Stable pairs with a twist and gluing morphisms for moduli of surfaces

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Abstract

We propose an alternative definition for families of stable pairs (X, D) over an arbitrary (possibly non-reduced) base in the case in which D is reduced, by replacing (X, D) with an appropriate orbifold pair $(\mathcal{X}, \mathcal{D})$. This definition of a stable family ends up being equivalent to previous ones, but has the advantage of being more amenable to the tools of deformation theory. Adjunction for $(\mathcal{X}, \mathcal{D})$ holds on the nose; there is no correction term coming from the different. This leads to the existence of functorial gluing morphisms for families of stable surfaces and functorial morphisms from $(n+1)$ dimensional stable pairs to n dimensional polarized orbispaces. As an application, we study the deformation theory of some surface pairs.

1 Introduction

Since the introduction of the space of stable curves by Deligne and Mumford [11], the theory of modular compactifications of moduli spaces of varieties has held a central role in algebraic geometry. The class of stable pairs (X, D), first introduced in dimension 2 by Kollár and Shepherd-Barron [31], give a natural generalization of stable curves to higher dimensions. Building on significant advances in the minimal model program, boundedness, properness, and projectivity of the coarse space has been proven for the space of stable pairs [4,7,16–18,26,28,30,31].

One subtle aspect is the notion of a flat family of stable pairs over an arbitrary base. The first difficulty is that the natural polarization $K_X + D$ on a stable pair

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is only a $\mathbb{Q}$-divisor, so the associated sheaf $\mathcal{O}_X(K_X + D)$ is not locally free. The second difficulty is that the family of divisors D is not flat when the coefficients are small. Kollár introduced moduli problems to address these difficulties in [22,27] respectively. However, infinitesimal deformations and obstruction theories for these moduli problems are not well understood in general since the moduli functor does *not* simply parametrize flat families of pairs $(X, D) \to B$ which are fiberwise stable (see e.g. [26, Chapter 6.1] and [27, Problem 13] for discussions).

In this paper we propose an alternative solution to the first difficulty for pairs (X, D) with D reduced, which is more amenable to the tools of deformation and obstruction theory.[1] Building upon the work of Abramovich and Hassett [2], we replace (X, D) with an associated *twisted stable pair* $(\mathcal{X}, \mathcal{D})$. Admissible families of stable pairs become just flat families of twisted stable pairs with no extra condition.

1.1 Advantages of twisted stable pairs

(1) We can compute admissible deformations and obstructions of stable pairs via usual deformations and obstructions of twisted stable pairs (Corollary 1.2 and Sect. 1.4).
(2) We extend the gluing formalism of [25, Theorem 5.13] to families over arbitrary bases giving *functorial* morphisms that describe the boundary of the moduli of stable surfaces (Theorem 1.4).
(3) Adjunction for twisted stable pairs is well behaved in families and induces a *functorial* morphism from the moduli of twisted stable pairs to the moduli of canonically polarized orbifolds in one dimension lower (Theorem 1.3).

1.2 Moduli of twisted stable pairs

One way to address the issue that $\mathcal{O}_X(K_X + D)$ is *not* a line bundle is to choose an appropriate reflexive power which is a line bundle; but different choices would result in different moduli problems. Therefore Kollár introduced the following, more canonical, condition on a flat family $\pi : (X, D) \to B$ where D is reduced: the sheaves $\omega_\pi^{[m]}(mD)$ are flat and commute with base change for *every* m (see [22]). This is *not* a fiberwise condition, but rather a global condition on the family. When the base is normal and the fibers are stable, Kollár's condition is equivalent to requiring that $K_{X/B} + D$ is $\mathbb{Q}$-Cartier on the total space but the condition is more subtle when the base is non-reduced [3].

In [2], Abramovich and Hassett showed that imposing Kollár's condition is equivalent to working with an associated orbifold $\mathcal{X} \to B$. When $D = 0$, they constructed a proper Deligne-Mumford stack parametrizing such canonically polarized twisted stable varieties. Our first result is to extend this approach, incorporating a non-zero reduced divisor D, and simplifying the construction of the moduli space of stable pairs in this setting:

Theorem 1.1 (Theorems 3.9 and 3.12) *There exists a proper Deligne-Mumford stack* $\mathcal{K}_{n,v}$ *parametrizing n-dimensional twisted stable pairs* $(\mathcal{X}, \mathcal{D})$ *of volume v where* $\mathcal{X}$

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[1] In this setting the family of divisors is automatically flat (see Corollary 3.6).

is an orbifold with no stabilizers in codimension one, $\mathcal{D}$ is a reduced divisor, $K_{\mathcal{X}} + \mathcal{D}$ is a Cartier ample divisor, and $(\mathcal{X}, \mathcal{D})$ has semi-log canonical singularities.

To incorporate the divisor into the framework of [2], we replace D with the morphism of sheaves $\mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D)$. In particular, we consider pairs $(\mathcal{X}, \phi : \omega_{\mathcal{X}} \to \mathcal{L})$ where $(\mathcal{X}, \mathcal{L})$ is a polarized orbispace (see Definition 2.10) and ϕ is a nonzero morphism which cuts out the divisor $\mathcal{D}$ as the support of its cokernel (see 3.1.1, 3.7 and 3.10).

The upshot, by taking the coarse space, is that twisted stable pairs $(\mathcal{X} \to B, \phi : \omega_{\mathcal{X}} \to \mathcal{L})$ are equivalent to flat families $\pi : (X, D) \to B$ with stable fibers and that satisfy Kollár's condition for the log canonical sheaves: the sheaves $\omega_\pi^{[m]}(mD)$ are flat and compatible with base change for every m.

Corollary 1.2 *Let $(\mathcal{X}, \mathcal{D})$ be a twisted stable pair with trivial stabilizers in codimension 1 and coarse space (X, D). Then infinitesimal deformations of the associated pair $(\mathcal{X}, \phi : \omega_{\mathcal{X}} \to \mathcal{L})$ are in bijection with infinitesimal deformations of the stable pair (X, D) that satisfy Kollár's condition.*

1.3 Applications to gluing and adjunction

The formalism of twisted stable pairs simplifies the adjunction formula. There is no longer a correction term coming from the singularities of X:

Theorem 1.3 (Proposition 4.7 and Corollary 4.8) *Let $(\mathcal{X}, \mathcal{D})$ be a twisted stable pair. Then $\mathcal{L}\big|_{\mathcal{D}} \cong \omega_{\mathcal{D}}$. Moreover, let $(\mathcal{X} \to B, \phi : \omega_{\mathcal{X}} \to \mathcal{L})$ be a twisted stable pair over an arbitrary base and suppose that $\mathcal{D} \to B$ is an S_2 morphism. Then $(\mathcal{D} \to B, \mathcal{L}\big|_{\mathcal{D}})$ is a family of canonically polarized orbifolds with semi-log canonical singularities.*

The first part of Theorem 1.3 is essentially saying that the different (see Sect. 4.2) of (X, D) is replaced by the orbifold structure of $(\mathcal{X}, \mathcal{D})$. We study the precise relation between the different on (X, D) and the stabilizers on $(\mathcal{X}, \mathcal{D})$ at the end of Sect. 4. For surfaces, the second part of Theorem 1.3 gives us a morphism from $\mathcal{K}_{2,v}$ to the moduli space of orbifold stable curves.

As an application, we show that in the case of surfaces, the gluing results of [25, Theorem 5.13] can be extended to the orbifold setting and can be done functorially for families over an arbitrary base. More precisely, we construct the algebraic stack $\mathcal{G}_{2,v}$ of gluing data consisting of triples $(\mathcal{X}, \mathcal{D}, \tau : \mathcal{D}^n \to \mathcal{D}^n)$ where $(\mathcal{X}, \mathcal{D})$ is an object of $\mathcal{K}_{2,v}$ as in Theorem 1.1, $\mathcal{D}^n$ is the normalization of $\mathcal{D}$, and $\tau : \mathcal{D}^n \to \mathcal{D}^n$ is a generically fixed point free involution which preserves the preimage of the nodes of $\mathcal{D}$. Denote by $\mathcal{K}_{2,v}^\omega$ the moduli stack of stable surfaces.

Theorem 1.4 (Theorems 5.17 and 5.20) *There is a morphism $\mathcal{G}_{2,v} \to \mathcal{K}_{2,v}^\omega$ which on the level of closed points sends a triple of gluing data as above to the stable surface given by gluing the coarse space $D \subset X$ along the involution τ as in [25, Theorem 5.13].*

Theorem 1.4 is an analogue for stable surfaces of the gluing morphisms that describe the boundary of the moduli space of stable curves $\overline{\mathcal{M}}_g$ [21]. In Sect. 5.3 we show that

there exists a finite stratification of $\mathcal{K}_{2,v}^{\omega}$ such that each boundary stratum is the image of a family of gluing data under the morphism in Theorem 1.4.

1.4 Applications to deformation problems

As a first application of our approach, we compute the admissible deformations and obstructions for a plt surface pair (X, D). If we denote with $(\mathcal{X}, \mathcal{D})$ its associated twisted stable pair, then we have equivalencies

$$\mathrm{Def}(\mathcal{X}, \omega_{\mathcal{X}} \to \mathcal{L}) \cong \mathrm{Def}(\mathcal{X}, \omega_{\mathcal{X}} \otimes \mathcal{L}^{-1} \to \mathcal{O}_{\mathcal{X}}) \cong \mathrm{Def}(\mathcal{X}, \mathcal{G} \to \mathcal{O}_{\mathcal{X}})$$

where $\mathcal{G} = \omega_{\mathcal{X}} \otimes \mathcal{L}^{-1}$. But by Lemma 4.12 when (X, D) is a plt surface, the surface $\mathcal{X}$ is Gorenstein. Therefore, $\mathcal{G}$ is a line bundle, so $\mathcal{G} \to \mathcal{O}_{\mathcal{X}}$ induces a morphism $\psi : \mathcal{X} \to [\mathbb{A}^1/\mathbb{G}_m]$. In particular,

$$\mathrm{Def}(\mathcal{X}, \omega_{\mathcal{X}} \to \mathcal{L}) = \mathrm{Def}(\psi).$$

The explicit description of the stabilizers of $\mathcal{X}$ around $\mathcal{D}$ shows that ψ is representable, so by [36] the space $\mathrm{Def}(\psi)$ is controlled by the cotangent complex $\mathbb{L}_{\mathcal{X}/[\mathbb{A}^1/\mathbb{G}_m]}$.

In Sect. 6 we use this approach to explicitly compute the admissible deformations and obstructions of a K3 surface with an A_1 singularity at a point p, and a divisor D which is *not* Cartier passing through the singular point. Using twisted stable pairs we prove that the Kollár's moduli space of stable pairs is smooth of dimension 19 at such a point and agrees with the moduli space of the quasi-polarized K3 pairs obtained by resolving the singularity. In forthcoming work we will extend this approach to study the deformations and obstructions for pairs in higher dimensions.

1.5 Outline

In Sect. 2 we recall the background we need on singularities of the MMP [20,24] and polarized orbispaces [2]. In Sect. 3 we give the definition of twisted stable pair (Definition 3.7) and we prove Theorem 1.1.

In Sect. 4 we study the local properties of polarized orbispaces in order to compare our moduli functor with the $\mathbb{Q}$-Gorenstein deformations used by [14] and to prove Theorem 1.3. In Sect. 5 we prove Theorem 1.4 on gluing morphisms. In Sect. 6 we give an example deformation theoretic computation using twisted stable pairs.

Conventions

We work over an algebraically closed field k of characteristic 0. Unless otherwise specified, all the stacks will be of finite type over k. A stack $\mathcal{X}$ has property $\mathcal{P}$ generically if there is an open embedding $U \to \mathcal{X}$ which intersects all the irreducible components of $\mathcal{X}$, such that the points of U have property $\mathcal{P}$. When an algebraic stack $\mathcal{X}$ admits a coarse space, unless otherwise stated we will denote its coarse space by X. When we say that a diagram of stacks commutes, we mean it 2-commutes.

2 Background on the minimal model program and polarized orbispaces

This section is divided into three subsections. In the first one, we begin by recalling the properties we need about the singularities of the MMP. Then we introduce the analogous singularities for a Deligne-Mumford (DM) stack. In the second subsection we recall the definition of the different. Finally, in the last subsection we recall the relevant constructions and definitions from [2].

2.1 Singularities of the MMP

In this subsection we recall the properties of the singularities that appear in the MMP, which are relevant for the rest of the paper. For a more detailed exposition, and for the definitions of lc, slc, plt, demi-normal and Du Bois, we refer to [20,25].

We begin by introducing the following notation:

Notation 2.1 We say that an open subset $U \subset X$ is *big* if the complement of U has codimension at least 2 in X.

Definition 2.2 [15] A coherent sheaf $\mathcal{F}$ is said to be reflexive if the natural map $\mathcal{F} \to (\mathcal{F})^{**}$ is an isomorphism. For an arbitrary sheaf $\mathcal{F}$, the double dual $\mathcal{F}^{**}$ is called the reflexive hull. More generally, the reflexive hull of the tensor powers are called the reflexive powers and denoted $\mathcal{F}^{[n]} := (\mathcal{F}^{\otimes n})^{**}$.

Reflexive sheaves are well behaved for S_2 schemes and for S_2 morphisms (see e.g. [15]). The following lemma is known to the experts. For convenience we include a proof.

Lemma 2.3 *Consider a flat S_2 morphism $\pi : \mathcal{X} \to B$ from a DM stack $\mathcal{X}$ to a scheme, and let $U \subseteq \mathcal{X}$ be an open substack which is big on each fiber. Let $\mathcal{E}$ be a reflexive sheaf on $\mathcal{X}$, and let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. Then the restriction map $\mathrm{Hom}_{\mathcal{X}}(\mathcal{F}, \mathcal{E}) \to \mathrm{Hom}_U(\mathcal{F}_{|U}, \mathcal{E}_{|U})$ is an isomorphism.*

Proof Up to replacing $\mathcal{X}$ with an atlas, we can assume it is a scheme, which we denote by X. Let $j : U \to X$ be the inclusion of U. Since $\mathcal{E}$ is reflexive and U is big along each fiber, from [15, Proposition 3.6.1] the morphism $\mathcal{E} \to j_*(j^*\mathcal{E})$ is an isomorphism. Therefore, by the adjunction between j_* and j^*, we have $\mathrm{Hom}_X(\mathcal{F}, \mathcal{E}) \cong \mathrm{Hom}_X(\mathcal{F}, j_*(j^*\mathcal{E})) \cong \mathrm{Hom}_U(j^*(\mathcal{F}), j^*(\mathcal{E}))$. □

Consider $f : X \to B$ a flat separated morphism of locally Noetherian DM stacks with S_2 and pure d-dimensional fibers. Let $\omega_f^{\cdot}$ be the relative dualizing complex.

Definition 2.4 We define the *relative canonical sheaf* $\omega_{X/B}$ or ω_f to be the sheaf $\mathcal{H}^{-d}(\omega_f^{\cdot})$.

Definition 2.5 Given $f : X \to B$ a flat morphism of DM stacks, we say that f is Gorenstein if $\omega_f^{\cdot} \simeq_{qis} \omega_f$ and ω_f is locally free. In general, we denote by $U(f) \subset X$ the f-Gorenstein locus, that is, the largest open subset where the morphism is Gorenstein.

We highlight the following useful observation:

Observation 2.6 *Let $f : \mathcal{X} \to B$ be a flat family of DM stacks and let $B' \to B$ a morphism. Consider the pullback $f' : \mathcal{X}' := \mathcal{X} \times_B B' \to B'$ and let $h : \mathcal{X}' \to \mathcal{X}$ be the first projection. Then $h^{-1}(U(f)) = U(f')$.*

When we assume that the fibers are Gorenstein in codimension 1, then $\omega_{X/B}$ agrees with the pushforward $\iota_* \omega_{U(f)/B}$ where $\iota : U(f) \hookrightarrow X$ the inclusion of the relative Gorenstein locus (see [33, Sect. 5]). In this case $\omega_{X/B}$ is in fact a reflexive sheaf [33, Proposition 5.6].

Next we generalize the definitions for singularities of pairs to DM stacks:

Definition 2.7 Consider a pair $(\mathcal{X}, \sum a_i \mathcal{D}_i)$ consisting of a DM stack $\mathcal{X}$ and reduced equi-dimensional closed substacks $\mathcal{D}_i$ of codimension 1 with $a_i \in \mathbb{Q}_{(0,1]}$. We say that the pair $(\mathcal{X}, \sum a_i \mathcal{D}_i)$ is *log canonical* or *lc* (resp. *semi-log canonical* or *slc*) if there is an étale cover $f : Y \to \mathcal{X}$ by a scheme such that $(Y, \sum a_i f^* \mathcal{D}_i)$ is log canonical (resp. semi-log canonical).

The following observation follows from [25, 2.40, 2.41 and Corollary 2.43] (see also [25, Proposition 2.15] and see [25, Definition 5.1] for the definition of demi-normal).

Observation 2.8 *Consider a pair $(X, \sum a_i D_i)$ consisting of a demi-normal scheme X and pure codimension 1 reduced subschemes D_i. Let $f : Y \to X$ be an étale surjective morphism, or a finite surjective morphism that is étale in codimension 1, with Y demi-normal. Then the pair $(X, \sum a_i D_i)$ is lc (resp. slc) if and only if $(Y, \sum a_i f^* D_i)$ is lc (resp. slc).*

The main consequences of Observation 2.8 are the following:

- Definition 2.7 does not depend on the choice of the étale cover Y;
- Consider a pair $(\mathcal{X}, \sum a_i \mathcal{D}_i)$ with $\mathcal{X}$ a demi-normal DM stack that is a scheme in codimension 2. Let X (resp. D_i) be the coarse space of $\mathcal{X}$ (resp. $\mathcal{D}_i$). Then $(X, \sum a_i D_i)$ is lc (resp. slc) if and only if $(\mathcal{X}, \sum a_i \mathcal{D}_i)$ is lc (resp. slc).

2.2 The different

In this subsection we recall the definition of different. We refer the reader to [25, Definition 2.34 and Chapter 4] for further details.

Suppose $(X, D + \Delta)$ is an lc pair, where D is a divisor with coefficient 1 and Δ is a $\mathbb{Q}$-divisor. If X and D are *smooth*, then the usual adjunction formula gives a canonical isomorphism

$$\mathcal{O}_X(K_X + D + \Delta)\big|_D \cong \mathcal{O}_D(K_D + \Delta\big|_D). \tag{1}$$

If either X or D are singular, Eq. (1) may no longer hold. However, there is a canonical correction term given by the *different*.

More precisely, suppose $(X, D + \Delta)$ is lc and let $\nu : D^n \to D$ be the normalization of D. Since $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, for m divisible enough one can compare $(m(K_X + D + \Delta))\big|_{D^n}$ and $m(K_{D^n} + \Delta\big|_D)$. Using this, one can define an effective divisor $\mathrm{Diff}_{D^n}(\Delta)$, such that

$$K_{D^n} + \Delta\big|_D + \mathrm{Diff}_{D^n}(\Delta) \sim_{\mathbb{Q}} (K_X + D + \Delta)\big|_{D^n}. \tag{2}$$

We will see that $\text{Diff}_{D^n}(\Delta)$ is an actual effective divisor on D^n, not just a divisor class.

There are two equivalent definitions for $\text{Diff}_{D^n}(\Delta)$ and both will be useful in the sequel.

2.2.1 The different, 1st definition

Writing D as a sum of its irreducible components and applying Eq. 2 to each component, one can see that it suffices to define $\text{Diff}_{D^n}(\Delta)$ in the case where D is irreducible. Consider then $p : Y \to X$ a log-resolution of $(X, D + \Delta)$, and let $T := p_*^{-1}D$. Then we have the following commutative diagram.

$$
\begin{array}{ccc}
T & \xrightarrow{\ i\ } & Y \\
{\scriptstyle p_T}\downarrow & & \downarrow{\scriptstyle p} \\
D^n & \xrightarrow{\ j\ } & X
\end{array}
$$

On Y, we can write $p^*(K_X + D + \Delta) = K_Y + T + F$ where F is a $\mathbb{Q}$-divisor. Then $i^*p^*(K_X + D + \Delta) = K_T + F\big|_T$. Now we can define the different as

$$
\text{Diff}_{D^n}(\Delta) := (p_T)_*(F\big|_T).
$$

We remark that this definition only depends on the points of codimension at most 2 on X, so to check properties of the different it suffices to consider the case in which X is a surface. Then from [25, Proposition 2.35] the different satisfies the following properties:

(1) $\text{Diff}_{D^n}(\Delta)$ does not depend on the choice of a log resolution, and
(2) $\text{Diff}_{D^n}(\Delta)$ is an effective divisor.

2.2.2 The different, 2nd definition

One can also define the different using the Poincaré residue map. We review the definition below. See [25, Definition 4.2] for more details. Let $Z \subseteq X$ to be the union of $\text{Supp}\,\Delta \cap D$ with the closed subset where either $\text{Supp}(D)$ or X are singular. Then on $X \smallsetminus Z$ there exists a *canonical* isomorphism

$$
\mathcal{R} : \omega_X(D)\big|_{(D \smallsetminus Z)} \to \omega_{D \smallsetminus Z}
$$

given by the Poincaré residue map. For m divisible enough so that $\omega_X^{[m]}(m(D + \Delta))$ is Cartier, consider the mth tensor power $\mathcal{R}^{\otimes m}$. Since the normalization morphism $\nu : D^n \to D$ is an isomorphism on the locus where D is smooth and since $\Delta|_{D \backslash Z} = 0$, $\mathcal{R}^{\otimes m}$ pulls back to a rational section of

$$
\mathcal{H}om_{D^n}\left(\nu^*(\omega_X^{[m]}(m(D + \Delta))), \omega_{D^n}^{[m]}\right)
$$

which we also denote by $\mathcal{R}^{\otimes m}$. Now $\omega_{D^n}^{[m]}$ is Cartier in codimension 1 so the rational section $\mathcal{R}^{\otimes m}$ defines a divisor Δ_m° in codimension 1 on D^n and we denote its closure by Δ_m. Then we can define the different by

$$\mathrm{Diff}_{D^n}(\Delta) := \frac{1}{m}\Delta_m.$$

2.3 Polarized orbispaces and Kollár families of $\mathbb{Q}$-line bundles

In this subsection, we recall the definitions from [2]. All our stacks are assumed to be of finite type over a field k unless otherwise noted.

Definition 2.9 [2, Definition 2.3.1] A *cyclotomic stack* is a separated DM stack $\mathcal{X}$ such that the stabilizers of the points of $\mathcal{X}$ are finite cyclic groups.

An important example of a cyclotomic stack is the *weighted projective stack* $\mathcal{P}(\rho_1, \ldots, \rho_n)$, defined as the stack quotient $\left[(\mathbb{A}^n \setminus 0)/\mathbb{G}_m\right]$ where $\mathbb{G}_m$ acts on the ith coordinate of $\mathbb{A}^n$ by weight $\rho_i > 0$. Moreover, any closed substack of a cyclotomic stack is cyclotomic.

We will consider *polarized orbispaces*, which are cyclotomic stacks analogous to projective varieties (see [2, Definitions 2.3.11, 2.4.1 and 4.1.1]):

Definition 2.10 Let $f : \mathcal{X} \to B$ be a flat proper equi-dimensional morphism from a cyclotomic stack to a scheme. Assume that each fiber of f is generically an algebraic space. Let $\pi : \mathcal{X} \to X$ be the coarse space and $f_X : X \to B$ the induced map. A *polarizing line bundle* is a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that:

(i) For every geometric point $\xi \in \mathcal{X}(\mathrm{Spec}(K))$, the action of $\mathrm{Aut}(\xi)$ on the fiber of $\mathcal{L}$ is effective, and
(ii) There is an f_X-ample line bundle M on X and an $N > 0$ such that $\mathcal{L}^N \cong \pi^*M$.

A pair $(\mathcal{X} \to B, \mathcal{L})$ as above is a *polarized orbispace*.

Remark 2.11 We do not require the fibers of f to be connected.

Remark 2.12 Line bundles satisfying Condition (i) of Definition 2.10 are called uniformizing line bundles in [2]. This condition is equivalent to requiring that the induced map to $B\mathbb{G}_m$ is representable [2, Proposition 2.3.10].

Note that a weighted projective stack with the line bundle $\mathcal{O}(1)$ is a polarized orbispace and by [2, Corollary 2.4.4], any polarized orbispace is a closed substack of a weighted projective stack.

Now one can define a category fibered in groupoids $\mathcal{O}rb^{\mathcal{L}}$ as follows. The objects $|\mathcal{O}rb^{\mathcal{L}}(B)|$ over a scheme B are polarized orbispaces $(\mathcal{X} \to B, \mathcal{L})$. A morphism of $(\pi : \mathcal{X} \to B, \mathcal{L}) \to (\pi' : \mathcal{X}' \to B', \mathcal{L}')$ over a map $g : B \to B'$ consists of a morphism $f : \mathcal{X} \to \mathcal{X}'$ and an isomorphism $\phi : f^*\mathcal{L}' \to \mathcal{L}$, such that the following diagram is cartesian

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;f\;} & \mathcal{X}' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{\;g\;} & B'. \end{array}$$

Theorem 2.13 [2, Proposition 4.2.1] *The stack $\mathcal{O}rb^{\mathcal{L}}$ is algebraic and locally of finite type.*

Remark 2.14 Contrary to our conventions, we do not claim that $\mathcal{O}rb^{\mathcal{L}}$ is of finite type.

For our purposes the relevance of polarized orbispaces lies in their relation with Kollár families of $\mathbb{Q}$-line bundles:

Definition 2.15 [2, Definition 5.2.1] A *Kollár family of $\mathbb{Q}$-line bundles* is the data of a pair $(f : X \to B, F)$ consisting of a morphism of schemes f and a coherent sheaf F on X satisfying the following conditions:

(1) f is flat with fibers that are reduced, S_2, and of equi-dimension n;
(2) For every fiber X_b, the restriction $F\big|_{X_b}$ is reflexive of rank 1;
(3) For every n the formation of $F^{[n]}$ commutes with base change for maps $B' \to B$, and
(4) For each X_b, there is an N_b divisible enough such that $F^{[N_b]}\big|_{X_b}$ is a line bundle.

Points (1), (2) and (4) do not pose any major difficulty as they are fiberwise conditions. However, point (3) is difficult to check and is not automatic (see [3]). In [2], Abramovich and Hassett give a stack theoretic characterization of families satisfying condition (3) which we now review.

First, observe that given a Kollár family of $\mathbb{Q}$-line bundles $(f : X \to B, F)$, one can consider the variety $\mathcal{P}(F) := \mathrm{Spec}_{\mathcal{O}_X}(\bigoplus_{n \in \mathbb{Z}} F^{[n]})$ lying over B. There is a natural action of $\mathbb{G}_m$ over B induced by the grading and taking the quotient $\mathcal{X}_F := [\mathcal{P}(F)/\mathbb{G}_m]$ gives a cyclotomic stack which is *flat* over B [2, Proposition 5.1.4]. Moreover, the fibers of g are reduced and S_2 with trivial stabilizers in codimension one. If we further assume that f is proper and that F_b is an ample $\mathbb{Q}$-line bundle for each $b \in B$, then the natural line bundle $\mathcal{O}(1)$ on $\mathcal{X}_F$ makes $(g : \mathcal{X}_F \to B, \mathcal{O}(1))$ into a polarized orbispace. Abramovich and Hassett show that this coresponence can be reversed:

Theorem 2.16 [2, Sect. 5] *Consider $(f : \mathcal{X} \to B, \mathcal{L})$ a polarized orbispace. Assume that for every $b \in B$ the fiber $\mathcal{X}_b$ is reduced and S_2 with trivial stabilizers in codimension one. Let $p : \mathcal{X} \to X$ be the coarse space. Then $(X \to B, p_*(\mathcal{L}))$ is a Kollár family of $\mathbb{Q}$-line bundles.*

In particular, consider a Kollár family of $\mathbb{Q}$-line bundles $(X \to \mathrm{Spec}(A), F)$ over a local Artin ring A and let $A' \to A$ be an extension of local Artin rings. Then the deformations of $(X \to \mathrm{Spec}(A), F)$ along $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ which satisfy the condition (4) of Definition 2.15 are identified with the deformations of the polarized orbispace $(g : \mathcal{X}_F \to \mathrm{Spec}(A), \mathcal{O}(1))$.

Definition 2.17 We will say that a polarized orbispace $(\mathcal{X} \to B, \mathcal{L})$ satisfying the assumptions of Theorem 2.16 is an *Abramovich-Hassett (or AH) family*. Given a

Kollár family of $\mathbb{Q}$-line bundles $(X \to B, F)$, we will call $(\mathcal{X}_F \to B, \mathcal{O}(1))$ the *associated AH family*.

Finally, [2] also proves the existence of a locally of finite type (but not necessarily of finite type) algebraic stack which parametrizes *canonically* polarized orbispaces [2, Definition 6.1.1]).

Definition 2.18 Following [2], we define the moduli space parametrizing AH families of canonically polarized orbispaces with at worst slc singularities by $\mathcal{K}^\omega_{\text{slc}}$. Furthermore, we denote by $\mathcal{K}^\omega_{n,v} \subseteq \mathcal{K}^\omega_{\text{slc}}$ the substack parametrizing those polarized orbispaces of dimension n and volume v.

3 The moduli space of twisted stable pairs

The goal of this section is to present a definition of a family of stable pairs over an arbitrary base, using polarized orbispaces. We then construct an algebraic stack $\mathcal{M}_{n,v}$ of these *twisted stable pairs*.

We start with the usual definition of a stable pair:

Definition 3.1 An slc pair (X, D) is a *stable pair* if $K_X + D$ is ample.

There are two obstacles one has to overcome in order to generalize this to a notion of families of stable pairs. The first is that the $\mathbb{Q}$-divisor $K_X + D$ is only defined up to rational equivalence. Moreover, the divisor D itself is an actual Weil divisor rather than just a divisor class: even when D is Cartier, the condition of the pair being slc is not invariant under linear equivalence. Thus one needs to find a suitable definition for a family of divisors, over an arbitrary base scheme B.

To address the first point, it is natural to consider Kollár families of $\mathbb{Q}$-line bundles $(X \to B, F)$ where F restricts to the reflexive sheaf $\mathcal{O}(K_X + D)$ along each fiber. As we saw in the previous section, this is equivalent to considering flat families of polarized orbispaces. To address the second point, we follow an idea originally due to Kollár (see [24, p. 21]) and used by Kovács and Patakfalvi in [30]: we replace the data of D with the morphism of sheaves $\mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D)$.

3.1 The stack of pairs $\mathcal{H}$

We begin by defining a category $\mathcal{H}$ fibered in groupoids over Sch/k, consisting of pairs of a polarized orbispace and a morphism of sheaves as above.

3.1.1 The objects of $\mathcal{H}$

For every scheme B, an object of $\mathcal{H}(B)$ consists of a pair $(f : \mathcal{X} \to B, \phi : \omega_{\mathcal{X}/B} \to \mathcal{L})$, with $(f : \mathcal{X} \to B, \mathcal{L})$ a polarized orbispace and ϕ a morphism such that:

(1) $f : \mathcal{X} \to B$ is a flat family of equi-dimensional demi-normal stacks, and
(2) for every $b \in B$, ϕ_b is an isomorphism at the generic points and codimension one singular points of $\mathcal{X}_b$;

Remark 3.2 Note that we do not assume that the fibers of f are connected. This will simplify the definition of the moduli of gluing data in Sect. 5.

3.1.2 The arrows of $\mathcal{H}$

Consider two schemes B_1 and B_2 and a morphism $a : B_1 \to B_2$. Suppose $\alpha := (f_1 : \mathcal{X}_1 \to B_1, \phi_1 : \omega_{\mathcal{X}_1/B_1} \to \mathcal{L}_1)$ and $\beta := (f_2 : \mathcal{X}_2 \to B_2, \phi_2 : \omega_{\mathcal{X}_2} \to \mathcal{L}_2)$ are objects in $\mathcal{H}(B_1)$ and $\mathcal{H}(B_2)$ respectively. An arrow $\Phi : \alpha \to \beta$ lying over a is the data of a morphism (μ, ν) of the objects $(\mathcal{X}_1 \to B_1, \mathcal{L}_1)$ and $(\mathcal{X}_2 \to B_2, \mathcal{L}_2)$ of $\mathcal{O}rb^{\mathcal{L}}$ such that following diagram is commutative:

$$
\begin{array}{ccc}
\mu^* \omega_{\mathcal{X}_2/B_2} & \longrightarrow & \omega_{\mathcal{X}_1/B_1} \\
\downarrow{\scriptstyle \mu^*(\phi_2)} & & \downarrow{\scriptstyle \phi_1} \\
\mu^* \mathcal{L}_2 & \xrightarrow{\ \nu\ } & \mathcal{L}_1.
\end{array}
$$

Here $\mu^* \omega_{\mathcal{X}_2/B_2} \to \omega_{\mathcal{X}_1/B_1}$ is a canonical morphism, defined as follows. By Observation 2.6, there is a morphism $U(f_1) \to U(f_2)$. Since $U(f_i) \to B_i$ is Gorenstein, this induces a unique canonical isomorphism (see [9, Theorem 3.6.1])

$$
\mu^*|_{U(f_1)}(\omega_{U(f_2)/B_2}) \to \omega_{U(f_1)/B_1}. \tag{3}
$$

Letting $\iota_j : U(f_j) \to \mathcal{X}_j$ be the inclusion, we have

$$
\iota_2^*(\omega_{\mathcal{X}_2/B_2}) = \iota_2^*(\iota_{2*}\omega_{U(f_2)/B_2}) \cong \omega_{U(f_2)/B_2},
$$

and so Eq. 3 induces a map

$$
\iota_1^* \mu^*(\omega_{\mathcal{X}_2/B_2}) \cong \mu^*|_{U(f_1)}(\iota_2^*(\omega_{\mathcal{X}_2/B_2})) \cong \mu^*|_{U(f_1)}(\omega_{U(f_2)/B_2}) \to \omega_{U(f_1)/B_1}.
$$

Then the push pull adjunction gives the map $\mu^*(\omega_{\mathcal{X}_2/B_2}) \to (\iota_1)_*(\omega_{U(f_1)/B_1}) = \omega_{\mathcal{X}_1/B_1}$.

Theorem 3.3 *The category $\mathcal{H}$ is an algebraic stack locally of finite type.*

Proof Consider the algebraic stack $\mathcal{O}rb^{\mathcal{L}}$, let $(\mathcal{X}, \mathcal{L}) \to \mathcal{O}rb^{\mathcal{L}}$ be the universal polarized orbispace, and assume that $\mathcal{X} \to \mathcal{O}rb^{\mathcal{L}}$ has relative dimension n. The locus $\mathcal{O}^{(1)} \to \mathcal{O}rb^{\mathcal{L}}$ where the fibers are S_2 and reduced is open by [13, Théorème 12.2.4]. Furthermore, the condition of having at worst nodal singularities is open, so there is an open substack $\mathcal{O}^{(2)} \hookrightarrow \mathcal{O}^{(1)}$ where the fibers of $\mathcal{X}^{(2)} := \mathcal{X} \times_{\mathcal{O}rb^{\mathcal{L}}} \mathcal{O}^{(2)} \to \mathcal{O}^{(2)}$ are demi-normal.

Now over $\mathcal{O}^{(2)}$, we can consider the relative canonical sheaf $\omega_{\mathcal{X}^{(2)}/\mathcal{O}^{(2)}}$ which is reflexive and equal to $\iota_*(\omega_{U/\mathcal{O}^{(2)}})$ where U is the relative Gorenstein locus of $\mathcal{X}^{(2)} \to \mathcal{O}^{(2)}$. Denoting by $\mathcal{L}^{(2)}$ the pullback of $\mathcal{L}$ to $\mathcal{X}^{(2)}$, we have that the Hom-stack $\mathcal{H}' := \underline{\mathrm{Hom}}_{\mathcal{O}^{(2)}}(\omega_{\mathcal{X}^{(2)}/\mathcal{O}^{(2)}}, \mathcal{L}^{(2)})$ is algebraic and locally of finite type by [32, Proposition 2.1.3]. We will check that $\mathcal{H}$ is a substack of $\mathcal{H}'$ by identifying $\mathcal{H}'$ with

the stack of pairs $(f : \mathcal{X} \to B, \phi : \omega_{\mathcal{X}/B} \to \mathcal{L})$. That is, for every scheme B there is an equivalence of categories

$$\mathrm{Hom}(B, \mathcal{H}') \to \{(f : \mathcal{X} \to B, \phi : \omega_{\mathcal{X}/B} \to \mathcal{L})\}$$

where $(\mathcal{X} \to B, \mathcal{L})$ is induced by a morphism $B \to \mathcal{O}^{(2)}$.

Indeed given a morphism $B \to \mathcal{H}'$, we can first compose it with the natural map $\mathcal{H}' \to \mathcal{O}^{(2)}$ to get a map $B \to \mathcal{O}^{(2)}$ and thus a family of demi-normal polarized orbispaces $(\mathcal{X} \to B, \mathcal{L})$. Then the universal property of the Hom stack identifies the category of liftings as in the dotted arrow below

$$
\begin{array}{ccc}
 & & \mathcal{H}' \\
 & \nearrow & \downarrow \\
B & \xrightarrow{\ \pi\ } & \mathcal{O}^{(2)}
\end{array}
$$
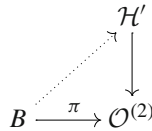
with maps $\phi_0 : \pi^*(\omega_{\mathcal{X}^{(2)}/\mathcal{O}^{(2)}}) \to \mathcal{L}$. Now since $\mathcal{L}$ is a line bundle, ϕ_0 factors uniquely through the reflexive hull of the source, so we obtain $\phi_0^{[1]} : \pi^*(\omega_{\mathcal{X}^{(2)}/\mathcal{O}^{(2)}})^{[1]} \to \mathcal{L}$. Notice that

$$\pi^*(\omega_{\mathcal{X}^{(2)}/\mathcal{O}^{(2)}})^{[1]} \cong \omega_{\mathcal{X}/B}$$

so setting $\phi := \phi_0^{[1]}$ gives a pair. We leave it to the reader to check this is an equivalence of categories.

For any B and any object $(f : \mathcal{X} \to B, \omega_{\mathcal{X}/B} \to \mathcal{L}) \in \mathcal{H}'(B)$, condition (1) is satisfied by construction since $(f : \mathcal{X} \to B, \mathcal{L})$ is an object of $\mathcal{O}^{(2)}(B)$. We must show that (2) imposes an algebraic condition. In fact, we will show that it cuts out $\mathcal{H}$ as an open substack of $\mathcal{H}'$. Let $(\mathcal{X}^{(3)}, \mathcal{L}^{(3)})$ denote the pullback of the universal polarized orbispace to $\mathcal{H}'$ and let $\Psi : \omega_{\mathcal{X}^{(3)}/\mathcal{H}'} \to \mathcal{L}^{(3)}$ be the universal morphism.

Since $\pi : \mathcal{X}^{(3)} \to \mathcal{H}'$ is Gorenstein in codimension one, ω_π is a line bundle in codimension one and a morphism of line bundles is an isomorphism if and only if it is surjective. Thus we need to show that requiring Ψ to be surjective at generic points and at the codimension one singular points of the fibers of π is an open condition.

We begin with the generic points. Consider $\mathcal{C} := \mathrm{Supp}(\mathrm{Coker}(\Psi))$. Since $\mathcal{C} \to \mathcal{X}^{(3)}$ is a closed embedding and $\mathcal{X}^{(3)} \to \mathcal{H}'$ is proper, $\mathcal{C} \to \mathcal{H}'$ is proper. We need to show that the locus where the fibers of $\mathcal{C} \to \mathcal{H}'$ have dimension at most $(n-1)$ is open. Since $\mathcal{C}$ and $\mathcal{H}'$ are tame DM stack, and formation of coarse spaces commutes with base change, it suffices to show that the locus where the fibers of the coarse moduli map $C \to H'$ have dimension at most $(n-1)$ is open. This follows from upper-semicontinuity of fiber dimension.

Finally we consider the codimension one singular points. Let $\mathcal{S} \subset \mathcal{X}^{(3)}$ be the singular locus of $\pi : \mathcal{X}^{(3)} \to \mathcal{H}'$. It is a closed substack of $\mathcal{X}^{(3)}$ so that the map to $\mathcal{H}'$ is proper. Let $\Psi_{\mathcal{S}}$ be the restriction of Ψ to $\mathcal{S}$ and $\mathcal{C}_{\mathcal{S}}$ be the support of its cokernel. Then it suffices to show that the locus where the fibers of $\mathcal{C}_{\mathcal{S}} \to \mathcal{H}'$ have dimension at

most $(n-2)$ is open. Since $\mathcal{C}_\mathcal{S}$ is a closed substack of $\mathcal{S}$, it is proper and the result again follows from upper-semicontinuity of fiber dimension applied to the coarse map.

Thus $\mathcal{H}$ is an open substack of the algebraic stack $\mathcal{H}'$. □

3.2 The family of divisors $\mathcal{D}$

Next we produce a family of divisors from the data of a pair $(f : \mathcal{X} \rightarrow B, \phi : \omega_{\mathcal{X}/B} \rightarrow \mathcal{L}) \in \mathcal{H}(B)$, and study its properties.

Lemma 3.4 *Suppose* $(f : \mathcal{X} \rightarrow B, \phi : \omega_{\mathcal{X}/B} \rightarrow \mathcal{L})$ *is an object of* $\mathcal{H}(B)$. *Then the morphism* $\phi \otimes \mathcal{L}^{-1} : \omega_{\mathcal{X}/B} \otimes \mathcal{L}^{-1} \rightarrow \mathcal{O}_\mathcal{X}$ *is injective.*

Proof The statement is étale local so we can replace $\mathcal{X}$ with an étale cover and assume it is a scheme. The sheaf $\omega_{\mathcal{X}/B} \otimes \mathcal{L}^{-1}$ is reflexive by [33, Proposition 5.6]. Therefore by [15, Proposition 3.5], if $j : U(f) \rightarrow \mathcal{X}$ is the open embedding of the Gorenstein locus,

$$\omega_{\mathcal{X}/B} \otimes \mathcal{L}^{-1} \cong j_* j^*(\omega_{\mathcal{X}/B} \otimes \mathcal{L}^{-1}) \text{ and } \mathcal{O}_\mathcal{X} \cong j_* j^*(\mathcal{O}_X).$$

This means that for every open subset $V \subseteq \mathcal{X}$ the restriction morphism $(\omega_{\mathcal{X}/B} \otimes \mathcal{L}^{-1})(V) \rightarrow (\omega_{\mathcal{X}/B} \otimes \mathcal{L}^{-1})(V \cap U(f)$ is an isomorphism. Therefore, it suffices to show that $(\omega_{\mathcal{X}/B} \otimes \mathcal{L}^{-1})(V \cap U(f)) \rightarrow \mathcal{O}_\mathcal{X}(V \cap U(f))$ is injective. Namely, we can assume without loss of generality that $\omega_{\mathcal{X}/B}$ is a line bundle.

Now the statement is local so suppose that $B = \mathrm{Spec}(A)$, $\mathcal{X} = \mathrm{Spec}(R)$ with $g : \mathrm{Spec}(R) \rightarrow \mathrm{Spec}(A)$ where R is a flat A-module, and $\omega_{\mathcal{X}/B} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_\mathcal{X}$ and denote $\phi \otimes \mathcal{L}^{-1}$ by ψ. Then $\psi : R \rightarrow R$ is an R-module homomorphism which is necessarily multiplication by some $a \in R$. Our goal is to show that a is not a zero divisor.

From the commutativity condition on morphisms and point (2) in the definition of objects of $\mathcal{H}$, the element a is not a zero divisor when restricted to each fiber of $\mathcal{X} \rightarrow B$. For any ring C and C-module M, we will denote by $\mathrm{Ass}_C(M)$ the associated primes of M, and by $\mathrm{Div}_C(M)$ the set of zero divisors for M. It is essential now to recall that $\mathrm{Div}_C(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_C(M)} \mathfrak{p}$. Then the following chain of implications finishes the proof:

$$a \notin \bigcup_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{Div}_R(R/\mathfrak{p}R) \Rightarrow a \notin \bigcup_{\mathfrak{p} \in \mathrm{Spec}(A),\, \mathfrak{q} \in \mathrm{Ass}_R(R/\mathfrak{p}R)} \mathfrak{q} \Rightarrow$$

$$\Rightarrow a \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(A),\, \mathfrak{q} \in \mathrm{Ass}_R(R/\mathfrak{p}R)} \mathfrak{q} \overset{(*)}{\Rightarrow} a \notin \bigcup_{\mathfrak{q} \in \mathrm{Ass}_R(R)} \mathfrak{q} \Rightarrow a \notin \mathrm{Div}_R(R),$$

where the arrow labelled with $(*)$ follows from [34, Theorem 23.2 (ii)] and its proof. □

Notation 3.5 The ideal sheaf given by taking the image of $\phi \otimes \mathcal{L}^{-1}$ in Lemma 3.4 will be denoted by $\mathcal{O}_\mathcal{X}(-\mathcal{D}_\phi)$ and the resulting closed substack will be denoted by $\mathcal{D}_\phi \subseteq \mathcal{X}$. Furthermore, we will drop the subscript ϕ when there is no risk of confusion.

Corollary 3.6 *Let* $(f : \mathcal{X} \to B, \phi : \omega_{\mathcal{X}/B} \to \mathcal{L})$ *be an object of* $\mathcal{H}(B)$ *and suppose that for each* $b \in B$, $\mathcal{X}_b$ *is Du Bois. Then* $\mathcal{D}_\phi \to B$ *is flat and commutes with base change.*

Proof We will drop the subscript ϕ in $\mathcal{D}_\phi$. Let $B' \to B$ be a morphism and consider the pullback $(\mathcal{X}' \to B', \phi' : \omega_{\mathcal{X}'/B'} \to \mathcal{L}')$. Let $g : \mathcal{X}' \to \mathcal{X}$ be the resulting morphism.

By the exact sequence of Tor and using that $\mathcal{X} \to B$ is flat, it suffices to show that the morphism $g^*(\mathcal{O}_\mathcal{X}(-\mathcal{D})) \to g^*\mathcal{O}_\mathcal{X} \cong \mathcal{O}_{\mathcal{X}'}$ is injective. Since the fibers are Du Bois, the relative canonical sheaf commutes with base change by [19, Corollary 1.5], so $g^*(\omega_{\mathcal{X}/B}) \cong \omega_{\mathcal{X}'/B'}$. Thus

$$g^*\big(\mathcal{O}_\mathcal{X}(-\mathcal{D})\big) \cong g^*\left(\omega_{\mathcal{X}/B} \otimes \mathcal{L}^{-1}\right) \cong g^*\big(\omega_{\mathcal{X}/B}\big) \otimes g^*(\mathcal{L})^{-1} \cong \omega_{\mathcal{X}'/B'} \otimes (\mathcal{L}')^{-1}.$$

By the commutativity condition in the definition of $\mathcal{H}$, the composition $\omega_{\mathcal{X}'/B'} \otimes (\mathcal{L}')^{-1} \cong g^*(\mathcal{O}_\mathcal{X}(-\mathcal{D})) \to \mathcal{O}_{\mathcal{X}'}$ is the map $\phi' \otimes (\mathcal{L}')^{-1}$ which is injective by Lemma 3.4 and identifies $g^*(\mathcal{O}_\mathcal{X}(-\mathcal{D}))$ with $\mathcal{O}_{\mathcal{X}'}(-\mathcal{D}')$. $\qquad\square$

3.3 The stack $\mathcal{M}_{n,v}$

We are now ready to introduce the stack of twisted stable pairs $\mathcal{M}_{n,v}$ as a category fibered in groupoids over Sch/k.

Definition 3.7 For B a scheme, an object of $\mathcal{M}_{n,v}(B)$ consists of a pair $(f : \mathcal{X} \to B, \phi : \omega_{\mathcal{X}/B} \to \mathcal{L}) \in \mathcal{H}(B)$ such that

(1) for every $b \in B$, $(\mathcal{X}_b, \mathcal{D}_b)$ is an slc pair of dimension n, and
(2) for every $b \in B$, the volume of $\mathcal{L}_b$ is v.[2]

Morphisms are given by morphisms in $\mathcal{H}$. We will call an object of $\mathcal{M}_{n,v}$ over B, a *twisted stable pair* over B.

Remark 3.8 Despite the fact that $\mathcal{M}_{n,v}$ is defined as a full subcategory of $\mathcal{H}$, it is not immediate that it is a substack. The issue is that condition (1) in Definition 3.7 is not compatible with base change a priori. However, by [25, Theorem 5.14] the fibers $\mathcal{X}_b$ of a twisted stable pair are Du Bois. Therefore, by Corollary 3.6, $\mathcal{D}$ is flat and commutes with base change so $\mathcal{M}_{n,v}$ is a category fibered in groupoids.

Theorem 3.9 *The stack* $\mathcal{M}_{n,v}$ *is algebraic and locally of finite type.*

Proof Since the stack of pairs $\mathcal{H}$ is algebraic by Theorem 3.3, it suffices to show that $\mathcal{M}_{n,v}$ is an open substack. Thus we will check that (1) and (2) in Definition 3.7 are open conditions.

Condition (1). Consider a family of pairs $(f : \mathcal{X} \to B, \mathcal{L}, \phi : \omega_{\mathcal{X}/B} \to \mathcal{L}) \in \mathcal{H}$. We need to show that the locus in B where $(\mathcal{X}_b, \mathcal{D}_b)$ is slc is open. It suffices then to show that it is constructible, and stable under generalization. Our proof is inspired by [26, Lemma 4.48] and [2, Lemma A.2.1]. See also [30, Lemma 5.10].

---

[2] The volume of a line bundle on $\mathcal{X}$ is defined to be the volume of the corresponding $\mathbb{Q}$-Cartier divisor on the coarse space.

We begin with some initial reductions. By Corollary 3.6, $\mathcal{D} \to B$ is flat for objects of $\mathcal{M}_{n,v}$ so $\mathcal{M}_{n,v}$ is contained in the substack $\mathcal{H}_1 \subset \mathcal{H}$ where $\mathcal{D} \to B$ is flat. By [13, Theorem 11.3.1], $\mathcal{H}_1 \subset \mathcal{H}$ is open. Furthermore, $\mathcal{M}_{n,v}$ is contained in the substack $\mathcal{H}_2 \subset \mathcal{H}_1$ where the fibers of $\mathcal{D} \to B$ are reduced. Since $\mathcal{D} \to B$ is flat and proper, then $\mathcal{H}_2 \subset \mathcal{H}_1$ is open by [13, Theorem 12.2.1]. Thus without loss of generality, we may suppose that $\mathcal{D} \to B$ is flat with reduced fibers.

Next we show that the locus in B where $(\mathcal{X}_b, \mathcal{D}_b)$ is slc is constructible. For that, we can assume that B is irreducible. Let η be the generic point of B and consider the generic fiber $(\mathcal{X}_\eta, \omega_{\mathcal{X}_\eta/\eta} \to \mathcal{L}_\eta)$. Let $\mathcal{Y}_\eta \to \mathcal{X}_\eta$ of $(\mathcal{X}_\eta, \mathcal{D}_\eta)$ be a log-resolution, which exists for DM stacks by functorial resolution of singularities (see for example [43]). Since B is locally of finite type over a field, this resolution can be spread out to give a simultaneous log-resolution $\pi : \mathcal{Y} \to \mathcal{X}\big|_U$ of $(\mathcal{X}\big|_U, \mathcal{D}\big|_U)$ for a suitable open subset $U \subseteq B$ using the method of [26, Lemma 4.48]. Then we have $\omega_{\mathcal{Y}/U} \otimes \mathcal{O}_{\mathcal{Y}}(\pi_*^{-1}(\mathcal{D}\big|_U)) \cong \pi^*(\mathcal{L}\big|_U) \otimes \mathcal{O}_{\mathcal{Y}}(\sum d_i \Delta_i)$ for appropriate irreducible π-exceptional divisors Δ_i and coefficients d_i. Then for every $b \in U$, the pair $(\mathcal{X}_b, \mathcal{D}_b)$ is slc if and only if $d_i \geq -1$ for every i which is a constructible condition by Chevalley's Theorem.

To show that being slc is stable under generalization, we proceed as in [30, Lemma 5.10]. Consider a family of demi-normal orbifolds $\mathcal{X} \to \mathrm{Spec}(R)$ over a DVR R and assume we have a divisor $\mathcal{D}$ such that $K_{\mathcal{X}} + \mathcal{D}$ is $\mathbb{Q}$-Cartier. After passing to an étale cover, we may assume $(\mathcal{X}, \mathcal{D})$ are schemes. If the closed fiber $(\mathcal{X}_p, \mathcal{D}_p)$ is slc, then by inversion of adjunction ([39, Corollary 2.11]) we conclude that the pair $(\mathcal{X}, \mathcal{D})$ is slc so the condition is stable under generalization.

Condition (2). We need to show that the locus in $\mathcal{H}$ where the fibers have volume v is open for every integer v. Since $\mathcal{H}$ is locally of finite type, we can restrict to an open connected substack $\mathcal{H}_0$ of finite type. Let $(\mathcal{X}_0, \mathcal{L}_0) \to \mathcal{H}_0$ be the pullback of the universal polarized orbispace and let $\pi : \mathcal{X}_0 \to X_0$ be the relative coarse space of $X_0 \to \mathcal{H}_0$. Since X_0 is of finite type, there exists an m such that $\mathcal{L}_0^{\otimes m} = \pi^* L$ for some line bundle L on X_0. It suffices to know that the self intersection $(L\big|_{(X_0)_b})^n$ is locally constant, which follows from the flatness of $X_0 \to \mathcal{H}_0$. $\qquad\square$

Observe that the stack $\mathcal{M}_{n,v}$ comes with a morphism to $\mathcal{O}rb^{\mathcal{L}}$. From [2, Corollary 5.3.7], there is an open embedding $\mathcal{K}^{\mathcal{L}} \to \mathcal{O}rb^{\mathcal{L}}$ parametrizing cyclotomic stacks which are S_2 and have no stabilizers in codimension one.

Definition 3.10 Let $\mathcal{K}_{n,v} := \mathcal{K}^{\mathcal{L}} \times_{\mathcal{O}rb^{\mathcal{L}}} \mathcal{M}_{n,v} \to \mathcal{M}_{n,v}$ be the open substack parametrizing orbispaces which have no stabilizers in codimension one. We call $\mathcal{K}_{n,v}$ the *Kollár component* of $\mathcal{M}_{n,v}$ and define the twisted stable pairs parametrized by $\mathcal{K}_{n,v}$ the *AH twisted stable pairs*.

By Theorem 2.16, the objects of $\mathcal{K}_{n,v}$ are the twisted stable pairs $(f : \mathcal{X} \to B, \phi : \omega_{\mathcal{X}} \to \mathcal{L})$ such that $(f : \mathcal{X} \to B, \mathcal{L})$ is the associated AH family of a Kollár family of $\mathbb{Q}$-line bundles. That is, they are twisted stable pairs that are also AH families as in Definition 2.17.

Notation 3.11 Let $\mathcal{K}_{n,v}^c$ denote the open substack of $\mathcal{K}_{n,v}$ parametrizing connected twisted stable pairs.

Theorem 3.12 *The stacks $\mathcal{K}_{n,v}^c$ and $\mathcal{K}_{n,v}$ are proper and DM.*

Proof It suffices to prove the desired claim for $\mathcal{K}_{n,v}^c$. We divide the proof into three steps.

$\mathcal{K}_{n,v}^c$ *is of finite type*: First, note that $\mathcal{K}_{n,v}^c$ is locally of finite type since it is an open substack of $\mathcal{M}_{n,v}$. Therefore, it suffices to show that $\mathcal{K}_{n,v}^c$ is quasi-compact and to do this it suffices to find a surjective morphism from a quasi-compact scheme. This follows from the results [16,22] (see also [23]). By [16, Theorem 1.1], there is a projective morphism of quasi-projective varieties $X \to B$, with a divisor $D \subseteq X$ such that every stable pair of dimension n and volume v appears as (X_b, D_b) for a certain $b \in B$. Up to replacing B with a resolution, we may assume that B is smooth. By taking the flattening stratification, we can assume that $X \to B$ is flat. Using boundedness and proceeding as in [22, Corollary 25], we can stratify further so that $(K_{X_b} + D_b)^{[m]}$ commutes with base change for every $m \in \mathbb{Z}$. In particular, we can consider the associated AH family $(f : \mathcal{X} \to B, \mathcal{L})$. On the locus $\mathcal{U}$ where f is Gorenstein and has trivial stabilizers, we have a morphism $\omega_f|_\mathcal{U} \to \mathcal{L}|_\mathcal{U}$ which extends uniquely to a morphism $\omega_f \to \mathcal{L}$ by Lemma 2.3. Since every point of $\mathcal{K}_{n,v}^c$ corresponds to the AH stack of a fiber of $X \to B$ and B is of finite type, it follows that $\mathcal{K}_{n,v}^c$ is as well.

$\mathcal{K}_{n,v}^c$ *is proper*: The valuative criterion for properness now follows from [26, Theorem 2.50 and Proposition 2.76 (2)].

$\mathcal{K}_{n,v}^c$ *is DM*: Consider an AH twisted stable pair $(\mathcal{X} \to \mathrm{Spec}(k), \omega_\mathcal{X} \to \mathcal{L})$, and let X be the coarse space of $\mathcal{X}$. Since $\mathcal{X}$ has no generic stabilizers, an automorphism of $\mathcal{X}$ which induces the identity on X is the identity. So it suffices to know that the pair (X, D) has finite automorphisms where D is the coarse space of $\mathcal{D}_\phi$. This is [30, Proposition 5.5]. $\qquad\square$

Remark 3.13 Note that the coarse space of $\mathcal{K}_{n,v}^c$ exists by the Keel-Mori theorem and is projective by the work of Kovács-Patakfalvi [30]. $\qquad\square$

4 Local and global structure of twisted stable pairs

The goal of this section is to study the twisted stable pairs of Definition 3.7. The section is divided into two subsection. In the first one we focus on the properties of an AH family, and we give conditions that are equivalent to having a morphism to an AH family. In the second section we study the local structure of a twisted stable pair in a neighbourhood of the divisor $\mathcal{D}$. By a careful analysis of the case that $\mathcal{X}$ is a surface, we show that the the only contribution to the different on a twisted stable pair comes from the double locus of $\mathcal{D}$ (see Proposition 4.7 and Corollary 4.8).

4.1 Global structure of AH families

In this subsection, we study the following questions. Given an AH family $\mathcal{X} \to B$ over a scheme B with coarse space X and a morphism $f : \mathcal{Y} \to X$ from a DM stack $\mathcal{Y}$, then:

(1) when can we lift f to a map $\mathcal{Y} \to \mathcal{X}$;

(2) when is such a lift to be an isomorphism?

Among other things, this will allow us to relate our definition of a family of twisted stable pairs over the spectrum of an Artin ring with the $\mathbb{Q}$-Gorenstein deformations of Hacking in [14] (see Corollary 4.5).

Proposition 4.1 *Suppose $(X \to B, F)$ is a Kollár family of $\mathbb{Q}$-line bundles with associated AH family $(\mathcal{X} \to B, \mathcal{L})$. Let $p : \mathcal{X} \to X$ be the coarse space map and consider a morphism $f : Y \to X$ from a scheme Y. Then the groupoid $\mathrm{Hom}_X(Y, \mathcal{X})$ of maps lifting f is equivalent to the following groupoid:*

- *objects are given by pairs $(\mathcal{G}, \phi)$ where $\mathcal{G}$ is a line bundle on Y, and $\phi : \bigoplus_{n \in \mathbb{Z}} f^*(F^{[n]}) \to \bigoplus_{n \in \mathbb{Z}} \mathcal{G}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_Y$-algebras, and*
- *morphisms between $(\mathcal{G}_1, \phi_1) \to (\mathcal{G}_2, \phi_2)$ are given by an isomorphism of line bundles $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ such that $\phi_2 = (\bigoplus_n \psi^{\otimes n}) \circ \phi_1$.*

Proof Recall that, as a stack over X, we have $\mathcal{X} = [\mathcal{P}(F)/\mathbb{G}_m] = [\mathrm{Spec}_{\mathcal{O}_X}(\bigoplus_{n \in \mathbb{Z}} F^{[n]}) /\mathbb{G}_m]$. Therefore for a scheme Y over X, the groupoid $\mathcal{X}(Y)$ is equivalent to the groupoid of $\mathbb{G}_m$-torsors $E \to Y$, with a $\mathbb{G}_m$-equivariant morphism $E \to \mathcal{P}(F)$ over X. This is equivalent to a $\mathbb{G}_m$-equivariant morphism $E \to \mathcal{P}(F) \times_X Y \cong \mathrm{Spec}_{\mathcal{O}_Y}(\bigoplus_{n \in \mathbb{Z}} f^*(F^{[n]}))$. There exists a unique line bundle $\mathcal{G}$ on Y such that the $\mathbb{G}_m$-torsor $E \to Y$ may be written as $\mathrm{Spec}_{\mathcal{O}_Y}(\bigoplus_{n \in \mathbb{Z}} \mathcal{G}^{\otimes n}))$. To conclude the proof it suffices to notice that $\mathbb{G}_m$-equivariant morphisms

$$\mathrm{Spec}_{\mathcal{O}_Y}\left(\bigoplus_{n \in \mathbb{Z}} \mathcal{G}^{\otimes n}\right)) \to \mathrm{Spec}_{\mathcal{O}_Y}\left(\bigoplus_{n \in \mathbb{Z}} f^*(F^{[n]})\right))$$

correspond to graded $\mathcal{O}_Y$-algebra homomorphisms $\bigoplus_{n \in \mathbb{Z}} f^*(F^{[n]}) \to \bigoplus_{n \in \mathbb{Z}} \mathcal{G}^{\otimes n}$. $\square$

Corollary 4.2 *Let $(p : X \to B, F)$ a Kollár family of $\mathbb{Q}$-line bundles with associated AH family $(\mathcal{X} \to B, \mathcal{L})$. Consider a DM stack $\mathcal{Y}$ with a morphism $f : \mathcal{Y} \to X$. Assume that there is a line bundle $\mathcal{G}$ on $\mathcal{Y}$ and an isomorphism of graded $\mathcal{O}_X$-algebras $\bigoplus_{n \in \mathbb{Z}} f_*(\mathcal{G}^{\otimes n}) \cong \bigoplus_{n \in \mathbb{Z}} F^{[n]}$. Then there is a morphism $g : \mathcal{Y} \to \mathcal{X}$ over X such that the following diagram is cartesian.*

$$\begin{array}{ccc} \mathcal{P}(\mathcal{G}) & \longrightarrow & \mathcal{P}(F) \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

Proof First observe that

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{Y}}\left(f^*\bigoplus_n F^{[n]}, \bigoplus_n \mathcal{G}^{\otimes n}\right) \cong \mathrm{Hom}_{\mathcal{O}_X}\left(\bigoplus_n F^{[n]}, f_*\bigoplus_n \mathcal{G}^{\otimes n}\right) \cong \mathrm{Hom}_{\mathcal{O}_X}\left(\bigoplus_n F^{[n]}, \bigoplus_n f_*(\mathcal{G}^{\otimes n})\right).$$

The graded morphism in the hypothesis gives an equivariant morphism of $\mathbb{G}_m$-torsors,

$$\mathcal{P}(\mathcal{G}) \to \mathrm{Spec}_{\mathcal{O}_\mathcal{Y}}(f^*(\bigoplus_n F^{[n]})) \cong \mathcal{P}(F) \times_X \mathcal{Y}$$

which is an isomorphism since every equivariant morphism between torsors is an isomorphism. The resulting $\mathbb{G}_m$-equivariant map $\mathcal{P}(\mathcal{G}) \to \mathcal{P}(F)$ induces $\mathcal{Y} \cong [\mathcal{P}(\mathcal{G})/\mathbb{G}_m] \to [\mathcal{P}(F)/\mathbb{G}_m] \cong \mathcal{X}$ and the diagram is cartesian by [38, Exercise 10.F]. □

Note that in Corollary 4.2, if we assume that $\mathcal{Y} \to X$ is the coarse space, we cannot conclude that the morphism $\mathcal{Y} \to \mathcal{X}$ is an isomorphism. For example, consider $\mathcal{Y} = B\mu_2$ with $\mathcal{G} = \mathcal{O}_{\mathcal{Y}}$ and $\mathcal{X} = X = \mathrm{Spec}(k)$ with $\mathcal{L} = \mathcal{O}_X$. The problem is that $\mathcal{Y}$ might have more stabilizers than $\mathcal{X}$. The following lemma shows that this is the only reason for the failure of $\mathcal{Y} \to \mathcal{X}$ to be an isomorphism.

Lemma 4.3 *In the situation of Corollary 4.2, suppose $\mathcal{Y}$ is separated with coarse space X. Then the morphism $g : \mathcal{Y} \to \mathcal{X}$ is the relative coarse space of the map $\mathcal{Y} \to B\mathbb{G}_m$ induced by $\mathcal{G}$. In particular, if $\mathcal{Y} \to B\mathbb{G}_m$ is representable, that is, if $\mathcal{G}$ is uniformizing, then $g : \mathcal{Y} \to \mathcal{X}$ is an isomorphism.*

Remark 4.4 We point the reader to [6, Theorem 3.1] for the definition and existence of relative coarse spaces.

Proof Let $\mathcal{Y} \to \mathcal{Y}' \to B\mathbb{G}_m$ the relative coarse space of the map $\mathcal{Y} \to B\mathbb{G}_m$. Then $\mathcal{Y}' \to B\mathbb{G}_m$ is representable and equips $\mathcal{Y}'$ with a line bundle $\mathcal{G}'$ that pulls back to $\mathcal{G}$. Next, we claim that f factors through $f' : \mathcal{Y}' \to X$. Indeed, let $b : \mathcal{Y}' \to X'$ be the coarse space of $\mathcal{Y}'$. By universal property of the coarse space, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{a} & \mathcal{Y}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{c} & X' \end{array} \, .$$

The maps a, b and f induce bijections on geometric points by definition of coarse moduli space, therefore c induces a bijection on geometric points. Moreover, since we are in characteristic zero, all stacks involved are tame, therefore by [6, Theorem 3.1], we have $a_*\mathcal{O}_{\mathcal{Y}} \cong \mathcal{O}_{\mathcal{Y}'}$, $f_*\mathcal{O}_{\mathcal{Y}} \cong \mathcal{O}_X$ and $b_*\mathcal{O}_{\mathcal{Y}'} \cong \mathcal{O}_{X'}$. Therefore, by functoriality of pushforward, $c_*\mathcal{O}_X \cong \mathcal{O}_{X'}$ and we conclude that c is an isomrphism and that $f' = c^{-1} \circ b$.

Furthermore, by commutativity of the above diagram, we have that $f'_*(\mathcal{G}')^{\otimes m} = f_*(\mathcal{G})^{\otimes m} = F^{[m]}$. Thus we may apply Corollary 4.2 to obtain a morphism $g' : \mathcal{Y}' \to \mathcal{X}$ and it suffices to prove g' is an isomorphism. In particular, without loss of generality we may assume that $\mathcal{Y} \to B\mathbb{G}_m$ is representable and we wish to prove that $g : \mathcal{Y} \to \mathcal{X}$ is an isomorphism.

To prove this claim, note first that the question is local over X, so from [38, Theorem 11.3.1] we can assume that $X = V/G$ and $\mathcal{Y} = [V/G]$ for a finite group G. Moreover, we can assume that $V = \mathrm{Spec}(A)$ is affine. Then $\mathcal{G}$ corresponds to a locally free A-module M with a G-action and the coarse space of $\mathcal{G} \cong [\mathrm{Spec}(\bigoplus_{n \in \mathbb{Z}} M^{\otimes n})/G]$ is given by

$$\mathrm{Spec}_{\mathcal{O}_X}\left(\bigoplus_{n\in\mathbb{Z}}((M^{\otimes n})^G)\right) = \mathrm{Spec}_{\mathcal{O}_X}\left(\bigoplus_{n\in\mathbb{Z}} f_*(\mathcal{G}^{\otimes n})\right) \cong \mathrm{Spec}_{\mathcal{O}_X}\left(\bigoplus_{n\in\mathbb{Z}} F^{[n]}\right) = \mathcal{P}(F).$$

Now, by [2, Proposition 2.3.10] the representability of $\mathcal{Y} \to B\mathbb{G}_m$ implies that for every $p \in \mathcal{Y}$, the action of $\mathrm{Aut}_{\mathcal{Y}}(p)$ on $\mathcal{G}_p$ is faithful. Then $\mathcal{P}(\mathcal{G}) = \mathrm{Spec}_{\mathcal{O}_{\mathcal{Y}}}(\bigoplus_{n\in\mathbb{Z}} \mathcal{G}^{\otimes n})$ is already an algebraic space so it is isomorphic to its coarse space: $\mathcal{P}(F) \cong \mathcal{P}(\mathcal{G})$. Thus, since $\mathcal{P}(F) \to \mathcal{X}$ is a smooth atlas and the diagram of Lemma 4.2 is cartesian, g is an isomorphism. □

Lemma 4.3 allows us to compare our definition of twisted stable family with $\mathbb{Q}$-Gorenstein deformations in the case where the divisor D is Cartier. Recall that to any $\mathbb{Q}$-Gorenstein variety, one may associate a *canonical covering stack* (see [14, Sect. 3.1]) which controls $\mathbb{Q}$-Gorenstein deformations.

Corollary 4.5 *Let (X, D) be an slc pair where D is a Cartier divisor. Then K_X is $\mathbb{Q}$-Cartier and the canonical covering stack $\mathcal{X}' \to X$ is isomorphic to the AH stack $\mathcal{X}$ for the $\mathbb{Q}$-line bundle $L = K_X + D$.*

Proof Consider the coarse space map $p : \mathcal{X}' \to X$ and let $\mathcal{G} := K_{\mathcal{X}'} + p^*D$. Then:

(1) $\mathcal{G}$ is a line bundle with $p_*(\mathcal{G}^{\otimes n}) = \mathcal{O}_X(K_X + D)^{[n]}$, and
(2) for every point $q \in \mathcal{X}'$, the action of $\mathrm{Aut}_{\mathcal{X}'}(q)$ on $\mathcal{G}_q$ is faithful.

Point (1) follows since both sides of the equality are reflexive and they agree on the big open set where p is an isomorphism. Point (2) holds because the action of $\mathrm{Aut}_{\mathcal{X}'}(q)$ on $(K_{\mathcal{X}'})_q$ is faithful by definition, the action on $(p^*D)_q$ is trivial, and the tensor product of a trivial and a faithful representation is a faithful representation. Thus we may apply Lemma 4.3 to conclude that $\mathcal{X} \cong \mathcal{X}'$. □

4.2 Local structure of twisted stable pairs

The goal of this subsection is to study the local structure of $\mathcal{X}$ along the divisor $\mathcal{D}$. First, we prove Theorem 1.3 (see Proposition 4.7 and Corollary 4.8) which says that on a twisted stable pair the singularities of $\mathcal{X}$ do not contribute to the different. Then we explore the relationship between the stack structure on $\mathcal{D}$ and the different on the coarse space of the pair (Lemmas 4.12 and 4.11).

We start with the local notion of a twisted pair:

Definition 4.6 A *twisted pair of relative dimension n* over B is a pair $(f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L})$ where $\mathcal{X}$ is an open substack of an $\mathcal{X}'$ with $(f' : \mathcal{X}' \to B, \phi' : \omega_{f'} \to \mathcal{L})$ a twisted stable pair of relative dimension n, $f = f'|_{\mathcal{X}}$ and ϕ pulled back from ϕ'.

Proposition 4.7 *Let $(f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L})$ be a twisted pair of relative dimension 2. Then $\mathcal{L}|_{\mathcal{D}_\phi} \cong \omega_{\mathcal{D}_\phi/B}$.*

Proof We drop the subscript ϕ in $\mathcal{D}_\phi$. The proof follows closely [25, p. 152].

Let $\iota : \mathcal{D} \to \mathcal{X}$ denote the natural closed embedding. From Grothendieck duality, since f is Cohen-Macauley and $\mathcal{D} \to B$ is Gorenstein, we have $\iota_* \omega_{\mathcal{D}/B} \cong \mathcal{E}xt^1(\iota_* \mathcal{O}_{\mathcal{D}}, \omega_f)$. Consider the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \to \mathcal{O}_{\mathcal{X}} \to \iota_* \mathcal{O}_{\mathcal{D}} \to 0.$$

Taking $\mathcal{H}om_{\mathcal{X}}(-, \omega_f)$ induces a morphism

$$\mathcal{H}om(\mathcal{O}_{\mathcal{X}}, \omega_f) \to \mathcal{H}om(\mathcal{O}_{\mathcal{X}}(-\mathcal{D}), \omega_f) \to \mathcal{E}xt^1(\iota_* \mathcal{O}_{\mathcal{D}}, \omega_f) \to \mathcal{E}xt^1(\mathcal{O}_{\mathcal{X}}, \omega_f) = 0$$

where the last term is 0 since $\mathcal{O}_{\mathcal{X}}$ is locally free. But from Lemma 3.4, the ideal $\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ is defined as $\omega_f \otimes \mathcal{L}^{-1}$, therefore

$$\mathcal{H}om(\mathcal{O}_{\mathcal{X}}(-\mathcal{D}), \omega_f) \cong \mathcal{H}om(\omega_f \otimes \mathcal{L}^{-1}, \omega_f) \cong \mathcal{H}om(\omega_f, \omega_f) \otimes \mathcal{L}.$$

Now from Lemma 2.3, taking U the f-Gorenstein locus, we have $\mathcal{H}om(\omega_f, \omega_f) \cong \mathcal{O}_{\mathcal{X}}$. Therefore the sequence above gives a surjective morphism $\mathcal{L} \to \iota_*(\omega_{\mathcal{D}/B})$. This induces a surjective morphism $\iota^*(\mathcal{L}) \to \omega_{\mathcal{D}/B}$, but a surjective morphism of line bundles is an isomorphism.   □

Corollary 4.8 *Let* $(\mathcal{X} \to \mathrm{Spec}(k), \omega_{\mathcal{X}} \to \mathcal{L})$ *be a twisted stable pair of any dimension. Then* $\mathcal{L}\big|_{\mathcal{D}} \cong \omega_{\mathcal{D}}$. *Moreover, when* $\mathcal{D}$ *is* S_2, *it is an slc canonically polarized orbispace.*

Proof By [29, Lemma 3.7.5], the canonical sheaf ω_Z is S_2 on any excellent scheme Z admitting a dualizing complex. Furthermore, by [8, Proposition 1.2.16], the property of being S_2 can be checked after finite étale base change. Therefore $\omega_{\mathcal{D}}$ is an S_2 sheaf on $\mathcal{D}$. Now, we may apply Proposition 4.7 on the codimension 2 points of $\mathcal{X}$ to see that $\omega_{\mathcal{D}}$ agrees with $\mathcal{L}\big|_{\mathcal{D}}$ in codimension 1 on $\mathcal{D}$. Since both sheaves are S_2, we conclude that $\mathcal{L}\big|_{\mathcal{D}} \cong \omega_{\mathcal{D}}$ via the canonical residue map of Proposition 4.7.

If $\mathcal{D}$ is S_2, since it is nodal in codimension 1, it is demi-normal. In particular by inversion of adjunction [39, Lemma 2.11] $\mathcal{D}$ is slc. Finally, $\omega_{\mathcal{D}}$ is a polarizing line bundle since it is the pullback of one by a closed embedding.   □

Remark 4.9 Usually when one considers questions about adjunction for stable pairs, one works with the normalization of the divisor $n : \mathcal{D}^n \to \mathcal{D}$. By Corollary 4.8, we have that $(\iota \circ n)^* \mathcal{L} = n^* \omega_{\mathcal{D}}$. Since $\mathcal{D}$ is nodal in codimension 1, this latter sheaf is isomorphic to $\omega_{\mathcal{D}^n}(G)$ where G is the preimage of the double locus of $\mathcal{D}$. Therefore, we may interpret the corollary as stating that the only contribution to the different on $\mathcal{D}^n$ comes from the double locus of $\mathcal{D}$.

Note that in general, we have an adjunction morphism $\mathcal{M}_{n,v} \to \mathcal{O}rb^{\mathcal{L}}$ given by

$$(\mathcal{X} \to B, \phi : \omega_{\mathcal{X}} \to \mathcal{L}) \mapsto (\mathcal{D}_\phi, \mathcal{L}\big|_{\mathcal{D}_\phi}).$$

Indeed $\mathcal{D}_\phi \to B$ is flat with reduced fibers (Corollary 3.6) and $\mathcal{L}\big|_{\mathcal{D}}$ is a polarizing line bundle. Consider the locus $\mathcal{U} \subseteq \mathcal{M}_{n,v}$ where $\mathcal{D}_\phi \to \mathcal{M}_{n,v}$ is S_2 which is open

by [13, Théorème 12.1.6] and the fact that the morphism is closed. Over $\mathcal{U}$, Corollary 4.8 gives us that $(\mathcal{D}_\phi \to B, \mathcal{L}|_{\mathcal{D}_\phi})$ is a family of canonically polarized slc orbispaces of dimension $(n-1)$ and so over this locus, the morphism above lands in the open substack of $\mathcal{O}rb^\omega$ of those orbispaces with slc singularities.

The following corollary follows from Proposition 4.7 and [25, Proposition 2.35, (3)]:

Corollary 4.10 *Let $(\mathcal{X}, \mathcal{D})$ be a twisted surface pair over $\mathrm{Spec}(k)$ and $p \in \mathcal{D}$ such that $\mathcal{D}$ is smooth at p. Then $\mathcal{X}$ is smooth at p.*

Proposition 4.7 tells us that in replacing a surface pair (X, D) with its associated AH pair $(\mathcal{X}, \mathcal{D})$, the different on D gets replaced by the stack structure on $\mathcal{D}$. Our goal now is to make explicit the relation between the stack structure on $\mathcal{D}$ and the different on D.

Lemma 4.11 *Consider (X, D) an lc surface and assume that $p \in D$ is a nodal point of D. Then $K_X + D$ is Cartier at p and X agrees with its AH stack in a neighbourhood of p.*

Proof From case 2 of the proof of [25, Corollary 2.32], it follows that the singularity of X at p is a cyclic quotient singularity. In particular, denoting the completed local ring $\widehat{\mathcal{O}_{X,p}}$ by R, there is a morphism $\pi : \mathrm{Spec}(k[\![x, y]\!]) \to \mathrm{Spec}(R)$ which is the quotient by a cyclic group G. The action of G is étale in codimension one so $\pi^* K_{\mathrm{Spec}(R)} = K_{\mathrm{Spec}(k[\![x, y]\!])}$ and the pair $(\mathrm{Spec}(k[\![x, y]\!]), \pi^* D)$ is lc from [25, Corollary 2.43]. But then $p^* D$ is nodal, and the group G preserves the two branches, so up to an analytic change of coordinates we may suppose that $D = (xy)$ and x, y are eigenvectors for G. Furthermore, the log canonical divisor is Cartier generated by $\frac{dx \wedge dy}{xy}$. Then the action of G on the log canonical divisor is trivial so the log canoical divisor descends to a Cartier divisor on $\mathrm{Spec}(R)$.   □

We are left with understanding $\mathcal{D}$ along the points where D is smooth:

Lemma 4.12 *Let (X, D) be an lc surface and $q \in D$ a closed point at which D is smooth. Assume that q appears in the different of (X, D) with multiplicity m. Let $(\mathcal{X}, \mathcal{D})$ be the twisted surface pair associated to (X, D) and let $p \in \mathcal{D}$ be the point lying over q with stabilizer group $G := \mathrm{Aut}_\mathcal{X}(p)$. Then we have the following.*

(1) *If p is a smooth point of $\mathcal{D}$, then $m = 1 - \frac{1}{|G|}$;*
(2) *If p is a node, then $m = 1$, $G = \mu_2$ and G acts by swapping the two branches of $\mathcal{D}$.*

Remark 4.13 *For case (1), m determines the stabilizer group since the stack is cyclotomic.*

Proof Since the question is local, we may pass to an open subset and assume that $\mathcal{X}$ has no stabilizers outside of p and $\mathcal{D}$ has no different outside of q.

For case (1), consider the following commutative diagram.

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\ \alpha\ } & \mathcal{X} \\ \scriptstyle\pi_D \downarrow & & \downarrow \scriptstyle\pi \\ D & \xrightarrow{\ a\ } & X \end{array} \qquad (4)$$

Letting n be the index of $K_X + D$, we have the following equalities:

- $\pi^*(n(K_X + D)) = n(K_{\mathcal{X}} + \mathcal{D})$ since π is an isomorphism in codimension one,
- $a^*(n(K_X + D)) = n(K_D + mq)$ by assumption on the different, and
- $\alpha^*(n(K_{\mathcal{X}} + \mathcal{D})) = nK_{\mathcal{D}}$ by Proposition 4.7.

Putting this together along with commutativity of the above diagram, we obtain

$$\pi_D^*(n(K_D + mq)) = \pi_D^* a^*(n(K_X + D)) = \alpha^* \pi^*(n(K_X + D)) = nK_{\mathcal{D}},$$

or simply $\pi_D^*(K_D + mq) = K_{\mathcal{D}}$. Up to completing around p, we may assume that coarse map $\mathcal{D} \to D$ is given by $\mathrm{Spec}(k[\![x]\!]) \to \mathrm{Spec}(k[\![x^r]\!])$ where $r = |G|$. In this case, one readily sees that $\pi_D^*(q) = rp$ and $\pi_D^* K_D = K_{\mathcal{D}} - (r-1)p$. Putting this together with the previous equality, we have then $K_{\mathcal{D}} = K_{\mathcal{D}} - (r-1)p + rmp$ which gives $m = \frac{r-1}{r}$.

For point (2), we may again replace $(\mathcal{X}, \mathcal{D})$ with an étale neighboourhood of p and assume the following:

(1) there is an lc surface pair (X', D') and a distinguished point $p' \in D'$;
(2) there is an action of G on X' preserving D' and fixing only q, and
(3) $X = X'/G$, $D = D'/G$ and the quotient map sends p' to p.

Since D is smooth, we know that G swaps the two branches of D'. Let H be the normal subgroup of G which preserves the branches of D'. Then the pair $(X'/H, D'/H)$ is lc, the map $\pi : X' \to X'/H$ is étale away from q, and D'/H is a nodal curve. But then from Lemma 4.11 the log canonical divisor L of $(X'/H, D'/H)$ is Cartier. Since π is étale in codimension one, $K_{X'} + D' \cong \pi^* L$. In particular, H acts trivially on $(K_{X'} + D')_p$. Then $H = \{1\}$ since the action of G on $(K_{X'} + D')_p$ is faithful (recall that $\mathcal{X} \to B\mathbb{G}_m$ is representable). Therefore, any non-zero element of G swaps the branches of D' and $G \cong \boldsymbol{\mu}_2$.

Finally, consider a diagram as in Diagram 4 above but with the top left corner replaced by the normalization $\mathcal{D}^n$ of $\mathcal{D}$. By Lemma 4.16 below, $\pi_D : \mathcal{D}^n \to D$ is an isomorphism. Moreover, by Proposition 4.7 we have $(K_{\mathcal{X}} + \mathcal{D})|_{\mathcal{D}} = K_{\mathcal{D}}$ so that $\alpha^*(K_{\mathcal{X}} + \mathcal{D}) = n^* K_{\mathcal{D}} = K_{\mathcal{D}^n} + p'$ where p' is the point of $\mathcal{D}^n$ lying over the node p of $\mathcal{D}$ by computation of the canonical of a nodal curve. Since $\pi_D : \mathcal{D}^n \to D$ is an isomorphism, we conclude that q has coefficient 1 in the different. $\square$

Remark 4.14 For other relations between the singularities of X along D and the different on D, one can consult [25, Theorem 3.36]. Indeed Lemma 4.12 is essentially a stacky rephrasing of parts of *loc. cit.*.

Example 4.15 Consider $(\mathbb{A}^2_{u,v}, D' := V(uv))$, with the action of $\mathbb{Z}/2\mathbb{Z}$ that sends $(u, v) \mapsto (-u, -v)$. Then D' is invariant and the quotient is the surface pair $(X :=$

$\text{Spec}(k[x, y, z]/xy - z^2)$, $D := V(z)$). The divisor D is still a nodal curve and the pair is lc since it is the quotient of an lc pair (see [25, Corollary 2.43]). Consider now the action of $G := \mu_2$ on X, which sends $(x, y, z) \mapsto (y, x, -z)$. The action is free in codimension one and preserves D so $(X/G, D/G)$ is again lc.

The log-canonical divisor on X/G is not Cartier. Indeed, it suffices to show that the generator of G acts nontrivially on a section of the log canonical divisor on X. By adjunction, $\omega_X = (\omega_{\mathbb{A}^3}(X))|_X$ so a section of ω_X is given by $(\frac{dx \wedge dy \wedge dz}{xy - z^2})|_X$ which is invariant under G. It follows that G acts on a generator for $\omega_X(D)$ by

$$\frac{dx \wedge dy \wedge dz}{xyz - z^3} \mapsto -\frac{dx \wedge dy \wedge dz}{zxy - z^3}$$

so the log canonical divisor is not Cartier on the quotient. Then the twisted pair associated to $(X/G, D/G)$ is $([X/G], [D/G])$, the divisor D/G is smooth, but $[D/G]$ is not. By Lemma 4.12, we may conclude that the fixed point p has different 1 on D/G.

One can argue indirectly that p must have different 1. Indeed the different is an effective divisor with coefficient at most 1. If it was not 1, then $(D/G, \text{Diff}_{D/G}(0))$ would be klt and $(X/G, D/G)$ would be plt by inversion of adjunction. Then from [25, Corollary 2.43], the pairs (X, D) and $(\mathbb{A}^2_{u,v}, V(uv))$ would be plt but this is not the case.

Finally we study how the normalization map $\mathcal{D}^n \to \mathcal{D}$ behaves around the node point $p \in \mathcal{D}$ in the situation of point (2) of Lemma 4.12.

Lemma 4.16 *Suppose that $(\mathcal{X}, \mathcal{D})$ is an lc twisted surface pair and $p \in \mathcal{D}$ is a node lying over a smooth point q of D. Let $\mathcal{D}^n$ be the normalization of $\mathcal{D}$. Then the composition $\mathcal{D}^n \to \mathcal{D} \to D$ is an isomorphism in a neighbourhood of each q.*

Proof It suffices to check the required map is an isomorphism after taking étale charts so without loss of generality, we may suppose that $\mathcal{D}$ is isomorphic to $[\text{Spec}(k[x, y]/xy)/\mu_2]$ with the action that sends $x \mapsto y$ and $y \mapsto x$. Since normalization commutes with étale base change, we have that the normalization $\mathcal{D}^n$ is the quotient

$$[\,\text{Spec}(k[x]) \sqcup \text{Spec}(k[y])/\mu_2\,]$$

with the action $(x, y) \mapsto (y, x)$. Then $\mathcal{D}^n$ is just isomorphic to the scheme $\text{Spec}(k[z])$ by the map $x, y \mapsto z$, and we can write the morphisms $\mathcal{D}^n \to \mathcal{D} \to D$ as

$$\text{Spec}(k[x]) \cong [\text{Spec}(k[x]) \sqcup \text{Spec}(k[y])/\mu_2] \to$$
$$\to [\text{Spec}(k[x, y]/xy)/\mu_2] \to \text{Spec}(k[x + y]) \cong \text{Spec}(k[t]).$$

The composition is the morphism induced by $t \to x + y \to x$ which is an isomorphism. $\square$

5 Gluing morphisms for families of twisted stable surfaces

In this section we produce gluing morphisms which describe the boundary of the moduli of twisted stable surfaces in terms of moduli of twisted stable surface pairs (see Theorem 1.4). More precisely, in Sect. 5.1, we prove that one can functorially glue a family of twisted stable surface pairs over an arbitrary base to obtain a family of twisted stable surfaces. In Sect. 5.2, we define an algebraic stack $\mathcal{G}_{2,b}$ of gluing data. Finally, in Sect. 5.3 we show that the image of $\mathcal{G}_{2,v} \to \mathcal{K}_{2,v}$ stratifies $\mathcal{K}_{2,v}$ into finitely many boundary components.

5.1 Gluing morphisms for twisted stable surfaces

We begin this subsection by recalling some of the results in [25, Chapter 5].

Consider an slc stable surface X' and let X be its normalization with conductor $D \subseteq X$. Since X' is nodal in codimension 1, the normalization $X \to X'$ induces a rational and generically fixed point free involution $D \dashrightarrow D$. By [25, Proposition 5.12], this gives a generically fixed point free involution on the normalization $\tau : D^n \to D^n$ which preserves the different. This can be summarized by the diagram

$$
\begin{array}{ccc}
D^n & \longrightarrow & X \\
\downarrow & & \downarrow \\
D^n/\tau & \longrightarrow & X'
\end{array}
$$

which expresses X' as the geometric quotient of X by the finite equivalence relation induced by τ. It follows from [25, Theorem 5.13] that the converse also holds. Namely, that one can recover X' from the data of a stable lc pair (X, D) and a generically fixed point free involution $D^n \to D^n$ which preserves the different. The goal of this subsection is to understand [25, Theorem 5.13] in terms of twisted stable surfaces, which will give us a generalization to families over arbitrary bases.

We begin by showing that the fixed point free involution τ and the diagram above lift to the associated AH stacks. Then we need to understand how to reconstruct the stack structure of the AH stack $\mathcal{X}'$ from the data of the AH stack $\mathcal{X}$ and the involution τ. This culminates in the construction of Diagram 5. Finally, in the main result of this section, Theorem 5.17, we show that each of the steps in the construction of Diagram 5 can be carried out functorially in families of gluing data.

Lemma 5.1 *Let $(\mathcal{X}, \mathcal{D})$ be a normal AH twisted stable surface pair, and let $\mathcal{D}^n$ be the normalization of $\mathcal{D}$. Denote the coarse space of $\mathcal{D}^n$ by D^n and let $\tau : D^n \to D^n$ be a different preserving involution on D^n. Then there is a unique involution $\sigma : \mathcal{D}^n \to \mathcal{D}^n$ lifting τ:*

$$\begin{array}{ccc} \mathcal{D}^n & \xrightarrow{\sigma} & \mathcal{D}^n \\ {\scriptstyle coarse\ space}\big\downarrow & & \big\downarrow{\scriptstyle coarse\ space} \\ D^n & \xrightarrow{\tau} & D^n \end{array}\ .$$

Proof It follows from Lemmas 4.16 and 4.12 that the coarse space morphism $\mathcal{D}^n \to D^n$ is an isomorphism away from the points where the different has a coefficient $0 < c < 1$. These correspond to smooth points of $\mathcal{D}$ with nontrivial stabilizer and it suffices check that we can extend the morphism along such points.

From Lemma 4.12, two points with stabilizers of different orders cannot be swapped. Locally around such points on the coarse space, we have an involution τ of $Y := \mathrm{Spec}(k[\![s]\!])$ that we wish to extend uniquely to an involution of the mth root stack $\mathcal{Y} := [\mathrm{Spec}(k[\![t]\!])/(t \mapsto \zeta_m t)]$ for some m. Here ζ_m is a primitive m^{th} root of unity and $t^m = s$. The required assertion follows from the universal property of root stacks (see [38, 10.3]). Such a root stack is the universal objects for triples (Y, L, v) where Y has a morphism $f : Y \to \mathrm{Spec}(k[\![s]\!])$, L is a line bundle on Y, and v is a section of L. Moreover, we require that there is an isomorphism $L^{\otimes m} \to f^*\mathcal{O}_{\mathrm{Spec}(k[\![s]\!])}$ sending $v^{\otimes m} \mapsto s$. □

Notation 5.2 Let $\nu : \mathcal{D}^n \to \mathcal{D}$ be the normalization morphism. We will denote the nodes of $\mathcal{D}$ by $\mathcal{N}$ and set $\Delta := \nu^{-1}(\mathcal{N})$. Then Δ is a scheme from Lemma 4.11 and Lemma 4.16.

Lemma 5.3 *Consider a normal twisted stable surface pair $(\mathcal{X}, \mathcal{D})$. Let $\mathcal{D}^n$ be the normalization of $\mathcal{D}$ and let $\tau : \mathcal{D}^n \to \mathcal{D}^n$ be an involution on $\mathcal{D}^n$ which is generically fixed point free and preserves Δ. Let X and D be the coarse spaces of $\mathcal{X}$ and $\mathcal{D}$ respectively. Consider the stable surface X' obtained from X, D and τ using [25, Theorem 5.13]. Let $\mathcal{X}'$ be the AH stack associated to $(X', \omega_{X'})$. Then $\mathcal{X}$ is the normalization of $\mathcal{X}'$ and there is a map $h : [\mathcal{D}^n/\tau] \to \mathcal{X}'$ which makes the following diagram commute.*[3]

$$\begin{array}{ccc} \mathcal{D}^n & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow \\ [\mathcal{D}^n/\tau] & \xrightarrow{h} & \mathcal{X}' \end{array}$$

Proof Recall that $\mathcal{X}'$ is a scheme away from a finite set of points. Therefore the normalization $\nu : \mathcal{Y} \to \mathcal{X}'$ has X as coarse space, and if we denote by $p : \mathcal{Y} \to X$ the coarse space map, $p_*(\nu^*\omega_{\mathcal{X}'}^{\otimes n}) \cong (\omega_X(D))^{[n]}$ for every n. Indeed both sides are reflexive and agree on a big open subset where p is an isomorphism by [25, 5.7.1]. But $\omega_{\mathcal{X}'}$ is a uniformizing line bundle for $\mathcal{X}'$ (see Remark 2.12), and ν is representable, so the morphism $\mathcal{Y} \to B\mathbb{G}_m$ induced by $\nu^*(\omega_{\mathcal{X}'})$ is also representable. It follows from Corollary 4.2 and Lemma 4.3 that $\mathcal{Y} \cong \mathcal{X}$. In particular we have a map $\mathcal{X} \to \mathcal{X}'$ which induces $f : \mathcal{D}^n \to \mathcal{X}'$. There is an open dense subset U of $\mathcal{D}^n$ where $\mathcal{D}^n$ is a scheme and the two maps $f|_U$, $(f \circ \tau)|_U : U \to \mathcal{X}'$ agree. By [10, Lemma 7.2],

---

[3] For properties of group quotients of DM stack, see [40].

the arrows f and $f \circ \tau$ agree and so they induce a map $h : [\mathcal{D}^n / \tau] \to \mathcal{Y}$ making the diagram commute. □

The canonical line bundle $\omega_{\mathcal{X}'}$ induces a map $\mathcal{X}' \to B\mathbb{G}_m$ and we obtain a map $[\mathcal{D}^n / \tau] \to B\mathbb{G}_m$ given by composition with the map h from Lemma 5.3. We wish to understand this map $[\mathcal{D}^n / \tau] \to B\mathbb{G}_m$ in terms of $\mathcal{X}$, $\mathcal{D}^n$ and τ. First, observe that τ induces a map $\psi : [\mathcal{D}^n / \tau] \to B\boldsymbol{\mu}_2$. We can identify $\mathrm{Pic}(B\boldsymbol{\mu}_2)$ with the characters of $\boldsymbol{\mu}_2$ and we denote by $\chi_{B\boldsymbol{\mu}_2}$ the line bundle corresponding to the sign representation. Given a morphism $\mathcal{Y} \to B\boldsymbol{\mu}_2$, we will denote by $\chi_{\mathcal{Y}}$ the pull back of $\chi_{B\boldsymbol{\mu}_2}$ to $\mathcal{Y}$.

Now, the involution τ naturally acts on $\omega_{\mathcal{D}^n}(\Delta)$ since τ preserves the different Δ. Thus $\omega_{\mathcal{D}^n}(\Delta)$ descends to a line bundle $\mathcal{G}$ on $[\mathcal{D}^n / \tau]$.

Proposition 5.4 *In the situation above, the line bundle corresponding to $[\mathcal{D}^n / \tau] \to B\mathbb{G}_m$, namely $h^* \omega_{\mathcal{X}'}$, is isomorphic $\mathcal{G} \otimes \chi_{[\mathcal{D}^n / \tau]}$.*

Proof Let $\pi : \mathcal{D}^n \to [\mathcal{D}^n / \tau]$ be the quotient map, let $\mathcal{H} := h^* \omega_{\mathcal{X}'}$, and let $\sigma : \boldsymbol{\mu}_2 \times \mathcal{D}^n \to \mathcal{D}^n$ be the action induced by τ. The key ingredient is [25, Proposition 5.8].

From the diagram of Lemma 5.3 and Proposition 4.7, we have $\pi^* \mathcal{H} \cong \omega_{\mathcal{D}^n}(\Delta)$. The line bundle $\mathcal{H}$ is uniquely determined by the isomorphism $\alpha : \sigma^*(\omega_{\mathcal{D}^n}(\Delta)) \to p_2^*(\omega_{\mathcal{D}^n}(\Delta))$ (see [40, Example 4.3]). To determine α, it suffices to restrict $\mathcal{H}$ to a dense open subset of $\mathcal{D}^n$: we can assume that $\mathcal{X}$ and $[\mathcal{D}^n / \tau]$ are schemes. Now from [25, Proposition 5.8], the sections of $\mathcal{H}$ and those of $\mathcal{G} \otimes \chi_{B\boldsymbol{\mu}_2}$ agree so $\mathcal{H} \cong \mathcal{G} \otimes \chi_{B\boldsymbol{\mu}_2}$. □

Notation 5.5 We will denote by $\mathcal{C}$ the relative coarse space of the map $[\mathcal{D}^n / \tau] \to B\mathbb{G}_m$ induced by $\mathcal{G} \otimes \chi_{[\mathcal{D}^n / \tau]}$.

Remark 5.6 Note that the map $h : [\mathcal{D}^n / \tau] \to \mathcal{X}'$ factors through a unique map $\mathcal{C} \to \mathcal{X}'$ by the universal property of relative coarse spaces. Indeed, the map $[\mathcal{D}^n / \tau] \to B\mathbb{G}_m$ is by definition the composition of h with the *representable* morphism $\mathcal{X}' \to B\mathbb{G}_m$ induced by $\omega_{\mathcal{X}'}$.

Example 5.7 Let C be a smooth curve in $\mathbb{P}^2$ of high degree and let $\tau : C \to C$ be a nontrivial involution. One could take for example $\{X_0^5 + X_1^5 + X_2^5 = 0\}$, with the restriction of the involution of $\mathbb{P}^2$ which sends a point $[a, b, c] \mapsto [b, a, c]$. Then the pair $(\mathbb{P}^2, C)$ is stable, and the involution τ acts on C with finitely many fixed points. By Kollár's gluing construction the triple $(\mathbb{P}^2, C, \tau)$ gives an slc pair X with normalization $\mathbb{P}^2$ and double locus C. This is an example where $[C/\tau]$ is not isomorphic to $\mathcal{C}$. Indeed, the quotient $[C/\tau]$ is not a scheme at the fixed points of the involution, however X is Gorenstein since it has pinch-points singularities at the fixed points of τ, so it is equal to its Abramovich-Hassett stack.

Now, since τ sends Δ to itself, we can factor $\Delta \to \mathcal{C}$ as $\Delta \to [\Delta/\tau] \to [\mathcal{D}^n/\tau] \to \mathcal{C}$ and we have the following commutative diagram.
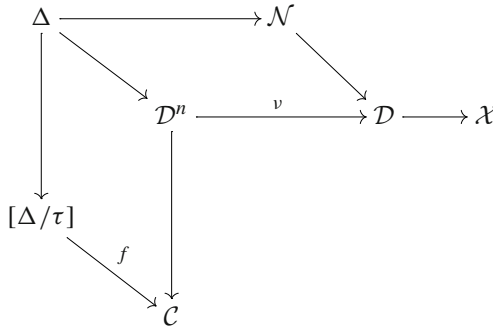

Lemma 5.8 *The map* $f : [\Delta/\tau] \to \mathcal{C}$ *is a closed embedding. In particular, the composition* $[\Delta/\tau] \to \mathcal{C} \to B\mathbb{G}_m$ *is representable.*

Proof Since $\Delta \to \mathcal{D}^n$ is a closed embedding, then $[\Delta/\tau] \to [\mathcal{D}^n/\tau]$ is also a closed embedding. Therefore the claim follows if we can prove that $\psi : [\mathcal{D}^n/\tau] \to \mathcal{C}$ is an isomorphism along $[\Delta/\tau]$. Let $\pi : \mathcal{D}^n \to [\mathcal{D}^n/\tau]$ be the quotient map.

If $q \in \Delta$ is such that $\tau(q) \neq q$, then $\pi(q)$ has no stabilizer so ψ is necessarily an isomorphism in a neighboourhood of $\pi(q)$. Assume then that $\tau(q) = q$. Formally locally around q, we can replace $\mathcal{D}^n$ with $\mathrm{Spec}(k[\![z]\!])$ and assume the involution τ sends $z \mapsto -z$ where z is a uniformizer for q. Since dz/z is a generator of $\omega_{\mathcal{D}^n}(\Delta)$, τ acts trivially on the fiber of $\omega_{\mathcal{D}^n}(\Delta)$ at q. Therefore, τ acts faithfully on the fiber of $\mathcal{G} \otimes \chi_{[\mathcal{D}^n/\tau]}$ at q and so by [2, Proposition 2.3.10], ψ is an isomorphism at $\pi(q)$. $\square$

We now define a DM stack $\mathcal{S}$ with two representable morphisms $g_1 : \mathcal{N} \to \mathcal{S}$ and $g_2 : [\Delta/\tau] \to \mathcal{S}$, such that the two morphisms $\Delta \to [\Delta/\tau] \to \mathcal{S}$ and $\Delta \to \mathcal{N} \to \mathcal{S}$ are isomorphic. We will denote with roman letters the coarse space of a DM stack, as in the conventions. We define S to be the pushout in algebraic spaces of the diagram below:

$$
\begin{array}{ccc}
\Delta & \longrightarrow & N \\
\downarrow & & \downarrow \\
\Delta/\tau & \longrightarrow & S.
\end{array}
$$
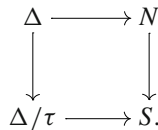
Thus S is a disjoint union of points. We determine a stack $\mathcal{S}$ with coarse space S by requiring that for every point $q \in S$, the group $\mathrm{Aut}_q(S)$ is $\boldsymbol{\mu}_2$ if there is a $\boldsymbol{\mu}_2$ stabilizer on a point of $g_1^{-1}(q) \cup g_2^{-1}(q)$ and trivial otherwise. Recall these are the only options for the stabilizers of $[\Delta/\tau]$ and $\mathcal{N}$. Now by Lemma 5.9, the morphisms g_i are uniquely determined.

Lemma 5.9 *Let* a, b *be positive integers with* a *even. Then, up to isomorphism, there are unique representable morphisms* $\mathrm{Spec}(k) \to B\boldsymbol{\mu}_b$ *and* $B\boldsymbol{\mu}_2 \to B\boldsymbol{\mu}_a$.

Proof Since k is algebraically closed, there is a unique μ_b-torsor over $\mathrm{Spec}(k)$, so there is a unique morphism $\mathrm{Spec}(k) \to B\mu_b$. For the claim about $B\mu_2$, consider the presentation of $B\mu_2$ given by $\mu_2 \rightrightarrows \mathrm{Spec}(k)$, where one arrow σ is the action of μ_2 on the point and the other one π_2 is the structure map. Note however, since μ_2 acts trivially on $\mathrm{Spec}(k)$ these are both just the structure map. Then a morphism $B\mu_2 \to B\mu_a$ is equivalent to the data of:

(1) A morphism $f : \mathrm{Spec}(k) \to B\mu_a$, and
(2) An isomorphism ψ in $B\mu_a(\mu_2) \cong B\mu_a(\mathrm{Spec}(k)) \times B\mu_a(\mathrm{Spec}(k))$ between $f \circ \sigma$ and $f \circ \pi_2$.

Moreover, ψ must satisfy the cocycle condition. More precisely, let $m : \mu_2 \times \mu_2 \to \mu_2$ be the multiplication and $p_{2,3}$ the projection onto the second factor $\mu_2 \times \mu_2 \to \mu_2$. Then we require that $p_{2,3}^* \psi \circ (\mathrm{Id}_{\mu_2} \times \sigma)^* \psi = m^* \psi$.

Now, there is a unique morphism $\mathrm{Spec}(k) \to B\mu_a$ from the previous point. An isomorphism ψ as above is the data of two automorphisms (α, β) in $B\mu_a(\mathrm{Spec}(k))$. We will denote by $\{\pm 1\}$ the two points of μ_2 and by $(1, 1), (1, -1), (-1, 1), (-1, -1)$ the four points of $\mu_2 \times \mu_2$. To fix the notation, α will be the automorphism over 1 and β the one over -1. Then the arrows $p_{2,3}, \mathrm{Id}_{\mu_2} \times \sigma$ and m behave as follows:

- $p_{2,3}(1, 1) = p_{2,3}(-1, 1) = 1$ and $p_{2,3}(1, -1) = p_{2,3}(-1, -1) = -1$;
- $\mathrm{Id}_{\mu_2} \times \sigma(1, 1) = \mathrm{Id}_{\mu_2} \times \sigma(1, -1) = 1$ and $\mathrm{Id}_{\mu_2} \times \sigma(-1, 1) = \mathrm{Id}_{\mu_2} \times \sigma(-1, -1) = -1$;
- $m(1, 1) = m(-1, -1) = 1$, whereas $m(1, -1) = m(-1, 1) = -1$.

In particular, the cocycle condition is the following equality of automorphisms over $\mu_2 \times \mu_2$:

$$(\alpha, \beta, \alpha, \beta) \circ (\alpha, \alpha, \beta, \beta) = (\alpha, \beta, \beta, \alpha).$$

Therefore we must have $\alpha \circ \beta = \beta$ which implies $\alpha = \mathrm{Id}$ and that $\beta \circ \beta = \mathrm{Id}$. Thus the only ψ which satisfy the cocycle condition are $(\mathrm{Id}, \mathrm{Id})$ and (Id, β) where β is the unique element of μ_a with $\beta^2 = \mathrm{Id}$ and $\beta \neq \mathrm{Id}$. This means that there are exactly two morphisms $B\mu_2 \to B\mu_a$. The first one is the composition $B\mu_2 \to \mathrm{Spec}(k) \to B\mu_a$ and the second one is induced by the morphism $\mu_2 \hookrightarrow \mu_a$ as in [38, Exercise 10.F]. Only the second is representable. $\qquad\square$

Proposition 5.10 *With notation as above, let $\mathcal{X}'$ be the AH stack associated to $(X' \to \mathrm{Spec}(k), \omega_{X'})$. Then there is a unique representable morphism $\mathcal{S} \to \mathcal{X}'$ which makes the diagram below commutative.*
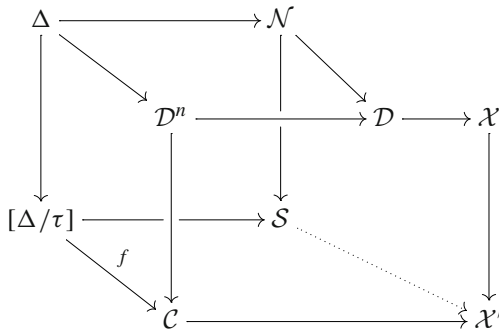
Remark 5.11 We observe that in general $\mathcal{S}$ is *not* the push-out of $[\Delta/\tau] \leftarrow \Delta \rightarrow \mathcal{N}$. Indeed, consider for instance the case where we have $\Delta = \operatorname{Spec}(k) \sqcup \operatorname{Spec}(k)$ and $\mathcal{N} = [\Delta/\tau] = \operatorname{Spec}(k)$. In this case $\mathcal{S} = \operatorname{Spec}(k)$, however it is not a push-out in algebraic stacks. For example, the groupoid of maps $\mathcal{S} \rightarrow B\mu_2$ has a unique object up to isomorphism, but one can compute that the 2-fiber product

$$\underline{\operatorname{Hom}}(\operatorname{Spec}(k), B\mu_2) \times_{\underline{\operatorname{Hom}}(\operatorname{Spec}(k)\sqcup\operatorname{Spec}(k),B\mu_2)} \underline{\operatorname{Hom}}(\operatorname{Spec}(k), B\mu_2)$$

has two isomorphism classes of objects. This is one of the major difficulties in extending the construction of this section to higher dimensions.

Proof It suffices to consider the following subdiagram:
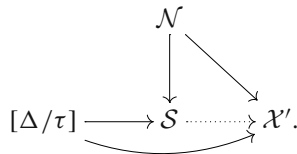
Note that $\mathcal{N} \rightarrow \mathcal{X}'$ is representable because $\mathcal{X} \rightarrow \mathcal{X}'$ is the normalization (Lemma 5.3) and $\mathcal{N} \rightarrow \mathcal{X}$ is a closed embedding. Moreover, $[\Delta/\tau] \rightarrow \mathcal{X}'$ is representable since f is a closed embedding (Lemma 5.8) and $\mathcal{C} \rightarrow \mathcal{X}'$ is representable (Remark 5.6).

Up to considering one connected component of $\mathcal{S}$ at a time, we can assume that $S \cong \operatorname{Spec}(k)$. Now, recall that the diagram on coarse spaces is a pushout and the diagram of solid arrows commutes. Then if we replace all the stacks above with their coarse spaces, we have an arrow $q : S = \operatorname{Spec}(k) \rightarrow X'$. In particular, if we denote $\mu_a := \operatorname{Aut}_{\mathcal{X}'}(q)$, we have a closed embedding $B\mu_a \rightarrow \mathcal{X}'$ (this follows from [38, Theorem 11.3.1]). The arrows $[\Delta/\tau] \rightarrow \mathcal{X}'$ and $\mathcal{N} \rightarrow \mathcal{X}'$ factor through $B\mu_a \rightarrow \mathcal{X}'$, so in the diagram above we can replace $\mathcal{X}'$ with $B\mu_a$. Now the claim follows from the description of $\mathcal{S}$ and Lemma 5.9. $\qquad\square$

We recall a theorem due to David Rydh on the existence of pinchings in algebraic stacks:

Theorem 5.12 [41, Theorem A.4] *Consider a diagram of solid arrows between algebraic stacks over B as below, with i a closed embedding and f a finite morphism.*

$$
\begin{array}{ccc}
\mathcal{X} & \overset{i}{\longhookrightarrow} & \mathcal{Y} \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\
\mathcal{X}' & \dashrightarrow & \mathcal{Y}' \\
& {\scriptstyle i'} &
\end{array}
$$

Then there are arrows i' and f' over S, such that the resulting diagram is a pushout in algebraic stacks. Moreover, i' is a closed embedding and f' is integral and an isomorphism away from $\mathcal{X}$. If we take the topological spaces of the algebraic stacks above, the corresponding square is a pushout in topological spaces. Finally, if $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}'$ are proper over B, then also $\mathcal{Y}'$ is proper over B (recall that for us all the algebraic stacks are of finite type).

Remark 5.13 The construction of $\mathcal{Y}'$ can be performed smooth locally. In particular, if we take $\mathrm{Spec}(A') \to \mathcal{Y}'$ a smooth morphism, and we pullback i', f', i and f through it, the corresponding diagram will be a pushout in schemes.

The following Lemma is the last technical result we need before proving Theorem 5.17.

Lemma 5.14 *The following diagram is a pushout in algebraic stacks:*

$$
\begin{array}{ccc}
\Delta & \longrightarrow & \mathcal{N} \\
\big\downarrow & & \big\downarrow \\
\mathcal{D}^n & \longrightarrow & \mathcal{D}
\end{array}
$$

Proof It suffices to check that it is a pushout étale locally on $\mathcal{D}$ so we may suppose that $\mathcal{D} = \mathrm{Spec}(R)$ is affine. In this case the diagram is a pushout by the definition of the conductor (see for example [42]).   □

Now we aim at constructing $\mathcal{X}'$ from the data of $\mathcal{D}$, τ and $\mathcal{X}$. The situation is sumarized in the following diagram, where the dashed arrows are pushouts, the wavy ones come from the universal property of a pushout, and all the arrows are representable. From the definition of $\mathcal{S}$, the morphism $[\Delta/\tau] \to \mathcal{S}$ is finite and the map $[\Delta/\tau] \to \mathcal{C}$ is a closed embedding from Lemma 5.8. Therefore from [41, Theorem A.4] (restated above as Theorem 5.12 for convenience) the two maps $[\Delta/\tau] \to \mathcal{S}$ and $[\Delta/\tau] \to \mathcal{C}$ have a pushout $\mathcal{C} \to \mathcal{Q}$ and $\mathcal{S} \to \mathcal{Q}$. Thus we have a morphism $\alpha : \mathcal{Q} \to \mathcal{X}'$. To check that α is representable, it suffices to check that it is representable after passing to the pullback to a smooth cover of $\mathcal{X}'$ by a scheme. Since pushouts commute with smooth base change (see [41, Theorem A.4, (iv)]), and the pushout of a closed embedding by a finite morphism between algebraic spaces is an algebraic space [25, Theorem 9.30][4], it suffices that $\mathcal{C} \to \mathcal{X}'$ and $\mathcal{S} \to \mathcal{X}'$ are representable, which is true by construction.

---

[4] Such pushouts are often called pinchings or Ferrand pushouts.

Then by Lemma 5.14, we have a morphism $\mathcal{D} \to \mathcal{Q}$ which is representable by [41, Theorem A.4]. One can check that it is also quasifinite, and $\mathcal{D}$ is proper, so $\mathcal{D} \to \mathcal{Q}$ is finite. Finally, again from [41, Theorem A.4], the two maps $\mathcal{D} \to \mathcal{X}$ and $\mathcal{D} \to \mathcal{Q}$ admit a pushout $\mathcal{Z}$. As before, we have a representable morphism $\Phi : \mathcal{Z} \to \mathcal{X}'$.

(5)

Proposition 5.15 *The morphism Φ is an isomorphism. In particular, $\mathcal{Z}$ is Gorenstein.*

Proof From the univeral property of the pushouts and coarse spaces, one can check that taking the coarse space commutes with pushouts. In particular, by [25, Chapter 5], the morphism Φ is an isomorphism on coarse spaces. Furthermore, $\mathcal{Z}$ is seminormal since it is the pushout of seminormal stacks.

Observe that Φ is an isomorphism away from a finite set of points $\{q_1, ..., q_r\}$. Let $p := q_i$ and let $m := |\operatorname{Aut}_{\mathcal{X}'}(p)|$ be the index of $K_{X'}$ at p. If $n := |\operatorname{Aut}_{\mathcal{Z}}(p)|$, the line bundle $\Phi^*(\omega_{\mathcal{X}'})^{\otimes n}$ descends to a line bundle in a neighborhood of $p \in X'$ and agrees with $\omega_{X'}^{[n]}$ in codimension one so the index m divides n. On the other hand, by representability, n divides m. Thus $m = n$ and the morphism $\Phi_p : \operatorname{Aut}_{\mathcal{Z}}(p) \to \operatorname{Aut}_{\mathcal{X}'}(p)$ is bijective. We conclude that Φ is a proper morphism of seminormal DM stacks such that $\mathcal{Z}(\operatorname{Spec}(k)) \to \mathcal{X}'(\operatorname{Spec}(k))$ is an equivalence so Φ is an isomorphism. $\square$

Example 5.16 Consider $(X, D) = (\mathbb{A}^2_{x,y}, V(xy))$, so D is a nodal curve and consider the involution $\tau : D \to D$ which sends $(x, y) \mapsto (-x, -y)$. Call the resulting slc surface X'.

Observe that $(\omega_X(D))\big|_D = \omega_D$ is generated by the section $\frac{dx \wedge dy}{xy}$ which is fixed by τ. Thus the action on $\omega_D \otimes \chi$ is non-trivial and $\mathcal{C} \cong [D^n/\tau]$. Then $\mathcal{N} = p$ is a node, $\Delta = q_1 \sqcup q_2$ lying above the node which are both fixed points of τ, and $\mathcal{S} = [p/\boldsymbol{\mu}_2] = B\boldsymbol{\mu}_2$. It follows that $\mathcal{Q} = [D/\tau]$ and the AH stack $\mathcal{X}'$ of X' is the pushout in stacks of $D \hookrightarrow \mathbb{A}^2$ and $D \to [D/\tau]$. In particular it has a $\boldsymbol{\mu}_2$ stabilizer at the origin. Therefore $\omega_{X'}$ has index 2 at the origin.

Using a computer algebra system, one can compute equations for X' directly from the presentation above. They are given by the following ideal in $k[a, b, c, d, e]$:

$$\left(cd - ae, bd - ce, ab - c^2, de - c^3, bc^2 - e^2, ac^2 - d^2\right).$$

Here X' is $\operatorname{Spec}(A)$ for A the subring of $k[x, y]$ defined by the map

$$(a, b, c, d, e) \mapsto \left(x^2, y^2, xy, x^2 y, xy^2\right).$$

We are now ready to prove the main theorem of this subsection, which states that the construction of $\mathcal{Z}$ above can be carried out in families over an arbitrary base.

Theorem 5.17 *Consider a twisted stable family of normal surface pairs* $(f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L})$. *Assume that* $\mathcal{D}_\phi \to B$ *has a simultaneous normalization, namely that there is a flat proper morphism* $\mathcal{D}^n \to B$ *and a morphism* $v : \mathcal{D}^n \to \mathcal{D}_\phi$ *such that for every* $b \in B$, *the map* $v_b : \mathcal{D}^n_b \to (\mathcal{D}_\phi)_b$ *is a normalization. Let* $\{\sigma_i : B \to \mathcal{D}^n\}$ *be disjoint sections which surject onto the locus where* $\mathcal{D}^n \to \mathcal{D}_\phi$ *is not an isomorphism. Finally let* $\tau : \mathcal{D}^n \to \mathcal{D}^n$ *be an involution over* B *which preserves the closed substack* $\bigsqcup \sigma_i(B)$ *and which is fiberwise generically fixed point free.*

Then there exists a DM stack $\mathcal{Z}$ *which fits in the commutative diagram below.*

$$
\begin{array}{ccc}
\mathcal{D}^n & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
[\mathcal{D}^n/\tau] & \longrightarrow & \mathcal{Z}
\end{array}
$$

Moreover, $\mathcal{Z} \to B$ *is flat and proper, and for every* $p \in B$, *the fiber* $\mathcal{Z}_p$ *is the AH stack of the stable surface obtained from gluing data* D^n_p, $\tau\big|_{D^n_p}$ *and* X_p *by* [25, Theorem 5.13].

Proof Throughout the proof, we will drop the subscript ϕ on $\mathcal{D}_\phi$. We will denote by Δ the closed substack, which is actually a scheme, given by the sections σ_i. Furthermore, we can assume that B is connected.

From Proposition 4.7 we have that $\mathcal{L}\big|_{\mathcal{D}^n} \cong \omega_{\mathcal{D}^n/B}(\Delta)$. It descends to a line bundle $\mathcal{G}$ on $[\mathcal{D}^n/\tau]$, and we let $[\mathcal{D}^n/\tau] \to \mathcal{C}$ be the relative coarse space of the map $[\mathcal{D}^n/\tau] \to B\mathbb{G}_{m,B}$ induced by $\mathcal{G} \otimes \chi_{[\mathcal{D}^n/\tau]}$. The induced morphism $\mathcal{C} \to B$ is flat and the construction of $\mathcal{C}$ commutes with arbitrary base change $B' \to B$ by [6, Proposition 3.4] and [5, Corollary 3.3].

Consider the quotient stack $[\Delta/\tau]$. Now, $\Delta \subseteq \mathcal{D}^n$ is a disjoint union of sections of $\mathcal{D}^n \to B$, so $\Delta \cong \bigsqcup B$. Since τ acts fiberwise and preserves Δ, and since B is connected, we have that $[\Delta/\tau] \cong [\Delta_b/\tau] \times B$ for any $b \in B$. In other words, $[\Delta/\tau]$ is a constant family over B. Similarly, if we denote by $\mathcal{N}$ the singular locus of $\mathcal{D} \to B$, then $\mathcal{N} \cong \mathcal{N}_b \times B$ and $\Delta \cong \Delta_b \times B$. For any fixed b, considering Δ_b, $[\Delta_b/\tau]$ and $\mathcal{N}_b$, we can construct $\mathcal{S}_b$ as in Proposition 5.10. Now we can define $\mathcal{S} := \mathcal{S}_b \times B$. Then $\mathcal{S}$ fits into a commutative diagram as below where the maps are the ones from Proposition 5.10 pulled back to the product. By construction its formation commutes with base change.

$$
\begin{array}{ccc}
\Delta & \longrightarrow & \mathcal{N} \\
\downarrow & & \downarrow \\
[\Delta/\tau] & \longrightarrow & \mathcal{S}
\end{array}
$$

Now we proceed by constructing the analogous pushouts from Diagram 5 and the discussion before it. Along the way, we will need to check that our pushouts, which

are pinchings as in Theorem 5.12, commute with arbitrary basechange $B' \to B$. This will be checked étale locally (see Remark 5.13) using Lemma 5.18.

First we have the following diagram which, as in Lemma 5.14, we claim that is a pushout:

$$
\begin{array}{ccc}
\Delta & \longrightarrow & \mathcal{N} \\
\downarrow & & \downarrow \\
\mathcal{D}^n & \longrightarrow & \mathcal{D}.
\end{array}
$$

Indeed, $\Delta, \mathcal{N}, \mathcal{D}^n$ and $\mathcal{D}$ are flat over B, so we can check that the diagram above is a pushout after pulling back along $\mathrm{Spec}(k) \to B$, but this is the content of Lemma 5.14.

Next we construct the analogous pushouts to the dashed ones in Diagram 5. This produces the desired proper morphism $\mathcal{Z} \to B$ which is flat and commutes with base change by the above discussion. So $\mathcal{Z}$ satisfies the claimed properties. □

In the proof of Theorem 5.17, we needed the following technical result to check that the gluing construction commutes with base change and produces a flat family.

Lemma 5.18 *Let R be a ring, and consider two homomorphisms of R-algebras $f : A' \to B'$ and $g : B \to B'$ with $A := A' \times_{B'} B$ their fiber product. Assume that B' is flat over R and that $(g, -f) : B \times A' \to B'$ is surjective. Then the square*

$$
\begin{array}{ccc}
A' \otimes_R S & \longrightarrow & B' \otimes_R S \\
\uparrow & & \uparrow \\
A \otimes_R S & \longrightarrow & B \otimes_R S
\end{array}
$$

is cartesian for any ring homomorphism $R \to S$. Moreover, if A' and B are flat over R, so is A.

Proof First note that the following sequence is exact:

$$
0 \to A \to B \times A' \xrightarrow{(g,-f)} B' \to 0.
$$

Since B' is flat over R, we also have that

$$
0 \to A \otimes_R S \to (B \times A') \otimes_R S \to B' \otimes_R S \to 0
$$

is exact. On the other hand, $(B \times A') \otimes_R S \cong (B \otimes_R S) \times (A' \otimes_R S)$, so $A \otimes_R S$ is the pullback of $B \otimes_R S \to B' \otimes_R S$ and $A' \otimes_R S \to B' \otimes_R S$, proving the first claim. The second claim follows by applying the long exact sequence of Tor_i to the first exact sequence in the proof above. □

5.2 Gluing data

In this subsection we package the information of a gluing data into an algebraic stack $\mathcal{G}_{2,v}$. Therefore, Theorem 5.17 produces a gluing morphism $\mathcal{G}_{2,v} \to \mathcal{K}_{2,n}^{\omega}$ which on the level of points agrees with the gluing morphism of [25, Theorem 5.13].

Proposition 5.19 *There is an algebraic stack $\mathcal{G}_{2,v}$ which parametrizes the following objects. Over a scheme B, the objects of $\mathcal{G}_{2,v}(B)$ are quadruples*

$$((f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L}); g : \mathcal{D}^n \to \mathcal{D}_\phi; \{\sigma_i : B \to \mathcal{D}^n\}; \tau : \mathcal{D}^n \to \mathcal{D}^n)$$

where:

(1) *$(f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L})$ is an object of $\mathcal{K}_{2,v}(B)$;*
(2) *$\mathcal{D}^n \to B$ is a flat family of orbifold smooth curves and g is a simultaneous normalization;*
(3) *For a certain n, we have n disjoint sections σ_i of $\mathcal{D}^n \to B$, such that $g \circ \sigma_i(b) \in (\mathcal{D}_\phi^{sing})_b$ and such that g is an isomorphism away from $\bigsqcup \sigma_i(B)$, and*
(4) *τ is a generically fixed point free involution which preserves $\bigsqcup \sigma_i(B)$.*

The morphisms are pullback diagrams which satisfy the obvious commutativity conditions.

Proof We will construct $\mathcal{G}_{2,v}$ one condition at the time. It suffices to construct $\mathcal{G}_{2,v}$ for a fixed choice of n of point (3), and then to construct $\mathcal{G}_{2,v}$ by taking an union over $n \in \mathbb{N}$. Therefore, from now on we consider the number of sections n as part of the data.

First, consider $\mathcal{D} \to \mathcal{K}_{2,v}$ the universal divisor. Consider the stack $\mathcal{O} := \mathcal{O}rb^{\mathcal{L}}$ parametrizing smooth polarized orbifold curves, and let $\mathcal{C}_{\mathcal{O}} \to \mathcal{O}$ be the universal curve. Consider now the Hom stack

$$\mathcal{H}_1 := \mathcal{H}om_{\mathcal{O} \times \mathcal{K}_{2,v}}(\mathcal{C}_{\mathcal{O}} \times \mathcal{K}_{2,v}, \mathcal{O} \times \mathcal{D}),$$

where for the definition and existence of this Hom stack we refer to [6, Theorem C.2]. Over $\mathcal{H}_1$ we have an universal curve $\mathcal{C}_1$, obtained from the morphism $\mathcal{H}_1 \to \mathcal{O}$. Consider then $\mathcal{H}_2 := \mathcal{H}om_{\mathcal{H}_1}(\mathcal{H}_1, \mathcal{C}_1)^{\times_{\mathcal{H}_1}^n}$, which is the stack that parametrizes n sections of $\mathcal{C}_1 \to \mathcal{H}_1$. Let $\mathcal{C}_2$ be the universal curve of $\mathcal{H}_2$, and finally consider $\mathcal{H}_3 := \mathcal{H}om_{\mathcal{H}_2}(\mathcal{C}_2, \mathcal{C}_2)$. Over a base B, the objects of $\mathcal{H}_3(B)$ are the following:

(1) $(f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L})$, an object of $\mathcal{K}_{2,v}(B)$;
(2) $\mathcal{C} \to B$ is a flat family of smooth orbifold curves and a map $g : \mathcal{C} \to \mathcal{D}$;
(3) n sections $\sigma_i : B \to \mathcal{C}$, and
(4) $\tau : \mathcal{C} \to \mathcal{C}$ a morphism.

Let $\mathcal{C}$ be the universal curve over $\mathcal{H}_3$, and let $\mathcal{D}_3 \to \mathcal{H}_3$ be the pulll-back of $\mathcal{D}$.

Now, according to [37, Corollary 1.6], there is an open substack of $\mathcal{H}_1$ which parametrizes representable morphisms g. So up to replacing $\mathcal{H}_1$ with this open substack, we can assume g to be representable. We will denote with $\pi : \mathcal{C} \to \mathcal{C}^{\text{r.c.}}$ the

relative coarse space of $\mathcal{C} \to \mathcal{H}_3$, and let $U \subseteq \mathcal{C}$ to be the locus where π is an isomorphism, and $\mathcal{C} \to \mathcal{H}_3$ is smooth. Having the sections σ_i to be disjoint, and to map to U, is an open condition. Then up to shriking $\mathcal{H}_3$, we can assume σ_i to be disjoint, and to map to U.

From the upper semicontinuity of the dimension of the fibers, we can also assume that the morphism $\mathcal{C} \to \mathcal{D}_3$ is quasi-finite. But then it is finite since it is also proper and representable. Consider now the morphism $\mathcal{O}_{\mathcal{D}_3} \to g_*\mathcal{O}_\mathcal{C}$, let $\mathcal{K}_1$ be its kernel, and let $\mathcal{S}_1$ be the support of $\mathcal{K}_1$. Then $\mathcal{S}_1 \to \mathcal{H}_3$ is proper. Thus from the upper-semicontinuity of the dimension of the fiber, applied to $\mathcal{S}_1 \to \mathcal{H}_3$, there is an open substack where $\mathcal{S}_1$ is empty so g is dominant. Since g is proper it must also be surjective on this subset. Next let $\mathcal{K}_2$ be the cokernel of $\mathcal{O}_{\mathcal{D}_3} \to g_*\mathcal{O}_\mathcal{C}$ and $\mathcal{S}_2$ its support. As before, by upper-semicontinuity there is an open substack where $\mathcal{S}_2$ is 0-dimensional and so g is generically an isomorphism. Since it is also a representable morphism of nodal curves, the locus where it is not an isomorphism must be contained in the nodes of $\mathcal{D}_3$.

To recap, we have now cut out an algebraic stack where g is a simultaneous normalization and $\sigma_i(B)$ are disjoint sections. We need to identify the locus where:

- $g\left(\sqcup_{i=1}^n \sigma_i(b)\right) = \mathcal{D}_b^{\mathrm{sing}}$ for every $b \in B$, that is, the image of the marked points is exactly the nodes;
- Fiber by fiber, τ is a generically fixed point free involution.

To address the first bullet point, consider $\mathcal{S}' \to \mathcal{D}$ the inclusion of the locus where g is not an isomorphism. This is a closed embedding, and consider the following cartesian diagram:

$$
\begin{array}{ccc}
G & \longrightarrow & \mathcal{S}' \\
{\scriptstyle h}\downarrow & & \downarrow \\
\sqcup_{i=1}^n \sigma_i(\mathcal{H}_3) & \longrightarrow & \mathcal{D}
\end{array}
$$

Then h will be a closed embedding. Requiring h to be an isomorphism is equivalent to the first bullet point. Then the flattening stratification guarantees that there is a well-defined closed substack where h is an isomorphism. In other terms, up to replacing $\mathcal{H}_3$ with a locally closed substack, we can assume that the first bullet point is satisfied.

Finally, being an isomorphism is an open condition, so there is an open substack of $\mathcal{H}_3$ where τ is an isomorphism. Observe now that if σ is an isomorphism of an orbifold nodal curve, such that it agrees with the identity on an open dense subset, then $\sigma = \mathrm{Id}$. Thus if τ fixes a generic point of $\mathcal{C}$, then it fixes the irreducible component that is its closure.

Consider then the following cartesian diagram:

$$
\begin{array}{ccc}
F & \longrightarrow & \mathcal{C} \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \mathrm{Diag}} \\
\mathcal{C} & \xrightarrow{(\mathrm{Id},\tau)} & \mathcal{C} \times_{\mathcal{H}_3} \mathcal{C}.
\end{array}
$$

F is the fixed locus of τ. We need to cut out the locus where F contains no irreducible components of $\mathcal{C}$. This is equivalent to $F \to \mathcal{H}_3$ being finite so by semi-continuity of fiber dimension, there is an open subset where τ fixes no generic points. Similarly, to ensure that $\tau^2 = \mathrm{Id}$, we can replace in the diagram above τ with τ^2. Let F' be the new fiber product we obtain. Then the locus where $\tau^2 = \mathrm{Id}$ is the locus where $\pi_1 : F' \to \mathcal{C}$ is surjective. Or in other terms, where the kernel of the map $\mathcal{O}_\mathcal{C} \to (\pi_1)_* \mathcal{O}_{F'}$ is zero. This is the locus where the support of $\mathrm{Coker}(\mathcal{O}_\mathcal{C} \to (\pi_1)_* \mathcal{O}_{F'})$ is empty: it is an open substack of $\mathcal{H}_3$. □

Putting this together with Theorem 5.17 we obtain:

Theorem 5.20 *There is a functorial gluing morphism $\mathcal{G}_{2,v} \to \mathcal{K}^\omega_{2,v}$ from the stack of gluing data to the stack of twisted stable surfaces.*

5.3 The boundary strata of $\mathcal{K}^\omega_{2,v}$

In this subsection, we show that there is a locally closed stratification of $\mathcal{K}^\omega_{2,v}$ of equinormalizable surfaces with equinormalizable double locus which are the images of the gluing morphisms above.

Lemma 5.21 *Let $f : \mathcal{Y} \to B$ be a proper family of generically reduced DM stacks over a base scheme B. Then there exists a locally closed stratification of B over which f is simultaneously normalizable.*

Proof Using Noetherian induction, it suffices to prove that there is a nonempty open subset of B where f is simultaneously normalizable. Up to replacing B with its reduced structure, we can assume that B is reduced. Then it is generically smooth, so up to further shrinking B we can assume that it is smooth and connected. Consider $\nu : \mathcal{Y}^n \to \mathcal{Y}$ the normalization. Up to shrinking B we can assume that $\mathcal{Y}^n \to B$ is flat. The generic geometric fiber is normal and the locus $\mathcal{U}$ in $\mathcal{Y}$ where the fibers are normal is open from [13, Théorème 12.1.6]. So its complement $\mathcal{Z} := \mathcal{Y} \setminus \mathcal{U}$ is closed, and since f is proper, $f(\mathcal{Z})$ is closed too. Then up to shrinking B we can assume that $\mathcal{Y}^n \to B$ has normal fibers.

But ν is an isomorphism on the smooth locus of the morphism $\mathcal{Y} \to B$, since B is normal. In particular, for every $b \in B$, the map $\mathcal{Y}^n_b \to \mathcal{Y}_b$ is an isomorphism at the generic points of $\mathcal{Y}_b$, and it is finite. So it is a simultaneous normalization. □

Definition 5.22 We let $\mathcal{K}^\circ \subset \mathcal{K}^\omega_{2,v}$ denote the open substack parametrizing normal polarized orbispaces and call the complement $\mathcal{K}^\omega_{2,v} \setminus \mathcal{K}^\circ$ the *boundary*.

Proposition 5.23 *There is a finite, locally closed stratification of $\mathcal{K}^\omega_{2,v} \setminus \mathcal{K}^\circ$ such that each stratum is the image of a family of gluing data under the gluing morphism $\mathcal{G}_{2,v} \to \mathcal{K}^\omega_{2,v}$. In particular, taking the scheme theoretic images of components under the gluing morphism stratifies the boundary $\mathcal{K}^\omega_{2,v} \setminus \mathcal{K}^\circ$ into a finite, locally closed union of boundary components.*

Proof Let $\mathcal{X} \to \mathcal{K}^\omega_{2,v}$ be the universal twisted stable surface. From Lemma 5.21 there is a locally closed embedding $\mathcal{S} \to \mathcal{K}^\omega_{2,v}$ where the fibers of $\mathcal{X} \to \mathcal{K}^\omega_{2,v}$ admit a

simultaneous normalization. Let $\mathcal{X}' := \mathcal{X} \times_{\mathcal{K}^{\omega}_{2,v}} \mathcal{S}$ be the pull back, and let $\mathcal{D}$ be the closed substack which cuts the fiberwise double locus. Again from Lemma 5.21, there is a locally closed embedding where the fibers of $\mathcal{D} \to \mathcal{S}$ are simultaneously normalizable. So up to further stratifying $\mathcal{S}$ we can assume that both $\mathcal{X}'$ and $\mathcal{D}$ admit a simultaneous normalization. Let $\nu : \mathcal{D}^n \to \mathcal{D}$ be the simultaneous normalization of $\mathcal{D} \to \mathcal{S}$.

Let $\mathcal{Z}$ be the singular locus of $\mathcal{D} \to \mathcal{S}$, and F its pull back through ν. Up to stratifying further $\mathcal{S}$, we can assume that $F \to \mathcal{S}$ is étale, and up to replacing $\mathcal{S}$ with an étale cover, we can assume that there are disjoint sections $\sigma_i : \mathcal{S} \to \mathcal{D}^n$ which surject to F.

The composition $\mathcal{D}^n \to \mathcal{X}'$ is generically a 2 to 1 cover onto its image so $\mathcal{D}^n$ is equipped with a rational involution τ. After further stratifying and applying Lemma 5.1 we can assume that τ is a morphism. One can check fiber by fiber to see that τ preserves $\sqcup \text{Im}(\sigma_i)$. This gives a family of gluing data whose image in $\mathcal{K}^{\omega}_{2,v}$ is this stratum. Finally, by [25, Theorem 5.13], every point of $\mathcal{K}^{\omega}_{2,v}$ corresponding to a non-normal twisted stable surface lies in such a stratum. $\qquad\square$

Remark 5.24 Proposition 5.23 only posits the existence of some finite stratification and says nothing about how to enumerate the strata, nor the components of $\mathcal{G}_{2,v}$. We include it to rule out pathological behaviour like the image of $\mathcal{G}_{2,v}$ being an infinite disjoint union of points. One hopes for a more functorial stratification described in terms of combinatorial and numerical data of the surfaces. Doing this will require a generalization of Kollár's theory of hulls and husks, e.g. [23], to cyclotomic stacks. One also hopes for a generalization of functorial gluing morphisms to higher dimensions which is a more subtle question (see for example Remark 5.11).

6 The deformation space of an elliptic K3 surface with a section and a fiber

The goal of this section is to give an explicit computation of the deformations and obstructions for Kollár families of stable pairs using the formalism of twisted stable pairs.

Consider a pair $(Y, E + F)$ consisting of a generic elliptic K3 surface with section E and smooth fiber F. This pair is of log general type and its log canonical model (X, D) is obtained by contracting the section $p : Y \to X$ and taking $D = p_*E$. Since p contracts a (-2)-curve, (X, D) is a Gorenstein stable pair with an A_1-singularity and D is a smooth curve passing through the singular point. Note in particular that D is not Cartier but $2D$ is.

Since E is rigid and F moves in a pencil, the pair $(Y, E + F)$ is parametrized by a smooth 19-dimensional moduli space sitting inside a $\mathbb{P}^1$-bundle over the 18-dimensional moduli space of U-polarized K3 surfaces where U is the standard hyperbolic plane (see e.g. [1, Sect. 2]). Thus one might expect that the moduli space of the stable pair (X, D) should agree with the 19-dimensional moduli space of $(Y, E + F)$ described above. It is not hard to see that this is the case set theoreti-

cally but *a priori* the infinitesimal deformation space of (X, D) might be larger (see Remark 6.1).

Analytically locally around the A_1-singularity, the pair (X, D) is isomorphic to the quotient of $(\mathbb{A}^2, \{x = 0\})$ by the action $(x, y) \mapsto (-x, -y)$ so the associated AH stack $\mathcal{X}$ is smooth with Cartier boundary divisor $\mathcal{D}$. In particular, the data of the divisor $\mathcal{D}$ is equivalent to the data of a morphism $q : \mathcal{X} \to \Theta := [\mathbb{A}^1/\mathbb{G}_m]$, and to understand the deformations and obstructions of $(\mathcal{X}, \mathcal{D})$ it suffices to compute the deformations and obstructions of q. Note also that X is Gorenstein, p is crepant, and the action fixes $dx \wedge dy$ so $\omega_X \cong \mathcal{O}_X$ and $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$ are both trivial.

A standard computation shows that the cotangent complex $\mathbb{L}_\Theta \simeq_{qis} [\mathcal{O}_\Theta \xrightarrow{\cdot x} \mathcal{O}_\Theta]$ in degrees $[0, 1]$ where x is the $\mathbb{G}_m$-invariant coordinate on $\mathbb{A}^1$. Moreover, the map $q : \mathcal{X} \to \Theta$ induced by the Cartier divisor $\mathcal{D}$ is flat so $q^*\mathbb{L}_\Theta[1] \simeq_{qis} \mathcal{O}_\mathcal{D}$. Since $\mathcal{X}$ is smooth over k, the transitivity triangle for the composition $\mathcal{X} \to \Theta \to \mathrm{Spec}(k)$ shifted by 1 yields

$$\Omega_\mathcal{X}^1[0] \to \mathbb{L}_{\mathcal{X}/\Theta} \to \mathcal{O}_\mathcal{D}[0] \xrightarrow{+1} .$$

In particular, $\mathcal{H}^i(\mathbb{L}_{\mathcal{X}/\Theta}) = 0$ if $i \neq 0$ and we have a short exact sequence

$$0 \to \Omega_\mathcal{X}^1 \to \mathcal{H}^0(\mathbb{L}_{\mathcal{X}/\Theta}) \to \mathcal{O}_\mathcal{D} \to 0$$

which identifies $\mathcal{H}^0(\mathbb{L}_{\mathcal{X}/\Theta})$ with the sheaf of logarithmic differentials $\Omega_\mathcal{X}^1(\log \mathcal{D})$.

We will see (Propositions 6.4 and 6.7) that $Def(\mathcal{X})$ and $Def(\mathcal{X}, \mathcal{D})$ are both smooth and 19-dimensional. One can interpret this as follows. $Def(\mathcal{X})$ is identified with the deformation space of (Y, E), and the 18-dimensional U-polarized deformation space of Y is identified with the deformations of the polarized pair $(\mathcal{X}, \mathcal{L})$ where $\mathcal{L}$ is the polarization of the twisted stable pair corresponding to (X, D). Finally the 19-dimensional deformation space of $(\mathcal{X}, \mathcal{D})$ over $Def(\mathcal{X}, \mathcal{D})$ can be identified with the $\mathbb{P}^1$-bundle described above.

Remark 6.1 Using the cotangent complex formalism, one can compute that the first-order deformation space of the coarse space X is 20-dimensional and can be identified with $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$. Moreover there are both local and global obstructions to deforming the non-Cartier subscheme D. After some work, one can show that the obstructions live in a 2-dimensional space $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_D)$, and that there is a map $obs : \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \to \mathrm{Ext}^1(\Omega_X, \mathcal{O}_D)$ which vanishes if and only if a first order deformation of X can be followed by a deformation of $D \subset X$. In this example, obs is in fact surjective. One can then check that the first-order deformation space of (X, D) is 19-dimensional, matching that of the twisted stable pair $(\mathcal{X}, \mathcal{D})$. We thus conclude that all first-order deformations of the pair (X, D) lift to deformations of $(\mathcal{X}, \mathcal{D})$ and in particular they all satisfy Kollár's condition. However, this is far from clear *a priori* and we only know how to check it by first computing the Kollár deformations using the twisted stable pair $(\mathcal{X}, \mathcal{D})$.

6.1 Deformation theory of $\mathcal{X}$

The goal of this subsection is to compute $\operatorname{Ext}^i(\Omega^1_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) \cong H^i(\mathcal{X}, T_{\mathcal{X}})$, where $T_{\mathcal{X}}$ denotes the tangent bundle of $\mathcal{X}$.

Let $f : \mathcal{X} \to X$ be the coarse space map. Since f is finite, surjective and cohomologically affine, $H^i(\mathcal{X}, T_{\mathcal{X}}) \cong H^i(X, f_*T_{\mathcal{X}})$ and the sheaf $f_*T_{\mathcal{X}}$ is reflexive. Consider now the contraction of the section $p : Y \to X$ where Y is the elliptic K3 surface with section E as above. The holomorphic symplectic form on Y gives an identification $T_Y \cong \Omega^1_Y$ so by [12, Theorem 1.4] we have that p_*T_Y is reflexive. Both these sheaves are also isomorphic to the tangent sheaf T_X over a big open subset of X so by reflexivity we have

$$p_*T_Y \cong T_X \cong f_*T_{\mathcal{X}}.$$

By the Leray spectral sequence we have $H^0(X, T_X) \cong H^0(Y, T_Y)$ and

$$0 \to H^1(X, T_X) \to H^1(Y, T_Y) \to H^0\big(X, R^1 p_*T_Y\big) \to H^2(X, T_X) \to H^2(Y, T_Y).$$

Hodge theory of K3 surfaces gives us that

$$h^i(Y, T_Y) = \begin{cases} 0 & i = 0 \\ 20 & i = 1 \\ 0 & i = 2 \end{cases}$$

Lemma 6.2 *With the notations above, we have $R^1 p_*T_Y = H^1(E, T_Y|_E)_x \cong k_x$ is a skyscraper sheaf supported at the singular point of X.*

Proof Since p induces an isomorphism $Y \setminus E \to X \setminus p$, $R^1 p_*T_Y$ is supported at x. Then

$$R^1 p_*T_Y = (R^1 p_*T_Y)_x^{\widehat{}} = \varprojlim_n H^1(E_n, T_Y|_{E_n})$$

by the theorem on formal functions where E_n is the nth infinitesimal thickening of E defined by ideal sheaf $\mathcal{I}^n$.

Since $E \cong \mathbb{P}^1$ and $E^2 = -2$, the normal bundle sequence for the regular embedding $i : E \to Y$ is

$$0 \to \mathcal{O}_{\mathbb{P}^1}(2) \to T_Y|_E \to \mathcal{O}_{\mathbb{P}^1}(-2) \to 0$$

which splits since $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{O}_{\mathbb{P}^1}(2)) = 0$, giving $T_Y|_E \cong \mathcal{O}_E(2) \oplus \mathcal{O}_E(-2)$ and $H^1(E, T_Y|_E) = 1$.

Consider the closed embedding $E_n \hookrightarrow E_{n+1}$ for $n \geq 1$. There is a short exact sequence $0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{E_{n+1}} \to \mathcal{O}_{E_n} \to 0$ with $\mathcal{I}^n/\mathcal{I}^{n+1} \cong \mathcal{O}_E(2n)$, the nth power of the conormal bundle $\mathcal{O}_E(2)$ of $E \subset Y$. Tensoring with $T_Y|_E$ we obtain

$$0 \to \mathcal{O}_E(2n + 2) \oplus \mathcal{O}_E(2n - 2) \to T_Y|_{E_{n+1}} \to T_Y|_{E_n} \to 0.$$

Since $H^1(\mathbb{P}^1, \mathcal{O}(2n+2) \oplus O(2n-2)) = 0$ for all $n \geq 1$, we conclude that the map $H^1(E_{n+1}, T_Y|_{E_{n+1}}) \to H^1(E_n, T_Y|_{E_n})$ is an isomorphism and so

$$\varprojlim_n H^1(E_n, T_Y|_{E_n}) = H^1(E, T_Y|_E) = k.$$

$\square$

Under the canonical isomorphism in Lemma 6.2, the edge map $H^1(Y, T_Y) \to H^0(X, R^1 p_* T_Y)$ can be identified with the restriction $H^1(Y, T_Y) \to H^1(E, T_Y|_E)$.

Lemma 6.3 *Let $g : Y \to \mathbb{P}^1$ be a generic elliptically fibered $K3$ surface with section E. Then*

(1) $H^1(Y, T_Y) \to H^1(Y, (T_Y|_E)$ *is surjective, and*
(2) $H^0(\Omega_g^1 \otimes \mathcal{O}_Y(E)) = 0.$

Proof By taking the long exact sequence in cohomology for $0 \to T_Y(-E) \to T_Y \to T_Y|_E \to 0$, for part (i) we need to check that $H^2(T_Y(-E)) = 0$. Using Serre duality, this is equivalent to

$$H^0(\Omega_Y^1(E)) = 0.$$

We have a short exact sequence of differentials

$$0 \to g^* \Omega_{\mathbb{P}^1}^1 \to \Omega_Y^1 \to \Omega_g^1 \to 0. \tag{6}$$

Moreover, ω_g is a line a bundle which agrees with Ω_g^1 on the big open set $U \subset Y$ where g is smooth. Since the singular fibers of g have nodal singularities, Ω_g^1 is torsion free and there is a canonical map $\Omega_g^1 \to \omega_g$ identifying ω_g with the reflexive hull. The cokernel F is a torsion sheaf supported at the nodes of the singular fibers. A local computation around each such node $p \in Y$ shows that $F_p = k$ is rank 1. Thus we have a short exact sequence

$$0 \to \Omega_g^1 \to \omega_g \to F \to 0 \tag{7}$$

where F is a direct sum of k_p over the 24 nodes of the 24 singular fibers.

Now, $\mathcal{O}_Y \cong \omega_Y \cong \omega_g \otimes g^*(\omega_{\mathbb{P}^1})$ so $\omega_g \cong g^*(\mathcal{O}_{\mathbb{P}^1}(2))$. Then $H^0(\omega_g(E)) = H^0(\mathcal{O}_{\mathbb{P}^1}(2) \otimes g_* \mathcal{O}(E))$ by projection formula. By [35, II.3.7] $g_* \mathcal{O}(E) \cong \mathcal{O}_{\mathbb{P}^1}$ so $H^0(\mathcal{O}_{\mathbb{P}^1}(2) \otimes g_* \mathcal{O}(E)) \cong k^{\oplus 3}$. Then twisting the exact sequence (7) with $\mathcal{O}(E)$ and computing cohomology we have

$$0 \to H^0(\Omega_g^1(E)) \to H^0(\omega_g(E)) \to H^0(F(E)).$$

To check that the morphism $H^0(\omega_g(E)) \to H^0(F(E))$ is injective, it suffices to check that for every section $s \in H^0(\omega_g(E))$ there is a nodal point p of a singular fiber where s does not vanish. We identified sections of $\omega_g(E)$ above with degree 2 polynomials

on $\mathbb{P}^1$ which vanish along at most 2 fibers, proving the claim as there are 24 singular fibers. Therefore $H^0(\Omega_g^1(E)) = 0$, proving (2).

Returning to (1), we twist (6) by E and note that $\Omega_{\mathbb{P}^1}^1 = \mathcal{O}_{\mathbb{P}^1}(-2)$ so $H^0(g^*\Omega_{\mathbb{P}^1}^1(E)) = H^0(\mathcal{O}_{\mathbb{P}^1}(-2)) = 0$ using projection formula and [35, II.3.7] as before. Then $H^0(\Omega_Y^1(E)) \to H^0(\Omega_g^1(E))$ is injective so $H^0(\Omega_Y^1(E)) = 0$ as required. $\qquad\square$

Putting this all together, we conclude the following:

Proposition 6.4 *The deformation space $Def(\mathcal{X})$ is smooth and 19-dimensional.*

Proof Using the long exact sequence

$$0 \to H^1(X, T_X) \to H^1(Y, T_Y) \to H^0(X, R^1 p_* T_Y) \to H^2(X, T_X) \to H^2(Y, T_Y) = 0$$

and surjectivity of $H^1(Y, T_Y) \to H^0(X, R^1 p_* T_Y)$ we have $H^1(X, T_X) \cong k^{\oplus 19}$ and $H^2(X, T_X) = 0$ so the deformation space is unobstructed. $\qquad\square$

6.2 Deformations theory of the twisted stable pair $(\mathcal{X}, \mathcal{D})$.

Following the discussion at the start of this section, we wish to compute $\mathrm{Ext}^i(\Omega_{\mathcal{X}}^1(\log \mathcal{D}), \mathcal{O}_{\mathcal{X}}) = H^i(X, T_{\mathcal{X}}(-\log \mathcal{D}))$. Consider the residue exact sequence

$$0 \to \Omega_{\mathcal{X}}^1 \to \Omega_{\mathcal{X}}^1(\log \mathcal{D}) \to \mathcal{O}_{\mathcal{D}} \to 0. \tag{8}$$

Lemma 6.5 $\mathrm{Ext}^2(\Omega_{\mathcal{X}}^1(\log \mathcal{D}), \mathcal{O}_{\mathcal{X}}) \cong H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1(\log \mathcal{D}))^\vee = 0.$

Proof The first equality is Serre duality. Consider the connecting homomorphism $k = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{D}}) \to H^1(\mathcal{X}, \Omega_{\mathcal{X}}^1)$. One can check that this map sends 1 to $c_1(\mathcal{L})$ where $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\mathcal{D})$. Since $\mathcal{D}$ is ample and $\mathcal{X}$ is proper, $c_1(\mathcal{L}) \neq 0$ so the connecting homomorphism is injective. Therefore $H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1) \to H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1(\log \mathcal{D}))$ is an isomorphism but $H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1) = H^0(X, f_*\Omega_{\mathcal{X}}^1) = H^0(X, p_*\Omega_Y^1) = H^0(Y, \Omega_Y^1) = 0$. $\qquad\square$

Lemma 6.6

$$\mathrm{Ext}^i(\mathcal{O}_D, \mathcal{O}_{\mathcal{X}}) \cong \begin{cases} 0 & i \neq 1, 2 \\ k & i = 1, 2 \end{cases}$$

Proof We apply the functor $RHom(-, \mathcal{O}_{\mathcal{X}})$ to the sequence $0 \to \mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_D \to 0$ using the fact that $\mathrm{Ext}^i(\mathcal{L}, \mathcal{O}_{\mathcal{X}}) = H^i(\mathcal{X}, \mathcal{L}^{-1})$ for line bundles $\mathcal{L}$. Moreover, since $f : \mathcal{X} \to X$ is cohomologically affine and X has rational singularities, $R^i f_* \mathcal{O}_{\mathcal{X}} = R^i p_* \mathcal{O}_Y = 0$ for $i > 0$ so $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = H^i(Y, \mathcal{O}_Y)$. We conclude that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = k$, and $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$. Putting this together, we have the following short exact sequences.

$$0 \to k \to H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) \to \mathrm{Ext}^1(\mathcal{O}_D, \mathcal{O}) \to 0 \tag{9}$$

$$0 \to H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D})) \to \mathrm{Ext}^2(\mathcal{O}_D, \mathcal{O}) \to k \to 0 \tag{10}$$

By Serre duality, $H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}(\mathcal{D})) = H^2(\mathcal{X}, \mathcal{O}_\mathcal{X}(-\mathcal{D}))$ and $H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}(\mathcal{D})) = H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}(-\mathcal{D}))$. Taking the long exact sequence of cohomology of $0 \to \mathcal{O}_\mathcal{X}(-\mathcal{D}) \to \mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{D} \to 0$ we have that $H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}(-\mathcal{D})) = 0$ and $H^2(\mathcal{X}, \mathcal{O}_\mathcal{X}(-\mathcal{D})) = k^{\oplus 2}$. Here we have used that $H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}) = 0$ and $H^1(\mathcal{D}, \mathcal{O}_\mathcal{D}) = 1$ since $\mathcal{D}$ is genus 1. Applying this to (9) and (10) finishes the proof. $\square$

Putting this all together, we conclude the following.

Proposition 6.7 *The deformation space of the twisted stable pair $Def(\mathcal{X}, \mathcal{D})$ is smooth and* 19-*dimensional.*

Proof Note that $\mathrm{Hom}(\Omega^1_\mathcal{X}(\log \mathcal{D}), \mathcal{O}_\mathcal{X}) = H^0(\mathcal{X}, T_\mathcal{X}(-\log \mathcal{D}))$ vanishes as it injects into $H^0(\mathcal{X}, T_\mathcal{X}) = 0$. Using this and applying $RHom(-, \mathcal{O}_\mathcal{X})$ to the short exact sequence (8), we obtain

$$0 \to \mathrm{Ext}^1(\mathcal{O}_\mathcal{D}, \mathcal{O}_\mathcal{X}) \to H^1(\mathcal{X}, T_\mathcal{X}(-\log \mathcal{D})) \to H^1(\mathcal{X}, T_\mathcal{X}) \to$$
$$\mathrm{Ext}^1(\mathcal{O}_\mathcal{D}, \mathcal{O}_\mathcal{X}) \to H^2(\mathcal{X}, T_\mathcal{X}(-\log \mathcal{D})) \to 0.$$

Applying the previous results, we obtain

$$0 \to k \to H^1(\mathcal{X}, T_\mathcal{X}(-\log \mathcal{D})) \to k^{\oplus 19} \to k \to 0$$

and $H^2(\mathcal{X}, T_\mathcal{X}(-\log \mathcal{D})) = 0$ so we conclude that $H^1(\mathcal{X}, T_\mathcal{X}(-\log \mathcal{D}))$ is 19-dimensional and $Def(\mathcal{X}, \mathcal{D}) = Def(\mathcal{X} \to \Theta)$ is unobstructed as claimed. $\square$

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