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Non-commutative deformations of simple objects in a category of perverse coherent sheaves

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Abstract
We define a category of perverse coherent sheaves as the abelian category corresponding to the category of modules under Bondal–Rickard equivalence which arises from a tilting bundle for a projective morphism. The purpose of this paper is to determine versal non-commutative deformations of simple collections in the categories of perverse coherent sheaves in some cases. In general we prove that the non-commutative structure algebra is recovered as the parameter algebra of the versal non-commutative deformation of the simple collection consisting of all simple objects over a closed point of the base space. In the case where the fiber dimensions are at most 1 and the structure sheaf is relatively acyclic, we determine the versal deformations of some partial simple collections consisting of vanishing simple objects. In particular it is proved that the parameter algebra of the versal non-commutative deformation is isomorphic to its opposite algebra in this case.

Keywords Non-commutative deformation · Simple collection · Perverse coherent sheaf · Bondal–Rickard equivalence · Tilting bundle · Contraction

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1 Introduction

Let k be an algebraically closed field of characteristic 0, and let $f : Y \to X$ be a projective morphism of Noetherian k-schemes such that $X = \text{Spec}(R)$ is an affine scheme. A locally free coherent sheaf P on Y is called a *tilting generator* if the following conditions are satisfied: (1) all higher direct images of $\mathcal{E}nd(P)$ for f vanishes, (2) P generates the derived category of quasi-coherent sheaves $D(\text{Qcoh}(Y))$ (see Definition 2.1). Then the derived Morita equivalence theorem of Bondal [5] and Rickard [26] tells us that there is an equivalence of triangulated categories $D^b(\text{coh}(Y)) \cong D^b(\text{mod-}A)$, where $A = f_*\mathcal{E}nd(P)$ is a coherent sheaf of associative $\mathcal{O}_X$-algebras. Let $\text{Perv}(Y/X)$ be the abelian subcategory of $D^b(\text{coh}(Y))$ corresponding to the category of coherent right A-modules (mod-A).

We would like to call $\text{Perv}(Y/X)$ the category of *perverse coherent sheaves*. By definition, the category of coherent sheaves $\text{coh}(Y)$ is more geometric and $\text{Perv}(Y/X)$ more algebraic. For example, for a k-valued point $x_0 \in X$, the set of simple objects in $\text{coh}(Y)$ above a simple object $\mathcal{O}_{x_0}$ in $\text{coh}(X)$ is identified as the set-theoretic fiber $f^{-1}(x_0)$ and is an infinite set in general. On the other hand, such a set in $\text{Perv}(Y/X)$ is finite since A is coherent.

The original perverse sheaves [3] are complexes in a derived category of constructible sheaves which correspond to sheaves of regular holonomic D-modules by the Riemann–Hilbert correspondence [20,25]. It was defined by generalizing the intersection homology theory of Goresky and MacPherson [14] which was discovered in the pursuit of homology theory for singular spaces which behaves better under the Poincare duality. Our definition is similar in that the perverse coherent sheaves are complexes of coherent sheaves which correspond to sheaves over associative algebras by the Bondal–Rickard equivalence. The perverse coherent sheaves possess more algebraic nature due to the construction, and we can expect that they behave better under certain problems.

In this paper, we consider the multi-pointed non-commutative deformations of sets of simple objects [24] in $\text{Perv}(Y/X)$, and determine the versal deformations in some cases.

We first determine the versal deformation of the simple collection consisting of all simple objects over a closed point of the base space in general and prove that we can recover the associative structure algebra:

Theorem 1.1 (= Theorem 6.1) *Let $\{s_j\}_{j=1}^m$ be the set of all simple objects in Perv(Y/X) above a closed point $x_0 \in X$. Let $\hat{P}$ be the completion of the direct sum of all indecomposable projective objects $\{P_i\}_{i=1}^m$ in the category of perverse coherent sheaves Perv$(\hat{Y}/\hat{X})$ on the completion $\hat{Y} = Y \times_X \hat{X}$ above x_0. Then $\hat{P}$ is the versal deformation of the simple collection $\bigoplus_{j=1}^m s_j$ with the parameter algebra $\hat{A} = f_*\mathcal{E}nd(\hat{P})$.*

Next we determine the versal deformation of a partial collection consisting of vanishing simple objects in the case of perverse coherent sheaves of Bridgeland [7] and Van den Bergh [28], and prove that we can recover the *contraction algebra* of Donovan and Wemyss [10]:

Theorem 1.2 (= Theorem 6.2) *Assume that the dimension of the fibers of f are at most 1, and that $Rf_*\mathcal{O}_Y = \mathcal{O}_X$. Let C be the scheme theoretic closed fiber of f above x_0, and let C_i $(i = 1, \dots, r)$ be the irreducible components of C.*

(A) *Let $\{P_i\}_{i=0}^r$ and $\{s_j\}_{j=0}^r$ be the sets of indecomposable projective objects and simple objects in ${}^{-1}$Perv$(\hat{Y}/\hat{X})$ defined in Sect. 3 (A). Let $\hat{P} = \bigoplus_{i=0}^r P_i$, and let Q be the kernel of the natural homomorphism p: $f^*f_*\hat{P} \to \hat{P}$. Let I be the two-sided ideal of $\hat{A} = f_*\mathcal{E}nd(\hat{P})$ generated by endomorphisms of $\hat{P}$ which can be factored in the form $\hat{P} \to P_0 \to \hat{P}$. Then $Q[1] \in {}^{-1}$Perv$(\hat{Y}/\hat{X})$ is the versal deformation of the simple collection $\bigoplus_{j=1}^r s_j$ with the parameter algebra End$(Q[1]) \cong \hat{A}/I$.*

(B) *Let $\{P'_i\}_{i=0}^r$ and $\{s'_j\}_{j=0}^r$ be the sets of indecomposable projective objects and simple objects in ^{0}Perv$(\hat{Y}/\hat{X})$ defined in Sect. 3 (B). Let $\hat{P}' = \bigoplus_{i=0}^r P'_i$, and let Q' be the cokernel of the natural homomorphism p': $f^*f_*\hat{P}' \to \hat{P}'$. Let I' be the two-sided ideal of $\hat{A}' = f_*$End$(\hat{P}')$ generated by endomorphisms of $\hat{P}'$ which can be factored in the form $\hat{P}' \to P'_0 \to \hat{P}'$. Then $Q' \in {}^{0}$Perv$(\hat{Y}/\hat{X})$ is the versal deformation of the simple collection $\bigoplus_{j=1}^r s'_j$ with the parameter algebra End$(Q') \cong \hat{A}'/I'$.*

As a corollary we prove that $\hat{A}/I$ is isomorphic to its opposite ring:

Corollary 1.3 (=Corollary 6.3) *$A/I \cong (A/I)^o$, the opposite ring.*

The reason for this is that we have $s_j = s'_j[1]$ for $j \neq 0$, hence $Q \cong Q'$. But we note that $\hat{P} \neq \hat{P}'$ and ${}^{-1}$Perv$(\hat{Y}/\hat{X}) \neq {}^{0}$Perv$(\hat{Y}/\hat{X})$. We also note that Y need not to be Gorenstein and f need not to be crepant.

The *contraction algebra* $\hat{A}/I \cong \hat{A}'/I'$ is an important invariant of f. For example, Donovan and Wemyss conjectured that, in the case of a flopping contraction of a smooth threefold Y, the contraction algebra determines the singularity of X (see [15, 17,18] for the development).

We also have an alternative proof of the following result of Donovan and Wemyss [10,11,13], see also Bodzenta and Bondal [4]:

Corollary 1.4 (= Corollary 6.5) *Assume in addition that f is a birational morphism. Then f is an isomorphism outside the fiber over x_0 in a neighborhood of the fiber if and only if the parameter algebra $\hat{A}/I$ of the versal deformation of the simple collection $\bigoplus_{j=1}^r s_j$ is finite dimensional as a k-vector space.*

We assume that all schemes and morphisms in this paper are defined over a fixed algebraically closed base field k of characteristic 0.

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2 Perverse coherent sheaves

Definition 2.1 Let $f: Y \to X$ be a projective morphism between noetherian schemes such that $X = \mathrm{Spec}(R)$ is an affine scheme. A locally free coherent sheaf P on Y is said to be a *tilting generator* for f if the following conditions are satisfied:

(1) $R^p f_* \mathcal{E}nd(P) = 0$ for $p > 0$.
(2) For $a \in D(\mathrm{Qcoh}(Y))$, if $\mathrm{Hom}(P, a[p]) \cong 0$ for all p, then $a \cong 0$.

Theorem 2.2 (Bondal [5] and Rickard [26]) *Let $f: Y \to X = Spec(R)$ be a projective morphism between noetherian schemes, let P be a tilting generator for f, and let $A = f_* \mathcal{E}nd(P)$ be the endomorphism algebra. Then A is an associative $\mathcal{O}_X$-algebra which is coherent as an $\mathcal{O}_X$-module, and there is an equivalence of triangulated categories*

$$\Phi: D^b(coh(Y)) \to D^b(mod\text{-}A)$$

given by an exact functor $\Phi(\bullet) = RHom(P, \bullet)$. The quasi-inverse functor $\Psi: D^b(mod\text{-}A) \to D^b(coh(Y))$ is given by $\Psi(\bullet) = \bullet \otimes^L_A P$.

The theorem connects (algebraic) geometry and (non-commutative) algebra.

We mainly consider the case where X is a spectrum of a noetherian complete local ring. This is justified by the following proposition:

Proposition 2.3 *Let $f: Y \to X = Spec(R)$ be a projective morphism between noetherian schemes, let $x \in X$ be a closed point, let $\hat{X}$ be the completion of X at x, let $\hat{Y} = Y \times_X \hat{X}$ be the fiber product, and let $\hat{f}: \hat{Y} \to \hat{X}$ be the induced morphism.*

(1) *Let P be a tilting generator for f, and let $A = f_* \mathcal{E}nd(P)$ be the endomorphism algebra. Let $\hat{P}$ be the locally free sheaf on $\hat{Y}$ induced from P, and let $\hat{A} = \hat{f}_* \mathcal{E}nd(\hat{P})$ be the endomorphism algebra. Then $\hat{P}$ is a tilting generator for $\hat{f}$, and $\hat{A} = \mathcal{O}_{\hat{X}} \otimes_{\mathcal{O}_X} A$.*
(2) *Let $\bigoplus_{j=1}^r s_j$ be a simple collection on Y whose support is contained in a fiber $f^{-1}(x)$. Then its non-commutative deformations on Y are the same as those on $\hat{Y}$ (please see Sect. 5 for the definitions of non-commutative deformations).*

We define the abelian category of perverse coherent sheaves:

Definition 2.4 Let $f: Y \to X = \mathrm{Spec}(R)$ be a projective morphism between noetherian schemes, and let P be a tilting generator for f. The category of *perverse coherent* *sheaves* Perv(Y/X) for f is the abelian category corresponding to the category of finitely generated right A-modules (mod-A) by Ψ. In other words,

$$\mathrm{Perv}(Y/X) = \left\{ a \in D^b(\mathrm{coh}(Y)) \mid \mathrm{Hom}(P, a[p]) \cong 0 \text{ for } p \neq 0 \right\}.$$

Thus the standard t-structure on $D^b(\text{mod-}A)$ is transferred to a t-structure on $D^b(\mathrm{coh}(Y))$ whose heart is the category of perverse coherent sheaves.

We would like to call an object in Perv(Y/X) a *perverse coherent sheaf*. It is not a sheaf nor perverse. This is a generalization of such a category defined in the seminal paper by Bridgeland [7] and extended by Van den Bergh [28]. We note that the category Perv(Y/X) depends on the choice of the tilting generator P. Indeed we obtain ${}^p\mathrm{Perv}(Y/X)$ with different perversities $p \in \mathbf{Z}$ in their papers for different choices of P. For example, if L is an invertible sheaf on Y, then the auto-equivalence of $D^b(\mathrm{coh}(Y))$ defined by $\bullet \mapsto \bullet \otimes L$ sends the tilting generator P and the category of perverse coherent sheaves Perv(Y/X) to different ones.

By definition, we have the following:

Proposition 2.5 *Let $f: Y \to X = \mathrm{Spec}(R)$ be a projective morphism between noetherian schemes, let P be a tilting generator for f, and let $\mathrm{Perv}(Y/X)$ be the corresponding category of perverse coherent sheaves. Then $P \in \mathrm{Perv}(Y/X)$, and P become its* projective generator*, i.e., P is a projective object in this abelian category and $\mathrm{Hom}(P, a) \cong 0$ for $a \in \mathrm{Perv}(Y/X)$ implies that $a \cong 0$.*

The category of perverse coherent sheaves Perv(Y/X) has a different nature from the category of coherent sheaves coh(Y) in the sense that there are only finitely many points, or simple objects, above a point of a base space X.

The author learned the following lemma, which is well-known to experts, from Dr. Kohei Yahiro:

Lemma 2.6 *Let $X = \mathrm{Spec}\, R$ be the spectrum of a noetherian complete local ring R whose residue field is isomorphic to the base field, and let A be an associative R-algebra which is finitely generated as an R-module. Then the numbers of mutually non-isomorphic simple objects and mutually non-isomorphic indecomposable projective objects are finite in the abelian category (mod-A). Moreover their numbers, say m, are equal. Let $s_1, \ldots, s_m$ (resp. $P_1, \ldots, P_m$) be such simple objects (resp. indecomposable projective objects). Then $\dim \mathrm{Hom}(P_i, s_j) = \delta_{ij}$ after possible permutations of indexes.*

Proof Let J be the Jacobson radical of A, i.e., the intersection of all maximal right ideals. Since J is a fully invariant right submodule of A, it is invariant under automorphisms by left multiplications, hence a two-sided ideal. Then $\bar{A} = A/J$ becomes an associative Artin algebra. $\bar{A}$ is semi-simple, and is isomorphic as an $\bar{A}$-module to a direct sum of finitely many simple modules. It follows that simple $\bar{A}$-modules and indecomposable projective $\bar{A}$-modules are the same, hence the assertion of the lemma for $\bar{A}$.

A simple A-module is the same as a simple $\bar{A}$-module with the natural A-module structure. On the other hand, since A is J-adically complete, the map from the set of finitely generated projective A-modules to the set of finitely generated projective $\bar{A}$-modules given by $P \mapsto P \otimes_A \bar{A}$ is bijective [1, III-2.12]. Therefore the lemma is proved. □

If $X = \mathrm{Spec}(R)$ is the spectrum of a complete local ring, then the original tilting generator P is a direct sum of the indecomposable projective objects P_i. The reduced sum $\bar{P} = \bigoplus_{i=1}^m P_i$ is also a tilting generator which gives the same category of perverse coherent sheaves. We call $\bar{P}$ the *reduced tilting generator*.

3 The case of Bridgeland and Van den Bergh

We recall the definition of perverse coherent sheaves by Bridgeland [7] and Van den Bergh [28].

Let $f: Y \to X$ be a projective morphism between noetherian k-schemes. Assume the following conditions:

(1) $X = \mathrm{Spec}(R)$ for a complete local ring R whose residue field is isomorphic to k.
(2) The dimension of the closed fiber of f is equal to 1.
(3) $Rf_*\mathcal{O}_Y = \mathcal{O}_X$.

Let C be the scheme theoretic closed fiber of f, and let C_i $(i = 1, \dots, r)$ be the irreducible components of C. By the assumption that $R^1 f_*\mathcal{O}_Y = 0$, we have $C_i \cong \mathbf{P}^1$ for all i. Let

$$\bar{\mathcal{C}} = \{c \in D^b(\mathrm{coh}(Y)) \mid Rf_*c = 0\}$$

and $\mathcal{C} = \bar{\mathcal{C}} \cap \mathrm{coh}(Y)$. By the spectral sequence

$$E_2^{p,q} = R^p f_* H^q(c) \Rightarrow R^{p+q} f_* c$$

we deduce that, for $c \in D^b(\mathrm{coh}(Y))$, we have $c \in \bar{\mathcal{C}}$ if and only if $H^p(c) \in \mathcal{C}$ for all p.

Remark 3.1 (1) $\mathcal{C}$ is an abelian category. Let $h: c_0 \to c_1$ be a morphism in $\mathcal{C}$, i.e, a homomorphism of coherent sheaves such that $R^i f_* c_j = 0$ for $i, j = 0, 1$. Let $c'_0 = \mathrm{Ker}(h)$ and $c'_1 = \mathrm{Coker}(h)$ in the category of coherent sheaves. Then we claim that $c'_j \in \mathcal{C}$ for $j = 0, 1$.
Indeed $c'_0 \subset c_0$ implies $f_*c'_0 = 0$. Since $f_*\mathrm{Im}(h) = 0$ and $R^1 f_* c_0 = 0$, we have $R^1 f_* c'_0 = 0$. c'_1 is treated similarly.
(2) But $\bar{\mathcal{C}} \not\cong D^b(\mathcal{C})$. For example, assume that $f: Y \to X$ is a contraction of a smooth rational curve C in a smooth threefold whose normal bundle is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Then C is rigid, and $\mathcal{C}$ is equivalent to the category of k-vector spaces generated by $\mathcal{O}_C(-1)$. On the other hand, $\mathrm{Hom}^3(\mathcal{O}_C(-1), \mathcal{O}_C(-1)) \cong k$ in $D^b(\mathrm{Coh}(Y))$.

We define the following categories of perverse coherent sheaves.

(A) Let $P_0 = \mathcal{O}_Y$, and let L_i for $i = 1, \dots, r$ be line bundles on Y such that $(L_i, C_j) = \delta_{ij}$. We define locally free sheaves $\tilde{P}_i$ on Y by exact sequences:

$$0 \to \mathcal{O}_Y^{\oplus r_i} \to \tilde{P}_i \to L_i \to 0 \tag{3.1}$$

such that the induced homomorphisms

$$\mathrm{Hom}(\mathcal{O}_Y^{\oplus r_i}, \mathcal{O}_Y) \to \mathrm{Ext}^1(L_i, \mathcal{O}_Y)$$

are surjective. Then $\tilde{P} = \bigoplus_{i=0}^r \tilde{P}_i$ is a tilting generator, and the corresponding category of perverse coherent sheaves is denoted by ${}^{-1}\mathrm{Perv}(Y/X)$.

The number -1 is the *perversity*; we have $\mathcal{C}[1] \subset {}^{-1}\mathrm{Perv}(Y/X)$. More precisely, we have

$$\begin{aligned} &{}^{-1}\mathrm{Perv}(Y/X) \\ &= \{E \in D^b(\mathrm{coh}(Y)) \mid f_*H^{-1}(E) = 0,\, R^1 f_* H^0(E) = 0,\, \mathrm{Hom}(H^0(E), \mathcal{C}) = 0 \\ &\qquad H^p(E) = 0 \text{ for } p \neq 0, -1\}. \end{aligned} \tag{3.2}$$

Let $\tilde{P}_i \cong P_i \oplus P_0^{\bar{r}_i}$ be a decomposition into indecomposable sheaves. Let $s_0 = \mathcal{O}_C$ and $s_j = \mathcal{O}_{C_j}(-1)[1]$ for $j > 0$. Then $\{P_i\}_{i=0}^r$ and $\{s_j\}_{j=0}^r$ are the sets of indecomposable projective objects and simple objects in ${}^{-1}\mathrm{Perv}(Y/X)$ [28, Propositions 3.5.7].

(B) Let $P'_0 = \mathcal{O}_Y$, and define locally free sheaves $\tilde{P}'_i$ by exact sequences

$$0 \to \tilde{P}'_i \to \mathcal{O}_Y^{\oplus r'_i} \to L_i \to 0 \tag{3.3}$$

such that the induced homomorphisms

$$\mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_Y^{\oplus r'_i}) \to \mathrm{Hom}(\mathcal{O}_Y, L_i)$$

are surjective. Then $\tilde{P}' = \bigoplus_{i=0}^r \tilde{P}'_i$ is a tilting generator, and the corresponding category of perverse coherent sheaves is denoted by ${}^{0}\mathrm{Perv}(Y/X)$.

The number 0 is the perversity; we have $\mathcal{C} \subset {}^{0}\mathrm{Perv}(Y/X)$. More precisely, we have

$$\begin{aligned} &{}^{0}\mathrm{Perv}(Y/X) \\ &= \{E \in D^b(\mathrm{coh}(Y)) \mid f_*H^{-1}(E) = 0,\, \mathrm{Hom}(\mathcal{C}, H^{-1}(E)) = 0,\, R^1 f_* H^0(E) = 0 \\ &\qquad H^p(E) = 0 \text{ for } p \neq 0, -1\}. \end{aligned} \tag{3.4}$$

Let $\tilde{P}'_i \cong P'_i \oplus (P'_0)^{\bar{r}'_i}$ be a decomposition into indecomposable sheaves. Let $s'_0 = \omega_C[1]$, the shift of the dualizing sheaf of C, and $s'_j = \mathcal{O}_{C_j}(-1)$ for $j > 0$. Then $\{P'_i\}_{i=0}^r$ and $\{s'_j\}_{j=0}^r$ are the sets of indecomposable projective objects and simple objects in ${}^{0}\mathrm{Perv}(Y/X)$ [28, Propositions 3.5.8].

We have $P_i' \cong P_i^* = \mathrm{Hom}(P_i, \mathcal{O}_Y)$. Hence $A' \cong A^o$, the opposite algebra where the addition is the same but the multiplication is reversed.

4 Other examples

We consider divisorial contractions in this section.

4.1 Contraction of a projective space

Let $f: Y \to X = \mathrm{Spec}(R)$ be a projective birational morphism from a smooth variety to a variety with an isolated singularity whose exceptional locus is a prime divisor $E \cong \mathbf{P}^{n-1}$ with normal bundle $N_{E/Y} \cong \mathcal{O}_E(-d)$ for some $d > 0$. X has a terminal singularity if $d < n$, and f is crepant if $d = n$. The line bundles $\mathcal{O}_E(i)$ can be extended to line bundles $P_i = \mathcal{O}_Y(i)$ for integers i.

Proposition 4.1 *$P = \bigoplus_{i=0}^{n-1} P_i$ is a tilting generator of $D^b(coh(Y))$.*

Proof For any small positive number ϵ, the pair $(Y, (1-\epsilon)E)$ is log terminal. By [21, Theorem 1.2.5], we have $R^p f_* \mathcal{O}_Y(i) = 0$ for $p > 0$ and $i > -n$, because $\mathcal{O}_E(K_Y + E) \cong \mathcal{O}_E(-n)$. Therefore $R^p f_*(P_i^* \otimes P_j) = 0$ for $p > 0$ and $0 \le i, j \le n-1$.

By [6, Lemma 4.2.4] or [28, Lemma 3.2.2], $D^b(\mathrm{coh}(Y))$ is generated by the $\mathcal{O}_Y(i)$ for $0 \le i \le n-1$. □

We denote by $\mathrm{Perv}(Y/X)$ the corresponding category of perverse coherent sheaves.

Proposition 4.2 *$s_j = \Omega_E^j(j)[j]$ for $0 \le j \le n-1$ are the simple objects of $Perv(Y/X)$ above the singular point of X such that $Hom(P_i, s_j) \cong k^{\delta_{ij}}$.*

Proof By [2], there is a resolution of the diagonal $\Delta_E \subset E \times E$:

$$\begin{aligned} &0 \to \mathcal{O}_E(-n+1) \boxtimes \Omega_E^{n-1}(n-1) \to \cdots \\ &\cdots \to \mathcal{O}_E(-1) \boxtimes \Omega_E^1(1) \to \mathcal{O}_E \boxtimes \mathcal{O}_E \to \mathcal{O}_{\Delta_E} \to 0. \end{aligned}$$

We define an integral functor Φ: $D^b(\mathrm{coh}(E)) \to D^b(\mathrm{coh}(E))$ by

$$\Phi(\bullet) = p_{1*}(p_2^* \bullet \otimes \mathcal{O}_{\Delta_E}).$$

Since $\Phi(\mathcal{O}_E(-i)) \cong \mathcal{O}_E(-i)$, we deduce that $\mathcal{O}_E(-i)$ is quasi-isomorphic to a complex

$$\begin{aligned} &\mathcal{O}_E(-n+1) \otimes R\Gamma(E, \Omega_E^{n-1}(n-1-i)) \to \cdots \\ &\cdots \to \mathcal{O}_E(-1) \otimes R\Gamma(E, \Omega_E^1(1-i)) \to \mathcal{O}_E \otimes R\Gamma(E, \mathcal{O}_E(-i)) \end{aligned}$$

where the last term is put at degree 0. Therefore, for $0 \le i, j \le n-1$, we have

$$R\Gamma(E, \Omega_E^j(j-i)) \cong \begin{cases} 0 & \text{if } i \ne j \\ k[-j] & \text{if } i = j. \end{cases}$$

Thus the proposition is proved. □

We can obtain a similar result for the Grassmannian variety $G(r, n)$ if we use [19] instead of [2].

X has only one quotient singularity. Let $\tilde{X}$ be the associated Deligne–Mumford stack. Then there is a fully faithful functor $D^b(\mathrm{coh}(Y)) \to D^b(\mathrm{coh}(\tilde{X}))$ if and only if $K_Y \le f^*K_X$ [22,23]. This inequality is equivalent to saying that X is not terminal, or $d \ge n$.

An associative algebra A is said to be *homologically homogeneous* if the homological dimension of all simple objects are equal. If A is homologically homogeneous, then A has finite global dimension and is Cohen–Macaulay [8]. Van den Bergh [28] defined that $X = \mathrm{Spec}(R)$ has a *non-commutative crepant resolution* if there is a reflexive R-module M such that $A = \mathrm{End}(M)$ is homologically homogeneous.

These facts correspond to the following:

Proposition 4.3 *$A = f_*\mathcal{E}nd(P)$ is Cohen–Macaulay if and only if $d \ge n$.*

Proof A is Cohen–Macaulay if and only if A satisfies the condition $\mathrm{Ext}^p(A, \omega_X) = 0$ for $p > 0$, where ω_X is the canonical sheaf of X. Since $R^p f_*\mathcal{E}nd(P) = 0$ for $p > 0$, we have $Rf_*\mathcal{E}nd(P) \cong A$. By the Grothendieck duality theorem, our condition is equivalent to that $R^p f_*\mathcal{E}nd(P)(K_Y) = 0$ for $p > 0$, where we have $\mathcal{O}_Y(K_Y) \cong \mathcal{O}_Y(d-n)$. Since $\mathcal{E}nd(P)$ is a direct sum of the $\mathcal{O}_Y(m)$ for $-n+1 \le m \le n-1$, it follows that $\mathcal{E}nd(P)(K_Y)$ is a direct sum of the $\mathcal{O}_Y(m)$ for $d-2n+1 \le m \le d-1$. We have $R^p f_*\mathcal{O}_Y(m) = 0$ for $p > 0$ and $m \ge -n+1$ by the proof of Proposition 4.1. On the other hand, since $H^{n-1}(E, \mathcal{O}_E(-n)) \cong k$, we have $R^{n-1} f_*\mathcal{O}_Y(-n) \ne 0$. Hence we obtain our assertion. □

Example 4.4 Assume that $n = 2$. By the above lemma, A is Cohen–Macaulay if and only if $d \ge 2$. The homological dimension is calculated in [30, Theorem 3.2 and Corollary 3.4]:

(1) $h.d.(s_0) = 1$ if $d = 1$ and $= 2$ if $d \ge 2$.
(2) $h.d.(s_1) = 2$ if $d = 2$ and $= 3$ if $d \ge 3$.

For example, we have $\mathrm{Hom}^3(\mathcal{O}_E(-1)[1], \mathcal{O}_E) = \mathrm{Hom}^2(\mathcal{O}_E(-1), \mathcal{O}_E) \cong \mathrm{Hom}(\mathcal{O}_E, \mathcal{O}_E(-1 + (K_Y, E)))^* \ne 0$ if $(K_Y, E) > 0$.

4.2 Contraction of a singular quadric surface

Let X be a singularity of dimension 3 defined by an equation

$$xy + z^2 + w^3 = 0$$

in the completion of $\mathbf{C}^4$ at the origin, and let $f: Y \to X$ be the blowing up of the origin. Y is smooth, and the exceptional divisor E of f is a singular quadric surface with a singular point Q.

Let $\mathcal{O}_E(m)$ for $m \in \mathbf{Z}$ be a reflexive sheaf of rank 1 on E corresponding to a Weil divisor mC_1, where C_1 is a line on the cone E. mC_1 is a Cartier divisor if and only if m is even. Let $\hat{\Omega}^1_E$ be the double dual of the sheaf of Kähler differentials on E. It is a reflexive sheaf of rank 2 on E with a short exact sequence

$$0 \to \hat{\Omega}^1_E \to \mathcal{O}_E(-1)^{\oplus 2} \oplus \mathcal{O}_E(-2) \to \mathcal{O}_E \to 0$$

because $E \cong \mathbf{P}(1,1,2)$. It follows that, for the double dual of the sheaf of differential 2-forms, we have $\hat{\Omega}^2_E \cong \mathcal{O}_E(-4)$.

Let L be a line bundle of Y such that $L|_E = \mathcal{O}_E(2)$, let $S \in |L|$ be a generic member, and let $C_2 = S \cap E \in |\mathcal{O}_E(2)|$. We take a generic curve C_0 on S such that $C_0 \cap C_2 = C_1 \cap C_2$ scheme theoretically.

It is known that there is a non-trivial extension

$$0 \to \mathcal{O}_E(1) \to Q_1 \to \mathcal{O}_E(1) \to 0$$

on E such that Q_1 is a locally free sheaf of rank 2 [24]; Q_1 is defined by the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_E(-1) & \longrightarrow & Q_1(-L) & \longrightarrow & \mathcal{O}_E(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_E(-1) & \longrightarrow & \mathcal{O}_E^{\oplus 2} & \longrightarrow & \mathcal{O}_E(1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{C_2}(1) & \xrightarrow{=} & \mathcal{O}_{C_2}(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Similarly there is a locally free sheaf P_1 of rank 2 on Y such that $P_1|_E \cong Q_1$ defined by the following exact sequence

$$0 \to P_1(-S) \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_S(C_0) \to 0$$

where the right hand side arrow is obtained as a composition of surjective homomorphisms $\mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_S^{\oplus 2} \to \mathcal{O}_S(C_0)$.

We denote by $P_0 = \mathcal{O}_Y$ and $P_2 = L$. Let $s_j = \hat{\Omega}^j_E(j)[j]$ for $j = 0, 1, 2$, where $\hat{\Omega}^1_E(1)$ denotes the double dual of $\hat{\Omega}^1_E \otimes \mathcal{O}_E(1)$. Thus $s_0 = \mathcal{O}_E$ and $s_2 = \mathcal{O}_E(-2)[2]$.

Proposition 4.5 (1) *The sum $P = \bigoplus_{i=0}^{2} P_i$ is a tilting generator of $D^b(coh(Y))$.*
(2) *$\{s_0, s_1, s_2\}$ is the set of simple objects in the category of perverse sheaves $Perv(Y/X)$ for $f: Y \to X$ defined by P such that* $\dim Hom(P_i, s_j) = \delta_{ij}$.

Proof (1) We have $R^p f_* \mathcal{H}om(P_i, P_j) = 0$ for $p > 0$ and for all i, j except $i = j = 1$, because $H^p(E, \mathcal{O}_E(i)) = 0$ for $p > 0$ and $i \geq -3$ by the vanishing theorem [21] Theorem 1.2.5) and $\mathcal{O}_E(-E) \cong \mathcal{O}_E(2)$.

We prove that $R^p f_* \mathcal{H}om(P_1, P_1) = 0$ for $p > 0$. There is an exact sequence

$$0 \to \mathcal{O}_E \to \mathcal{H}om(Q_1, \mathcal{O}_E(1)) \to \mathcal{O}_E \to \mathcal{E}xt^1(\mathcal{O}_E(1), \mathcal{O}_E(1)) \to 0 \tag{4.1}$$

where we have $\mathcal{E}xt^1(\mathcal{O}_E(1), \mathcal{O}_E(1)) \cong \mathcal{O}_Q$ for the singular point Q of E. Since $H^p(E, \mathrm{Ker}(\mathcal{O}_E \to \mathcal{O}_Q)) = 0$ for all p, we deduce that $H^p(E, \mathcal{H}om(Q_1, \mathcal{O}_E(1))) = 0$ for $p > 0$. Therefore $H^p(E, \mathcal{H}om(Q_1, Q_1)) = 0$ for $p > 0$. We can also check that, for $p > 0$ and $m > 0$, we have $H^p(E, \mathrm{Ker}(\mathcal{O}_E(2m) \to \mathcal{O}_Q(2m))) = 0$, $H^p(E, \mathcal{H}om(Q_1, \mathcal{O}_E(1))(2m)) = 0$, and $H^p(E, \mathcal{H}om(Q_1, Q_1)(2m)) = 0$. Therefore we have $R^p f_* \mathcal{H}om(P_1, P_1) = 0$ for $p > 0$.

We prove that P_0, P_1, P_2 generate $D^b(\mathrm{coh}(Y))$. First, $\mathcal{O}_Y(iL)$ for $i = 0, 1, 2$ generate $D^b(\mathrm{coh}(Y))$ by [6, Lemma 4.2.4] or [28, Lemma 3.2.2]. By the exact sequence

$$0 \to \mathcal{O}_Y(L) \to \mathcal{O}_Y(2L) \to \mathcal{O}_S(4C_0) \to 0$$

we deduce that $\mathcal{O}_Y, \mathcal{O}_Y(L), \mathcal{O}_S(4C_0)$ generate $D^b(\mathrm{coh}(Y))$. Then by

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(L) \to \mathcal{O}_S(2C_0) \to 0$$
$$0 \to \mathcal{O}_S(2C_0) \to \mathcal{O}_S(3C_0)^{\oplus 2} \to \mathcal{O}_S(4C_0) \to 0$$

we deduce that $\mathcal{O}_Y, \mathcal{O}_Y(L), \mathcal{O}_S(3C_0)$ generate $D^b(\mathrm{coh}(Y))$. Finally, by

$$0 \to P_1 \to \mathcal{O}_Y(L)^{\oplus 2} \to \mathcal{O}_S(3C_0) \to 0$$

we conclude that P_0, P_1, P_2 generate $D^b(\mathrm{coh}(Y))$.

(2) We prove that $R\mathrm{Hom}(P_i, s_j) \cong k^{\delta_{ij}}$. Then it follows that $s_j \in \mathrm{Perv}(Y/X)$ and $\dim \mathrm{Hom}(P_i, s_j) = \delta_{ij}$. For $j = 0$, we have

$$R\mathrm{Hom}(P_0, s_0) \cong R\Gamma(E, \mathcal{O}_E) \cong k$$
$$R\mathrm{Hom}(P_2, s_0) \cong R\Gamma(E, \mathcal{O}_E(-2)) \cong 0.$$

Since $R\Gamma(E, \mathcal{O}_E(-1)) \cong 0$, we also have $R\mathrm{Hom}(P_1, s_0) \cong R\Gamma(E, Q_1^*) \cong 0$.

For $j = 2$, we have

$$R\mathrm{Hom}(P_0, s_2) \cong R\Gamma(E, \mathcal{O}_E(-2)[2]) \cong 0$$
$$R\mathrm{Hom}(P_2, s_2) \cong R\Gamma(E, \mathcal{O}_E(-4)[2]) \cong k.$$

Since $R\Gamma(E, \mathcal{O}_E(-3)) \cong 0$, we also have $R\mathrm{Hom}(P_1, s_2) \cong 0$.

For $j = 1$, we have a distinguished triangle

$$s_1[-1] \to \mathcal{O}_E^{\oplus 2} \oplus \mathcal{O}_E(-1) \to \mathcal{O}_E(1) \to s_1.$$

Since $\mathrm{Hom}(P_0, \mathcal{O}_E^{\oplus 2}) \cong \Gamma(E, \mathcal{O}_E^{\oplus 2}) \to \mathrm{Hom}(P_0, \mathcal{O}_E(1)) \cong \Gamma(E, \mathcal{O}_E(1))$ is an isomorphism, we have $R\mathrm{Hom}(P_0, s_1) \cong 0$. We also have $R\mathrm{Hom}(P_2, s_1) \cong 0$, because $R\Gamma(E, \mathcal{O}_E(-i)) \cong 0$ for $i = 1, 2, 3$.

We have $R\mathrm{Hom}(P_1, \mathcal{O}_E) \cong R\mathrm{Hom}(P_1, \mathcal{O}_E(-1)) \cong 0$. Hence we have isomorphisms

$$R\mathrm{Hom}_E(Q_1, \mathcal{O}_E(1)) \cong R\mathrm{Hom}(P_1, \mathcal{O}_E(1)) \cong R\mathrm{Hom}(P_1, s_1).$$

Q_1 is a versal non-commutative deformation of $\mathcal{O}_E(1)$ on E [24]). Therefore we have $\mathrm{Hom}_E(Q_1, \mathcal{O}_E(1)) \cong k$ and $\mathrm{Hom}^1_E(Q_1, \mathcal{O}_E(1)) \cong 0$. By duality, we have $\mathrm{Hom}^2_E(Q_1, \mathcal{O}_E(1)) \cong \mathrm{Hom}_E(\mathcal{O}_E(5), Q_1)^* \cong 0$. Our claim is proved. ◻

Let $\mathcal{D}$ be the left orthogonal complement of an exceptional object $s_2 = \mathcal{O}_E(-2)[2]$ in $D^b(\mathrm{coh}(Y))$:

$$\mathcal{D} = \{a \in D^b(\mathrm{coh}(Y)) \mid \mathrm{Hom}(a, s_2[p]) = 0 \ \forall p\}.$$

We can extend the concept of the tilting generators for triangulated categories such as $\mathcal{D}$, and consider the categories of perverse coherent sheaves.

Proposition 4.6 *$P_{\mathcal{D}} = P_0 \oplus P_1$ is a tilting generator of the triangulated category $\mathcal{D}$.*

Proof We prove that $P_0, P_1 \in \mathcal{D}$, and that $P_{\mathcal{D}}$ generate $\mathcal{D}$. We already know that $\mathrm{Hom}(P_{\mathcal{D}}, P_{\mathcal{D}}[p]) = 0$ for $p \neq 0$.

By the vanishing theorem, we have $R\Gamma(E, \mathcal{O}_E(i)) = 0$ for $i \geq -3$. Thus $R\mathrm{Hom}(P_0, \mathcal{O}_E(-2)) \cong R\Gamma(E, \mathcal{O}_E(-2)) \cong 0$. By the duality, we have

$$R\mathrm{Hom}(P_1, \mathcal{O}_E(-2)) \cong R\mathrm{Hom}_E(Q_1, \mathcal{O}_E(-2)) \cong R\mathrm{Hom}_E(\mathcal{O}_E(2), Q_1[2])^* \cong 0.$$

Hence $P_0, P_1 \in \mathcal{D}$.

There is an exact sequence

$$0 \to P_2 \to P_0 \to \mathcal{O}_E \to 0.$$

Hence $P_0, P_1, \mathcal{O}_E$ generate $D^b(\mathrm{coh}(Y))$. Since we have $\omega_Y = \mathcal{O}_Y(-L)$, we have $\mathrm{Hom}(\mathcal{O}_E, a) \cong \mathrm{Hom}(a, \mathcal{O}_E(-2)[3])^*$ for any $a \in D^b(\mathrm{coh}(Y))$ by the Serre duality. Thus $a \in \mathcal{D}$ if and only if $R\mathrm{Hom}(\mathcal{O}_E, a) = 0$. Therefore P_0, P_1 generate $\mathcal{D}$. ◻

Let

$$\mathrm{Perv}(\mathcal{D}) = \{a \in \mathcal{D} \mid \mathrm{Hom}(P_{\mathcal{D}}, a[p]) = 0 \text{ for } p \neq 0\}$$

be the heart of a t-structure of $\mathcal{D}$ defined by the tilting generator $P_{\mathcal{D}}$. We will determine the set of all simple objects in $\mathrm{Perv}(\mathcal{D})$.

We have

$$\mathrm{Hom}(s_0, s_2[1]) = \mathrm{Hom}(\mathcal{O}_E, \mathcal{O}_E(-2)[3]) \cong \mathrm{Hom}(\mathcal{O}_E, \mathcal{O}_E)^* \cong k$$

by the Serre duality. Let

$$0 \to s_2 \to s_0' \to s_0 \to 0 \tag{4.2}$$

be the corresponding extension in $\mathrm{Perv}(Y/X)$, the category of perverse coherent sheaves defined by the tilting generator P in $D^b(\mathrm{coh}(Y))$. Let $s_1' = s_1$.

Proposition 4.7 *$\{s_0', s_1'\}$ is the set of all simple objects in $\mathrm{Perv}(\mathcal{D})$ such that $\mathrm{Hom}(P_i, s_j') \cong k^{\delta_{ij}}$ for $i, j = 0, 1$.*

Proof We first prove that $s_j' \in \mathrm{Perv}(\mathcal{D})$ for $j = 0, 1$. We have

$$\mathrm{Hom}(s_2, s_2[p]) \cong \mathrm{Hom}(\mathcal{O}_E, \mathcal{O}_E[p]) \cong \begin{cases} k & \text{for } p = 0 \\ 0 & \text{for } p \neq 0. \end{cases}$$

$$\mathrm{Hom}(s_0, s_2[p]) \cong \mathrm{Hom}(\mathcal{O}_E, \mathcal{O}_E[1-p])^* \cong \begin{cases} k & \text{for } p = 1 \\ 0 & \text{for } p \neq 1. \end{cases}$$

The extension sequence (4.2) yields a long exact sequence

$$0 \to \mathrm{Hom}(s_0', s_2) \to \mathrm{Hom}(s_2, s_2) \to \mathrm{Hom}(s_0, s_2[1]) \to \mathrm{Hom}(s_0', s_2)[1] \to 0$$

where the middle homomorphism is injective, hence bijective, by definition of the long exact sequence associated to an extension. Therefore $\mathrm{Hom}(s_0', s_2[p]) \cong 0$ for all p, hence $s_0' \in \mathrm{Perv}(\mathcal{D})$. We have

$$\mathrm{Hom}(\mathcal{O}_E(-1), s_2[p]) \cong \mathrm{Hom}(s_2, \mathcal{O}_E(-1)[3-p])^* \cong 0$$

$$\mathrm{Hom}(\mathcal{O}_E, s_2[p]) \cong \begin{cases} k & \text{for } p = 1 \\ 0 & \text{for } p \neq 1. \end{cases}$$

$$\mathrm{Hom}(\mathcal{O}_E(1), s_2[p]) \cong \mathrm{Hom}(\mathcal{O}_E, \mathcal{O}_E(1)[1-p])^* \cong \begin{cases} k^2 & \text{for } p = 1 \\ 0 & \text{for } p \neq 1. \end{cases}$$

Moreover $\mathrm{Hom}(\mathcal{O}_E(1), s_2[1]) \to \mathrm{Hom}(\mathcal{O}_E^{\oplus 2}, s_2[1])$ is an isomorphism. Hence we have $R\mathrm{Hom}(s_1, s_2) \cong 0$, and $s_1 \in \mathrm{Perv}(\mathcal{D})$.

We prove that $\mathrm{Hom}(P_i, s_j') \cong k^{\delta_{ij}}$ for $i, j = 0, 1$. Then it follows that the s_j' are simple. By Proposition 4.5, we have $R\mathrm{Hom}(P_0, s_2) \cong 0$. Hence $R\mathrm{Hom}(P_0, s_0') \cong R\mathrm{Hom}(P_0, s_0) \cong k$. Since $R\mathrm{Hom}(P_1, s_j) \cong 0$ for $j = 0, 2$, we have $R\mathrm{Hom}(P_1, s_0') \cong 0$. We also have $R\mathrm{Hom}(P_0, s_1) \cong 0$ and $R\mathrm{Hom}(P_1, s_1) \cong k$. □

Proposition 4.8 (1) *$f_* P$ is not reflexive, but $f_* P_{\mathcal{D}}$ is reflexive.*
(2) *$f_* \mathcal{E}nd(P)$ and $f_* \mathcal{E}nd(P_{\mathcal{D}})$ are not Cohen–Macaulay.*

Proof (1) We have $f_*P_2 \subsetneq f_*P_2(E) \cong f_*P_0 \cong \mathcal{O}_X$. Thus f_*P_2 is an ideal sheaf whose quotient is supported at the singular point of X, hence is not reflexive. On the other hand, $f_*P_1 \cong f_*P_1(E)$, hence f_*P_1 is reflexive.

(2) We have $R^2 f_*\mathcal{E}nd(P)(K_Y) \ne 0$, because $\mathcal{H}om(P_2, P_0) \otimes \mathcal{O}_Y(K_Y) \cong \mathcal{O}_Y(-2L)$. Hence $f_*\mathcal{E}nd(P)$ is not Cohen–Macaulay.

The second statement is more subtle. We consider the exact sequence (4.1) tensored with $\mathcal{O}_Y(-L)$. Since $R\Gamma(E, \mathcal{O}_E(-2)) \cong 0$, we have $H^1(E, \mathrm{Ker}(\mathcal{O}_E(-2) \to \mathcal{O}_Q(-2))) \ne 0$ and $H^2(E, \mathrm{Ker}(\mathcal{O}_E(-2) \to \mathcal{O}_Q(-2))) = 0$. Then we deduce that $H^1(E, \mathcal{H}om(Q_1, \mathcal{O}_E(-1))) \ne 0$ and $H^2(E, \mathcal{H}om(Q_1, \mathcal{O}_E(-1))) = 0$. Hence

$$H^1(E, \mathcal{H}om(Q_1, Q_1(-2))) \ne 0, H^2(E, \mathcal{H}om(Q_1, Q_1(-2))) = 0$$

while $H^p(E, \mathcal{H}om(Q_1, Q_1(2m))) \cong 0$ for $p > 0$ and $m \ge 0$. Therefore we have

$$R^1 f_*\mathcal{H}om(P_1, P_1)(K_Y) \ne 0$$

hence $f_*\mathcal{E}nd(P_{\mathcal{D}})$ is not Cohen–Macaulay. □

Therefore $f_*\mathcal{E}nd(P_{\mathcal{D}})$ is not homologically homogeneous as already proved in [28, Example A.1]. Indeed there is no non-commutative crepant resolution in this case [29, Lemma 4.2].

5 Non-commutative deformations

We recall the theory of multi-pointed non-commutative deformations of simple collections developed in [24].

Definition 5.1 The base ring is a direct product k^r of the base field k for a positive integer r. We deform a set of objects $\{F_i\}_{i=1}^r$ in a k-linear abelian category. It is said to be a *simple collection* if $\mathrm{End}(F) \cong k^r$ with $F = \bigoplus_{i=1}^r F_i$, i.e., $\mathrm{Hom}(F_i, F_j) \cong k^{\delta_{ij}}$.

The simple collections are defined in [24] as generalizations of simple sheaves. If $r = 1$ and F is a sheaf, then a simple collection is nothing but a simple sheaf. Simple sheaves behave well under deformations. For example, a stable sheaf is a simple sheaf. A set of simple objects in a k-linear abelian category is automatically a simple collection.

Example 5.2 [7] Let $f: Y \to X = \mathrm{Spec}(R)$ be a projective birational morphism from a smooth threefold whose exceptional locus C is a smooth rational curve with normal bundle $N_{C/Y} \cong \mathcal{O}_C(-1)^{\oplus 2}$. $\{\mathcal{O}_C, \mathcal{O}_C(-1)[1]\}$ is the set of simple objects above the singular point $x \in X$ in the category of perverse coherent sheaves ${}^{-1}\mathrm{Perv}(Y/X)$. For a point $y \in Y$ above x, there is an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_y \to \mathcal{O}_C(-1)[1] \to 0$$

in ${}^{-1}\mathrm{Perv}(Y/X)$. $\mathcal{O}_y$ becomes a stable object under a suitable Bridgeland stability condition determined by the values of the central charge on the set $\{\mathcal{O}_C, \mathcal{O}_C(-1)[1]\}$, and Y is the corresponding moduli space. If we take a different Bridgeland stability condition, then we obtain the flop of Y. We refer to [27] for related topics.

Definition 5.3 The category of r-pointed Artin algebras (Art_r) consists of k^r-algebras R with augmentations:

$$k^r \xrightarrow{e} R \xrightarrow{p} k^r$$

with $p \circ e = \mathrm{Id}$, which are finite dimensional as k-vector spaces and such that the ideals $M = \mathrm{Ker}(p\colon R \to k^r)$ are nilpotent.

A (multi-pointed) *non-commutative (NC) deformation* of a simple collection $F = \bigoplus_{i=1}^r F_i$ over a parameter algebra $R \in (\mathrm{Art}_r)$ is a pair (F_R, ϕ) consisting of an object F_R with a left R-module structure and an isomorphism $\phi\colon k^r \otimes_R F_R \to F$ such that F_R is flat over R.

Let $(\hat{\mathrm{Art}}_r)$ be the category of pro-objects $\hat{R}$ of (Art_r); $\hat{R}$ is a k^r-algebra with augmentation $k^r \to \hat{R} \to k^r$ such that $R_m = \hat{R}/M^{m+1} \in (\mathrm{Art}_r)$ for all $m \ge 0$ and $\bigcap_{m>0} M^m = 0$, where $M = \mathrm{Ker}(p\colon R \to k^r)$. We denote $\hat{R} = \varprojlim R_m$.

A *formal NC deformation* of F over $\hat{R}$ is a pair $(\{F_{R_m}\}_{m\ge 0}, \{\phi_m\}_{m\ge 0})$ consisting of a series of NC deformations F_{R_m} of F over $R_m = \hat{R}/M^{m+1}$ with isomorphisms $\phi_{m+1}\colon R_m \otimes_{R_{m+1}} F_{R_{m+1}} \to F_{R_m}$ and $\phi_0\colon F_{R_0} \to F$. Any NC deformation is considered to be a special case of a formal NC deformation whose parameter algebra is finite dimensional as a k-vector space.

A formal NC deformation $(\{F_{R_m}\}, \{\phi_m\})$ of F is said to be *versal* if the following conditions are satisfied:

(1) For any NC deformation (F_R, ϕ) of F, there are an integer m, a k^r-algebra homomorphism $g\colon R_m \to R$ and an isomorphism $F_R \cong R \otimes_{R_m} F_{R_m}$ which is compatible with $\{\phi_m\}$ and ϕ.
(2) The induced homomorphism $R_m/M_{R_m}^2 \to R/M^2$ is uniquely determined by (F_R, ϕ).

Definition 5.4 An *iterated non-trivial extension* of $F = \bigoplus_{i=1}^r F_i$ is a sequence of objects $\{G^n\}_{n=0}^N$ with $G^n = \bigoplus_{i=1}^r G_i^n$ such that

(1) $G_i^0 = F_i$ for all i.
(2) For each $n < N$, there are $i_1 = i_1(n)$ and $i_2 = i_2(n)$ such that $G_i^{n+1} = G_i^n$ for $i \ne i_1$ and that there is a non-trivial extension

$$0 \to F_{i_2} \to G_{i_1}^{n+1} \to G_{i_1}^n \to 0.$$

Theorem 5.5 *Let $\{G^n\}_{n=0}^N$ be an iterated non-trivial extension of a simple collection F, and let $R = End(G^N)$. Then G^N is an NC deformation of F over R.*

The point in the above theorem is that $\dim R = r + N$ as a k-vector space. A versal deformation can be constructed by iterated universal extensions:

Theorem 5.6 [24, Proposition 4.1 and Theorem 4.8] *Let* $F = \bigoplus_{i=1}^r F_i$ *be a simple collection such that* $\dim Ext^1(F, F) < \infty$. *Define a sequence of objects* $F^{(n)} = \bigoplus_{i=1}^r F_i^{(n)}$ *by universal extensions*

$$0 \to \bigoplus_{j=1}^r Ext^1(F_i^{(n)}, F_j)^* \otimes F_j \to F_i^{(n+1)} \to F_i^{(n)} \to 0.$$

Then the $F^{(n)}$ *can be obtained by iterated non-trivial extensions of* F, *and the inverse limit* $\varprojlim F^{(n)}$ *is a versal NC deformation of* F.

We note that the above exact sequences correspond to distinguished triangles

$$F_i^{(n+1)} \to F_i^{(n)} \to \bigoplus_{j=1}^r \mathrm{Hom}(F_i^{(n)}, F_j[1])^* \otimes F_j[1] \to F_i^{(n+1)}[1].$$

6 Versal deformations

The reduced tilting generator $\bar{P}$ at the end of Sect. 2 is recovered as a versal non-commutative deformation of the simple objects in the category of perverse coherent sheaves $\mathrm{Perv}(X/Y)$:

Theorem 6.1 *Let* $f : Y \to X$ *be a projective morphism between noetherian schemes, and let* P *be a tilting generator for* f. *Assume that* $X = Spec(R)$ *for a complete local ring* R *whose residue field is isomorphic to the base field. Let* $\{P_i\}_{i=1}^m$ *and* $\{s_j\}_{j=1}^m$ *be the sets of indecomposable projective objects and simple objects in* $Perv(Y/X)$. *Set* $\bar{P} = \bigoplus_{i=1}^m P_i$ *and* $\bar{A} = f_* \mathcal{E}nd(\bar{P})$. *Then* $\bar{P}$ *is the versal deformation of the simple collection* $\bigoplus_{j=1}^m s_j$ *with the parameter algebra* $\bar{A}$.

Proof Since X is the spectrum of a complete local ring, we have $\bar{P} = \varprojlim \bar{P}/\mathfrak{m}^n \bar{P}$ for the maximal ideal $\mathfrak{m} \subset R$. Since $\bar{A}/\mathfrak{m}^n \bar{A}$ is finite dimensional as a vector space over the base field, we deduce that $\bar{P}/\mathfrak{m}^n \bar{P}$ is obtained by iterated (trivial or non-trivial) extensions of the s_j in $\mathrm{Perv}(Y/X)$ by Theorem 2.2. By construction, we have

$$\dim \mathrm{Hom}(\bar{P}, s_j) = 1, \mathrm{Hom}(\bar{P}, s_j[1]) = 0$$

for all j. We will prove our assertion using these two cohomological properties.

Let

$$\cdots \to F^{m+1} \to F^m \to \cdots \to F^1 \to F^0 = \bigoplus_{j=1}^n s_j \tag{6.1}$$

be a sequence of surjective morphisms in $\mathrm{Perv}(Y/X)$ corresponding to the iterated extensions toward $\bar{P}$ which passes through the quotients $\bar{P}/\mathfrak{m}^n \bar{P}$; we have exact sequences

$$0 \to s_{j(m)} \to F^{m+1} \to F^m \to 0$$

for each m, where $j(m)$ depends on m.

The first property $\dim \mathrm{Hom}(\bar{P}, s_j) = 1$ implies that all the extensions are non-trivial. Indeed if there is a trivial extension during the course, then there exists j and m such that $\dim \mathrm{Hom}(F^m, s_j) \geq 2$. Since $\bar{P} \to F^m$ is surjective, we deduce that $\dim \mathrm{Hom}(\bar{P}, s_j) \geq 2$, a contradiction.

By the combination with the second property $\mathrm{Hom}(\bar{P}, s_j[1]) = 0$, we deduce that the formal deformation $\bar{P}$ is versal. Indeed let $G \to F^m$ for some m be any non-trivial extension by some s_j corresponding to a non-trivial morphism $F^m \to s_j[1]$; we have

$$0 \to s_j \to G \to F^m \to 0.$$

By an exact sequence

$$\mathrm{Hom}(\bar{P}, G) \to \mathrm{Hom}(\bar{P}, F^m) \to \mathrm{Hom}(\bar{P}, \bar{s}_j[1]) = 0$$

we infer that the morphism $\bar{P} \to F^m$ can be lifted to a morphism $\bar{P} \to G$, so that $\bar{P}$ dominates this non-trivial extension. Since a versal deformation can be obtained by a sequence of iterated non-trivial extensions, we conclude that $\bar{P}$ is a versal deformation. □

In the case of Bridgeland and Van den Bergh, the non-commutative deformations in the null category $\mathcal{C}$ can be described by the following theorem, which extends and recovers [4, §6], [10, §3.2 and Lemma 3.9] who work mainly in the birational case:

Theorem 6.2 *Let $f: Y \to X$ be as in Sect. 3.*

(A) *Let $\{P_i\}_{i=0}^r$ and $\{s_j\}_{j=0}^r$ be the sets of indecomposable projective objects and simple objects in ${}^{-1}Perv(Y/X)$ as in Sect. 3 (A). Then the reduced tilting generator $P = \bigoplus_{i=0}^r P_i$ is relatively generated by global sections, i.e., the natural homomorphism $p: f^* f_* P \to P$ is surjective. Let $Q = Ker(p)$ be the kernel. Let I be the two-sided ideal of $A = f_*\mathcal{E}nd(P)$ generated by endomorphisms of P which can be factored in the form $P \to P_0 \to P$. Then the following hold:*

(1) *$Q[1] \in {}^{-1}Perv(Y/X)$, and it is the versal deformation of the simple collection $\bigoplus_{j=1}^r s_j$.*
(2) *The parameter algebra of the versal deformation $Q[1]$ is given by the following formula*

$$End(Q[1]) \cong A/I.$$

(B) *Let $\{P_i'\}_{i=0}^r$ and $\{s_j'\}_{j=0}^r$ be the sets of indecomposable projective objects and simple objects in ${}^0Perv(Y/X)$ as in Sect. 3 (B). Let $P' = \bigoplus_{i=0}^r P_i'$ be the reduced tilting generator, let $p': f^* f_* P' \to P'$ be the natural homomorphism, and let $Q' = Coker(p')$ be the cokernel. Let I' be the two-sided ideal of $A' = f_*End(P')$ generated by endomorphisms of P' which can be factored in the form $P' \to P_0' \to P'$. Then the following hold:*

(1) *$Q' \in {}^0Perv(Y/X)$, and it is the versal deformation of the simple collection $\bigoplus_{j=1}^r s'_j$.*
(2) *The parameter algebra of the versal deformation Q' is given by the following formula*

$$End(Q') \cong A'/I'.$$

Proof (A) In the exact sequence (3.1), L_i is relatively generated by global sections and $R^1 f_* \mathcal{O}_Y = 0$. Hence $\tilde{P}_i$ is also relatively generated by global sections.

We have an exact sequence

$$0 \to Q \to f^* f_* P \to P \to 0 \tag{6.2}$$

in coh(Y). We have $f_* f^* f_* P \cong f_* P$ by the projection formula. Hence $f_* Q = 0$. Since $R^1 f_* \mathcal{O}_Y = 0$ and the fiber dimension is 1, we deduce that $R^1 f_* f^* f_* P = 0$. Hence $R^1 f_* Q = 0$. Thus $Rf_* Q = 0$ and $Q[1] \in {}^{-1}\mathrm{Perv}(Y/X)$.

As in the proof of Theorem 6.1, $Q[1]/\mathfrak{m}^n Q[1]$ for any n can be expressed by a series of (trivial or non-trivial) iterated extensions of the s_j for $0 \le j \le r$. We claim that s_0 does not appear in this series. This follows from the following facts:

$$\mathrm{Hom}(P_0, Q[1]) = R^1 f_* Q = 0, \mathrm{Hom}(P_0, s_0) \cong k, \mathrm{Hom}(P_0, s_j[1]) = 0 \ \forall j.$$

Indeed if s_0 appears in the series of extensions $\{F^m\}$ as in (6.1), then we have $\mathrm{Hom}(P_0, F^{m_0}) \neq 0$ for some m_0, since $\mathrm{Hom}(P_0, s_0) \cong k$. Since $\mathrm{Hom}(P_0, s_j[1]) = 0$, the natural homomorphisms

$$\mathrm{Hom}(P_0, F^{m+1}) \to \mathrm{Hom}(P_0, F^m)$$

are surjective for all $m \ge m_0$. Then $\mathrm{Hom}(P_0, Q[1]) = \varprojlim \mathrm{Hom}(P_0, Q[1]/\mathfrak{m}^n Q[1]) \neq 0$, a contradiction.

Now we prove that $Q[1]$ is the versal deformation of the simple collection $\bigoplus_{j=1}^r s_j$. By the argument of the proof of Theorem 6.1, it is sufficient to prove the following claim:

$$\mathrm{Hom}(Q[1], s_j) \cong k, \mathrm{Hom}(Q[1], s_j[1]) = 0 \text{ for } 1 \le j \le r.$$

Since $\mathrm{Hom}(P, s_j) \cong k$ and $\mathrm{Hom}(P, s_j[q]) = 0$ for $q \neq 0$, our claim is reduced to the assertion that $\mathrm{Hom}(f^* f_* P, s_j[t]) = 0$ for $t = -1, 0$ and $1 \le j \le r$. This is equivalent to saying that $\mathrm{Hom}(f^* f_* P, \mathcal{O}_{C_j}(-1)[t]) = 0$ for $t = 0, 1$.

Let $G_1 \to G_0 \to f_* P \to 0$ be an exact sequence with free $\mathcal{O}_X$-modules G_i for $i = 0, 1$. Then we have exact sequences $f^* G_1 \to G' \to 0$ and $0 \to G' \to f^* G_0 \to f^* f_* P \to 0$ for some G'. Since $Rf_* \mathcal{O}_{C_j}(-1) \cong 0$, we have $\mathrm{Hom}(f^* G_i, \mathcal{O}_{C_j}(-1)[t]) = 0$ for all i, j, t. Then $\mathrm{Hom}(G', \mathcal{O}_{C_j}(-1)) = 0$ and we obtain our claim.

We prove the second assertion that $\mathrm{End}(Q[1]) \cong A/I$. The exact sequence (6.2) in the category $\mathrm{coh}(Y)$ of coherent sheaves defines a distinguished triangle

$$f^* f_* P \to P \to Q[1] \to f^* f_* P[1]$$

in $D^b(\mathrm{coh}(Y))$. Since $R^1 f^* f_* P \cong 0$ and $\mathrm{Hom}(f^* f_* P, \mathcal{C}) \cong \mathrm{Hom}(f_* P, f_* \mathcal{C}) \cong 0$, we have $f^* f_* P \in {}^{-1}\mathrm{Perv}(Y/X)$ by (3.2). Therefore the above distinguished triangle yields an exact sequence

$$0 \to f^* f_* P \to P \to Q[1] \to 0$$

in the abelian category ${}^{-1}\mathrm{Perv}(Y/X)$. In particular, we have a surjective morphism $h: P \to Q[1]$ in ${}^{-1}\mathrm{Perv}(Y/X)$. The induced surjective morphism $P \to Q[1] \oplus s_0$ is a natural morphism from a versal deformation of a simple collection $\bigoplus_{j=0}^r s_j$ to another non-commutative deformation. Let $h_*: \mathrm{End}(P) \to \mathrm{End}(Q[1])$ be the corresponding surjective homomorphism of the parameter rings.

Let $I_1 = \mathrm{Ker}(h_*)$. We have to prove that $I_1 = I$. If an endomorphism $g: P \to P$ factors through P_0, then the composition $h \circ g: P \to Q[1]$ vanishes, since $\mathrm{Hom}(P_0, Q[1]) = 0$. Therefore $h_*(g) = 0$, and $I \subset I_1$.

Conversely, assume that $g \in I_1$, i.e., $h_*(g) = 0$. Then there is the following commutative diagram in the category $\mathrm{coh}(Y)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q & \longrightarrow & f^* f_* P & \xrightarrow{p} & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & Q & \longrightarrow & f^* f_* P & \xrightarrow{p} & P & \longrightarrow & 0.
\end{array}$$

By the diagram chasing, we find a homomorphism $\tilde{g}: P \to f^* f_* P$ such that g is factored as $g = p \circ \tilde{g}$.

Let $G_1 \to G_0 \to f_* P \to 0$ be an exact sequence with locally free G_i as in the first part of the proof, and let $G' = \mathrm{Im}(f^* G_1 \to f^* G_0)$. We claim that the homomorphism $\tilde{g}: P \to f^* f_* P$ can be lifted to $\tilde{g}_0: P \to f^* G_0$. Indeed, since $\mathrm{Hom}(P, f^* G_1[1]) = 0$, we obtain $\mathrm{Hom}(P, G'[1]) = 0$, because the fiber dimension of f is 1 and that $f^* G_1$ and G' are sheaves. Then the homomorphism $\mathrm{Hom}(P, f^* G_0) \to \mathrm{Hom}(P, f^* f_* P)$ is surjective. Therefore g is factored through a direct sum of P_0, hence $g \in I$. Therefore $I = I_1$, and the theorem is proved.

(B) We have exact sequences

$$\begin{gathered}
0 \to H_1 \to P' \to Q' \to 0 \\
0 \to H_2 \to f^* f_* P' \to H_1 \to 0
\end{gathered} \tag{6.3}$$

in $\mathrm{coh}(Y)$ for some sheaves H_1, H_2. We have $f_* f^* f_* P' \cong f_* P'$ by the projection formula. Hence $f_* f^* f_* P' \cong f_* H_1 \cong f_* P'$. Since $R^1 f_* f^* f_* P' = 0$ and the fiber dimension is 1, we deduce that $R^1 f_* H_1 = 0$. Hence $f_* Q' = 0$. Since $R^1 f_* P' = 0$, we have $R^1 f_* Q' = 0$. Therefore $R f_* Q' = 0$ and $Q' \in {}^0\mathrm{Perv}(Y/X)$.

As in the proof of case (A), Q' can be expressed by a series of iterated extensions of the s'_j for $1 \le j \le r$, since $\mathrm{Hom}(P'_0, Q') = 0$.

Now we prove that Q' is the versal deformation of the simple collection $\bigoplus_{j=1}^r s'_j$. It is sufficient to prove the following claim:

$$\mathrm{Hom}(Q', s'_j) \cong k, \mathrm{Hom}(Q', s'_j[1]) = 0, 1 \le j \le r.$$

We have $\mathrm{Hom}(P', s'_j) \cong k$ and $\mathrm{Hom}(P', s'_j[q]) = 0$ for $q \ne 0$. Since H_1 is a quotient of a direct sum of P'_0, we have $\mathrm{Hom}(H_1, s'_j) = 0$ for $1 \le j \le r$. Therefore we have our claim and the versality of Q'.

We prove the second assertion that $\mathrm{End}(Q') \cong A'/I'$. Since $R^1 f_* H_1 = 0$, we have also $H_1 \in {}^0\mathrm{Perv}(Y/X)$ by (3.4). Hence the first exact sequence of (6.3) is an exact sequence in ${}^0\mathrm{Perv}(Y/X)$. In particular, we have a surjective homomorphism $h' \colon P' \to Q'$ in ${}^0\mathrm{Perv}(Y/X)$, which is a homomorphism of non-commutative deformations. Let $h'_* \colon \mathrm{End}(P') \to \mathrm{End}(Q')$ be the corresponding homomorphism of the parameter rings of the deformations.

Let $I'_1 = \mathrm{Ker}(h'_*)$. We have to prove that $I'_1 = I'$. If an endomorphism $g \colon P' \to P'$ factors through P'_0, then the composition $h' \circ g \colon P' \to Q'$ vanishes, since $\mathrm{Hom}(P_0, Q') = 0$. Therefore $h'_*(g) = 0$, and $I' \subset I'_1$.

Conversely, assume that $g \in I'_1$. Then there is the following commutative diagram in the category $\mathrm{coh}(Y)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1 & \longrightarrow & P' & \longrightarrow & Q' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & H_1 & \xrightarrow{p_1} & P' & \longrightarrow & Q' & \longrightarrow & 0.
\end{array}$$

By the diagram chasing, we find a homomorphism $g_1 \colon P' \to H_1$ such that g is factored as $g = p_1 \circ g_1$. Since $f_* f^* f_* P' \cong f_* H_1$ and $R^1 f_* f^* f_* P' = 0$, we have $Rf_* H_2 = 0$, hence $H_2 \in {}^0\mathrm{Perv}(Y/X)$. Thus the second sequence in (6.3) is also exact in ${}^0\mathrm{Perv}(Y/X)$. Since P' is a projective object, we have $\mathrm{Hom}(P', H_2[1]) = 0$. Hence $\mathrm{Hom}(P', f^* f_* P') \to \mathrm{Hom}(P', H_1)$ is surjective, and g_1 is lifted to $\tilde{g} \colon P' \to f^* f_* P'$.

Let $G_1 \to G_0 \to f_* P \to 0$ be an exact sequence with free sheaves G_i as in part (A) of the proof, and let $G' = \mathrm{Im}(f^* G_1 \to f^* G_0)$. Since $R^1 f_* f^* G_1 = 0$, we have $R^1 f_* G' = 0$, and $G' \in {}^0\mathrm{Perv}(Y/X)$. Then $\mathrm{Hom}(P', G'[1]) = 0$, and $\mathrm{Hom}(P', f^* G_0) \to \mathrm{Hom}(P', f^* f_* P')$ is surjective, and $\tilde{g}$ is lifted to a morphism through a direct sum of P'_0. Thus $I'_1 \subset I'$, and this completes the proof. □

Corollary 6.3 (1) *$Q \cong Q'$.*
(2) *$A/I \cong A'/I' \cong (A/I)^o$, where the last term is an opposite ring.*

Proof (1) We have $s_j \cong s'_j[1]$ for $1 \le j \le r$ in $D^b(\mathrm{coh}(Y))$. Though the non-commutative deformations of the s_j and $s'_j[1]$ are considered in different abelian categories, their deformations are the same. Indeed the extension group $\mathrm{Ext}^1(a, b)$ for $a, b \in D^b(\mathrm{coh}(Y))$ is independent of the abelian categories containing a, b. The corresponding distinguished triangles in $D^b(\mathrm{coh}(Y))$ determine the extensions. Therefore we have $Q \cong Q'$.

(2) We have $A/I \cong A'/I'$ by (1). On the other hand, we have an order-reversing bijection $A \to A'$ which sends an endomorphism $g: P \to P$ to its transpose ${}^t g: P^* \to P^*$. If g is factored as $P \to \mathcal{O}_Y \to P$, then ${}^t g$ is factored as $P^* \to \mathcal{O}_Y \to P^*$. Therefore the isomorphism $A^o \to A'$ induces an isomorphism $(A/I)^o \to A'/I'$. □

Example 6.4 [10, Example 1.3] Let $X = \mathrm{Spec} k[[u, v, x, y]]/(u^2 + v^2 y - x(x^2 + y^3))$ and let $f: Y \to X$ be a small crepant resolution. Then $A/I \cong k\langle\langle x, y\rangle\rangle/(xy + yx, x^2 - y^3)$.

The following recovers a result of Donovan and Wemyss [12, Theorem 4.7]:

Corollary 6.5 *Assume in addition that f is a birational morphism. Then f is an isomorphism outside the closed fiber if and only if the parameter algebra of the versal deformation A/I of the simple collection $\bigoplus_{j=1}^r s_j$ is finite dimensional as a vector space over the base field.*

Proof We prove that the cosupport $\mathrm{Supp}(A'/I')$ of I' coincides with the discriminant locus $D \subset X$ of f, the set of scheme theoretic points on X over which f is not an isomorphism. Then it follows that A'/I' is finite dimensional if and only if D consists of an isolated point.

If $x \notin D$, then f is an isomorphism near x. Then $f^* f_* P' \to P'$ is an isomorphism near x, and $Q' = 0$ near x. Therefore $x \notin \mathrm{Supp}(A'/I')$.

Conversely, assume that $x \in D$. Since the fiber $f^{-1}(x)$ is positive dimensional and P' has a negative degree along the fiber, it follows that P' is not generated by relative global sections. Then $Q' \neq 0$ near x, and $x \in \mathrm{Supp}(A'/I')$. □

Question 6.6 Let $C \cong \mathbf{P}^1$ be a smooth rational curve embedded in a 3-dimensional complex manifold Y. If C is contractible complex analytically by a proper bimeromorphic morphism $f: Y \to X$ which is an isomorphism on $Y \setminus C$, then the parameter algebra of the versal non-commutative deformation of C in Y is finite dimensional by the corollary. Conversely, one can ask the following question: if the parameter algebra of the versal non-commutative deformation of C in Y is finite dimensional, then is C contractible by a proper bimeromorphic morphism? An example by Clemens [9] shows that it is not sufficient to consider only commutative deformations (cf. [16]).

References

1. Bass, H.: Algebraic K-Theory. W. A. Benjamin, Inc., New York (1968)
2. Beilinson, A.A.: Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. Funkt. Anal. i Priloz. **12**, 68–69 (1978)
3. Beilinson, A.A., Bernstein, J., Deligne, P.: Faisceaux pervers. Astérisque, vol. 100. Société Mathématique de France, Paris (1982). (**in French**)
4. Bodzenta, A., Bondal, A.: Flops and Spherical Functors. arXiv:1511.00665
5. Bondal, A.: Representations of associative algebras and coherent sheaves. Izv. Akad. Nauk SSSR Ser. Mat. 53(1), 25–44 (1989) (**Russian**); translation in Math. USSR-Izv. 34 (1), 23–42 (1989)

6. Bondal, A., Van den Bergh, M.: Generators and representability of functors in commutative and non-commutative geometry. Mosc. Math. J. **3**(1), 1–36 (2003)
7. Bridgeland, T.: Flops and derived categories. Invent. Math. **147**(3), 613–632 (2002)
8. Brown, K.A., Hajarnavis, C.R.: Homologically homogeneous rings. Trans. Am. Math. Soc. **281**, 197–208 (1984)
9. Clemens, H.: The infinitesimal Abel–Jacobi mapping and moving the O(2)+O(-4) curve. Duke Math. J. **59**(1), 233–240 (1989)
10. Donovan, W., Wemyss, M.: Noncommutative deformations and flops. Duke Math. J. **165**(8), 1397–1474 (2016)
11. Donovan, W., Wemyss, M.: Twists and braids for general 3-fold flops. J. Eur. Math. Soc. **21**(6), 1641–1701 (2019)
12. Donovan, W., Wemyss, M.: Contractions and deformations. Am. J. Math. **141**(3), 563–592 (2019)
13. Donovan, W., Wemyss, M.: Noncommutative enhancements of contractions. Adv. Math. **344**, 99–136 (2019)
14. Goresky, M., MacPherson, R.: Intersection homology theory. Topology **19**(2), 135–162 (1980)
15. Hua, Z.: Contraction algebra and singularity of three-dimensional flopping contraction. Math. Z. **290**, 431–443 (2018)
16. Hua, Z., Keller, B.: Cluster Categories and Rational Curves. arXiv:1810.00749
17. Hua, Z., Toda, Y.: Contraction algebra and invariants of singularities. Int. Math. Res. Not. **2018**(10), 3173–3198 (2018)
18. Hua, Z., Zhou, G.-S.: Noncommutative Mather–Yau theorem and ita applications to Calabi–Yau algebras and homological minimal model program. arXiv:1803.06128
19. Kapranov, M.M.: On the derived category of coherent sheaves on Grassmann manifolds. Izv. Akad. Nauk SSSR Ser. Mat. **48**(1), 192–202 (1984); Math. USSR-Izv. **24**(1), 183–192 (1985)
20. Kashiwara, M.: Faisceaux constructibles et systemes holonomes d'equations aux derivees partielles lineaires a points singuliers reguliers. Sem. Goulaouic-Schwartz, expose 19 (1979–80)
21. Kawamata, Y., Matsuda, K., Matsuki, K.: Introduction to the minimal model problem. In: Algebraic Geometry Sendai 1985, Advanced Studies in Pure Mathematics, vol. 10, pp. 283–360. Kinokuniya and North-Holland (1987)
22. Kawamata, Y.: D-equivalence and K-equivalence. J. Differ. Geom. **61**(1), 147–171 (2002)
23. Kawamata, Y.: Birational geometry and derived categories. Surv. Differ. Geom. **22**(1), 291–317 (2017)
24. Kawamata, Y.: On multi-pointed non-commutative deformations and Calabi–Yau threefolds. Compos. Math. **154**, 1815–1842 (2018)
25. Mebkout, Z.: Sur le probleme de Riemann–Hilbert. Lecture Notes in Physics, vol. 126, pp. 99–110. Springer, Berlin (1980)
26. Rickard, J.: Morita theory for derived categories. J. Lond. Math. Soc. (2) **39**(3), 436–456 (1989)
27. Toda, Y.: Moduli stacks of semistable sheaves and representations of Ext-quivers. Geom. Topol **22**(5), 3083–3144 (2018)
28. Van den Bergh, M.: Three-dimensional flops and noncommutative rings. Duke Math. J. **122**(3), 423–455 (2004)
29. Van den Bergh, M. Non-commutative Crepant Resolutions. The legacy of Niels Henrik Abel, pp. 749–770, Springer, Berlin (2004)
30. Wemyss, M.: The $GL(2, \mathbf{C})$ McKay correspondence. Math. Ann. **350**(3), 631–659 (2011)