\mathcal{D}_{E[H],S}$. In this case $\mathcal{D}_{E[G],S}$ has a structure of a crossed product $\mathcal{D}_{E[H],S} * G/H$. For arbitrary S, $\mathcal{D}_{E[G],S}$ is not always a subring of $\mathcal{D}_{E[G]}$.

Let $\phi \in H^1(H;\mathbb{R})$. We denote by ϕ^G the G-orbit in $H^1(H;\mathbb{R})$. Then $\mathcal{D}_{E[G],\phi}$ is a right $\mathcal{D}_{E[G],\phi^G}$-module. Let $N = \ker\phi$. As in (4) we have

$$\begin{array}{ccc} E[H] & \hookrightarrow & \mathcal{D}_{E[H]} \\ \downarrow & & \downarrow^{\alpha_\phi} \\ \widehat{E[H]}^{\phi} & \hookrightarrow^{\beta_\phi} & \widehat{\mathcal{D}_{E[N]} * H/N}^{\phi} \end{array}, \qquad (7)$$

which induces another commutative diagram

$$\begin{array}{ccc} E[G] & \hookrightarrow & \mathcal{D}_{E[G]} \\ \downarrow & & \downarrow^{\tilde{\alpha}_\phi} \\ \widehat{E[H]}^{\phi} \otimes_{E[H]} E[G] & \hookrightarrow^{\tilde{\beta}_\phi} & \widehat{\mathcal{D}_{E[N]} * H/N}^{\phi} \otimes_{\mathcal{D}_{E[H],\phi^G}} \mathcal{D}_{E[G],\phi^G} \end{array}, \tag{8}$$

where $\tilde{\alpha}_\phi$ and $\tilde{\beta}_\phi$ are homomorphisms of right $E[G]$-modules defined in the following way. Fix a right transversal Q of H in G. Then $\tilde{\beta}_\phi$ is defined on a basic tensor by

$$\tilde{\beta}_\phi(b \otimes q) = \beta_\phi(b) \otimes q.$$

In order to define $\tilde{\alpha}_\phi$, we write an element $a \in \mathcal{D}_{E[G]}$ as $a = \sum_{q \in Q} a_q q$, with $a_q \in \mathcal{D}_{E[H]}$, and define

$$\tilde{\alpha}_\phi(a) = \sum_{q \in Q} \alpha_\phi(a_q) \otimes q.$$

Observe that with this new notation we also have

$$\mathcal{D}_{E[G],\phi} = \{x \in \mathcal{D}_{E[G]} : \ \tilde{\alpha}_\phi(x) \in \operatorname{Im} \tilde{\beta}_\phi\}. \tag{9}$$

5.3 Continuity of $\| \ \|_\phi$

Let $\phi \in H^1(G; \mathbb{R})$ and $x \in \mathcal{D}_{E[G]}$. Then we put

$$\|x\|_\phi = \|\alpha_\phi(x)\|_\phi.$$

Proposition 5.4 *Let $x \in \mathcal{D}_{E[G]}$. Then the map $H^1(G; \mathbb{R}) \to \mathbb{R}$ defined by*

$$\phi \mapsto \|x\|_\phi$$

is continous.

Proof Let G/K be the maximal torsion-free abelian quotient of G. Let R be a subring of $\mathcal{D}_{E[G]}$ generated by $\mathcal{D}_{E[K]}$ and G. Then the ring $\mathcal{D}_{E[G]}$ is isomorphic to the classical Ore ring of fractions of R. Thus, there are $y \in R$ and $0 \neq z \in R$ such that $x = yz^{-1}$. Since $\|x\|_\phi = \|y\|_\phi \|z\|_\phi^{-1}$, it is enough to prove the proposition in the case $x \in R$. Thus, let us assume that $x \in R$.

Let A be a transversal of K in G. Then we can write $x = \sum_{a \in A_0} x_a a$, where A_0 is a finite subset of A, and, for each $a \in A_0$,$x_a \in \mathcal{D}_{E[K]}$. Observe that

$$\|x\|_\phi = \max\{\|a\|_\phi : \ a \in A_0\} = \max\{2^{-\phi(a)} : \ a \in A_0\}.$$

This clearly implies that $\|x\|_\phi$ is a continuous function in ϕ. ◻

5.4 Invertibility over Novikov rings

Let H be a normal subgroup of G of finite index and $\phi \in H^1(H;\mathbb{R})$. In this subsection we want to give a sufficient condition for $x \in \mathcal{D}_{E[G],\phi^G}$ to have its inverse in $\mathcal{D}_{E[G],\phi}$.

Let G_0 be a subgroup of G containing H and let Q be a transversal of H in G_0. Observe that

$$\phi^{G_0} = \{\phi^g : g \in G_0\} = \{\phi^g : g \in Q\} = \phi^Q.$$

We can decompose any $x \in \mathcal{D}_{E[G_0]}$ as $x = \sum_{q \in Q} x_q q$ with $x_q \in \mathcal{D}_{E[H]}$. The (Q,ϕ)**-norm** of x is defined by

$$\|x\|_{\phi,Q} = \max\{\|x_q\|_\psi \|q^{|Q|}\|_\phi^{\frac{1}{|Q|}} : \psi \in \phi^Q, q \in Q\}.$$

By the definition, $\| \ \|_{\phi,Q}$ has the following properties.

Lemma 5.5 *Let $z_1, z_2 \in \mathcal{D}_{E[H]}$ and $q \in Q$. Then*

(1) $\|z_1 z_2 q\|_{\phi,Q} \le \|z_1\|_{\phi,Q} \|z_2 q\|_{\phi,Q}$.
(2) $\|z_1 q\|_{\phi,Q} = \|z_1\|_{\phi,Q} \|q\|_{\phi,Q}$.

Observe that if $\phi \in H^1(G_0;\mathbb{R}) \subseteq H^1(H;\mathbb{R})$ is a restriction of some character of G_0, then $\|x\|_{\phi,Q} = \|x\|_\phi$, and so, in this case $\| \ \|_{\phi,Q}$ is multiplicative. However, if ϕ is an arbitrary character of $H^1(H;\mathbb{R})$, then $\| \ \|_{\phi,Q}$ is not multiplicative in general. This motivates the notion of the **defect of** $\| \ \|_{\phi,Q}$.

$$\operatorname{def}_Q(\phi) = \max\left\{\frac{\|q_1 q_2\|_{\phi,Q}}{\|q_1\|_{\phi,Q}\|q_2\|_{\phi,Q}} : q_1, q_2 \in Q\right\}.$$

Observe that if $q_1 \in H$, then by Lemma 5.5, $\|q_1 q_2\|_{\phi,Q} = \|q_1\|_{\phi,Q}\|q_2\|_{\phi,Q}$,. Thus, $\operatorname{def}_Q(\phi)$ is always at least 1. We have the following consequence of Proposition 5.4.

Corollary 5.6 *Let H be a normal subgroup of finite index of G, $H \le G_0 \le G$ and Q a transversal of H in G_0. Let $x \in \mathcal{D}_{E[G_0]}$. Then the following functions on $H^1(H;\mathbb{R})$,*

$$\phi \mapsto \|x\|_{\phi,Q} \text{ and } \phi \mapsto \operatorname{def}_Q(\phi),$$

are continuous.

We will use the following properties of $\| \ \|_{\phi,Q}$.

Proposition 5.7 *Let H be a normal subgroup of finite index of G, $H \le G_0 \le G$ and Q a transversal of H in G_0. Let $\phi \in H^1(H;\mathbb{R})$. Then for every $w, z \in \mathcal{D}_{E[G_0]}$,*

$$\|z+w\|_{\phi,Q} \le \max\{\|z\|_{\phi,Q}, \|w\|_{\phi,Q}\} \text{ and } \|z \cdot w\|_{\phi,Q} \le \|z\|_{\phi,Q} \cdot \|w\|_{\phi,Q} \cdot \operatorname{def}_Q(\phi).$$

Proof If $g \in G_0$, let $\overline{g} \in Q$ be such that $Hg = H\overline{g}$. We write $z = \sum_{q\in Q} z_q q$ and $w = \sum_{q\in Q} w_q q$, with $z_q, w_q \in \mathcal{D}_{E[H]}$. Then

$$z + w = \sum_{q\in Q}(z_q + w_q)q \text{ and } z \cdot w = \sum_{q\in Q}\left(\sum_{q=\overline{q_1 q_2}} z_{q_1}(w_{q_2})^{q_1^{-1}} q_1 q_2\right).$$

Let $\psi \in \phi^Q$. Since $\|z_q + w_q\|_\psi \le \max\{\|z_q\|_\psi, \|w_q\|_\psi\}$, we obtain that $\|z+w\|_{\phi,Q} \le \max\{\|z\|_{\phi,Q}, \|w\|_{\phi,Q}\}$.

Observe that

$$\begin{aligned} \|z_{q_1}(w_{q_2})^{q_1^{-1}} q_1 q_2\|_{\phi,Q} &\overset{\text{Lemma 5.5}}{\le} \|z_{q_1}\|_{\phi,Q}\|w_{q_2}\|_{\phi,Q}\|q_1 q_2\|_{\phi,Q} \\ &\le \|z_{q_1}\|_{\phi,Q}\|q_1\|_{\phi,Q}\|w_{q_2}\|_{\phi,Q}\|q_2\|_{\phi,Q} \operatorname{def}_Q(\phi) \\ &\overset{\text{Lemma 5.5}}{=} \|z_{q_1} q_1\|_{\phi,Q}\|w\|_{\phi,Q} \operatorname{def}_Q(\phi) \le \|z\|_{\phi,Q}\|w\|_{\phi,Q} \operatorname{def}_Q(\phi). \end{aligned}$$

Therefore $\|z \cdot w\|_{\phi,Q} \le \|z\|_{\phi,Q} \cdot \|w\|_{\phi,Q} \cdot \operatorname{def}_Q(\phi)$. □

Corollary 5.8 *Let H be a normal subgroup of finite index of G, $H \le G_0 \le G$ and Q a transversal of H in G_0. Let $\phi \in H^1(H; \mathbb{R})$ and let $w, y \in \mathcal{D}_{E[G_0],\phi^Q}$. Assume that w is invertible in $\mathcal{D}_{E[G_0],\phi^Q}$ and*

$$\|y\|_{\phi,Q} \cdot \|w^{-1}\|_{\phi,Q} < \operatorname{def}_Q(\phi)^{-2}.$$

Then $w + y \ne 0$ and $(w + y)^{-1} \in \mathcal{D}_{E[G_0],\phi}$.

Proof By Proposition 5.7,

$$(w + y)w^{-1} = 1 - z \text{ with } \|z\|_{\phi,Q} < \operatorname{def}_Q(\phi)^{-1}.$$

In particular $w + y \ne 0$.

Let us put $\epsilon = \|z\|_{\phi,Q} \operatorname{def}_Q(\phi)$. Then $\epsilon < 1$ and, by Proposition 5.7,

$$\|z^n\|_{\phi,Q} \le \frac{\epsilon^n}{\operatorname{def}_Q(\phi)}.$$

Thus, if we write

$$z^n = \sum_{q\in Q} z_{q,n} q, \quad \text{with } z_{q,n} \in \mathcal{D}_{E[H],\phi^Q},$$

then we obtain that for every $\psi \in \phi^Q$,

$$\|z_{q,n}\|_\psi \le \frac{\|z^n\|_{\phi,Q}}{\|q^{|Q|}\|_\phi^{\frac{1}{|Q|}}} = \frac{\epsilon^n}{\operatorname{def}_Q(\phi)\|q^{|Q|}\|_\phi^{\frac{1}{|Q|}}}. \tag{10}$$

Consider

$$v = \sum_{q\in Q}\left(\sum_{n=0}^{\infty} z_{q,n}\right)\otimes q,$$

and observe that, by (10), $v \in \operatorname{Im}\tilde{\beta}_\psi$. On the one hand we have that

$$\begin{aligned} v(1-z) &= \left(\sum_{q\in Q}\left(\lim_{k\to\infty}\sum_{n=0}^{k} z_{q,n}\right)\otimes q\right)(1-z) \\ &= \left(\lim_{k\to\infty}\tilde{\beta}_\psi\left(\sum_{n=0}^{k} z^n\right)\right)(1-z) = \lim_{k\to\infty}\tilde{\beta}_\psi(1-z^{k+1}) = 1\otimes 1. \end{aligned}$$

On the other hand,

$$\tilde{\alpha}_\psi((1-z)^{-1})(1-z) = \tilde{\alpha}_\psi(1) = 1\otimes 1.$$

Thus, $\tilde{\alpha}_\psi((1-z)^{-1}) = v$. By (9), we conclude that $(1-z)^{-1} \in \mathcal{D}_{E[G_0],\phi}$, and so, $(w+y)^{-1} \in \mathcal{D}_{E[G_0],\phi}$. □

5.5 A description of $\mathcal{D}_{E[G]}$.

For any $x \in \mathcal{D}_{E[G]}$ and any normal subgroup H of finite index in G we put

$$U_H(x) = \{\phi \in H^1(H;\mathbb{R}) : x \in \mathcal{D}_{E[G],\phi}\}.$$

Informally, $U_H(x)$ consists of the set of characters of H such that x can be represented as a matrix over $\widehat{E[H]}^{\phi}$.

Lemma 5.9 *Let $H_2 \le H_1$ be two normal subgroups of G of finite index. Let A be a transversal of H_1 in G. Consider $x \in \mathcal{D}_{E[G]}$ and write $x = \sum_{a\in A} x_a a$ with $x_a \in \mathcal{D}_{E[H_1]}$. Then*

$$U_{H_2}(x) = \bigcap_{a\in A} U_{H_2}(x_a).$$

Proof Let $\phi \in H^1(H_2;\mathbb{R})$. By the definition,

$$\mathcal{D}_{E[G],\phi} = \sum_{g\in G}\mathcal{D}_{E[H_2],\phi}\, g \quad\text{and}\quad \mathcal{D}_{E[H_1],\phi} = \sum_{g\in H_1}\mathcal{D}_{E[H_2],\phi}\, g.$$

Therefore, $\mathcal{D}_{E[G],\phi} = \sum_{a\in A}\mathcal{D}_{E[H_1],\phi}\, a$. Thus, $x \in \mathcal{D}_{E[G],\phi}$ if and only if $x_a \in \mathcal{D}_{E[H_1],\phi}$ for all $a \in A$. Hence, $U_{H_2}(x) = \bigcap_{a\in A} U_{H_2}(x_a)$. □

Since G is RFRS, there exists a chain $G = H_0 > H_1 > \cdots$ of finite index normal subgroups of G with trivial intersection such that H_{i+1} contains a normal subgroup K_i of H_i satisfying H_i/K_i is torsion free abelian. The chain $\{H_i\}$ satisfying this property is called **witnessing**. We fix a witnessing chain $\{H_i\}$ in G. Let $\mathcal{K}_{E[G]}$ denotes the set of all $x \in \mathcal{D}_{E[G]}$ such that for some $k \geq 0$, $U_{H_n}(x)$ is $(G, \{H_i\})$-rich for every $n \geq k$.

In this section we prove the following theorem. This is the main result of Appendix.

Theorem 5.10 *We have that* $\mathcal{K}_{E[G]} = \mathcal{D}_{E[G]}$.

First let us see that $\mathcal{K}_{E[G]}$ is a subring of $\mathcal{D}_{E[G]}$. Indeed, if $a, b \in \mathcal{K}_{E[G]}$, using Lemma 5.1, we obtain that there exists $k \geq 0$ such that for every $n \geq k$ there is a G-invariant rich subset U_n of $H^1(H_n; R)$ with $a, b \in \mathcal{D}_{E[G],U_n}$. Since $\mathcal{D}_{E[G],U_n}$ is a subring of $\mathcal{D}_{E[G]}$, $a + b, ab \in \mathcal{D}_{E[G]}$. Hence $\mathcal{K}_{E[G]}$ a subring of $\mathcal{D}_{E[G]}$.

Thus, in order to show that $\mathcal{K}_{E[G]} = \mathcal{D}_{E[G]}$, we have to prove that for any $0 \neq x \in \mathcal{K}_{E[G]}$, $x^{-1} \in \mathcal{K}_{E[G]}$. First let us consider the case where $x \in E[G]$.

Proposition 5.11 *Let* $0 \neq x \in E[G]$. *Then* x *is invertible in* $\mathcal{K}_{E[G]}$.

Proof Write $x = \sum_{g\in G} \alpha_g g$ and denote by $\operatorname{supp} x = \{g \in G : \alpha_g \neq 0\}$. We will show that $x^{-1} \in \mathcal{K}_{E[G]}$ by induction on $|\operatorname{supp} x|$. The base of induction is clear. Let us assume that $|\operatorname{supp} x| > 1$. There exists $k \geq 0$ such that

$$|\{gH_k : g \in \operatorname{supp} x\}| = 1 \text{ and } |\{gH_{k+1} : g \in \operatorname{supp} x\}| \geq 2.$$

Let A be a transversal of H_{k+1} in H_k. Hence, there exists $g \in G$ such that we can write

$$x = \sum_{a\in A} x_a a g, \text{ with } x_a \in E[H_{k+1}].$$

Since $g, g^{-1} \in \mathcal{K}_{E[G]}$, without loss of generality we may assume that $g = 1$. In particular, $x \in E[H_k]$.

For each $i \geq k$ we fix a transversal Q_i of H_i in H_k. For any $a \in A$, we put

$$V_{i,a} = \{\phi \in H^1(H_i; \mathbb{R}) : \|x - x_a a\|_{\phi,Q_i} \cdot \|(x_a a)^{-1}\|_{\phi,Q_i} < \operatorname{def}_{Q_i}(\phi)^{-2}\}.$$

Let $V_i = \bigcup_{a\in A} V_{i,a}$.

Claim 5.12 *For each* $i \geq k$, *the set* V_i *is rich in* $H^1(H_i; \mathbb{R})$.

Proof First observe that Corollary 5.6 implies that $V_{i,a}$, and so, V_i are open in $H^1(H_i; \mathbb{R})$. Let ϕ be an irrational character of $H^1(H_k; \mathbb{R})$. Since $\{H_i\}$ is a witnessing chain and ϕ is irrational, $\ker \phi \leq H_{k+1}$. Therefore, there exists $a \in A$ such that

$$\|x - x_a a\|_{\phi,Q_i} = \|x - x_a a\|_\phi < \|(x_a a)\|_\phi = \frac{1}{\|(x_a a)^{-1}\|_\phi} = \frac{1}{\|(x_a a)^{-1}\|_{\phi,Q_i}}.$$

Since $\mathrm{def}_{Q_i}(\phi) = 1$, we obtain that $\phi \in V_{i,a}$ for all $i \geq k$, and so V_i contains all irrational characters of H_k. Now the claim follows from Lemma 5.2. □

By the inductive assumption, $x_a a$ is invertible in $\mathcal{K}_{E[G]}$. Thus, there exists $n \geq k$ such that for every $i \geq n$ and $a \in A$, $U_{H_i}((x_a a)^{-1})$ is rich in $H^1(H_i, \mathbb{R})$. We put

$$W_i = \bigcap_{q \in Q_i} \left(V_i \cap \bigcap_{a \in A} U_{H_i}((x_a a)^{-1}) \right)^q .$$

By Lemma 5.1, W_i is rich. Let $\phi \in W_i$. Observe that W_i is H_k-invariant. Hence $\phi^{Q_i} \subseteq V_i \cap \bigcap_{a \in A} U_{H_i}((x_a a)^{-1})$. There exists $a \in A$ such that $\phi \in V_{i,a}$. Observe that $x - x_a a, x_a a, (x_a a)^{-1} \in \mathcal{D}_{E[H_k],\phi^{Q_i}}$. By Corollary 5.8, $x^{-1} \in \mathcal{D}_{E[H_k],\phi} \subseteq \mathcal{D}_{E[G],\phi}$. Thus, $W_i \subseteq U_{H_i}(x^{-1})$ and we are done. □

Now, we consider the general case.

Proof of Theorem 5.10 We will show that $x^{-1} \in \mathcal{K}_{E[G]}$ for every $0 \neq x \in \mathcal{K}_{E[G]}$ by induction on the level $l(x)$ of x, that is defined as follows.

$$l(x) = \min\{n - k : \ x \in \mathcal{D}_{E[H_k]} \text{ and } U_{H_i}(x) \text{ is rich for every } i \geq n\}.$$

Consider first the case $l(x) \leq 0$. Then $x \in \mathcal{D}_{E[H_k]}$ and $U_{H_i}(x)$ is rich for every $i \geq k$. Let H_k/K be the maximal torsion-free abelian quotient of H_k. Let R be the subring of $\mathcal{D}_{E[H_k]}$ generated by $\mathcal{D}_{E[K]}$ and H_k. Since $\mathcal{D}_{E[H_k]}$ is the classical ring of quotients of R, we can write $x = yz^{-1}$ with non-zero $y, z \in R$. Let A be a transversal of K in H_k. Then there are finite subsets A_0 and B_0 of A such that

$$y = \sum_{a \in A_0} y_a a, \ z = \sum_{a \in B_0} z_a a \ \text{ with non-zero } \ y_a, z_a \in \mathcal{D}_{E[K]}.$$

Let ϕ be an irrational character of H_k. Observe that ϕ takes different values on the elements of A_0 and on the elements of B_0. Therefore, there are unique $a_\phi \in A_0$ and $b_\phi \in B_0$ such that

$$\phi(a_\phi) = \min\{\phi(a) : \ a \in A_0\} \text{ and } \phi(b_\phi) = \min\{\phi(b) : \ b \in B_0\}.$$

Claim 5.13 *Let ϕ be an irrational character of H_k and $w = (y_{a_\phi} a_\phi)(z_{b_\phi} b_\phi)^{-1}$. Then $\|x\|_\phi = \|w\|_\phi > \|x - w\|_\phi$. Moreover, if $x \in \mathcal{D}_{E[H_k],\phi}$, then $w \in E[H_k]$.*

Proof The claim follows directly from the definitions. □

Let

$$T = \{w_{a,b} = (y_a a)(z_b b)^{-1} : \ a \in A_0, b \in B_0\} \cap E[H_k].$$

Since $T^{-1} \subseteq \mathcal{K}_{E[G]}$ (Proposition 5.11), there exists n such that $U_{H_i}(w^{-1})$ is rich for every $w \in T$ and $i \geq n$.

For each $i \geq n$ let Q_i be a transversal of H_i in H_k. For each $w \in T$ and $i \geq n$ we put

$$V_{i,w} = \{\phi \in H^1(H_i; \mathbb{R}) : \|x - w\|_{\phi,Q} \cdot \|w^{-1}\|_{\phi,Q} < \operatorname{def}_{Q_i}(\phi)^{-2}\}$$

and $V_i = \cup_{w \in T} V_{i,w}$. Observe that V_i are open and if $\phi \in H^1(H_k; \mathbb{R})$, $\operatorname{def}_{Q_i}(\phi) = 1$. Thus, by Claim 5.13, for all $i \geq n$, V_i contains all the irrational characters of $(U_{H_k}(x))^o$. Since $(U_{H_k}(x))^o$ is rich, Lemma 5.2 implies that V_i is rich for $i \geq n$.

For each $i \geq n$ we define

$$W_i = \bigcap_{q \in Q_i} \left(V_i \cap U_{H_i}(x) \cap \bigcap_{w \in T} U_{H_i}(w^{-1}) \right)^q .$$

By Lemma 5.1, W_i is rich. Let $\phi \in W_i$. Observe that W_i is H_k-invariant. Hence $\phi^{Q_i} \subseteq V_i \cap \bigcap_{w \in T} U_{H_i}(w^{-1})$. There exists $w \in T$ such that $\phi \in V_{i,w}$. Observe that $x - w, w, (w)^{-1} \in \mathcal{D}_{E[H_k],\phi^{Q_i}}$. By Corollary 5.8, $x^{-1} \in \mathcal{D}_{E[H_k],\phi} \subset \mathcal{D}_{E[G],\phi}$. Thus, $W_i \subseteq U_{H_i}(x^{-1})$. Thus, $x^{-1} \in \mathcal{K}_{E[G]}$.

Now, we assume that $l(x) > 0$ and that the non-zero elements of $\mathcal{K}_{E[G]}$ of level less than of $l(x)$ are invertible in $\mathcal{K}_{E[G]}$. There are n and k such that $l(x) = n - k$, $x \in \mathcal{D}_{E[H_k]}$ and $U_{H_i}(x)$ is rich for every $i \geq n$.

Let A be a transversal of H_{k+1} in H_k. Hence, we can write

$$x = \sum_{a \in A} x_a a g, \quad \text{with } x_a \in \mathcal{D}_{E[H_{k+1}]}.$$

By Lemma 5.9, for every $a \in A$, $x_a \in \mathcal{K}_{E[G]}$ and $l(x_a) < l(x)$.

For each $i \geq k$ we fix a transversal Q_i of H_i in H_k. For any $a \in A$ we put

$$V_{i,a} = \{\phi \in H^1(H_i; \mathbb{R}) : \|x - x_a a\|_{\phi,Q_i} \cdot \|(x_a a)^{-1}\|_{\phi,Q_i} < \operatorname{def}_{Q_i}(\phi)^{-2}\}.$$

Let $V_i = \bigcup_{a \in A} V_{i,a}$. Arguing as in the proof of Claim 5.12, we obtain that all V_i are rich. By the inductive assumption, $x_a a$ is invertible in $\mathcal{K}_{E[G]}$. Thus, there exists $n \geq k$ such that for every $i \geq n$ and $a \in A$, $U_{H_i}((x_a a)^{-1})$ is rich in $H^1(H_i, \mathbb{R})$. We put

$$W_i = \bigcap_{q \in Q_i} \left(V_i \cap U_{H_i}(x) \cap \bigcap_{a \in A} U_{H_i}((x_a a)^{-1}) \right)^q .$$

By Lemma 5.1, W_i is rich. Let $\phi \in W_i$. Observe that W_i is H_k-invariant. Hence $\phi^{Q_i} \subseteq V_i \cap \bigcap_{a \in A} U_{H_i}((x_a a)^{-1})$. There exists $a \in A$ such that $\phi \in V_{i,a}$. Observe that $x - x_a a, x_a a, (x_a a)^{-1} \in \mathcal{D}_{E[H_k],\phi^{Q_i}}$. By Corollary 5.8, $x^{-1} \in \mathcal{D}_{E[H_k],\phi} \subseteq \mathcal{D}_{E[G],\phi}$. Thus, $W_i \subseteq U_{H_i}(x^{-1})$ and we are done. □

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