Special Kähler structures, cubic differentials and hyperbolic metrics

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Abstract

We obtain necessary conditions for the existence of special Kähler structures with isolated singularities on compact Riemann surfaces. We prove that these conditions are also sufficient in the case of the Riemann sphere and, moreover, we determine the whole moduli space of special Kähler structures with fixed singularities. The tool we develop for this aim is a correspondence between special Kähler structures and pairs consisting of a cubic differential and a hyperbolic metric.

Keywords Special Kaehler · Constant curvature metrics · Cubic differential

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1 Introduction

For reader's convenience, let us recall the definition of the affine special Kähler structure, which is the main object of study of this article.

Definition 1 [7] An *(affine) special Kähler structure* on a manifold Σ is a quadruple (g, I, ω, ∇), where (Σ, g, I, ω) is a Kähler manifold with Riemannian metric g, complex structure I, and symplectic form $\omega(\cdot, \cdot) = g(I\cdot, \cdot)$, and ∇ is a flat symplectic torsion-free connection on the tangent bundle $T\Sigma$ such that

$$\left(\nabla_X I\right) Y = \left(\nabla_Y I\right) X \tag{2}$$

holds for all vector fields X and Y.

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If I is fixed, which is always assumed to be the case below, we say for simplicity that (g, ∇) is a special Kähler structure.

The notion of a special Kähler structure has its origin in physics [6,9] and is the natural structure of the base of an algebraic integrable system [7]. In particular, algebraic integrable systems appear naturally in gauge theory [8,14,17,19,20], where a special instance of an algebraic integrable system—the Seiberg–Witten curve—plays a central rôle in the (physical) Seiberg–Witten theory. Very recently, special Kähler structures on Riemann surfaces have been extensively studied from the perspective of $\mathcal{N} = 2$ superconformal field theory, see [1] and references therein.

Examples of special Kähler structures can be found in [3,5,7,10,12,18]. An elementary introduction to special Kähler geometry on Riemann surfaces can be found in [4].

Note that a *complete* special Kähler metric is necessarily flat [2,15]. Besides, singularities of fibers of an algebraic integrable system lead to singularities of the corresponding special Kähler structure. This motivates studies of *singular* special Kähler metrics as a natural structure on bases of algebraic integrable systems.

Associated to a special Kähler structure is the period map τ, which takes values in the Siegel upper half-space [7]. If Σ is a Riemann surface, which is assumed to be the case below unless otherwise stated explicitly, τ takes values in the upper half-plane $\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$, which is endowed with the standard hyperbolic metric $g_{\mathcal{H}} = \left(\mathrm{Im}\, z\right)^{-2} |dz|^2$.

Definition 3 Let Σ be a Riemann surface. For a special Kähler structure (g, ∇) on Σ with the period map τ we call $\tilde{g} := \tau^* g_{\mathcal{H}}$ *the associated hyperbolic metric*.

Notice that τ depends on certain choices and, moreover, is defined locally only (or, equivalently, on the universal covering of Σ), however $\tilde{g}$ is well-defined. Also, $\tilde{g}$ may either degenerate or be singular at isolated points, hence, strictly speaking, $\tilde{g}$ is a metric outside of some discrete subset of Σ. This is not a concern for us, since we are interested in singular special Kähler structures, which involves singular metrics anyway.

In Definition 3 and below, a metric is said to be *hyperbolic*, if its Gaussian curvature is constant and equals -1. If $\tilde{g}$ is any hyperbolic metric on Σ, we say that $\tilde{g}$ *represents a divisor* $\Sigma_{j=1}^{n}\, (\alpha_j - 1)\, p_j$ *with* $0 \le \alpha_j \ne 1$ if the following holds: If $\alpha_j = 0$, then $\tilde{g}$ has a *cusp singularity* at p_j; If $\alpha_j > 0$, $\tilde{g}$ has a *conical singularity of order* $\alpha_j - 1$, i.e. $\tilde{g}$ has a conical angle of $2\pi\alpha_j$ at p_j (See the precise explanation in Corollary 30).

Recall [7] that for any special Kähler structure we can also construct *the associated cubic form* Ξ, which is a holomorphic section of K_{Σ}^{3}, where K_{Σ} is the canonical bundle of Σ. Throughout this manuscript we assume that Ξ is non-zero. This means that we exclude special Kähler structures (g, ∇), where g is flat and ∇ is the Levi–Civita connection of g.

Thus, to any special Kähler structure on a Riemann surface we can associate a pair $(\tilde{g}, \Xi)$ as above. Our main result, Theorem 8 below, states, roughly speaking, that for any pair $(\tilde{g}, \Xi)$ consisting of a hyperbolic metric possibly singular at isolated points and a meromorphic cubic form we can construct a special Kähler structure, whose associated hyperbolic metric and associated cubic form are $\tilde{g}$ and Ξ respectively.

The precise statement requires some notations, which are introduced next. Denote by (r, θ) the polar coordinates on $\mathbb{C}^*$, where $\theta \in (0, 2\pi)$, and put $\rho := \log r$,

$$\mathbb{1}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad I_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For any $k \in \mathbb{Z}$ and $b \in \mathbb{C}\backslash\{0\}$ the following

$$
\begin{aligned}
g_k &= -|b|r^k \log r \, |dz|^2, \\
\omega_{k,\nabla} &= \frac{1}{2}\left(kI_2 + \begin{pmatrix} \operatorname{Im}\left(\frac{b}{|b|}e^{ik\theta}\right) & -1 + \operatorname{Re}\left(\frac{b}{|b|}e^{ik\theta}\right) \\ 1 + \operatorname{Re}\left(\frac{b}{|b|}e^{ik\theta}\right) & -\operatorname{Im}\left(\frac{b}{|b|}e^{ik\theta}\right) \end{pmatrix} \rho^{-1} \right) d\theta \\
&+ \frac{1}{2}\left(k\mathbb{1}_2 + \begin{pmatrix} 1 - \operatorname{Re}\left(\frac{b}{|b|}e^{ik\theta}\right) & \operatorname{Im}\left(\frac{b}{|b|}e^{ik\theta}\right) \\ \operatorname{Im}\left(\frac{b}{|b|}e^{ik\theta}\right) & 1 + \operatorname{Re}\left(\frac{b}{|b|}e^{ik\theta}\right) \end{pmatrix} \rho^{-1} \right) d\rho
\end{aligned}
\tag{4}
$$

is a special Kähler structure on the punctured disc $\{0 < r < 1\}$ [3]. Here $\omega_{k,\nabla}$ is the connection one-form of ∇ with respect to the trivialization (∂_x, ∂_y), where $z = x + yi$ is the standard coordinate on $\mathbb{C}$. By [3, Thm. 5], (4) together with flat cones

$$g^c_\beta = r^\beta |dz|^2, \qquad \omega^c_{\beta,\nabla} = \omega_{LC} = \frac{\beta}{2}\left(\mathbb{1}_2 \, d\rho + I_2 \, d\theta\right), \tag{5}$$

where $\beta \in \mathbb{R}$, are local models of isolated singularities of affine special Kähler structures in complex dimension one provided the associated cubic form is meromorphic.

Definition 6 [3, Def. 6] We say that a special Kähler structure (g, ∇) on the punctured disc has *a conical singularity* of order $\frac{1}{2}\beta$ at the origin, if (g, ω_∇) is asymptotic to $(g^c_\beta, \omega^c_{\beta,\nabla})$. We say that (g, ∇) has *a logarithmic singularity* of order $\frac{1}{2}k$, $k \in \mathbb{Z}$, at the origin, if (g, ω_∇) is asymptotic to $(g_k, \omega_{k,\nabla})$.

Remark 7 Geometrically, (5) can be thought of as follows:

- If $\beta > -2$, (5) is a cone of total angle $\pi(\beta + 2)$.
- If $\beta = -2$, (5) is a cylindrical end with the origin at infinity.
- If $\beta < -2$, (5) is a conical end of total angle $-\pi(\beta + 2)$, where the origin is at infinity.

In other words, in the case of conical singularity the corresponding special Kähler metric is either *locally conical*, *asymptotically cylindrical*, or *asymptotically conical* respectively.

With this at hand, we can state our main result as follows.

Theorem 8 *Let Ξ be a meromorphic cubic differential on a Riemann surface Σ (not necessarily compact) with the divisor $(\Xi) = \sum_{p\in\Sigma} \operatorname{ord}_p \Xi \cdot p$. Let also $\tilde{g}$ be a hyperbolic metric on Σ representing a divisor D. Then there is a unique special Kähler structure (g, ∇) on Σ whose associated hyperbolic metric and associated cubic form are $\tilde{g}$ and Ξ respectively. Moreover, (g, ∇) is smooth on $\Sigma_0 := \Sigma\backslash\big(supp(\Xi)\cup supp D\big)$ and the following also holds:*

(i) *A cusp singularity p of $\tilde{g}$ is a logarithmic singularity of (g, ∇) of order $\frac{1}{2}(\mathrm{ord}_p \Xi + 1)$;*

(ii) *A conical singularity p of $\tilde{g}$ of order α is a conical singularity of (g, ∇) of order $\frac{1}{2}(\mathrm{ord}_p \Xi - \alpha)$;*

(iii) *(g, ∇) has a conical singularity of order $\frac{1}{2}\mathrm{ord}_p \Xi$ at a point $p \in supp(\Xi)\backslash supp D$.*

A somewhat more precise version of this result is Theorem 31, which is proved in Sect. 3.

We would like to point out that the correspondence of Theorem 8 is pretty much explicit. To demonstrate this, pick any local holomorphic coordinate z and write $\tilde{g} = e^{2v}|dz|^2$ and $\Xi = \Xi_0(z)\,dz^3$. Then the special Kähler metric of Theorem 8 is given by

$$g = 4\,|\Xi_0|e^{-v}|dz|^2.$$

Using [3, (9), (11)], one can also obtain an explicit formula for a connection one-form of ∇ in terms of v and Ξ_0. The details are provided at the end of Sect. 2.

Furthermore, pick integers $k \geq 1$, $\ell \in [0, k]$, a k–tuple $\mathfrak{p} = (p_1, \ldots, p_k)$ of pairwise distinct points on Σ as well as a k–tuple $\mathfrak{b} = (\beta_1, \ldots, \beta_k)$ of real numbers. If (g, ∇) is a special Kähler structure on Σ away from $\{p_1, \ldots, p_k\}$, then each p_j is an isolated singularity of the associated cubic form Ξ. It turns out that in general Ξ may have essential singularities at some of p_j's (see Example 32), however in the definition below, we assume that Ξ is meromorphic, i.e., each p_j is a pole of Ξ at worst.

Definition 9 We call

$$\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}) := \big\{(g, \nabla) \mid (g, \nabla) \text{ is a special Kähler structure on } \Sigma \text{ such that}$$
$$\Xi \text{ is meromorphic, } \Xi \not\equiv 0, \text{ and } \mathrm{ord}_{p_j}(g, \nabla) = \tfrac{1}{2}\beta_j \big\}/\mathbb{R}_{>0}$$

the moduli space of special Kähler structures with fixed singularities (or, simply the moduli space of special Kähler structures for short), where $\mathrm{ord}_{p_j}(g, \nabla)$ is the order of (g, ∇) at p_j, the first ℓ points of $\mathfrak{p}$ are of conical type, and the remaining points are all of logarithmic type. In particular, $\beta_{\ell+1}, \ldots, \beta_k$ are integers, if $\ell = 0$ all points are of logarithmic type, whereas for $\ell = k$ all points are of conical type. Notice that the group $\mathbb{R}_{>0}$ acts on the set of special Kähler structures via $\lambda \cdot (g, \nabla) = (\lambda g, \nabla)$.

In addition, we call

$$\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b}) := \big\{g \mid \exists \nabla \text{ such that } [g, \nabla] \in \mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}) \big\}/\mathbb{R}_{>0}$$

the moduli space of special Kähler metrics.

Notice that at this point both $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ and $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ are defined as sets only. We justify the name by introducing a topology on these sets in Sect. 5 below.

Theorem 10 *Let Σ be a compact Riemann surface of genus γ. If $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is non-empty, then the following inequalities hold:*

$$4(\gamma - 1) < \beta_1 + \cdots + \beta_k,$$
$$[\beta_1] + \cdots + [\beta_\ell] + \beta_{\ell+1} + \cdots + \beta_k \leq 6(\gamma - 1) + k - \ell, \tag{11}$$

where $[\beta]$ is the greatest integer not exceeding β.

If $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}) \neq \varnothing$, then it is homeomorphic to an open dense subset of a sphere of an odd dimension $2N + 1$. In this case the space $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is homeomorphic to a Zariski open subset of $\mathbb{C}P^N$. In the special case $\ell = k$, i.e., all singularities are of conical type, $\mathcal{M}_k^k(\mathfrak{p}, \mathfrak{b})$ is homeomorphic to S^{2N+1} and $\mathcal{R}_k^k(\mathfrak{p}, \mathfrak{b})$ is homeomorphic to $\mathbb{C}P^N$. In particular these moduli spaces are compact.

For $\Sigma = \mathbb{P}^1$ the space $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is non-empty if and only if (11) *holds. In this case,*

$$N = -6 + k - \ell - \sum_{j=1}^{k} [\beta_j]. \tag{12}$$

The proof of this theorem can be found in Sect. 5.

We also establish necessary and sufficient conditions for the existence of special Kähler structures on elliptic curves as well as describe the corresponding moduli spaces in Corollary 47.

While proving our main statements we obtain also other results, which may be of some interest. In particular, as already mentioned above we construct an example of a special Kähler structure whose associated cubic form has essential singularities, see Example 32. To the best of our knowledge this is the first example of an associated cubic form with an essential singularity.

We also describe all special Kähler structures compatible with a fixed metric, see Sect. 4.

Furthermore, let (g, ∇) be a special Kähler structure on a compact Riemann surface with finitely many prescribed singularities. Then the map which assigns to (g, ∇) the associated cubic form Ξ is injective, see Theorem 37 for a more precise statement. This is surprising, since there is no reason to believe that Ξ, which a priori encodes the difference between the Levi–Civita and the flat symplectic connections only, should determine the whole special Kähler structure (with prescribed singularities). Moreover, this is a truly global statement in the sense that the corresponding local statement is clearly false.

Finally, in the last section we construct compactifications of the moduli spaces $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ and $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ in the case $\ell < k$.

2 Preliminaries

Let $\Omega \subset \mathbb{C}$ be any domain, which is viewed as being equipped with a holomorphic coordinate $z = x + yi$ and the flat Euclidean metric $|dz|^2 = dx^2 + dy^2$. We assume that any element of $H^1(\Omega; \mathbb{R})$ can be represented by a co-closed 1-form.

Write a special Kähler metric g on Ω in the form

$$g = e^{-u}|dz|^2.$$

Using the global trivialisation of $T\Omega$ provided by the real coordinates (x, y) the connection ∇ is described by its connection 1-form $\omega_\nabla \in \Omega^1\big(\Omega; \mathfrak{gl}(2, \mathbb{R})\big)$. A computation shows [4] that ω_∇ can be written in the form

$$\omega_\nabla = \begin{pmatrix} \omega_{11} & -*\omega_{11} \\ *\omega_{22} & \omega_{22} \end{pmatrix}, \tag{13}$$

where

$$2\,\omega_{11} = e^u (dh + 2\,\psi) - du, \qquad 2\,\omega_{22} = -e^u (dh + 2\,\psi) - du. \tag{14}$$

Here $*$ denotes the Hodge star operator with respect to the flat metric, h is a smooth function, and ψ is a 1-form. These data are subject to the equation

$$\Delta h = 0, \qquad (d + d^*)\psi = 0, \qquad \Delta u = |dh + 2\psi|^2 e^{2u}, \tag{15}$$

where $\Delta = \partial_{xx}^2 + \partial_{yy}^2$. Moreover, given any triple (h, u, ψ) satisfying (15) the metric $g = e^{-u}|dz|^2$ together with ω_∇, which is given by (13) and (14), constitutes a special Kähler structure on Ω (with its complex structure inherited from $\mathbb{C}$).

If Ω is the punctured disc B_1^*, any closed and co-closed 1-form can be written as $a\varphi$, where φ is a generator of $H^1(B_1^*; \mathbb{R})$. For example, we can fix

$$\varphi = \frac{y\,dx - x\,dy}{x^2 + y^2} = -d\big(\arg(x + iy)\big).$$

Hence, a special Kähler structure on the punctured disc can be described in terms of solutions of the following equations

$$\Delta h = 0, \qquad \Delta u = |dh + a\varphi|^2 e^{2u}, \tag{16}$$

where $h, u \in C^\infty(B_1^*)$ and $a \in \mathbb{R}$.

If (h, u, a) is a solution of (16), the associated holomorphic cubic form of the corresponding special Kähler structure is

$$\Xi = \Xi_0\,dz^3 = \frac{1}{2}\Big(\frac{a}{2z} - \frac{\partial h}{\partial z}i\Big)\,dz^3.$$

Remark 17 Tracing through the description of special Kähler structures in terms of solutions of (16) as given in [4], it is easy to see that the function h is defined only up to a constant. In other words, if c is any real constant, (h, u, a) and $(h + c, u, a)$ determine equal special Kähler structures.

A straightforward computation shows that $|dh+a\varphi|^2 = 16|\Xi_0|^2 = 16|\Xi|^2$. Hence, the second equation of (16) can be written as

$$\Delta u = 16 |\Xi|^2 e^{2u}, \tag{18}$$

which implies in particular that the Gaussian curvature of the special Kähler metric $g = e^{-u}|dz|^2$ equals $8|\Xi|_g^2 \geq 0$. Furthermore, write $\Xi_0(z) = \mathring{\Xi}_0(z) + Az^{-1}$, where $A \in \mathbb{C}$ is the residue of Ξ_0 at the origin, and denote by H a primitive of $\mathring{\Xi}_0$. Notice that H is well-defined up to a constant. Define

$$h := -4 \operatorname{Im} H - 4 \operatorname{Im} A \, \log|z| \quad \text{and} \quad a := 4 \operatorname{Re} A. \tag{19}$$

Using $\partial_z \operatorname{Im} H = \frac{1}{2i}\partial_z H$ we compute

$$\frac{1}{2}\left(\frac{a}{2z} - \frac{\partial h}{\partial z}i\right) = \Xi_0(z).$$

The upshot of this computation is that Ξ_0 determines and is determined by h and a.

Slightly more generally, let $\Omega = \tilde{\Omega}\backslash\{p_1, \ldots, p_k\}$, where $\tilde{\Omega}$ is a simply connected domain in $\mathbb{C}$. Then any closed and co-closed 1-form ψ representing a non-trivial cohomology class can be written as $\sum_{j=1}^{k} a_k\varphi_k$, where $\varphi_k := -d\left(\arg\left(z - p_k\right)\right)$. It is easy to see that the above discussion can be repeated verbatim in this case too leading to the following result.

Proposition 20 *Let $\Omega = \tilde{\Omega}\backslash\{p_1, \ldots, p_k\}$, where $\tilde{\Omega}$ is a simply connected domain in $\mathbb{C}$. Any pair (u, Ξ) satisfying (18) determines a special Kähler structure on Ω such that the corresponding associated cubic form is Ξ. Conversely, any special Kähler structure on Ω determines a solution of (18).* $\square$

For the sake of clarity, let us spell the correspondence in the above proposition. Thus, if $(u, \Xi = \Xi_0 \, dz^3)$ is a solution of (18), put $g = e^{-u}|dz|^2$. Also, write $\Xi_0(z) = \mathring{\Xi}_0(z) + \sum_j A_j(z - p_j)^{-1}$, where A_j is the residue of Ξ_0 at p_j. If H is a primitive of $\mathring{\Xi}_0$, put

$$h := -4 \operatorname{Im} H - 4 \sum_{j=1}^{k}\left(\operatorname{Im} A_j\right) \log|z - p_j| \quad \text{and} \quad a_j := 4 \operatorname{Re} A_j.$$

Then the corresponding special Kähler structure is given by (13) and (14) with $\psi = \sum_j a_j\varphi_j$.

3 Special Kähler structures and the period maps

Let $(\Sigma, g, I, \omega, \nabla)$ be a special Kähler structure, where $\dim_{\mathbb{C}} \Sigma = n$. Denote by $\mathcal{U}$ the corresponding affine structure. This means that $\mathcal{U}$ is a covering of M by open sets; Moreover, each $U \in \mathcal{U}$ is equipped with a $2n$-tuple of holomorphic functions $(z_1, \ldots, z_n; w_1, \ldots, w_n)$, where $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ are conjugate special

holomorphic coordinates on U [7]. If $\tilde{U} \in \mathcal{U}$ is another open set equipped with $(\tilde{z}, \tilde{w})$, then we have a relation

$$\begin{pmatrix} z \\ w \end{pmatrix} = P \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix},$$

where $P \in Sp(2n; \mathbb{R})$ and $a, b \in \mathbb{C}^n$ are some constants.

Denote

$$\tau_{jk} = \frac{\partial w_k}{\partial z_j}.$$

Then the matrix $\tau := (\tau_{jk})$ is symmetric and $\operatorname{Im} \tau$ is positive definite. In fact, $\omega = \frac{i}{2} \sum \operatorname{Im} \tau_{jk} dz_j \wedge d\bar{z}_k$. In particular, we have a holomorphic map

$$\tau : U \to \mathcal{H}_n := \left\{ Z \in M_n(\mathbb{C}) \mid Z^t = Z, \ \operatorname{Im} Z \text{ is positive definite} \right\}$$

whose target space is the Siegel upper half-space.

Recall that the group $Sp(2n, \mathbb{R})$ acts on $\mathcal{H}_n$ via

$$P \cdot Z = (AZ + B)(CZ + D)^{-1}, \quad \text{where } P = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and the unique $Sp(2n, \mathbb{R})$-invariant metric is given by $g_{\mathcal{H}_n} = tr\left((Y^{-1}dZ)(Y^{-1}d\bar{Z})\right)$, where $Y = \operatorname{Im} Z$.

If $\tilde{\tau}$ is a map corresponding to the chart $\tilde{U}$, then the corresponding period maps are related by

$$\tau = (D\tilde{\tau} + C)(A\tilde{\tau} + B)^{-1} = \tilde{P} \cdot \tau, \quad \text{where} \quad \tilde{P} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2n, \mathbb{R}).$$

Hence, $\tau^* g_{\mathcal{H}_n}$ does not depend on the choice of an affine patch.

While the pull-back metric is defined in any dimension, the case $n = 1$ has some special features. Indeed, in this case Σ is a Riemann surface, $\mathcal{H} = \mathcal{H}_1$ is the upper half-plane so that τ is a local biholomorphism except perhaps at isolated points. Hence, $\tilde{g}$ is non-degenerate on Σ outside of some discrete subset. Moreover, the subset where $\tilde{g}$ degenerates is easy to describe, see Proposition 26 below.

More importantly, in the case $n = 1$ the metric $g_{\mathcal{H}_1}$ coincides with the standard hyperbolic metric $(\operatorname{Im} z)^{-2} |dz|^2$. Hence, the pull-back metric $\tilde{g}$ is also hyperbolic where it is non-degenerate.

Remark 21 Recall, that a holomorphic map $\tau : \Sigma \to \mathcal{H}$, which may be multi-valued, is called a developing map of a hyperbolic metric $\tilde{g}$, if $\tilde{g} = \tau^* g_{\mathcal{H}}$. Hence, the very definition yields that the period map of a special Kähler structure is a developing map of the associated hyperbolic metric.

Example 22 Consider the following local example: Σ is the punctured unit disc in $\mathbb{C}$ equipped with the metric $g = -\log |z| \, |dz|^2$, which is special Kähler. Then z is

a special holomorphic coordinate with the conjugate given by $w = 2i(z \log z - z)$. Hence, the period map is $\tau = 2i \log z$. Of course, τ is multivalued, but all values of τ are related by translations by real numbers and therefore $\tau^* g_{\mathcal{H}}$ is well defined and equals $(|z| \log |z|)^{-2}|dz|^2$, which is the standard Poincaré metric on the punctured disc.

Example 23 Let Σ be the upper half-plane $\mathcal{H}$ equipped with the following special Kähler structure [7, Rem. 1.20]

$$g = y \, |dz|^2, \qquad \omega_{\nabla} = \frac{1}{y} \begin{pmatrix} dy & dx \\ 0 & 0 \end{pmatrix},$$

where $z = x + yi$ is a coordinate on $\mathcal{H}$.

It is easy to check that $(-iz, -\frac{i}{2}z^2)$ is a pair of special holomorphic conjugate coordinates. Hence, $\tau(z) = z$, which means that $\tau^* g_{\mathcal{H}} = g_{\mathcal{H}}$.

It will be useful below to have a relation between Ξ and τ. Thus, if Z is a special holomorphic coordinate, we have

$$\Xi = \frac{1}{4} \frac{d\tau}{dZ} \, dZ^3. \tag{25}$$

Then, for an arbitrary holomorphic coordinate z we obtain

$$\Xi = \Xi_0 \, dz^3 = \frac{1}{4} \frac{d\tau}{dZ} \, dZ^3 = \frac{1}{4} \frac{d\tau}{dz} \frac{dz}{dZ} \left(\frac{dZ}{dz}\right)^3 dz^3 = \frac{1}{4} \frac{d\tau}{dz} \left(\frac{dZ}{dz}\right)^2 dz^3,$$

which yields in turn

$$\frac{d\tau}{dz} = 4 \, \Xi_0 \cdot \left(\frac{dZ}{dz}\right)^{-2}. \tag{25}$$

Notice in particular, that we have the following statement, which will be useful below.

Proposition 26 *Let p be a regular point of a special Kähler structure on a Riemann surface. Then the associated hyperbolic metric degenerates at p if and only if $\Xi(p) = 0$.* $\qquad \square$

The next result is the key ingredient in the proof of our main result, Theorem 8.

Lemma 27 *Let Ω be as in Proportion 20.*

(i) *Let $(g = e^{-u}|dz|^2, \nabla)$ be a special Kähler structure on Ω. Then the associated hyperbolic metric, which is defined on $\Omega \backslash \Xi^{-1}(0)$, is given by $\tilde{g} = e^{2v}|dz|^2$, where*

$$v = u + \log |\Xi_0| + 2 \log 2. \tag{28}$$

(ii) *Given any hyperbolic metric $\tilde{g} = e^{2v}|dz|^2$ and any holomorphic cubic form $\Xi = \Xi_0(z) \, dz^3$ on Ω, there is a unique special Kähler structure (g, ∇) on $\Omega \backslash \Xi^{-1}(0)$ such that $g = e^{-u}|dz|^2$, where u is determined by (28), and Ξ is the associated cubic form.*

Remark 29 We would like to point out that in the statement of Lemma 27, the domain Ω is allowed to have no punctures, i.e., $k = 0$ is allowed.

Proof of Lemma 27 Notice that since Ξ_0 is holomorphic, $\log|\Xi_0|$ is harmonic on $\Omega\backslash\Xi^{-1}(0)$. Since by Proportion 20 the pair (u, Ξ_0) satisfies (18), for $v := u + \log|\Xi_0| + 2\log 2$ we have

$$\Delta v = \Delta u = 16\,|\Xi_0|^2 e^{2v - 2\log|\Xi_0| - 4\log 2} = e^{2v}.$$

Hence, $\tilde{g} = e^{2v}|dz|^2$ is a metric of constant curvature -1 on $\Omega\backslash\Xi^{-1}(0)$.

Furthermore, we claim that $\tau^* g_{\mathcal{H}} = \tilde{g}$. To see this, notice that if Z is a special holomorphic coordinate (in a neighbourhood of some point), we have

$$g = e^{-u}|dz|^2 = (\text{Im }\tau)|dZ|^2 = (\text{Im }\tau)\,|\partial_z Z|^2\,|dz|^2 = (\text{Im }\tau)\frac{4\,|\Xi_0|}{|\partial_z\tau|}\,|dz|^2.$$

Here the last equality follows from (25). Hence,

$$u = \log|\partial_z\tau| - \log(\text{Im }\tau) - \log|\Xi_0| - 2\log 2 \qquad \Leftrightarrow \qquad v = \log|\partial_z\tau| - \log(\text{Im }\tau),$$

which yields in turn

$$\tilde{g} = e^{2v}|dz|^2 = \frac{|\partial_z\tau|^2}{(\text{Im }\tau)^2}|dz|^2 = \tau^* g_{\mathcal{H}_1}.$$

This clearly proves (i).

The last part, (ii), is obtained essentially by reading the above computation backwards. That is, if $\tilde{g} = e^{2v}|dz|^2$ is a metric of constant curvature -1, we have $\Delta v = e^{2v}$. Using this, it is easy to check that for any holomorhic function Ξ_0 the function

$$u := v - \log|\Xi_0| - 2\log 2$$

satisfies (18). Appealing to Proposition 20, we obtain (ii). □

Corollary 30 *Let g be a special Kähler metric on the punctured disc B_1^* such that the associated holomorphic cubic form Ξ has order $n \in \mathbb{Z}$ at the origin. Let $\tilde{g}$ be the associated hyperbolic metric. Then the following holds:*

(i) *g is conical of order $\beta/2$ if and only if $\tilde{g}$ is conical of order $n - \beta \in (-1, +\infty)$, i.e.,*

$$g = r^{\beta}(C + o(1))\,|dz|^2 \qquad \Longleftrightarrow \qquad \tilde{g} = r^{2(n-\beta)}(C' + o(1))\,|dz|^2;$$

(ii) *g has a logarithmic singularity if and only if $\tilde{g}$ has a cusp, i.e.,*

$$g = -r^{n+1}\log r\,(C + o(1))\,|dz|^2 \qquad \Longleftrightarrow \qquad \tilde{g} = \frac{C' + o(1)}{(r\log r)^2}\,|dz|^2.$$

□

The inequality $n - \beta > -1$ claimed in (i) has been established in [12, Thm. 1.1]. Of course, this also follows from the classifaction of isolated singularities for metrics of constant negative curvature.

Theorem 31 *Let Σ be a Riemann surface (not necessarily compact) and $\Sigma_0 \subset \Sigma$ be an open subset. For any holomorphic cubic form Ξ and any smooth hyperbolic metric $\tilde{g}$ on Σ_0 there is a unique special Kähler structure (g, ∇) on $\Sigma_0 \backslash \Xi^{-1}(0)$ whose associated hyperbolic metric and associated cubic form are $\tilde{g}$ and Ξ respectively.*

If Ξ is meromorphic on Σ with the divisor $(\Xi) = \sum_{p \in \Sigma} \mathrm{ord}_p \Xi \cdot p$ and $\tilde{g}$ represents a divisor D, then for the special Kähler structure (g, ∇) on $\Sigma_0 := \Sigma \backslash (supp(\Xi) \cup supp D)$ as above the following holds:

(i) *A cusp singularity p of $\tilde{g}$ is a logarithmic singularity of (g, ∇) of order $\frac{1}{2}(\mathrm{ord}_p \Xi + 1)$;*

(ii) *A conical singularity p of $\tilde{g}$ of order α is a conical singularity of (g, ∇) of order $\frac{1}{2}(\mathrm{ord}_p \Xi - \alpha)$;*

(iii) *(g, ∇) has a conical singularity of order $\frac{1}{2}\mathrm{ord}_p \Xi$ at a point $p \in supp(\Xi) \backslash supp D$.*

Proof Pick a point $p \in \Sigma$ and an open set U together with a holomorphic coordinate z centered at p. If $p \notin supp(\Xi) \cup supp D$, we may think of U as a disc $\{|z| < 1\}$. Otherwise, U can be chosen to be the punctured disc.

By Lemma 27, $\tilde{g} = e^{2v}|dz|^2$ and $\Xi = \Xi_0(z)\, dz^3$ determine a unique special Kähler structure (g, ∇) on U, where $g = e^{-u}|dz|^2$ with $u = v - \log|\Xi_0| - 2\log 2$. Moreover, u satisfies (18). Since this construction of (g, ∇) involves a local coordinate, (g, ∇) a priori depends on this choice. We prove, however, that it is in fact immaterial, i.e., different choices yield equal special Kähler structures.

To this end, choose another holomorphic coordinate $\hat{z}$ on U. If $\hat{z} = f(z)$, where f is holomorphic, the local representations $\hat{\Xi}_0(\hat{z})\, d\hat{z}^3$ and $\Xi_0(z)\, dz^3$ of Ξ are related by $\hat{\Xi}_0(\hat{z}) = \Xi_0(z)\big(f'(z)\big)^{-3}$. Also, for the flat metric $g_1 = |d\hat{z}|^2$ and the corresponding Laplacian $\Delta_1 = \partial^2_{\hat{x}\hat{x}} + \partial^2_{\hat{y}\hat{y}}$ we have

$$g_1 = |f'(z)|^2|dz|^2 \quad \text{and} \quad \Delta_1 = |f'(z)|^{-2}\Delta.$$

Multiply (18) by $|f'(z)|^{-2}$ to obtain

$$\Delta_1 u = |f'(z)|^{-2}\,|\Xi_0(z)|_0^2\, e^{2u} = |f'(z)|^4\,|\hat{\Xi}_0(\hat{z})|_0^2\, e^{2u},$$

where the subscript "0" indicates the norm induced by $|dz|^2$. Furthermore, for $\hat{u} := u + 2\log|f'(z)|$ we compute

$$\Delta_1\,\hat{u} = \Delta_1\,u = |\hat{\Xi}_0|_1^2\, e^{2\hat{u}}.$$

Hence, for the unique special Kähler structure $(\hat{g}, \hat{\nabla})$ determined by $(\hat{u}, \Xi)$ in the coordinate $\hat{z}$ as in Proposition 20, we have

$$\hat{g} = e^{-\hat{u}}|d\hat{z}|^2 = e^{-u}|dz|^2 = g.$$

Since a special Kähler metric and the associated cubic form determine the flat symplectic connection uniquely, we conclude that (g, ∇) and $(\hat{g}, \hat{\nabla})$ coincide (more precisely, this means $(g, \nabla) = f^*(\hat{g}, \hat{\nabla})$). By the construction, $(\hat{g}, \hat{\nabla})$ is the special Kähler structure determined by Ξ and the hyperbolic metric

$$\exp\!\big(2\hat{u} + 2\log|\hat{\Xi}_0(\hat{z})|\big)\,\big|d\hat{z}\big|^2 = e^{2v}|dz|^2 = \tilde{g},$$

where the above equality follows from the definition of $\hat{u}$. Thus, the choice of the local coordinate used in Proposition 20 is immaterial as claimed. This proves the existence of a special Kähler structure for given Ξ and $\tilde{g}$.

The uniqueness of the special Kähler structure corresponding to $(\tilde{g}, \Xi)$ follows immediately from the corresponding local statement. The other properties claimed follow directly from Corollary 30.   □

Example 32 (A special Kähler structure whose associated cubic form has an essential singularity) Let $\Xi(z) := e^{1/z}dz^3$ be a cubic holomorphic form on $\mathbb{C}^*$. Ξ may be thought of as a holomorphic cubic form on $\mathbb{P}^1$ with two singularities: one essential and the other one of degree -6. Pick a hyperbolic metric singular at any 3 points $w_1, w_2, w_3 \in \mathbb{P}^1$. By Theorem 31 we obtain a special Kähler structure on $\mathbb{P}^1$ with at least three and at most five singularities depending on the number of points in $\{w_1, w_2, w_3\} \cap \{0, \infty\}$ such that Ξ is the associated cubic form. To the best of our knowledge, this is the first example of a special Kähler structure whose associated cubic form has essential singularities.

Let (g, ∇) be a special Kähler structure on the punctured disc whose associated cubic form $\Xi = \Xi_0\,dz^3$ has an essential singularity at the origin. Assume for simplicity of exposition that the origin is a regular point for the associated hyperbolic metric $\tilde{g} = e^{2v}|dz|^2$. The existence of such structures follows by Theorem 31 just like in the example above. By (28) we have

$$g = e^{-u}|dz|^2 = 4e^{-2v}|\Xi_0|\,|dz|^2.$$

Notice that e^{-2v} has a positive limit at the origin, whereas by the great Picard Theorem $|\Xi_0|$ takes any positive value near the origin. Hence, in this case the behavior of g is highly irregular near the origin.

4 Special Kähler metrics versus special Kähler structures

Theorem 31 allows us to construct inequivalent special Kähler structures such that the corresponding Riemannian metrics are equal. Indeed, fix a pair $(\tilde{g}, \Xi)$ as in Theorem 31 and let g be the corresponding special Kähler metric. It is then clear from (28) that the pair $(\tilde{g}, \lambda\,\Xi)$ leads to the metric $|\lambda| \cdot g$, where $\lambda \in \mathbb{C}^*$. Hence, specializing to $|\lambda| = 1$ we obtain a family of special Kähler structures parameterized by S^1 such that all corresponding Riemannian metrics are equal.

Example 33 Fix arbitrarily a hyperbolic metric $\tilde{g}$ on the punctured unit disc B_1^*. Choose a holomorphic cubic differential Ξ on B_1^* such that Ξ is of order -3 at the origin. Observe that the leading coefficient ξ_{-3} in the expansion $\Xi_0(z) = \xi_{-3}\, z^{-3} + \xi_{-2}\, z^{-2} + \cdots$ is independent of the choice of a local coordinate. Hence, the family $\{\lambda\, \Xi \mid |\lambda| = 1\}$ consists of holomorphic cubic differentials that are pairwise inequivalent even up to a change of coordinates. Hence, for the corresponding family of special Kähler structures (g, ∇_λ) the metric is independent of λ and the corresponding structures are pairwise inequivalent.

Proposition 34 *Let Σ be a Riemann surface.*

(i) *Let (g, ∇) and $(\hat{g}, \hat{\nabla})$ be two special Kähler structures on Σ, whose associated cubic forms are Ξ and $\hat{\Xi}$ respectively. If $g = \hat{g}$, then*

$$\hat{\Xi} = \lambda \cdot \Xi, \tag{35}$$

where $\lambda \in \mathbb{C}$ is of absolute value 1;

(ii) *If (g, ∇) is a special Kähler structure on Σ whose associated cubic form is Ξ, then for each $\lambda \in S^1$ there is a unique special Kähler structure (g, ∇_λ), whose associated cubic form is $\lambda\, \Xi$.*

Proof Clearly, to prove (i) it is enough to check (35) in a neighborhood of a regular point $p \in \Sigma$. Thus, let z be a local holomorphic coordinate in a neighborhood U of p.

If (g, ∇) and $(\hat{g}, \hat{\nabla})$ are two special Kähler structures with the associated cubic forms $\Xi = \Xi_0\, dz^3$ and $\hat{\Xi} = \hat{\Xi}_0\, dz^3$ respectively such that $g = \hat{g}$, then (18) implies $|\Xi_0| = |\hat{\Xi}_0|$. Since both Ξ_0 and $\hat{\Xi}_0$ are holomorphic on U, there is $\lambda \in S^1 \subset \mathbb{C}$, such that $\Xi_0 = \lambda \cdot \hat{\Xi}_0$. This proves (i).

Claim (ii) follows from Theorem 31 by setting $\tilde{g}$ to be the associated hyperbolic metric of (g, ∇). $\qquad\square$

5 A necessary and sufficient condition for the existence of special Kähler structures on compact Riemann surfaces

Just like in the introduction, pick integers $k \geq 1$, $\ell \in [0, k]$, a k–tuple $\mathfrak{p} = (p_1, \ldots, p_k)$ of pairwise distinct points on Σ as well as a k–tuple $\mathfrak{b} = (\beta_1, \ldots, \beta_k)$ of real numbers, where $\beta_{\ell+1}, \ldots, \beta_k$ are integers. Denote

$$D = D(\mathfrak{p}, \mathfrak{b}) := -\sum_{j=1}^{\ell} [\beta_j] p_j - \sum_{j=\ell+1}^{k} (\beta_j - 1) p_j, \qquad L = L(\mathfrak{p}, \mathfrak{b}) := \mathcal{O}\big(K_\Sigma^3 + D\big), \quad \text{and}$$

$$H(\mathfrak{p}, \mathfrak{b}) := \Big\{ \Xi \in \mathrm{H}^0(L) \mid \Xi \neq 0, \ \mathrm{ord}_{p_j} \Xi = \beta_j - 1 \text{ for } \ell + 1 \leq j \leq k \Big\},$$

where K_Σ is the canonical bundle of Σ. Recall also that $[\beta]$ denotes the greatest integer not exceeding β. In other words, $H(\mathfrak{p}, \mathfrak{b})$ consists of all non-trivial meromorphic cubic differentials Ξ which are holomorphic on $\Sigma \setminus \{p_1, \ldots, p_k\}$ and satisfy

$$\mathrm{ord}_{p_j} \Xi \geq [\beta_j] \text{ if } j \leq \ell \qquad \text{and} \qquad \mathrm{ord}_{p_j} \Xi = \beta_j - 1 \text{ if } j > \ell. \tag{36}$$

For each $j > \ell$ choose a local holomorphic coordinate z_j centered at p_j and consider the holomorphic map

$$f_j \colon H^0(L) \to \mathbb{C} \quad \text{defined by} \quad f_j(\Xi) = a_j \; \Leftrightarrow \; \Xi = \left(a_j z_j^{\beta_j - 1} + \cdots\right) dz_j^3,$$

where dots denote the higher order terms. Then $H(\mathfrak{p}, \mathfrak{b})$ is the subset of $H^0(L)$ where each f_j does not vanish. Hence, $H(\mathfrak{p}, \mathfrak{b})$ is Zarisky open.

Theorem 37 *Let Σ be a compact Riemann surface of genus γ. Then $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}) \neq \varnothing$ if and only if the following two conditions hold*:

(i) $4(\gamma - 1) < \beta_1 + \cdots + \beta_k$;
(ii) $H(\mathfrak{p}, \mathfrak{b}) \neq \varnothing$.

Moreover, the map that assigns to a special Kähler structure (g, ∇) as above its associated cubic form $\Xi \in H(\mathfrak{p}, \mathfrak{b})$ is a bijection.

Proof If $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}) \neq \varnothing$, then by the quantitative relationship between the special Kähler metric and the associated cubic form in Corollary 30, the associated cubic form Ξ of any special Kähler structure (g, ∇) such that $[g, \nabla] \in \mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ lies in $H(\mathfrak{p}, \mathfrak{b})$, hence (ii) holds.

Furthermore, since $\operatorname{ord}_{p_j} \Xi - \beta_j \geq -1$, the associated hyperbolic metric $\tilde{g}$ has either a conical singularity with positive angle or a cusp at each p_j. Let $p_{k+1}, \ldots, p_m$ be further points on Σ such that $\tilde{g}$ is singular. By Corollary 30,(i) each p_j with $j \geq k+1$ is conical singularity of $\tilde{g}$ of order $\operatorname{ord}_{p_j} \Xi > 0$. Hence, by the Gauß–Bonnet theorem applied to $\tilde{g}$ we have

$$\sum_{j=1}^{k} \left(\operatorname{ord}_{p_j} \Xi - \beta_j\right) + \sum_{j=k+1}^{m} \operatorname{ord}_{p_j} \Xi + 2 - 2\gamma < 0.$$

Hence, we obtain

$$\sum_{j=1}^{k} \beta_j > \sum_{j=1}^{m} \operatorname{ord}_{p_j} \Xi + 2 - 2\gamma = 4(\gamma - 1).$$

It remains to show that (i) and (ii) yield a special Kähler structure. Indeed, notice that (i) and (ii) imply

$$\sum_{j=1}^{k} \left(\operatorname{ord}_{p_j} \Xi - \beta_j\right) + \sum_{\substack{\Xi(Q)=0 \\ Q \notin \{p_1, \ldots, p_k\}}} \operatorname{ord}_Q \Xi + 2 - 2\gamma < 0.$$

Hence, by [13] there exists a hyperbolic metric $\tilde{g}$ which has conical singularities at each zero q of Ξ of order $\operatorname{ord}_q \Xi$, and has either a conical singularity or a cusp at each p_j for all $1 \leq j \leq k$. The proof is finished by appealing to Theorem 31. □

Denote by $\pi : \mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}) \to \mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ the natural projection, which has been studied in Sect. 4. In particular, each fiber of π is isomorphic to the circle.

We have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}) & \xrightarrow{\;\Xi\;} & H(\mathfrak{p}, \mathfrak{b})/\mathbb{R}_{>0} \\[4pt]
\pi \downarrow & & \downarrow \\[4pt]
\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b}) & \xrightarrow{\;\xi\;} & H(\mathfrak{p}, \mathfrak{b})/\mathbb{C}^*,
\end{array}
\tag{38}
$$

where slightly abusing notations Ξ stays for the map, which assigns to a special Kähler structure its associated cubic form, and ξ is just the induced map. By Proportion 34 and Theorem 10 both Ξ and ξ are bijections. This can be used to define topologies on $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ and $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$. Indeed, $H(\mathfrak{p}, \mathfrak{b})$ is naturally a subset of a vector space $H^0(L)$, which can be equipped with a topology by introducing a Hermitian inner product (notice that the origin is not contained in $H(\mathfrak{p}, \mathfrak{b})$).

Proof of Theorem 10 The proof consists of the following parts.

1. The first inequality of (11) coincides with Theorem 37 (i). The second one of (11) follows by combining the following facts: $\deg K_\Sigma^3 = 6\gamma - 6$, Ξ is holomorphic outside $\{p_j : 1 \le j \le k\}$, and (36).

2. By Theorem 37, $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is homeomorphic to an open subset of the unit sphere in the complex vector space $H^0(L)$. Hence, $\dim \mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is odd if $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}) \ne \varnothing$. Likely, $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is homeomorphic to a Zariski open subset of $\mathbb{P}H^0(L)$. Moreover, if $\ell = k$, then $H(\mathfrak{p}, \mathfrak{b}) = H^0(L)\backslash\{0\}$. Hence, if $H^0(L)$ is non-trivial, $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ and $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ can be identified with $S(H^0(L))$ and $\mathbb{P}H^0(L)$ respectively.

3. We prove that the space $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is non-empty for $\Sigma = \mathbb{P}^1$ provided (11) holds. Indeed, given a $\mathbb{Z}$-divisor $D = m_1[z_1] + \cdots + m_n[z_n]$ $(z_1, \ldots, z_n \in \mathbb{C} \subset \mathbb{P}^1)$ of degree -6, the meromorphic cubic differentials whose associated divisor coincides with D have the form Const. $(z-z_1)^{m_1} \ldots (z-z_n)^{m_n} dz^3$. If $z_1 = \infty \in \mathbb{P}^1$, then the cubic differentials equals Const. $(z-z_2)^{m_2} \ldots (z-z_n)^{m_n} dz^3$. Hence, given a k-tuple of points on $\Sigma = \mathbb{P}^1$ as well as a k-tuple $\mathfrak{b} = (\beta_1, \ldots, \beta_k)$ of real numbers such that (11) holds, there exists a meromorphic cubic differential Ξ on $\mathbb{P}^1$ such that (36) holds and Ξ is holomorphic on $\mathbb{P}^1\backslash\{p_1, \ldots, p_k\}$. It follows from Theorem 37 that the space $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is non-empty.

4. Finally, by Riemann–Roch, in the case $\Sigma = \mathbb{P}^1$ the complex dimension of $H^0(L) = H^0(K_\Sigma^3 + D)$ is $N + 1$, where N is given by (12).

$\square$

Remark 39 It is clear from the proof of Theorem 10 that the case $\Sigma = \mathbb{P}^1$ is somewhat special due to the fact that it is easy to describe when $H^0(L)$ is non-trivial. In general, the non-triviality of $H^0(L)$ depends on the complex structure on Σ and the fixed singularities $(\mathfrak{p}, \mathfrak{b})$ of the special Kähler structures under consideration.

Corollary 40 *Let (g, ∇) be a special Kähler structure on $\mathbb{P}^1$ such that the associated cubic form Ξ is non-trivial and meromorphic. Then (g, ∇) must have at least three singularities.*

Proof Combining (11) and the trivial inequality $\sum_{j=1}^{k} \beta_j \leq \ell + \sum_{j=1}^{k} [\beta_j]$, we obtain

$$4(\gamma - 1) < \sum_{j=1}^{n} \beta_j \leq 6(\gamma - 1) + k.$$

Since $\gamma = 0$ for $\mathbb{P}^1$, we conclude $-4 < -6 + k$, i.e. $k \geq 3$.   □

Remark 41 For any $n \in \mathbb{Z}$ the metric $g = r^n |dz|^2$ is flat on $\mathbb{C}\backslash\{0\}$, hence, can be thought of as a special Kähler structure on $\mathbb{P}^1$ singular at most at 2 points, namely 0 and ∞. Notice that the corresponding cubic form is trivial, hence this example does not contradict Corollary 40.

Example 42 Let $\mathcal{R}_{24}^0$ denote the moduli space of all special Kähler metrics with 24 singular points all of logarithmic type of order zero. $\mathcal{R}_{24}^0$ fibers over $\mathrm{Sym}^{24}(\mathbb{P}^1)\backslash\{\text{Diagonal subset}\}$, where each fiber is homeomorphic to a Zariski open subset of $\mathbb{C}P^{18}$. Hence, $\mathcal{M}_{24}^0$ has complex dimension 42. If we also mod out by the natural action of $\mathrm{PGL}(2, \mathbb{C})$, the resulting space is of complex dimension 39. This space is of interest for elliptic K3 surfaces [11].

In what follows below we would like to describe which metrics actually appear as associated hyperbolic metrics of some special Kähler structure from $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$. Thus, let $\mathcal{R}_{\mathrm{hyp}}$ be the set of all hyperbolic metrics on Σ with isolated singularities. We have a natural map

$$T: \mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}) \to \mathcal{R}_{\mathrm{hyp}}, \qquad T(g, \nabla) = \tilde{g},$$

where $\tilde{g}$ is the hyperbolic metric associated with (g, ∇).

Proposition 43 *For any compact Riemann surface the image of T is isomorphic to $H(\mathfrak{p}, \mathfrak{b})/\mathbb{C}^*$. Moreover, if $\Sigma = \mathbb{P}^1$, the image of T consists of those hyperbolic metrics $\tilde{g}$, which satisfy the following: There exist $r \geq 0$ points $q_1, \ldots, q_r \in \mathbb{P}^1\backslash\{p_1, \ldots, p_k\}$ as well as $m \in \mathbb{Z}^k$ and $n \in \mathbb{Z}_{>0}^r$ such that the following holds:*

(a) *$\tilde{g}$ is smooth on $\mathbb{P}^1\backslash\{p_1, \ldots, p_k, q_1, \ldots, q_r\}$.*
(b) *For all $j \leq \ell$ we have $m_j > \beta_j - 1$ and*

$$\sum_{j=1}^{\ell} m_j + \sum_{j=\ell}^{k}(\beta_j - 1) + \sum_{j=1}^{r} n_j = -6. \tag{44}$$

(c) *If $j \leq \ell$, then $\mathrm{ord}_{p_j} \tilde{g} = m_j - \beta_j$.*
(d) *If $j > \ell$, then $\tilde{g}$ has a cusp singularity at p_j.*
(e) *Each q_j is a conical singularity of $\tilde{g}$ of order n_j.*

Proof Using Theorem 10, consider T as a map $T: H(\mathfrak{p}, \mathfrak{b})/\mathbb{R}_{>0} \to \mathcal{R}_{\mathrm{hyp}}$. By Proportion 34, this yields an injective map $H(\mathfrak{p}, \mathfrak{b})/\mathbb{C}^* \to \mathcal{R}_{\mathrm{hyp}}$ with the same image. Hence, we obtain that the image of T is isomorphic to $H(\mathfrak{p}, \mathfrak{b})/\mathbb{C}^*$.

Furthermore, think of $\mathbb{P}^1$ as the affine complex line $\mathbb{C}$ compactified by a point at infinity. Without loss of generality we can assume that none of p_j equals ∞. Then for $\Xi = \Xi_0 \, dz^3 \in H(\mathfrak{p}, \mathfrak{b})$ we have the following expression:

$$\Xi_0 = A(z-p_1)^{m_1} \ldots (z-p_\ell)^{m_\ell} (z-p_{\ell+1})^{\beta_{\ell+1}-1} \ldots (z-p_k)^{\beta_k-1} (z-q_1)^{n_1} \ldots (z-q_r)^{n_r}. \tag{45}$$

Here $A \neq 0$ is a constant, m_j is an integer satisfying $m_j > \beta_j - 1$, n_j is a positive integer, and $q_1, \ldots, q_r$ are those zeros of Ξ, which are not contained in $\{p_1, \ldots, p_k\}$. Moreover, (44) holds since Ξ is regular at ∞.

If $[g, \nabla] \in \mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$, where $g = e^{-u}|dz|^2$, then by (28) we obtain that $\tilde{g} = e^{2v}|dz|^2$ is smooth on $\mathbb{P}^1 \backslash \{p_1, \ldots, p_k, q_1, \ldots, q_r\}$. Moreover, Theorem 31 yields (c)–(e).

Conversely, given a hyperbolic metric $\tilde{g}$ satisfying (a)–(e), define Ξ_0 by (45) and put $\Xi := \Xi_0 \, dz^3$. Then Theorem 31 yields a special Kähler structure $[g, \nabla] \in \mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ whose associated hyperbolic metric is $\tilde{g}$. □

We note in passing that it is possible to define topologies, or even smooth structures, on $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ and $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ directly along the lines of [16]. This would then require to prove that the map Ξ is a homeomorphism, which seems to be excessive for our modest aims.

6 Existence of special Kahler metrics on Riemann surfaces with positive genera

Corollary 46 *Let Σ be a compact Riemann surface with genus $\gamma > 0$. Then $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is non-empty if and only if the following three conditions hold*:

(i) *$\beta_1 + \cdots + \beta_k > 4\gamma - 4$,*
(ii) *the line bundle $L = L(\mathfrak{p}, \mathfrak{b})$ has a non-trivial holomorphic section, and*
(iii) *for all $\ell + 1 \leq j \leq k$ we have $\dim_\mathbb{C} H^0(L) > \dim_\mathbb{C} H^0(L - p_j)$.*

Moreover, under the above conditions, $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is homeomorphic to an open dense subset of a sphere of dimension $2N + 1$, where $N := \dim_\mathbb{C} H^0(L) - 1$. The space $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is homeomorphic to a Zariski open subset of $\mathbb{C}P^N$.

Proof Suppose that these three conditions hold. By the last two ones, there exists a meromorphic cubic differential Ξ which is holomorphic outside $\{p_1, \ldots, p_k\}$ and satisfies $\operatorname{ord}_{p_j} \Xi \geq [\beta_j]$ for $1 \leq j \leq \ell$ and $\operatorname{ord}_{p_j} \Xi = \beta_j - 1$ for $\ell + 1 \leq j \leq k$. The conclusion follows from the first condition and Theorem 37.

Suppose that $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is non-empty. The first two conditions follows from Theorem 37. We show the third one by contradiction. Suppose that there exist $\ell + 1 \leq j \leq k$ such that

$$\dim_\mathbb{C} H^0(L) \leq \dim_\mathbb{C} H^0(L - p_j).$$

Then we find $H^0(L - p_j) = H^0(L)$. We pick a special Kähler structure (g, ∇) in $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ with associated cubic differential Ξ. By Theorem 37, we know that Ξ has

order $\beta_j - 1$ at p_j. On the other hand, since Ξ belongs to $H^0(L) = H^0(L - p_j)$, the order of Ξ at p_j should be greater than or equal β_j. This is a contradiction.   □

In the case Σ is an elliptic curve (compact Riemann surface of genus one), the statement of the above corollary can be made more explicit.

Corollary 47 *For an elliptic curve E the space $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is non-empty if and only if the following three conditions hold:*

(i) $\beta_1 + \ldots + \beta_k > 0$;
(ii) *The line bundle $L = L(\mathfrak{p}, \mathfrak{b})$ has a non-trivial holomorphic section;*
(iii) *If deg $L = 1$, then for all $\ell + 1 \leq j \leq k$, the divisor $-D(\mathfrak{p}, \mathfrak{b}) + p_j$ is not equivalent to zero.*

Moreover, under these conditions, $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is homeomorphic to an open dense subset of a sphere of dimension $2N + 1$, where $N := \dim_{\mathbb{C}} H^0(L) - 1$. The space $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is homeomorphic to a Zariski open subset of $\mathbb{C}P^N$.   □

Proof While a proof of this corollary could be obtained from Corollary 46, we prefer a more direct approach.

Thus, suppose there exists a special Kähler structure (g, ∇) as in the statement of this corollary. The first two condtions follow from Theorem 37 directly. Suppose deg $L = 1$. Then $H^0(L)$ has dimension one by the Riemann–Roch theorem. If there exists $\ell + 1 \leq j \leq k$ such that the divisor $-D + p_j := -D(\mathfrak{p}, \mathfrak{b}) + p_j$ is equivalent to zero, then there exists an elliptic function f such that $(f) = -D + p_j$ and $H^0(L)$ is generated by $f \, dz^3$, where dz is a nowhere vanishing holomorphic one-form on E. Furthermore, the associated cubic differential Ξ equals Const. $f \, dz^3$ and has order β_j at p_j. This is a contradiction, which finishes the proof of the "only if" part.

Assume that (i)–(iii) hold. We divide the proof of the "if" part into the following two cases.

Case 1. Suppose deg $L = 0$. Then L is trivial, in particular L has a non-trivial holomorphic section, and there exists an elliptic function f on E such that $(f) = -D$. By Theorem 37, there exists a special Kähler structure (g, ∇) whose associated cubic differential is $f \, dz^3$.

Case 2. Suppose $d := \deg L > 0$. By the Abel–Jacobi theorem, we can find a point $q \in E = \mathbb{C}/\Lambda$ and an elliptic function f on E such that $(f) = dq - D$. If $q \notin \{p_{\ell+1}, \ldots, p_k\}$, then by Theorem 37 there exists such a special Kähler structure (g, ∇) whose associated cubic differential is $f \, dz^3$, where dz is a non-where vanishing holomorphic 1-form on E. Hence, it remains to consider the case $q \in \{p_{\ell+1}, \ldots, p_k\}$.

Subcase 2.1. Suppose $d \geq 2$. Since there exist $q_1, \ldots, q_d$ in $E \setminus \{p_{\ell+1}, \ldots, p_k\}$ such that $q_1 + \cdots + q_d \equiv dq \pmod{\Lambda}$, we can find an elliptic function f on E such that $(f) = q_1 + \cdots + q_d - D$. We are done.

Subcase 2.2. Suppose $d = 1$. Then there exists $\ell + 1 \leq j \leq k$ such that $-D + p_j \sim 0$. However, this possibility is excluded by (iii).

We conclude this section by proposing a conjecture about the existence of certain cubic differentials and special Kähler structures on compact Riemann surfaces of genera greater than one.

Conjecture 48 *Let γ be an integer greater than 1 and $\mathfrak{b} = (\beta_1, \ldots, \beta_k)$ be as in Definition 9. Assume that the numerical condition (11) holds. Then there exists a compact Riemann surface Σ of genus γ and a k-tuple $\mathfrak{p} = (p_1, \ldots, p_k)$ of pairwise distinct points in Σ such that $H(\mathfrak{p}, \mathfrak{b})$—and consequently also $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$—is non-empty.*

7 On compactifications of the moduli spaces

A consequence of Theorem 10 is that the moduli spaces $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ and $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ are non-compact provided there is at least one logarithmic singularity, i.e., if $\ell < k$. The purpose of this section is to describe compactifications of these moduli spaces.

Notice first, that we have the following natural stratification

$$H^0(L) \setminus \{0\} = \bigcup_{\mathfrak{m} \in \mathbb{Z}_{\geq 0}^{k-\ell}} H(\mathfrak{p}, \mathfrak{b}_\mathfrak{m}), \tag{49}$$

where $\mathfrak{b}_\mathfrak{m} := \mathfrak{b} + (0, \mathfrak{m})$. Clearly, $H(\mathfrak{p}, \mathfrak{b}) = H(\mathfrak{p}, \mathfrak{b}_0)$ is the stratum with the highest dimension. For example, in the case $\Sigma = \mathbb{P}^1$ we have

$$\dim_\mathbb{C} H(\mathfrak{p}, \mathfrak{b}_m) = -6 + k - \ell - \sum_{j=1}^{k} [\beta_j] - \sum_{j=1}^{k-\ell} m_j$$

provided $H(\mathfrak{p}, \mathfrak{b}_m)$ is non-empty.

Moreover, this stratification is in fact finite. Indeed, if γ is the genus of Σ and $\Xi \in H(\mathfrak{p}, \mathfrak{b}_\mathfrak{m})$, then

$$6(\gamma - 1) = \sum_{p \in supp(\Xi)} \mathrm{ord}_p \Xi \geq \sum_{j=1}^{k} \mathrm{ord}_{p_j} \Xi \geq \sum_{j \leq \ell}(\beta_j - 1) + \sum_{j > \ell}(\beta_j + m_{j-\ell} - 1)$$

$$= |\mathfrak{b}| + |\mathfrak{m}| - k,$$

which implies the upper bound $|\mathfrak{m}| \leq k - 6(\gamma - 1) - |\mathfrak{b}|$.

Notice also that each stratum is invariant under the action of $\mathbb{C}^*$.

Starting from a different perspective, consider the set $\overline{\mathcal{M}}_k^\ell(\mathfrak{p}, \mathfrak{b})$ which consists of all special Kähler structures on Σ satisfying the following:

- The associated cubic form Ξ is non-trivial and meromorphic;
- For $j \leq \ell$ we have a conical singularity at p_j and $\mathrm{ord}_{p_j}(g, \nabla) = \frac{1}{2}\beta_j$;
- For $j > \ell$ we have a logarithmic singularity at p_j and $\mathrm{ord}_{p_j}(g, \nabla) \geq \frac{1}{2}\beta_j$.

Just like above, we do not distinguish two special Kähler structures if they differ by a rescaling of the metric.

Notice that the only difference in the definitions of $\overline{\mathcal{M}}_k^\ell(\mathfrak{p}, \mathfrak{b})$ and $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$ is that $\frac{1}{2}\beta_j$ is only *a lower bound* for the order of the logarithmic singularity at p_j ($j > \ell$).

Furthermore, $\overline{\mathcal{M}}_k^\ell(\mathfrak{p}, \mathfrak{b})$ admits a stratification akin to (49), namely

$$\overline{\mathcal{M}}_k^\ell(\mathfrak{p}, \mathfrak{b}) = \bigcup_{\mathfrak{m} \in \mathbb{Z}_{\geq 0}^{k-\ell}} \mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}_\mathfrak{m}) \tag{50}$$

with the top stratum being $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$. Moreover, the map Ξ extends as a bijective map

$$\bar{\Xi}\colon \overline{\mathcal{M}}_k^\ell(\mathfrak{p}, \mathfrak{b}) \to \mathrm{H}^0(L)\backslash\{0\}/\mathbb{R}_{>0}$$

such that each $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b}_\mathfrak{m})$ is mapped bijectively to $H(\mathfrak{p}, \mathfrak{b}_\mathfrak{m})$. In particular, (50) consists of a finite number of strata. Just like in the case of $\mathcal{M}_k^\ell(\mathfrak{p}, \mathfrak{b})$, we use $\bar{\Xi}$ to endow $\overline{\mathcal{M}}_k^\ell(\mathfrak{p}, \mathfrak{b})$ with a topology.

Clearly, we can also construct a compactification of $\mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b})$ in a similar manner. Namely, defining

$$\overline{\mathcal{R}}_k^\ell(\mathfrak{p}, \mathfrak{b}) := \bigcup_{\mathfrak{m}\in\mathbb{Z}_{\geq 0}^{k-\ell}} \mathcal{R}_k^\ell(\mathfrak{p}, \mathfrak{b}_\mathfrak{m})$$

we obtain a bijection

$$\bar{\xi}\colon \overline{\mathcal{R}}_k^\ell(\mathfrak{p}, \mathfrak{b}) \to \mathrm{H}^0(L)\backslash\{0\}/\mathbb{C}^* = \mathbb{P}\mathrm{H}^0(L)$$

fitting into the commutative diagram

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_k^\ell(\mathfrak{p}, \mathfrak{b}) & \xrightarrow{\;\bar{\Xi}\;} & S(\mathrm{H}^0(L)) \\
\pi \downarrow & & \downarrow \\
\overline{\mathcal{R}}_k^\ell(\mathfrak{p}, \mathfrak{b}) & \xrightarrow{\;\bar{\xi}\;} & \mathbb{P}\mathrm{H}^0(L),
\end{array}
$$

cf. (38). Here $S(\mathrm{H}^0(L))$ denotes the sphere of unite radius in $H^0(L)$ with respect to some norm.

Summarizing, we obtain the following.

Proposition 51 *The map $\bar{\Xi}$ establishes a natural bijective correspondence between $\overline{\mathcal{M}}_k^\ell(\mathfrak{p}, \mathfrak{b})$ and the unit sphere in $\mathrm{H}^0(L)$. Likely, $\bar{\xi}$ establishes a natural bijective correspondence between $\overline{\mathcal{R}}_k^\ell(\mathfrak{p}, \mathfrak{b})$ and $\mathbb{P}\mathrm{H}^0(L)$.* $\square$

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